Example: Laplace Equation Problem

University of Pennsylvania - Math 241
Umut Isik

We would like to find the steady-state temperature of the first quadrant when we keep the axes at the following temperatures:

$$u(x,0) = 1 \text{ for } 0 < x < 1$$
$$u(x,0) = 0 \text{ for } x > 1$$
$$u(0,y) = 0 \text{ for all } y > 0$$

So we need to solve the boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$
$$u(x,0) = 1 \text{ for } 0 < x < 1 \tag{2}$$
$$u(x,0) = 0 \text{ for } x > 1$$
$$u(0,y) = 0 \text{ for all } y > 0 \tag{3}$$

We shall use the Fourier transform. But since we have only half the real line as our domain (for x), we need to use the sine or cosine Fourier transform. When we apply the cosine or sine Fourier transform to the equation, we want to get a simpler differential equation for $U_c = \mathcal{F}_c\{u(x,y)\}$ (or $U_s = \mathcal{F}_s\{u(x,y)\}$ if we are taking the sine transform); where the transform is taken with respect to x. To this end, we need to see what the Fourier sine transform of the second derivative of u with respect to x is in terms of U_c (or U_s). But this is easy: we just need to use integration by parts. If f is a function that is absolutely integrable and that converges to zero as x goes to ∞ (we assume that f is absolutely integrable so that the integrals we take below exist; in general we assume all our functions are absolutely integrable – recall that absolutely integrable means that the integral of the absolute value of f on the whole interval we are concerned with exists):

$$\begin{aligned} \mathcal{F}_c\{f'\} = \int_0^\infty \cos \alpha x f'(x) dx = (f(x). \cos \alpha x)\Big|_0^\infty + \alpha \int_0^\infty f(x) \sin \alpha x dx \\ = f(0) + \alpha \mathcal{F}_s\{f\} \end{aligned} \tag{4}$$

Similarly for the sine transform:

$$\begin{aligned} \mathcal{F}_s\{f'\} = \int_0^\infty \sin \alpha x f'(x) dx = (f(x). \sin \alpha x)\Big|_0^\infty - \alpha \int_0^\infty f(x) \cos \alpha x dx \\ = -\alpha \mathcal{F}_c\{f\} \end{aligned} \tag{5}$$

Now we can combine these two to get:

$$\begin{aligned} \mathcal{F}_c\{f''\} &= f'(0) - \alpha^2 \mathcal{F}_c\{f\} \\ \text{and } \mathcal{F}_s\{f''\} &= -\alpha f(0) - \alpha^2 \mathcal{F}_s\{f\} \end{aligned} \tag{6}$$

If we used the cosine transform, i.e. we applied the cosine transform to the equation, we would get:

$$\mathcal{F}_c\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\} = \mathcal{F}_c\{\frac{\partial^2 u}{\partial x^2}\} + \mathcal{F}_c\{\frac{\partial^2 u}{\partial y^2}\} = \frac{\partial u}{\partial x}(0,y) - \alpha^2 U_c + \frac{\partial^2 U_c}{\partial y^2} = 0 \tag{7}$$

which is not very nice because we don't know what to do with the first derivative of u. Whereas:

$$\mathcal{F}_s\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\} = \mathcal{F}_s\{\frac{\partial^2 u}{\partial x^2}\} + \mathcal{F}_s\{\frac{\partial^2 u}{\partial y^2}\} = u(0,y) - \alpha^2 U_s + \frac{\partial^2 U_s}{\partial y^2} = 0 \tag{8}$$

which is much better because one of our assumptions for u in the boundary value problem is that $u(0,y) = 0$. So we should prefer the sine transform in this case. (Note that instead of this boundary problem, we looked at the problem with the y axis being isolated, we would prefer the cosine transform). Transforming also the boundary consitions using the sine transform, we have a new boundary value problem for $U = U_s$.

$$\frac{\partial^2 U}{\partial y^2} - \alpha^2 U = 0 \tag{9}$$

$$U(0,y) = 0 \tag{10}$$

$$U(\alpha,0) = \int_0^\infty u(x,0)\sin\alpha x dx = \int_0^1 50\sin\alpha x dx = 50\frac{-1}{\alpha}\sin\alpha \tag{11}$$

We know the solution to the above differential equation. It must be of the form:

$$U = c_1(\alpha)\cosh\alpha y + c_2(\alpha)\sinh\alpha y \tag{12}$$

Plugging in the boundary condition, we see that:

$$U(\alpha,0) = c_1(\alpha) = -50\frac{\sin\alpha}{\alpha} \tag{13}$$

We cannot say anything about c_2 by looking at the boundary conditions. However, we know that the function U must be bounded as $\alpha \to \infty$. We can argue that this should be true because U must physically make sense. But we can also see this mathematically. Indeed, U is the transform of an absolutely integrable function, so it must be bounded as $\alpha \to \infty$. Now, recalling the definitions:

$$\cosh(\alpha y) = \frac{e^{\alpha y} + e^{-\alpha y}}{2} \text{ and } \sinh\alpha y = \frac{e^{\alpha y} - e^{-\alpha y}}{2} \tag{14}$$

we see that the only way to have U bounded is to have the $e^{\alpha y}$'s cancel. So we must have $c_2(\alpha) = -c_1(\alpha)$. Therefore we have:

$$U(\alpha,y) = -50\frac{\sin\alpha}{\alpha}e^{-\alpha y} \tag{15}$$

Finally, taking the inverse Fourier sine transform of U to get u, we have:

$$u(x,y) = -\frac{2}{\pi}\int_0^\infty 50\frac{\sin\alpha}{\alpha}e^{-\alpha y} sin(x\alpha) d\alpha \tag{16}$$