Lecture 4: 27 May

Last time: • $\{\text{linear functions } \mathbb{R}^n \longrightarrow \mathbb{R}^m\} \longleftrightarrow \{m \times n \text{ matrices}\}$

the matrix corresponding to a linear function was the matrix whose columns were the values of the function on $e_1, \dots, e_n$ (the standard basis)

• rank of a matrix = dimension of the image

= number of linearly independent column vectors

(because the column vectors span the image)

= number of linearly independent rows

$$\text{rank}\, A + \underset{\substack{\nearrow \\ \text{dimension of the space of solutions to } A\vec{v}=0 \\ (\text{inside } \mathbb{R}^n)}}{\text{nullity}}\, A = n$$

Today: <u>determinants</u>:

$\det A$ is a number you associate to a square, meaning $n \times n$ matrix A.

2×2 case: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad - bc$

$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2$

3×3 case: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

eg: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix}$ $\quad \det A = 1\cdot(1\cdot5 - 2\cdot1) - 2(0\cdot5 - 2\cdot1) + 3(0\cdot1 - 1\cdot1)$

$= 3 + 4 - 3 = 4$

The general definition is an "inductive" or "recursive" definition in the sense that we use the definition of determinant for $n \times n$ matrices to define det for $(n+1) \times (n+1)$ matrices.

Define the (i,j)th cofactor of A to be the determinant of the matrix obtained by removing the ith row and the jth column; multiplied by $(-1)^{i+j}$. Denote this C_{ij}

eg for $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix}$ $\quad C_{11} = \det\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}(-1)^{1+1}$

$C_{12} = \det\begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix}\cdot(-1)^{1+2}$

etc

for an $n \times n$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

that's it; only the first row. Make sure to remember the signs.

You can do the same along any row or column of the matrix to get the same answer. The signs are like a chess board

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

But this is just a definition What matters is that det has very good properties.

before that: what is $\det I = ?$

$\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = ?$

(*) <u>Coolest property</u>: for $n \times n$ matrices A and B

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

(determinant is "multiplicative".)

It takes a bit of calculation to see why this is true. So we won't do that.

Most other properties easily follow from this one.

(1) $\det(A^T) = \det A$.

(2) [redacted] If B is obtained from A by multiplying a row (or column) by a number k, then $\det B = k \cdot \det A$

(3) If B is obtained from A by interchanging two rows (or columns), then $\det B = -\det A$

(4) If two rows of A are identical, then $\det A = 0$

(5) If a whole row or column is 0, then $\det A = 0$

(6) If B is obtained by adding a multiple of a row of A to another, then $\det B = \det A$.

(7) det of an upper or lower triangular matrix A is the product of the diagonal entries.

<u>proofs</u>: (1) calculation, can be seen from the definition.

(2) B is $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & \bigcirc & \\ & & 1 & & & \\ & & & k & & \\ & \bigcirc & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \cdot A$. By $\star$, $\det B = k \det A$

(3) B is $\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & \bigcirc & \\ & & 1 & & \\ & \bigcirc & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \cdot A$ if interchanging first two rows

$\nearrow$

$\det = -1$.

(4) By 3, $\det A = -\det A$

$2 \det A = 0 \Rightarrow \det A = 0$

(5) true by (2). ($k=0$)

(6) B is $\begin{pmatrix} 1 & k & & O \\ 0 & 1 & & \\ & & 1 & \\ O & & & \ddots \ 1 \end{pmatrix} \cdot A$ if we are adding k times the second to the first row.

$\nwarrow \det = 1$

(7) can see from definition.

In our previous example $A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix}$ we found $\det A = 4$.

Look at $A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}$ $\det A = 1 + 2 - 3 = 0$

Look at rank A_1. We know how to calculate this using row operations. It is 3. rank $A_1 - 3$

and rank $A_2 = 2$

Look at $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2$ $\quad$ rk $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 2$

$\det \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = 0$ $\quad$ rk $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = 1$

observation: det $= 0$ when rank is not n.

recall: rank A + nullity $A = n$

so in the case when $\det A \neq 0$, rank $A = n$, nullity $A = 0$

this means no nontrivial solution to $A\vec{v} = 0$.

