

Commutants mod normed ideals and their Calkin algebras

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The number $k_y(\tau)$ which measures the obstruction to quasicentral approximate units is a key invariant in the study of normed ideal perturbations. Recently $\mathcal{E}(\tau; \mathcal{I})$ the commutant mod the normed ideal appears to be a structure related to the invariant. The Calkin algebras $\mathcal{E}/\mathcal{K}(\tau; \mathcal{I})$ of these commutants are relatives of the usual Calkin algebra which may share with the usual Calkin algebra and other coronas an amenability to model theoretic methods.

normed ideals of compact operators

$$(J, ||_J)$$

$$(\mathcal{C}_p, ||_p) \quad p\text{-class } ||T||_p = \left(\sum_j \lambda_j^p \right)^{1/p}$$

$$(\mathcal{C}_p^-, ||_p^-) \quad \text{Lorentz } (p, 1)$$

$$||T||_p^- = \sum_j \lambda_j j^{-1+1/p}$$

$$(\mathcal{C}_\infty^-, ||_\infty^-) \quad \text{Macaev ideal}$$

$$\lambda_1 \geq \lambda_2 \geq \dots \quad \text{eigenvalues of } (T^*T)^{1/2}$$

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the modulus of quasicentral approximation

$\mathcal{T} = (T_j)_{1 \leq j \leq n}$ bounded operators, I normed ideal

$k_{\mathcal{I}}(\mathcal{T}) =$ smallest $C \in [0, \infty]$ for which $\exists A_m \uparrow I$
 $0 \leq A_m \leq I$ finite rank, $\max_{1 \leq j \leq n} \| [A_m, T_j] \|_{\mathcal{I}} \xrightarrow{m \rightarrow \infty} C$

$\mathcal{I} = \mathcal{C}_p \quad k_p(\mathcal{T}) \quad , \quad \mathcal{I} = \mathcal{C}_p^- \quad k_p^-(\mathcal{T})$

$\mathcal{I} = \mathcal{C}_\infty^- \quad k_\infty^-(\mathcal{T})$

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general properties of $k_J(\tau)$

1° $p \longrightarrow k_p^-(\tau)$

decreasing function of $p \in [1, \infty]$

2° there is $p_0 \in [1, \infty]$ so that

$$p \in [1, p_0) \implies k_p^-(\tau) = \infty$$

$$p \in (p_0, \infty] \implies k_p^-(\tau) = 0$$

3° $p \in (1, \infty] \implies k_p(\tau) \in \{0, \infty\}$

4° $\tau - \tau' \in J \implies k_J(\tau) = k_J(\tau')$

assuming finite rank ops. dense in J

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5°. $\tau = \tau^*$ and finite rank dense in $\mathcal{I}$, then:

$$k_2(\tau) > 0$$

$$\Updownarrow \exists Y_j = Y_j^* \in \mathcal{I}^{\text{dual}}, 1 \leq j \leq n$$

$$\text{Tr} \left(\underbrace{\sum_j [T_j, Y_j]}_{\tau_1 + B(\mathcal{H})_+} \right) > 0$$

$[(\tau_\infty^-)^{\text{dual}}] = \tau_1^+$ habitat of Dixmier trace

$$\|T\|_1^+ = \sup_m \frac{\lambda_1 + \lambda_2 + \dots + \lambda_m}{1 + 1/2 + \dots + 1/m} \quad]$$

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$k_\gamma(\tau)$ a "Size J "-dimensional measure of τ

dimension p ~~—————~~ $J = \mathcal{C}_p^-$

τ n -tuple of commuting Hermitian operators

$$(k_n^-(\tau))^n = \gamma_n \int_{\mathbb{R}^n} m(\lambda) d\lambda_n(\lambda)$$

multiplicity function of Lebesgue
absolutely continuous part of
spectral measure of τ

application: τ, τ' n -tuples of commuting Hermitian

$\tau - \tau' \in \mathcal{C}_n^- \Rightarrow$ Lebesgue absolutely continuous parts
unitarily equivalent

($n=1$ case Cor. of Kato-Rosenblum Thm.)

