Lecture 3 26 May

- homework session time/place.

Last time:
- linear independence, span, basis
- using row operations to solve systems of linear equations (which was the same as manipulating the equations)

one more example (see notes from last lecture)

Linear functions: Recall: matrices give functions $\mathbb{R}^n \to \mathbb{R}^m$ (for an $m \times n$ matrix)

A linear function $f: \mathbb{R}^n \to \mathbb{R}^m$ is a function $f: \mathbb{R}^n \to \mathbb{R}^m$ such that, for all $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

$$f(v+w) = f(v) + f(w)$$
$$f(\lambda v) = \lambda f(v)$$

We saw that matrices give linear functions $f(v) = A.v$

we have
$$A(v+w) = Av + Aw$$
$$A(\lambda v) = \lambda Av .$$

Say we have a linear function $f: \mathbb{R}^n \to \mathbb{R}^m$. Knowing what it does to the standard basis (or any other basis) will tell us what it does to all vectors. We saw last time that every vector can be described uniquely in terms of basis elements (stick with the standard basis for now $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n$

For every $v \in \mathbb{R}^n$, we have $\left(v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right)$

$$v = a_1 e_1 + b_2 e_2 + \ldots + a_n e_n$$

To find $f(v)$, assuming we know $f(e_1), f(e_2), \ldots, f(e_n)$ we use linearity

$$f(v) = f(a_1 e_1 + \ldots + a_n e_n) = a_1 f(e_1) + a_2 f(e_2) + \ldots + a_n f(e_n)$$

now, we are going to write this down as a matrix, that is we are going to write A such that $f(v) = Av$.

First, an observation:

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Look at $A e_1 = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$

similarly $Ae_i =$ ith column of A

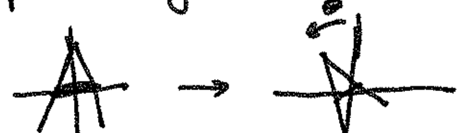
So if we set A to be the matrix with columns $f(e_1), \dots, f(e_n)$ we would have $Ae_1 = f(e_1)$, $Ae_2 = f(e_2), \dots, Ae_n = f(e_n)$

But then
$$\begin{aligned} Av = A(b_1 e_1 + \dots + b_n e_n) &= b_1 A e_1 + \dots + b_n A e_n \\ &= b_1 f(e_1) + \dots + b_n f(e_n) \\ &= f(b_1 e_1 + \dots + b_n e_n) \\ &= f(v) \end{aligned}$$

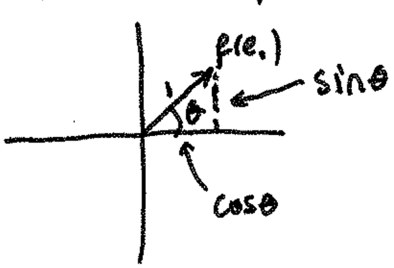
for every $v = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

So A is the matrix that gives the linear function f.

<u>eg.</u>: Look at the rotation matrix from last time. We'll write it down now.
Say f rotates the plane by θ in the counterclockwise (positive) direction.



Look at $f(e_1)$ $\quad f(e_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. $\quad f(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$



so rotation is given by
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We found that we can get linear functions from matrices and matrices from linear functions. So there is a 1-1 correspondence

$$\{\text{linear functions } \mathbb{R}^n \to \mathbb{R}^m\} \longleftrightarrow \{m \times n \text{ matrices}\}$$

This is how matrices come up in mathematics.
We'll learn to use matrices for various things but this is why we study them.

If we have linear functions $f: \mathbb{R}^p \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^s$ and ~~[illegible]~~ we have ~~[illegible]~~ matrices $\underset{m\times p}{B}$ for f and $\underset{s\times m}{A}$ for g

if we compose the functions $g \circ f(v) = g(f(v))$ the matrix that corresponds to this is the product of A and B

$$\underset{s\times m}{A} \cdot \underset{m\times p}{B} \quad \text{is an } s\times p \text{ matrix given by}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{sm} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mp} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} & \cdots \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2m}b_{m2} & \cdots \\ \vdots & \vdots & \\ a_{s1}b_{11} + a_{s2}b_{21} + \cdots + a_{sm}b_{m1} & a_{s1}b_{12} + a_{s2}b_{22} + \cdots + a_{sm}b_{m2} & \cdots \end{pmatrix}$$

the (i,j)th place in $A \cdot B$ is the dot product of the ith row of A and the jth column of B.

eg:
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1.1 + 2.2 & 1.(-1) + 2.1 \\ 3.1 + 4.2 & 3.(-1) + 4.1 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 1 \\ 11 & 1 \end{pmatrix}$$

- We can also add matrices. Say $A = (a_{ij})$ $B = (b_{ij})$ both $m \times n$ matrices, then

$$A + B = (a_{ij} + b_{ij}) \qquad A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \cdots \\ a_{21} + b_{21} & a_{22} + b_{22} \cdots \\ \vdots & \vdots \end{pmatrix}$$

just add them componentwise.