when $\det A \neq 0$ and $\operatorname{rank} A = n$, $\operatorname{nullity} A = 0$.
We are not losing information because nothing non-zero goes to 0. This means we could go back by a matrix we will call A^{-1}.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \arrow[l, bend left, "A^{-1}"] \end{tikzcd}$$

<u>Inverse of a matrix</u>:

The inverse of a $(n\times n)$ matrix A is a $(n\times n)$ matrix B such that

$$A\cdot B = B\cdot A = I \qquad (\text{then } B = A^{-1})$$

We just understood that the following are equivalent:

<u>Theorem</u>: The following are equivalent for an $n\times n$ matrix A

(1) There is an inverse to A. ("A is invertible")

(2) $\det A \neq 0$

(3) A has rank n

(4) A has nullity 0 (equivalently $A\vec{v}=0$ has no solutions except $\vec{v}=0$)

<u>formal proof</u>: (1) ⇒ (2). We have $A\cdot(A^{-1}) = I$

then $\det(A\cdot A^{-1}) = \det(A)\cdot\det(A^{-1}) = \det I = 1$

so $\det A$ cannot be 0 and $\det(A^{-1}) = \dfrac{1}{\det A}$

(2)⇒(3) We can do row operations to get an upper triangular matrix from A. While doing these, we may change the value of det (when multiplying a row by a number etc) but won't change whether $\det A = 0$ or not. When we have the upper triangular, all the diagonal entries are non-zero because otherwise $\det A = 0$.

(3)⇔(4) by rank + nullity = n

(3) and (4) ⇒(1) Since A has full rank, it is onto. It is 1-1 because if $Av = Aw$, then $A(v-w)=0$ so $v=w$ by $\operatorname{null} A = 0$.

If we have an invertible matrix A, and we are trying to solve $A\vec{x}=\vec{b}$, there is a unique solution

$$A\vec{x}=\vec{b}$$
$$A^{-1}A\vec{x}=A^{-1}\vec{b}$$
$$\vec{x}=A^{-1}\vec{b}\ .$$

How to find the inverse:

for 2x2: $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad A^{-1}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

check.

In general, we can show that:

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix} C_{11} & C_{12} \cdots & C_{1n} \\ \vdots & & \\ C_{n1} & C_{n2} \cdots & C_{nn} \end{pmatrix}^{T}$$

here, the C_{ij} are the cofactors we defined before. We won't use this, but it's good to know it is there.

To find the inverse of A, we will write

$$\left(A \;\middle|\; \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \right)$$

and will do row operations to get $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}=I$ on the left side

what we end up with on the right side is A^{-1}.

Remark: this is much more efficient computationally! In the computer, row operations give a polynomial time algorithm, whereas each cofactor takes $(n-1)!$ calculations!!!

eg: $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 5 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_3 \cdot 2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 10 & -4 & 0 & 2 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 9 & -4 & -1 & 2 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow R_2 \cdot 9]{R_1 \leftarrow R_1 \cdot 9} \left(\begin{array}{ccc|ccc} 9 & 0 & -9 & 9 & 0 & 0 \\ 0 & -18 & 9 & 0 & 9 & 0 \\ 0 & 0 & 9 & -4 & -1 & 2 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + R_3} \left(\begin{array}{ccc|ccc} 9 & 0 & 0 & 5 & -1 & 2 \\ 0 & -18 & 0 & 4 & 10 & -2 \\ 0 & 0 & 9 & -4 & -1 & 2 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \leftarrow R_2/-18 \\ R_3 \leftarrow R_3/9}]{R_1 \leftarrow R_1/9} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/9 & -1/9 & 2/9 \\ 0 & 1 & 0 & 4/-18 & 10/-18 & -2/-18 \\ 0 & 0 & 1 & -4/9 & -1/9 & 2/9 \end{array}\right)$$

The right-hand block is marked with an underbrace: $\underbrace{}_{A^{-1}}$

Hopefully it's correct. A^{-1}. You can check by multiplying this with A.

Some properties of inverses: • the inverse is unique

- $(A^{-1})^{-1} = A$. (because $(A^{-1}) . A = I$ so A is the inverse of A^{-1}.

- $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$ (check: $(A \cdot \underbrace{B)\,(B^{-1}}_{I} A^{-1}) = A \cdot I \cdot A^{-1}$
$= A \cdot A^{-1} = I$)