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$p = \infty \quad \mathcal{J} = \mathcal{L}_{\infty}^-$ Macaev ideal case

$$h_{\infty}^-(z) \leq 2 \|z\| \log(zn+1) \quad (\text{finite!})$$

$$h_{\infty}^-(z \otimes I_{\mathcal{H}}) = h_{\infty}^-(z) \quad (\text{multiplicity, irrelevant!})$$

$$\mathcal{L}_{\infty}^- \subsetneq \mathcal{J} \implies h_{\mathcal{J}}(z) = 0 \quad (\text{all } z!)$$

$$\begin{array}{l} S_1, \dots, S_n \text{ isometries} \\ \text{with orthogonal ranges} \\ (\text{Cuntzian}) \end{array} \implies h_{\infty}^-(S_1, \dots, S_n) > 0 \quad \text{if } n > 1$$

h_{∞}^- and entropy

1° Kolmogorov-Sinai entropy

θ measure-preserving ergodic automorphism of
 (Ω, Σ, μ) , $\mu(\Omega) = 1$

U_{θ} induced unitary operator in L^2

Φ multiplication operators in L^2 by
 measurable functions with finite range

$$H_p(\theta) = \sup_{\substack{\varphi \subset \Phi \\ \text{finite}}} h_{\infty}^-(\varphi \cup \{U_{\theta}\})$$

$$h_p(\theta) \cong h(\theta)$$

K-S entropy

2°. Avez entropy

G group with generators $g_1, \dots, g_n$

μ finitary probability measure on G

$$h(G, \mu) > 0 \implies h_\infty(\lambda(g_1), \dots, \lambda(g_n)) > 0$$

Avez entropy of
random walk on G
defined by μ

λ left regular representation
of G on $L^2(G)$

Further results on h_∞ for Gromov hyperbolic G ,
entropy of subshifts in papers of Rui Okayasu.

G group with finite generator K

$$h_g(\lambda(K)) = \begin{cases} 0 \\ \text{finite} > 0 \\ \infty \end{cases} \quad \begin{array}{l} \text{does not depend} \\ \text{on choice of } K \end{array}$$

$$h_p(\lambda(K)) > 0 \iff G \text{ Yamasaki } p\text{-hyperbolic}$$

recent result using Kellera's - Monod-Randam The

$$h_{\infty}^-(\lambda(K)) = 0 \implies G \text{ supramenable}$$

G supramenable

J.M. Rosenblatt

$$\emptyset \neq A \subset G \Rightarrow \exists \mu: \mathcal{P}(G) \rightarrow [0, \infty]$$

μ finitely additive, $\mu(A) = 1$

μ G invariant

Kellerhals - Monod - Rørdam:

G supramenable, countable

$\Updownarrow$
there is no injective Lipschitz map

$$\varphi: F_2 \rightarrow G$$

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Problem: G supramenable $\Rightarrow k_{\infty}^{-}(\lambda(K)) = 0$?
(i.e. supramenable $\Leftrightarrow k_{\infty}^{-} = 0$)

Remark: it is possible that $k_{\infty}^{-}(\lambda(K)) = 0$
and G supramenable are both equivalent
to some subexponential growth condition on G .

some uses of $k_y(\tau)$

adaptation to normed ideals of the
Voiculescu Noncommutative Weyl-v. Neumann Thm.

