Lecture 1. May 24

- switch sections if possible (4)
- course policies, grading (Quiz 25, midterm 30, 45 final)
  - do the homework, ask questions, math help
- course overview: linear algebra, ODEs, vector calculus.

What Is Linear algebra about: Brief intro:

example of a System of linear equations

$$\begin{aligned} 30x_1 + 20x_2 + 40x_3 &= 55 \\ x_1 + 2x_2 + 4x_3 &= 4.5 \\ 6x_1 + 0x_2 + 2x_3 &= 4 \end{aligned}$$

solution: $x_1 = 1/2$, $x_2 = 1$, $x_3 = 1/2$

in matrix-vector form:

$$\underbrace{\begin{pmatrix} 30 & 20 & 40 \\ 1 & 2 & 4 \\ 6 & 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 55 \\ 4.5 \\ 4 \end{pmatrix}}_{\vec{b}}$$

we're trying to solve: $A\vec{x} = \vec{b}$.

We solved it, fine but there are other questions: most important:
- how many solutions?, also: what does the matrix do (not so interesting in this case, but will be interesting in others)

A matrix is a machine that turns vectors into other vectors
$m \times n$ matrix, takes n-vectors, gives m-vectors.

Some background and notation:

Sets: (for notation) a set is a collection of things

$A = \{a, b, c, d\}$

$P = \{\text{people in this classroom}\} = \{\text{Umut}, \dots\}$

Some sets we like: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ ⟵ complex numbers

$\mathbb{N}$: $\{0, 1, 2, \dots\}$ $\mathbb{Z}$: $\{\dots, -2, -1, 0, 1, 2, \dots\}$ $\mathbb{Q}$: fractions $\mathbb{R}$: real numbers

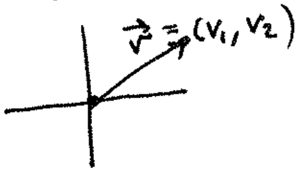
notation:

$$E = \{ x \in \mathbb{Z} \mid x = 2y \text{ for some } y \in \mathbb{Z} \}$$ even numbers

$$\text{or } E = \{ x \mid x \in \mathbb{Z} \text{ and } x = 2y \text{ for some } y \in \mathbb{Z} \}$$

Vectors: arrows from the origin or n-tuples of numbers

when you draw:



when you write:

$$(v_1, v_2) \in \mathbb{R}^2$$

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$$

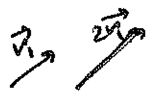
$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

"n-space" ⟶ $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, \dots, n\}$

- you can add vectors 

$$\vec{v_1} + \vec{v_2} = (x_1 + x_2, y_1 + y_2)$$

for $\vec{v_1} = (x_1, y_1)$ $v_2 = (x_2, y_2)$

- multiply them by scalar 

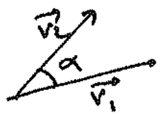
$$2\vec{v_1} = (2x_1, 2y_1)$$

- take dot product of two of these

$$\vec{v_1} \cdot \vec{v_2} = x_1 x_2 + y_1 y_2$$

Same for n-vectors.

can also write vectors as $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ etc.



$\vec{v_1} \cdot \vec{v_2} = \|\vec{v_1}\| \cdot \|\vec{v_2}\| \cos\alpha$

length of a vector. $\|v_1\| = \sqrt{x_1^2 + y_1^2}$ distance to the origin.

$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

"Standard basis" we'll explain

$$\underbrace{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix}0\\0\\1\\0\\\vdots\\0\end{pmatrix}}_{e_3}, \dots, \underbrace{\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}}_{e_n}$$

can write every vector uniquely in terms of these.

What is $\mathbb{R}^n$ (similarly $\mathbb{Q}^n, \mathbb{C}^n$), the things you can do with it (add vectors and multiply w/ scalar) are packaged in this def:

A vector space over $\mathbb{R}$ is a set V where we can

- add elements of V
- multiply elements of V with numbers in $\mathbb{R}$

to get elements in V.

Such that the following properties hold:

(1) $x + y = y + x$ (commutativity, but forget about all these names, they don't matter)

(2) $x + (y + z) = (x + y) + z$ (associativity)

(3) there is a $\vec{0}$ vector. $\vec{0} + x = x + \vec{0} = x$ for all x

(4) for each $x \in V$, there is $-x$ s.t. $x + (-x) = \vec{0}$

(5) for each $k \in \mathbb{R}$, $x \in V$, $y \in V$, $k(x+y) = kx + ky$

(6) $(k_1 + k_2)x = k_1 x + k_2 x$

(7) $k_1(k_2 x) = (k_1 k_2) x$

(8) $1x = x$

Similarly, we have vector spaces over $\mathbb{Q}$, $\mathbb{C}$ just replace $k \in \mathbb{R}$ with $k \in \mathbb{Q}$ or $\mathbb{C}$.

just some big definition. what matters is that everything we do with $\mathbb{R}^n$ will work with all the other examples of vector spaces.