- We can multiply a matrix by a scalar.

$$\lambda A = (\lambda a_{ij}) = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \cdots & \lambda a_{m} \\ \vdots & \vdots & \\ \lambda a_{m1} & \lambda a_{m2} \cdots & \lambda a_{mn} \end{pmatrix}$$

note that matrix addition is commutative. but matrix multiplication is not. Look at $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
← first switch x and y then project onto x axis.

$$B \cdot A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
← project onto x axis then swich x and y

picture!

Transpose of a matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

A^T has the entry a_{ji} for the i,j th place

eg: $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Properties:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ (change order!!!)
- $kA^T = (kA)^T$

Rank of a matrix :

For a linear function $f: \mathbb{R}^n \to \mathbb{R}^m$ (which we now know corresponds to a matrix A which is $m \times n$), define the *image* to be the set of vectors in $\mathbb{R}^m$ that come from vectors in $\mathbb{R}^n$ via f.

$$\operatorname{Im} f = \{\vec{w} \in \mathbb{R}^m \mid \text{there is a } \vec{v} \in \mathbb{R}^n \text{ such that } f(\vec{v}) = \vec{w}\}$$

we can also write $\operatorname{Im} f = \operatorname{Im} A = f(\mathbb{R}^n)$

eg: Image of the projection map onto the x axis :

$P: \mathbb{R}^2 \to \mathbb{R}^2 \qquad P(x,y) = (x,0)$

is given by $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the image is all of the x axis.

sitting in $\mathbb{R}^2$ as a subspace.

eg: image of the rotation map: $rot_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is all of $\mathbb{R}^2$.

eg: Image of the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$ is a plane in $\mathbb{R}^3$

why?. Because $\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

so we have $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

now
$$A\begin{pmatrix} x \\ y \end{pmatrix} = A\left(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = xA\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yA\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

this shows that the image is spanned by the columns of A.

This is true in general. Say A is an $m \times n$ matrix & for any $v = x_1 e_1 + \dots + x_n e_n$

$$Av = x_1 Ae_1 + \dots + x_n Ae_n$$

so everything in the image of A is a linear combination of the columns of A.

<u>Definition</u>: The rank of A is the dimension of the image of A.

This means that the rank is the dimension of the subspace spanned by the column vectors of A. which is the number of linearly independent column vectors of A.

<u>FACT</u>: This is equal to the number of linearly independent rows of A.

so $\operatorname{rank}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \qquad \operatorname{rank}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = 2$

$\operatorname{rank}\begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 6 \\ 3 & -1 & 5 \end{pmatrix} = ?$ trickier! We do row operations to see how many rows we can eliminate.

the ones we can eliminate were the ones that were linearly dependent on the other rows (the ones that were linear combinations of the other rows)

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 6 \\ 3 & -1 & 5 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 6 \\ 0 & -4 & 8 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & 8 \\ 0 & -4 & 8 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we can see clearly that the first two rows are linearly independent.
so the rank is 2.

<u>Definition</u>: The dimension of the subspace

$$\text{Kernel } A = \{ v \in \mathbb{R}^n \mid Av = 0 \}$$

is called the <u>nullity</u> of A.

This is the dimension of the space of solutions to $A\vec{x} = 0$

Or in other words "the number of parameters the solution depends on".

<u>Important theorem</u>: Let A be an $m \times n$ matrix then:

$$\text{rank } A + \text{nullity } A = n$$

now, for example, we can tell that the solutions to $A\vec{x} = 0$ for

$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 6 \\ 3 & -1 & 5 \end{pmatrix}$ (above) depend on 1 parameter.

$$\text{nullity } A = n - \text{rank } A = 3 - 2 = 1$$

Determinants: is a number you associate to a square matrix.

for 2×2: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

for 3×3: $$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Cofactor description: for a matrix A, if you remove the ith row and jth column and take the determinant, and then you multiply this by $(-1)^{i+j}$ you get what is called C_{ij}

$$\begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{pmatrix} \qquad C_{11} = (-1)^{1+1}\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$$

& $$\begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{pmatrix} \qquad C_{12} = (-1)^{1+2}\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} \qquad \text{etc.}$$

the signs go like a chess board.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

for an $n\times n$ matrix:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \dots + a_{1n}C_{1n}$$

that's it. only the first row.

But this says nothing about determinant. It's the properties that matter!