A C^* -algebra with $1, X_1, \dots, X_n$ generators

ρ_1, ρ_2 $*$ -representations on $\mathcal{H}$, $\text{Ker } \rho \circ \rho_j = 0, j=1,2$

($\rho: B(\mathcal{H}) \rightarrow B/\mathcal{K}(\mathcal{H})$ canonical)

$$\underbrace{k_j(\rho_j(\{X_1, \dots, X_n\})) = 0 \quad j=1,2}$$



$$\exists U \text{ unitary } \|U \rho_1(X_k) U^* - \rho_2(X_k)\|_2 < \varepsilon, 1 \leq k \leq n$$

τ n -tuple of commuting Hermitian operators

$\mathcal{J} = \mathcal{L}_n^-$, representation of $C(\sigma(\tau))$ given
by functional calculus

spectral measure of τ Lebesgue singular

$\Leftrightarrow \exists \delta$ diagonal n -tuple, $\tau - \delta \in \mathcal{L}_n^-$.

$\mathcal{E}(\mathcal{T}; \mathcal{J})$ the commutant of $\mathcal{T}$ mod $\mathcal{J}$

$$\mathcal{T} = \mathcal{T}^* = (T_j)_{1 \leq j \leq n} \subset \mathcal{B}(\mathcal{H}), \quad (\mathcal{J}, \| \cdot \|_{\mathcal{J}})$$

$$\mathcal{E}(\mathcal{T}; \mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J}, 1 \leq j \leq n\}$$

$$\|X\|_{\mathcal{J}} = \|X\| + \max_{1 \leq j \leq n} \|[X, T_j]\|_{\mathcal{J}}$$

Banach $*$ -algebra

$$\mathcal{K}(\mathcal{T}; \mathcal{J}) = \mathcal{E}(\mathcal{T}; \mathcal{J}) \cap \mathcal{K}$$

the compact ideal of $\mathcal{E}(\mathcal{T}; \mathcal{J})$

$$\mathcal{E}/\mathcal{K}(\mathcal{T}; \mathcal{J}) = \mathcal{E}(\mathcal{T}; \mathcal{J})/\mathcal{K}(\mathcal{T}; \mathcal{J})$$

the Calkin algebra of $\mathcal{E}(\mathcal{T}; \mathcal{J})$

if $J = K$ and $C^*(\mathcal{Z}) \cap K = 0$

$\mathcal{E}/K(\mathcal{Z}; K) =$ the Paschke Dual
of $C^*(\mathcal{Z})$

in general $\mathcal{E}/K(\mathcal{Z}; J)$ is not a smooth subalgebra
of $\mathcal{E}/K(\mathcal{Z}; K)$

$\mathcal{E}/K(\mathcal{Z}; J)$ can have much richer K -theory
than $\mathcal{E}/K(\mathcal{Z}; K)$.

$K_0(\Sigma(\tau; J))$ simple examples

τ n -tuple of commuting Hermitian operators
with $\sigma(\tau) = [0, 1]^n$. Since $[0, 1]^n$ is contractible
we have $K_0(\Sigma(\tau; J)) = 0$.

$\mathcal{H}_{ac}(\tau)$ absolutely continuous part of τ
 $(\tau|_{\mathcal{H}_{ac}(\tau)})'$ commutant of $\tau|_{\mathcal{H}_{ac}(\tau)}$

$K_0((\tau|_{\mathcal{H}_{ac}(\tau)})')$ identifies with

$\mathcal{F}(\tau) \sim f: [0, 1]^n \rightarrow \mathbb{Z}$ Lebesgue measurable

$|f(x)| \leq (\text{multiplicity } \tau|_{\mathcal{H}_{ac}(\tau)})$ and

$f|_{(\text{multiplicity } \tau|_{\mathcal{H}_{ac}(\tau)})^{-1}(\infty)} = 0$

$$1^\circ \quad n=1, \quad \mathcal{J} = \mathcal{C}_1, \quad \mathcal{C} = \{T\}$$

$$K_0(\mathcal{E}(T; \mathcal{C}_1)) \cong K_0((T | \mathcal{H}_{ac}(T))')$$

P projection in $M_n(\mathcal{E}(T; \mathcal{C}_1))$

$[P]_0 \rightsquigarrow$ multiplicity function of
Lebesgue absolutely continuous
part of $P(T \otimes I_n)P$

$$2^\circ \quad n=1, \quad \mathcal{J} \neq \mathcal{C}_1 \text{ (i.e. } \mathcal{C}_1 \subsetneq \mathcal{J}), \quad \mathcal{C} = \{T\}$$

$$K_0(\mathcal{E}(T; \mathcal{J})) = 0$$

3° $n \geq 3$, $J = \mathcal{C}_n^-$, assume $\mathcal{H}_{ac}(\tau) = \mathcal{H}$

$$K_0(\xi(\tau; \mathcal{C}_n^-)) \simeq K_0((\tau)') \oplus \mathcal{X}_{\text{unknown}}$$