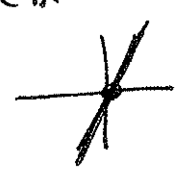
<u>examples</u> · $\mathbb{R}^n$ check the axioms. · $\mathbb{C}$ is a vector space over $\mathbb{R}$.

- set P of all polynomials
- set P_2 of all polynomials of order 2
- $\{f : \mathbb{R} \to \mathbb{R}\}$ set of functions $\mathbb{R} \to \mathbb{R}$
- $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\} =: \mathcal{C}^0(\mathbb{R})$
- same for differentiable :

A subspace $W \subset V$ is a subset that is closed under addition and multiplication.

- $P \subset \{f : \mathbb{R} \to \mathbb{R}\}$ (last example)
- line in $\mathbb{R}^2$ $\{(x,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$

- another line $\{(x,2x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$



- not a subspace:

- there is no 0
- addition is not "closed"
- scalar mult is not closed

or

- scalar multiplication is not closed.

<u>Span</u>: The span of vectors $\vec{v_1}, \dots, \vec{v_k} \in \mathbb{R}^n$ is the set of all linear combinations of $\vec{v_1}, \dots, \vec{v_k}$

$$\operatorname{Span}\{\vec{v_1}, \dots, \vec{v_k}\} = \left\{ \sum_{i=1}^{n} \lambda_i \vec{v_i} \;\middle|\; \lambda_i \in \mathbb{R} \text{ for all } i = 1, \dots, k \right\}$$

examples: $\mathrm{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} = \left\{\begin{pmatrix}\lambda_1\\\lambda_1\end{pmatrix} \middle| x \in \mathbb{R}\right\} \subset \mathbb{R}^2$

$$\mathrm{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}\lambda_1\\\lambda_2\\0\end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{R}\right\} \subset \mathbb{R}^3$$

the xy plane.

$\mathrm{Span}\{e_1, \dots, e_n\} = \mathbb{R}^n$. $\quad \mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} = \mathbb{R}^2$

$\mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}5\\7\end{pmatrix}\right\} = \mathbb{R}^2$

$\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$ is always a subspace

Linear independence:

$\vec{v}_1, \dots, \vec{v}_n$ are said to be linearly independent if the only values of $\lambda_1, \dots, \lambda_n$ satisfying

$$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n = 0$$

are $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

examples:

- $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent.

  indeed: $\lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \implies \lambda_1 = 0$ and $\lambda_2 = 0$

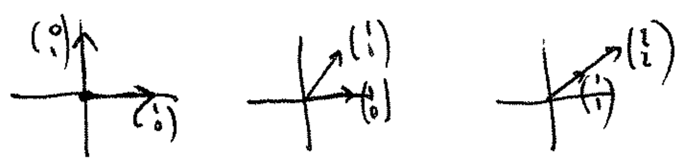
- $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ are also linearly independent

  $\lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\lambda_1 + \lambda_2\\\lambda_2\end{pmatrix} = 0 \implies \lambda_1 = 0$ and $\lambda_2 = 0$

- $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\2\end{pmatrix}$ are not linearly independent

  since $-2\begin{pmatrix}1\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0$

pictures:

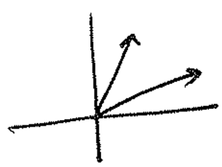


Remarks: (1) Always think of the pictures!

(2) ~~inner~~ dot product not part of a vector space that's extra.

A basis for a vector space is a subset of elements that are linearly independent and span the whole space

Span $\{\vec{v}_1, \ldots, \vec{v}_n\} = \mathbb{R}^n$ and $\vec{v}_1, \ldots, \vec{v}_n$ linearly independent.

picture:  ← basis:

- I can get every element
- no redundancy.

that is $\vec{v}_1, \ldots, \vec{v}_n$ is a basis if I can write every element in a unique way in terms of $\vec{v}_1, \ldots, \vec{v}_n$.

(explain)

The dimension of a vector space is the size of a basis.

$\mathbb{R}^n$ has dimension n.

- what's a basis for $\mathbb{R}^n$?
- what's a basis for polynomials of degree 2?
- what's a basis for all polynomials?