Lecture 7: Differential equations.

an ordinary differential equation looks like:

$$y'' - 4y = 12x$$

$$\frac{d^2y}{dx^2} - 4y = 12x$$

there usually are boundary conditions: like $y(0)=4$, $y'(0)=1$

In general an nth order ODE looks like:

$$(*) \qquad a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

with conditions

$$(**) \qquad y(x_0)=y_0,\ y'(x_0)=y_1,\ \dots,\ y^{(n-1)}(x_0) = y_{n-1}.$$

Since all the conditions <u>are at the same point x_0</u>, this is called an <u>initial value problem</u>.

<u>Theorem</u>: Let a's and g be continuous on an interval I. Let $a_n(x) \neq 0$ for every x in this interval. $x_0 \in I$. Then there is a unique solution to the equation $(*)$ subject to the conditions $(**)$.

We won't think about this much, but it's good to know this.

<u>example</u>: $3y''' + 5y'' - y' + 7y = 0 \quad y(1)=0, y'(1)=0, y''(1)=0$

(all conditions are at $x_0=1$, and the coefficients are never 0, so this has a unique solution on all of $\mathbb{R}$.

$y=0$ is already a solution. So that is the only one.

when the conditions are at different points, the theorem does not hold and anything can happen.

Anyway, let's start solving:

eg: $y' - y = 0$ ← this is a "homogeneous" equation because the right side is 0.

with condition $y(0) = 1$.

the solution is $y_0 = c_1 e^x$. Indeed, try: $c_1 e^x - c_1 e^x = 0$ ..

for $y(0) = 1$, we need $c_1 = 1$

eg: $y' - y = x$ ← this is a "non-homogeneous" equation
$y' - y = 0$ is the homogeneous part of the equation.

try: $y = Ax + B$ $\quad y' = A$

$$y' - y = A - Ax - B = x$$

so $A = -1 \quad B = -1$

so one solution is $y = -x - 1$

observe that if we add a solution to $y' - y = \underline{\underline{0}}$ to this, it will still be a solution

$$(y + y_0)' - (y + y_0) = \underbrace{y' - y}_{x} + \underbrace{y_0' - y_0}_{0} = x$$

so the general solution is $y = -x - 1 + c_1 e^x$

If we have a boundary condition, like $y(0) = 0$ then we could find the value of c_1 (=1 in this case)

<u>eg</u>: Look at $y'' - y = 0$

the solutions are $y_1 = e^x$ and $y_2 = e^{-x}$. Any linear combination of these two is also a solution:

$$\underbrace{y_0 = c_1 y_1 + c_2 y_2}_{\text{this is called the "general solution" of the equation.}} = c_1 e^x + c_2 e^{-x}$$

- For an nth order homogeneous ODE, there should be n solutions.

We see that if we have solutions $y_1, \dots, y_n$, then any linear combination

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

is also a solution.

This is called the "superposition principle".

now look at $y'' - y = x$

try $y = Ax + B$, $y'' - y = 0 - Ax - B = x$

so $B = 0$ $A = -1$

so $y_p = -x$ is a solution

p for "particular".

To find the general solution, look at $y'' - y = 0$ the homogeneous equation. We know this has solution

$$y_0 = c_1 e^x + c_2 e^{-x}$$

so the general solution is

$$y = y_p + y_0 = -x + c_1 e^x + c_2 e^{-x}$$