Lecture 6 More eigenvalue problems, diagonalization

Review of last time.

Let A be an $n\times n$ matrix. We said that if $v \overset{\neq 0}{\in} \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ are such that $Av = \lambda v$, then λ is an eigenvalue and v is an eigenvector.

To find the eigenvalues and eigenvectors of A, we looked at:

$$Av = \lambda v \iff Av - \lambda I v = 0 \iff (A-\lambda I)v = 0.$$

$A-\lambda I$ must be singular, so $\det(A-\lambda I) = 0$

This helps us find the λ's, then we can solve for the v's.

eg.: $A = \begin{pmatrix} 1 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)\left((2-\lambda)^2-1\right) - 6\left(0\cdot(2-\lambda) - 1\cdot 0\right) + 0\cdot(\dots)$$

$$= (1-\lambda)(4-4\lambda+\lambda^2-1) = (1-\lambda)(\lambda-3)(\lambda-1)$$
$$= -(\lambda-3)(\lambda-1)^2$$

two eigenvalues: 3 and 1
with multiplicities, 1 and 2

$\lambda_1 = 3$

$A-3I = \begin{pmatrix} -2 & 6 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ We are trying to solve $(A-3I)v = 0$

do row operations

if $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ then $-2v_1 + 6v_2 = 0$, $-v_2 + v_3 = 0$

$\xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} -2 & 6 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

so $v_1 = 3v_2$, $v_2 = v_3$

pick: $v_3 = 1$

$K_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 1}$:

$$A - 1.I = \begin{pmatrix} 0 & 6 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 0 & 6 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$v_2 = 0$

$v_2 + v_3 = 0$ so $v_3 = 0$

v_1 can be anything

$$K_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

we could not find a K_3.

So, even though the eigenvalue had multiplicity two, there is only one eigenvector for it.

Another example : $A = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$

$$\det(A - \lambda I) = -(\lambda - 11)(\lambda - 8)^2$$

so $\lambda = 11$ or 8

$\underline{\lambda_1 = 1}$: $(A - 11I) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3/3]{R_2 \leftarrow 2R_2} \begin{pmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2/3} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so $v_2 = v_3$ and $v_1 = v_2$

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

<u>for $\lambda = 8$</u>:

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

good news! rank = 1
nullity = 2
we will find two independent eigenvectors

$$\xrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so $v_1 + v_2 + v_3 = 0$

~~pick~~

the eigenvectors will look like $\begin{pmatrix} v_1 \\ v_2 \\ -v_1 - v_2 \end{pmatrix}$

pick $v_1 = 0$, $v_2 = 1$
$v_1 = 1$, $v_2 = 0$

$$K_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Diagonalization:

We say A is diagonalizable if there is an invertible matrix P and a diagonal matrix D such that

$$P^{-1} A P = D$$

Let's analyze this situation in 3×3 case. Say $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ and P is invertible such that

$$P^{-1} A P = D$$

then $AP = PD$

Look at $APe_i = PDe_i$, $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith place

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \underbrace{\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{\text{ith column of } P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = d_i \begin{pmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{pmatrix}$$

so the ith column of P must be an eigenvector and

~~the~~ d_i must be an eigenvalue!

So, if we can find n eigenvectors, then we can set

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} K_1 & K_2 & \cdots & K_n \end{pmatrix}$$

<u>but</u>: P must be invertible (so that we can write $P^{-1}AP = D$)

so the K's must be linearly independent.

Thus:

<u>Theorem</u>: A $\overset{n\times n}{\text{matrix}}$ A is diagonalizable if and only if it has n linearly independent eigenvectors.

<u>example</u>: earlier, we found that $A = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$

has eigenvalues $\underset{\text{multiplicity 2}}{8}$ and 11 and eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

so we write: $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 11 \end{pmatrix}$

we have to make sure that the order of the eigenvalues and the corresponding K's are the same.

eg: $A = \begin{pmatrix} -4 & -5 \\ 8 & 10 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} -4-\lambda & -5 \\ 8 & 10-\lambda \end{pmatrix}$$

$$= (-4-\lambda)(10-\lambda) + 40$$

$$= \lambda^2 - 6\lambda - 40 + 40 = \lambda(\lambda - 6)$$

so $\lambda_1 = 0$ and $\lambda_2 = 0$

<u>for $\lambda_1 = 0$</u>, $A - 0I = A = \begin{pmatrix} -4 & -5 \\ 8 & 10 \end{pmatrix}$

$$\xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -4 & -5 \\ 0 & 0 \end{pmatrix} \qquad K_1 = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

<u>for $\lambda_2 = 6$</u> $A - 6I = \begin{pmatrix} -10 & -5 \\ 8 & 4 \end{pmatrix}$

$$\xrightarrow[R_2 \leftarrow R_2/4]{R_1 \leftarrow R_1/5} \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

so: $D = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$ $\quad P = \begin{pmatrix} 5 & 1 \\ -4 & -2 \end{pmatrix}$

Cool trick:

Find A^{10} for $A = \begin{pmatrix} -4 & -5 \\ 8 & 10 \end{pmatrix}$

solution: We already know that $PAP^{-1} = D$ for

$P = \begin{pmatrix} 5 & 1 \\ -4 & -2 \end{pmatrix}$ $D = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$

we have $PAP^{-1} = D \implies \underbrace{P^{-1}P}_{I}A\underbrace{P^{-1}P}_{I} = P^{-1}DP$

$$A = P^{-1}DP$$

so $A^{10} = \underbrace{P^{-1}D\overbrace{PP^{-1}}^{I}DP \ldots P^{-1}DP}_{10 \text{ times}} = P^{-1}D^{10}P$

so $A^{10} = P^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 6^{10} \end{pmatrix}P$