Math 240 Extra Credit Assignment

In this assignment, we are going to prove some basic theorems in linear algebra. The exercises start fairly easy and become harder. You will find the whole exercise challenging, but if you complete it, I believe that you will feel very satisfied with it. Feel free to look in the book, or linear algebra texts for definitions and explanations for things you may not immediately understand. You can also ask me questions about the exercises. In the end, you should write your own answers and explain all the steps in your thinking in a clear way. To get credit for a question, your answers should be precise and completely correct, there will not be any partial credit for questions.

Recall that a vector space is a set with two operations on it: addition and scalar multiplication; satisfying a bunch of axioms that can be found in section 7.6 of the book. Given a vector space V, there are many ways to get other vector spaces from it. One way was to take a subset $S \subset V$ of vectors and take the span of S. The span of S is the set of all finite linear combinations of elements of S.

$$\mathrm{Span}(S) = \left\{ \lambda_1 s_1 + \dots \lambda_k s_k \;\middle|\; k \in \mathbb{N},\, \lambda_i \in \mathbb{R} \text{ and } s_i \in S \right\}$$

Exercise 1. Let V be a vector space and $S \subset V$ is a subset of V. Show that $\mathrm{Span}(S)$ is a subspace.

Exercise 2. Let V be a vector space and W a subspace. What is $\mathrm{Span}(W)$? Prove that your answer is correct.

Recall the notion of linear independence. A the elements of a subset $S \subset V$ are said to be linearly independent if there is no non-trivial finite linear combination

$$\lambda_1 s_1 + \cdots + \lambda_k s_k = 0$$

of them equalling zero. We say non-trivial because if we take all the coefficients in the linear combination to be 0, then we would trivially get 0. Also recall that S is called a basis of V if the elements of S are linearly independent and $\mathrm{Span}(S) = V$.

Exercise 3. Show that if S is a basis for V, then every element of V can be written in a unique way as a finite linear combination of elements of S.

One interesting question is whether every vector space has a basis. It is easy to see that vector spaces like $\mathbb{R}^n$ have a basis, in fact we know that $\mathbb{R}^n$ has a finite basis, consisting of n elements. Recall that the size of a basis for a vector space is the same for any to different bases for the same space. This size is called the dimension of the space. So the dimension of R^n is n. We saw, however, some vector spaces that do not have a finite basis, which means they are *infinite dimensional.* Weird.

Exercise 4 (Harder exercise)**.** Recall that the set of continuous functions on the unit interval

$$\mathcal{C} = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

is a vector space. Prove that this vector space does not have a finite basis. (hint: assume that there was a finite basis and then show that there is an element you cannot get as a linear combination of your 'basis').

Back to our question: does every vector space have a (not necessarily finite) basis? We will not prove the answer to this question in this assignment. But I want to tell you that if you want to show that every vector space has a basis, then you have to make use of an interesting thing called Zorn's lemma. Zorn's lemma is another version of something called the axiom of choice, which says that if you are given an infinite collection of sets, you can choose one element in each of them and make a set of representatives for this collection of sets. It seems obvious that you should be able to do this, but assuming this to be true leads to a proof of the funny statement that you can cut a ball into finitely many pieces and put it back together to get two balls of the same size. Weird.

Anyway, another way to make vector spaces is to take products of vector spaces. The product of two vector spaces V and W is defined to be the set:

$$V \times W = \{(v, w) \mid v \in V \text{ and } w \in W\}$$

To make $V \times W$ into a vector space, we need to define addition and scalar multiplication on it. Define $(v_1, w_1) + (v_2, w_2) = (v_1 + w_1, v_2 + w_2)$ and $\lambda(v, w) = (\lambda v, \lambda w)$. You can check that these operations satisfy all the axioms of a vector space.

Exercise 5. Knowing the dimensions of V and W, what is the dimension of $V \times W$? Prove your answer (hint: start with a basis for V and a basis for W and construct a basis for $V \times W$ and prove that it really is a basis; then count the number of elements in it).

