Lecture 7: Differential equations.

An ordinary differential equation looks like:

\[ y'' - 4y = 12x \]

\[ \frac{d^2y}{dx^2} - 4y = 12x \]

There usually are boundary conditions: like \( y(0) = 4 \), \( y'(0) = 1 \)

In general an \( n \)-th order ODE looks like:

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \]

with conditions:

\[ y(x_0) = y_0, \ y'(x_0) = y_1, \ldots, \ y^{(n-1)}(x_0) = y_{n-1} \]

Since all the conditions are at the same point \( x_0 \),
this is called an initial value problem.

**Theorem:** Let \( a \)'s and \( g \) be continuous on an interval \( I \).

Let \( a_n(x) \neq 0 \) for every \( x \) in this interval, \( x_0 \in I \).

Then there is a unique solution to the equation \((*)\)
subject to the conditions \((**\))

We won't think about this much, but it's good to know this.

**Example:** \( 3y''' + 5y'' - y' + 7y = 0 \)

\( y(1) = 0, y'(1) = 0, y''(1) = 0 \)

(all conditions are at \( x_0 = 1 \), \( \mathbb{R} \) is bounded and the coefficients are never 0, so this has a unique solution on all of \( \mathbb{R} \).

\( y = 0 \) is already a solution. So that is the only one.
when the conditions are at different points, the theorem does not hold and anything can happen.

Anyway, let's start solving:

\[ y' - y = 0 \quad \text{this is a "homogeneous" equation because the right side is 0.} \]

with condition \( y(0) = 1 \).

the solution is \( y_0 = c_1 e^x \). Indeed, try: \( c_1 e^x - c_1 e^x = 0 \).

For \( y(0) = 1 \), we need \( c_1 = 1 \).

\[ y' - y = x \quad \text{this is a "non-homogeneous" equation.} \]

\( y' - y = 0 \) is the homogeneous part of the equation.

Try: \( y = Ax + B \) \quad \( y' = A \)

\[ y' - y = A - A x + B = x \]

so \( A = -1 \) \quad \( B = 1 \).

So one solution is \( y = -x + 1 \).

Observe that if we add a solution to \( y' - y = 0 \) to this, it will still be a solution:

\[ (y + y_0)' - (y + y_0) = \underbrace{y' - y}_{x} + \underbrace{y_0' - y_0}_{0} = x \]

so the general solution is \( y = -x + 1 + c_1 e^x \).

If we have a boundary condition, like \( y(0) = 0 \),

then we could find the value of \( c_1 \) (=1 in this case).
eg: look at \( y'' - y = 0 \)
the solutions are \( y_1 = e^x \) and \( y_2 = e^{-x} \). Any linear combination of these two is also a solution:
\[
y_0 = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{-x}
\]

This is called the "general solution" of the equation.

- For an \( n \)th order homogeneous ODE, there should be \( n \) solutions. We see that if we have solutions \( y_1, \ldots, y_n \), then any linear combination
\[
y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n
\]
is also a solution.
This is called the "superposition principle".

Now look at \( y'' - y = x \)

Try \( y = Ax + B \), \( y'' - y = 0 - Ax - B = x \)
so \( B = 0 \) \( A = -1 \)

so \( y_p = -x \) is a solution

\( p \) for "particular".

To find the general solution, look at \( y'' - y = 0 \) the homogeneous equation. We know this has solution
\[
y_0 = c_1 e^x + c_2 e^{-x}
\]
so the general solution is
\[
y = y_p + y_0 = -x + c_1 e^x + c_2 e^{-x}
\]