Lecture 4: 27 May

Last time:

- \( \{ \text{linear functions } \mathbb{R}^n \to \mathbb{R}^m \} \leftrightarrow \{ \text{m} \times \text{n matrices} \} \)
- The matrix corresponding to a linear function was the matrix whose columns were the values of the function on \( e_1, \ldots, e_n \) (the standard basis).
- Rank of a matrix = dimension of the image = number of linearly independent column vectors (because the column vectors span the image) = number of linearly independent rows
- \( \text{rank } A + \text{nullity } A = n \)

Dimension of the space of solutions to \( A \mathbf{x} = 0 \) (inside \( \mathbb{R}^n \))

Today: Determinants:

- \( \text{det } A \) is a number you associate to a matrix \( A \).
- \( n \times n \) square meaning \( n \times n \)

**2x2 case:** \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

\( \text{det } A = ad - bc \)

\[
\text{det} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \times 4 - 2 \times 3 = 4 - 6 = -2
\]

**3x3 case:** \( A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \)

\( \text{det } A = a_{11} \left( a_{22} a_{33} - a_{23} a_{32} \right) - a_{12} \left( a_{21} a_{33} - a_{23} a_{31} \right) + a_{13} \left( a_{21} a_{32} - a_{22} a_{31} \right) \)
\[ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix} \]

\[ \det A = 1 \cdot (5 - 2) - 2 \cdot (0 - 2 \cdot 1) + 3 \cdot (0 - 1 \cdot 1) \]

\[ = 3 + 4 - 3 = 4 \]

The general definition is an "inductive" or "recursive" definition in the sense that we use the definition of determinant for \(n \times n\) matrices to define \(\det\) for \((n+1) \times (n+1)\) matrices.

Define the \(ij\)th cofactor of \(A\): to be the determinant of the matrix obtained by removing the \(i\)th row and the \(j\)th column; multiplied by \((-1)^{i+j}\). Denote this \(C_{ij}\).

Example for \(A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix}\)

\[ C_{11} = \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} (-1)^{1+1} \]

\[ C_{12} = \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} (-1)^{1+2} \]

etc

for an \(n \times n\) matrix \(A = (a_{ij})\)

\[ \det A = a_{11} C_{11} + a_{12} C_{12} + \ldots + a_{nn} C_{nn} \]

that's it; only the first row. Make sure to remember the signs. You can do the same along any row or column of the matrix to get the same answer. The signs are like a chess board

\[ \begin{pmatrix} + & - & + & \ldots \\ - & + & - & \ldots \\ + & - & + & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

But this is just a definition. What matters is that \(\det\) has very good properties.

Before that: what is \(\det I = ?\)

\[ \det \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = ? \]
Cool property: for \( n \times n \) matrices \( A \) and \( B \)

\[
\det(A \cdot B) = \det(A) \cdot \det(B)
\]

(determinant is "multiplicative").

It takes a bit of calculation to see why this is true. So we won't do that.

Most other properties easily follow from this one.

1. \( \det(AB) = \det(A) \cdot \det(B) \)

2. If \( B \) is obtained from \( A \) by multiplying a row (or column) by a number \( k \), then \( \det(B) = k \cdot \det(A) \)

3. If \( B \) is obtained from \( A \) by interchanging two rows (or columns), then \( \det(B) = -\det(A) \)

4. If two rows of \( A \) are identical, then \( \det(A) = 0 \)

5. If a whole row or column is \( 0 \), then \( \det(A) = 0 \)

6. If \( B \) is obtained by adding a multiple of a row of \( A \) to another, then \( \det(B) = \det(A) \)

7. \( \det \) of an upper or lower triangular matrix \( A \) is the product of the diagonal entries.

Proofs: 
1. Calculation, can be seen from the definition.

2. \( B \) is 
\[
\begin{pmatrix}
1 & k & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\( \cdot A \). By \( \star \), \( \det(B) = k \cdot \det(A) \)

3. \( B \) is 
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]
\( \cdot A \)

\( \Rightarrow \det = -1 \).
(4) By 3, \[ \det A = -\det A, \]
\[ 2 \det A = 0 \implies \det A = 0 \]

(5) true by (2) \((k=0)\)

(6) \(B\) is 
\[
\begin{pmatrix}
1 & k \\
0 & 1 \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]
If we are adding \(k\) times the second to the first row,

\[ \det = 1 \]

(7) can see from definition.

In our previous example \(A_1= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix} \) we found \(\det A = 4 \).

Look at \(A_2= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix} \) \(\det A = 1 + 2 - 3 = 0 \)

Look at rank \(A_1\). We know how to calculate this using row operations. It is 3. \(\text{rank } A_1 = 3\)
and \(\text{rank } A_2 = 2\)

Look at \(\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2\) \(\text{rk} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 2\)
\(\det \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = 0\) \(\text{rk} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = 1\)

Observation: \(\det = 0\) when rank is not \(n\).