Recall that a linear transformation between two vector spaces is a function $f : V \to W$ that respects addition and multiplication in the sense that $f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(\lambda v_1) = \lambda f(v_1)$ for every v_1, $v_2 \in V$ and $\lambda \in \mathbb{R}$. We saw in class how linear transformations between R^n and R^m correspond to $m \times n$ matrices. Recall that the kernel of a linear transformation is the soluion space to $f(v) = 0$.

$$\operatorname{Ker}(f) = \{v \in V \mid f(v) = 0\}$$

And the image is

$$\operatorname{Im}(f) = \{w \in W \mid \text{ there is a } v \in V \text{ such that } f(v) = w\}$$

Exercise 6. Show that the kernel of a linear transformation $f : V \to W$ is a subspace of V and the image of f is a subspace of W

Now we are going to define the quotient of one vector space by another. Let V be a vector space and W be a subspace of V. We are going to define a set Q which we are going to turn into a vector space in a moment. Elements of Q are written as

$$v + W$$

for elements $v \in V$. But we say that $v_1 + W$ and $v_2 + W$ are equal, $v_1 + W = v_2 + W$. if $v_1 - v_2 \in W$. So, in a way, the big W allows us to pull out elements of W from it, and two elements in Q are equal if they are equivalent in this way. So we have elements in Q being represented by elements in V as $v+W$ but more than one element in V represent the same element in Q. All those that differ by elements of W represent the same element in Q.

What is addition in Q? It is defined as you would expect:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

The issue is that we can represent the same element in Q by different elements in V as $v + W$. We need to check that this addition is well defined in the sense that picking different elements of V as representatives for the same element in Q does not change the answer of the addition. Indeed, let $v_1 + W = v_2 + W$ and $v_3 + W = v_4 + W$, then we have $v_1 = v_2 + w_1$ for some w_1 in W and $v_3 = v_4 + w_2$ for some $w_2 \in W$. Then

$$(v_1+W)+(v_3+W) = v_1+v_3+W = v_2+w_1+v_4+w_2+W = v_2+v_4+w_1+w_2+W = (v_2+v_4)+W$$

So the result of the addition does not depend on the representatives chosen. This makes the addition operation well-defined.

Define the scalar multiplication in Q as

$$\lambda(v + W) = \lambda v + W$$

Exercise 7. Show that the multiplication operation in Q is well-defined.

Q is called the quotient space and is usually denoted by V/W. Of course, we do need to check that these operations make Q into a vector space by checking all the axioms to really know that it is a vector space.

Define the map $q : V \to Q$ by sending $q(v) = v + W \in Q$.

Exercise 8. Find the kernel of the map q. Prove your answer. (you first need to see what is the 0 element in Q)

Let $f : V \to W$ be a linear transformation. You showed that the kernel is a subspace. So we now know how to construct the quotient space $V/\operatorname{Ker}(f)$. We want to define a new linear transformation $\bar{f} : V/\operatorname{Ker}(f) \to W$ by setting

$$\bar{f}(v + \operatorname{Ker}(f)) = f(v)$$

Contemplate on this definition for a moment...

Exercise 9. Show that $\bar{f}$ is well defined. That is, show that the value of $f(v + \operatorname{Ker}(f))$ does not depend on the representative v chosen for the element $v + \operatorname{Ker}(f)$. Recall that different v's can give the same element in the quotient space $V/\operatorname{Ker}(f)$. Also explain why $\bar{f}$ is a linear transformation.

$\bar{f}$ is a linear transformation from $V/\operatorname{Ker}(f)$ to W, but we can also consider it as a linear transformation $\bar{f} : V/\operatorname{Ker}(f) \to \operatorname{Im}(f)$ by forgetting about the vectors in W which are not in the image in the first place.

Here is an important statement:

Exercise 10. Show that $\bar{f} : V/\operatorname{Ker}(f) \to \operatorname{Im}(f)$ is one to one and onto. This means that $V/\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$ are equivalent as vector spaces. (the technical term is *isomorphic*).

Exercise 11 (pretty hard)**.** Show that for V a vector space and W a subspace, we have

$$\dim(V/W) = \dim(V) - \dim(W)$$

Here is our coup de gras, the rank-nullity theorem that we used in class so many times:

Exercise 12. Recall that the rank of a linear transformation is the dimension of its image. The nullity of a linear transformation is the dimension of its kernel. Using the previous exercises, show that if $f : V \to W$ is a linear transformation, then

$$\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim(V)$$