Eigenvalues and eigenvectors: Given a matrix A $(n \times n)$

It is useful in linear algebra (and in applications) to find vectors $\vec{v}$ and numbers λ such that

$$A\vec{v} = \lambda \vec{v}$$

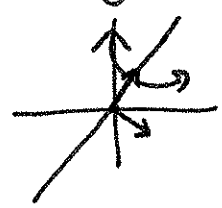
What does this mean?. It means that the action of A on $\vec{V}$ is just rescaling.

In this case, $\vec{v}$ is called an eigenvector

λ is called an eigenvalue

eg: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

recall: this was swapping x and y, equivalently, was reflection along the y=x line



has two eigenvalues and eigenvectors.

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_1 = -1 \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

of course, we could have picked $v_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ or any $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$. but this is just the same vector rescaled.

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 1 \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

eg: $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (skews the plane in the x direction.)

has $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_1 = 2 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

also $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 1 \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

• sometimes there are no eigenvalues and eigenvectors.

eg: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for say $\theta = 30°$

this is rotation, so no vector is just scaled (at least for $\theta = 30°$)

(what values of θ would ~~make~~ give some eigens?)

• sometimes there is only one eigenvalue $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

but there are two eigenvectors for the same $\lambda = 2$.

• sometimes there is only one eigenvalue and only one eigenvector.

for example $A = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$ we'll see how.

Let's look at the situation. Say we have a matrix A $(n\times n)$ and we want to find its eigenvalues and eigenvectors.

Say A is 3×3.

We are looking for the λ's and $\vec{v}$'s such that

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

we have four unknowns (v_1, v_2, v_3 and λ) but only three equations! We have to get smart. Here's how:

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

for λ to be an eigenvalue, there must be a $\vec{v}$ such that $(A-\lambda I)\vec{v} = 0$. In other words, $A-\lambda I$ must be ("non-invertible") singular. Which, we know from previous lectures, is equivalent to $\det(A-\lambda I) = 0$.

So, to find the eigenvalues, we look at the λ's that make $\det(A-\lambda I) = 0$. We can then find the eigenvectors.

<u>example</u>: Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$$

we have

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-1-\lambda)(8-\lambda) - 2(-7) \\ &= \lambda^2 - 7\lambda - 8 + 14 \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda-1)(\lambda-6) \end{aligned}$$

this is called the characteristic polynomial of the matrix A.

so $\lambda_1 = 1$ and $\lambda_2 = 6$ are the eigenvalues.

<u>for $\lambda = 1$</u>, $A - \lambda_1 I = \begin{pmatrix} -1-1 & 2 \\ -7 & 8-1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & 7 \end{pmatrix}$

clearly a singular matrix, but we already knew that!

$(A - \lambda_1 I)v = 0$

means

$$\begin{pmatrix} -2 & 2 \\ -7 & 7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 2 \\ -7 & 7 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{R_1}{2}]{R_3 \leftarrow \frac{R_3}{7}} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

so $-v_1 + v_2 = 0$ $\quad v_1 = v_2$.

We have a choice, pick $v_1 = 1$, then $v_2 = 1$ $\quad K_I \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

check: $\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ good.

<u>for $\lambda = 6$</u>

$$A - \lambda_2 I = \begin{pmatrix} -1-6 & 2 \\ -7 & 8-6 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ -7 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} -7 & 2 \\ 0 & 0 \end{pmatrix}$$ so $-7v_1 + 2v_2 = 0$

$2v_2 = 7v_1$

we again have a choice, pick $v_2 = 7$, $v_1 = 2$
but we could have picked any multiple.

so $K_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

Check: $\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 12 \\ 42 \end{pmatrix} = 6 \cdot \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

λ_2 K_2

good.

Now look at $A = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{pmatrix} = (3-\lambda)(7-\lambda)+4$$
$$= \lambda^2 - 10\lambda + 21 + 4$$
$$= \lambda^2 - 10\lambda + 25$$
$$= (\lambda-5)^2$$

there is only one eigenvalue. $\lambda = 5$ but it has multiplicity 2. (because $\det(A-\lambda I) = (\lambda-5)^2$.

Let's find the eigenvectors:

$$A - \lambda I = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$$

Immediately, we can see that this matrix has rank = 1 nullity = 1.

so there is only a one parameter family of solutions to $(A-\lambda I)v = 0$

indeed after row ops $\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$

so $-v_1 + 2v_2 = 0$

$2v_2 = v_1$ $\qquad K_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

this is the only eigenvector.