4° $n=2$, $J = \mathcal{C}_2$

$$K_0(\xi(\tau; \mathcal{C}_2)) \longrightarrow L'_{\text{real}}([0,1]^2, d\lambda)$$

$$[P]_0 \rightsquigarrow \mathfrak{P}(T_1 + iT_2)P \quad \text{Pinus principal function}$$

nontrivial homomorphism
infinite range

Homomorphisms in 1°, 3°, 4° "canonical"

do not change by replacing $\tau \equiv \tau' \pmod J$

$\mathcal{E}/\mathcal{K}(\mathcal{E}; \mathcal{J})$

many similarities between

$\mathcal{K}$	$\mathcal{B}$	$\mathcal{B}/\mathcal{K}$ and
$\mathcal{K}(\mathcal{E}; \mathcal{J})$	$\mathcal{E}(\mathcal{E}; \mathcal{J})$	$\mathcal{E}/\mathcal{K}(\mathcal{E}; \mathcal{J})$

I. Assume $\mathcal{R}^{*}$ dense in $\mathcal{J}$ and $\mathcal{J}^d$ and $k_{\mathcal{J}}(\mathcal{E}) = 0$
then:

$$\mathcal{E}(\mathcal{E}; \mathcal{J}) = \text{bidual of } \mathcal{K}(\mathcal{E}; \mathcal{J})$$

$$\mathcal{E}(\mathcal{E}; \mathcal{J}) = \mathcal{M}(\mathcal{K}(\mathcal{E}; \mathcal{J}))$$

* $\mathcal{R}$ denotes the finite rank operators

II. Assume J reflexive and $h_J(\tau) = 0$, then:
 $\xi(\tau; J)$ has unique predual.

III. Assume R dense in J and $h_J(\tau) = 0$, then:
 $p(\xi(\tau; J))$ is a C^* -subalgebra of B/K
 and is isometrically isomorphic to $\xi/K(\tau)$
 via the canonical algebraic isomorphism.

IV. Assume R dense in J and $k_J(z) < \infty$, then:
 $p(\mathcal{E}(z; J))$ is a C^* -algebra and is isomorphic
 as a Banach $*$ -algebra to $\mathcal{E}/\mathcal{K}(z; J)$ via
 the canonical algebraic isomorphism.
 Moreover R is dense in $\mathcal{K}(z; J)$.

Corollary 1. Assume R dense J , $k_J(z) < \infty$ then:
 $K_0(\mathcal{K}(z; J)) \simeq \mathbb{Z}$, $K_1(\mathcal{K}(z; J)) = 0$.

Corollary 2. For every n -tuple τ , $p(\mathcal{E}(\tau; \mathcal{C}_\infty^-))$ is a C^* -algebra canonically isomorphic to $\mathcal{E}/\mathcal{K}(\tau; \mathcal{C}_\infty^-)$ as a Banach $*$ -algebra. If $k_\infty^-(\tau) = 0$ the isomorphism is isometric. Moreover $\mathcal{R}$ is dense in $\mathcal{K}(\tau; \mathcal{C}_\infty^-)$.

V. Assume $\mathcal{R}$ is dense in $\mathcal{I}$ and $k_\mathcal{I}(\tau) = 0$, then $\mathcal{E}/\mathcal{K}(\tau; \mathcal{I})$ is countably degree-1 saturated C^* -algebra.
(in the sense of Farah-Hart).

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