Recall: \(\text{rank } A + \text{nullity } A = n\)

so in the case when \(\det A \neq 0\), \(\text{rank } A = n\), \(\text{nullity } A = 0\)
this means no non-trivial solution to \(A^T \mathbf{y} = 0\).
when \( \det A \neq 0 \) and \( \text{rank } A = n \), \( \text{nullity } A = 0 \).

We are not losing information because nothing non-zero goes to 0. This means we could go back \( \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n \xrightarrow{A^{-1}} \mathbb{R}^n \) by a matrix we will call \( A^{-1} \).

**Inverse of a matrix:**

The inverse of a \((n \times n)\) matrix \( A \) is a \((n \times n)\) matrix \( B \) such that

\[
A \cdot B = B \cdot A = I \quad \text{(then } B = A^{-1})
\]

We just understood that the following are equivalent:

**Theorem:** The following are equivalent for an \( n \times n \) matrix \( A \)

1. There is an inverse to \( A \). ("\( A \) is invertible")
2. \( \det A \neq 0 \)
3. \( A \) has rank \( n \)
4. \( A \) has nullity 0 (equivalently \( A^T \cdot 0 \) has no solutions except \( \mathbf{0} \))

**Formal proof:** \((1) \Rightarrow (2)\). We have \( A \cdot (A^{-1}) = I \)

then \( \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}) = \det I = 1 \)

so \( \det A \) cannot be 0 and \( \det(A^{-1}) = \frac{1}{\det A} \)

\((2) \Rightarrow (3)\). We can do row operations to get an upper triangular matrix from \( A \). While doing these, we may change the value of \( \det \) (when multiplying a row by a number etc.) but won't change whether \( \det A = 0 \) or not. When we have the upper triangular, all the diagonal entries are non-zero because otherwise \( \det A = 0 \).

\((3) \Leftrightarrow (4)\) by rank + nullity = \( n \)

\((3) \text{ and } (4) \Rightarrow (2)\). Since \( A \) has full rank, it is onto. It is 1-1 because if \( Av = Aw \), then \( A(v-w) = 0 \) so \( v = w \) by null \( A \neq 0 \).
If we have an invertible matrix $A$, and we are trying to solve $AX = b$, there is a unique solution

$$AX = b$$

$$A^{-1}AX = A^{-1}b$$

$$X = A^{-1}b$$

**How to find the inverse:**

For $2 \times 2$:  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

check.

In general, we can show that:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^{T}$$

here, the $C_{ij}$ are the cofactors we defined before.

We won't use this, but it's good to know it is there.

To find the inverse of $A$, we will write

$$\begin{pmatrix} A & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

and will do row operations to get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ on the left side.

What we end up with on the right side is $A^{-1}$.

Remark: this is much more efficient computationally!

In the computer, row operations give a polynomial time algorithm, whereas each cofactor takes $(n-1)!$ calculations!!!
\[
\begin{align*}
\text{eg: } A &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix} \\
R_3 &\rightarrow R_3 - 2R_1 \\
&\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
R_3 &\rightarrow R_3 - 2R_1 \\
&\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & -1 & 5 \end{pmatrix} \\
R_2 &\rightarrow R_2 - 2 \Rightarrow 2 \\
&\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 5 \end{pmatrix} \\
R_3 &\rightarrow R_3 - R_2 \\
&\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \\
R_3 &\rightarrow R_3 - R_2 \\
&\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \\
R_1 &\rightarrow R_1 + R_3 \\
&\rightarrow \begin{pmatrix} 9 & -9 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \\
R_2 &\rightarrow R_2 - R_3 \\
&\rightarrow \begin{pmatrix} 9 & -9 & 9 \\ 0 & 18 & 0 \\ 0 & 0 & -4 \end{pmatrix} \\
R_1 &\rightarrow R_1 + R_3/9 \\
&\rightarrow \begin{pmatrix} 5/9 & -7/9 & 2/9 \\ 0 & 1 & 0 \\ 0 & -4/9 & -7/9 \end{pmatrix} \\
&\rightarrow \begin{pmatrix} 5/9 & -7/9 & 2/9 \\ 0 & 1 & 0 \\ 0 & -4/9 & -7/9 \end{pmatrix} \\
\text{Hopefully it's correct.} \quad A^{-1} \quad \text{You can check by multiplying this with A.}
\end{align*}
\]

Some properties of inverses:

1. \((A^{-1})^{-1} = A\)  
   (because \((A^{-1}) \cdot A = I\) so \(A\) is the inverse of \(A^{-1}\)).

2. \((A \cdot B)^{-1} = B^{-1} \cdot A^{-1}\)  
   (check: \((A \cdot B) \cdot \left( B^{-1} \cdot A^{-1} \right) = A \cdot I \cdot A^{-1} \) \(= A \cdot A^{-1} = I \))