Lecture 9: Last time: • we saw how we can solve homogeneous linear equations with constant coefficients.

• then we saw how to do non-homogeneous versions of these by guessing the answer to find a particular solution, (with undetermined coefficients)

Funny situation.

eg: $y'' - 5y' + 4y = 8e^x$

• first find y_c by solving $y'' - 5y' + 4y = 0$

auxiliary: $m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$y_c = c_1 e^{4x} + c_2 e^x$

• for y_p, try $y = Ae^x$. $Ae^x - 5Ae^x + 4e^x = 0$

so we can't get $8e^x$ because e^x is already a solution to the homogeneous version.

try $y = Axe^x$ $\quad y' = Ae^x + Axe^x$

$y'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$

then $y'' - 5y + 4y = 2Ae^x \cancel{+Axe^x} - 5Ae^x \cancel{-5Axe^x} \cancel{+4Axe^x} = 8e^x$

$= 2Ae^x - 5Ae^x = 8e^x \qquad A = \frac{-8}{3}$

$y = -\frac{8}{3}xe^x + c_1 e^{4x} + c_2 e^x$

double trouble:

$y'' - 2y' + y = e^x$

$y_c = c_1 e^x + c_2 x e^x$

can't try e^x or xe^x for y_p because they are both solutions to the homogeneous version.

$y_p = Ax^2 e^x$ will work.

Cauchy-Euler equation:

Let's try to solve:

eg: $x^2 y'' - 2xy' - 4y = 0$ with boundary conditions $y(1) = 1$ and $y(2) = 16$

can you guess the solution?.

try $y = x^m$ $y' = mx^m$ $y'' = m(m-1)x^{m-2}$

plug in: $x^2y'' - 2xy' - 4y = x^2 m(m-1)x^{m-2} - 2xmx^{m-1} - 4x^m = 0$

$= x^m(m(m-1) - 2m - 4)$

$= x^m(m^2 - m - 2m - 4)$

$= x^m(m^2 - 3m - 4)$

$= x^m(m-4)(m+1)$

so $m_1 = 4$ or $m_2 = -1$

$y = c_1 x^4 + c_2 x^{-1}$.

To find c_1 and c_2, plug in: $y(1) = c_1 + c_2 = 1$ $y(2) = 16c_1 + \frac{c_2}{2}$ $\Rightarrow$ $c_1 = 1$, $c_2 = 0$

You can guess what we are going to do next. We will look at a situation where we will have ~~dist~~ repeated roots and then complex roots.

eg: $4x^2y'' + 8xy' + y = 0$

$$y = x^m \qquad 4x^2 m(m-1)x^{m-2} + 8x\, m x^{m-1} + x^m = 0$$

$$x^m\left(4m(m-1) + 8m + 1\right) = 0$$

$$x^m \underbrace{\left(4m^2 + 4m + 1\right)}_{\text{auxiliary eqn.}} = 0$$

$$x^m (2m+1)^2 = 0$$

so $m = \frac{1}{2}$ is a double root.

the solutions are: $y_1 = x^{\frac{1}{2}} \qquad y_2 = x^{\frac{1}{2}} \ln x$

plug in and check!

<u>Complex roots</u>:

eg: $4x^2y'' + 17y = 0 \qquad y(1) = -1 \qquad y'(1) = -\frac{1}{2}$

auxiliary eqn is $4m^2 - 4m + 17 = 0$

$$m = \underset{\substack{\shortparallel \\ 1/2}}{\alpha} \pm i\underset{\substack{\shortparallel \\ 2}}{\beta} = \frac{1}{2} \pm 2i$$

$$y = x^\alpha\left(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)\right).$$

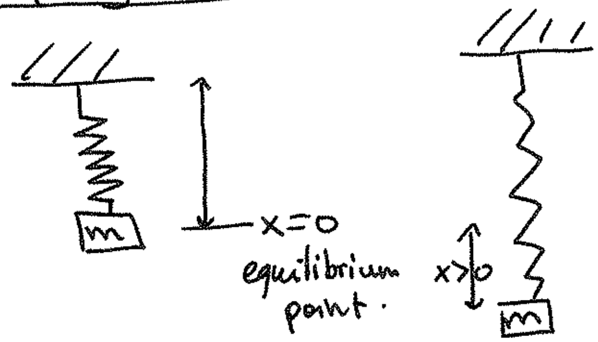
then plug in to find c_1 and c_2.

what happens if we have higher degree equations ?

try: $x^3 y''' + 5x^2 y'' + 7x y' + 8y = 0$

we use the same method, we just have three roots and three equations.

<u>Spring-mass model</u>:



the equation is:

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right) x = 0.$$

write $\omega^2 = \frac{k}{m}$ for ease of notation later.

<u>eg</u>: Say we pull a string down by 1 unit and let it go:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad x(0) = 1 \text{ and } x'(0) = 0$$

(it's not moving when we start)

$m^2 + \omega^2 = 0 \qquad m = 0 \pm i\omega$

ω = "angular velocity"

$x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

$x(0) = 1 \Rightarrow c_1 1 + c_2 \cdot 0 = 1$ so $c_1 = 1$

$x'(0) = 0 \Rightarrow c_2 = 0$.

Now, we could modify the equations to model different physical situations with the spring system.

Say the medium has some resistance which is proportional to the velocity of the mass. Then the eqn would look like

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

the auxiliary equation is $m^2 + 2\lambda m + \omega^2$.

then $m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$ $\quad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$

what are the solutions in 3 cases

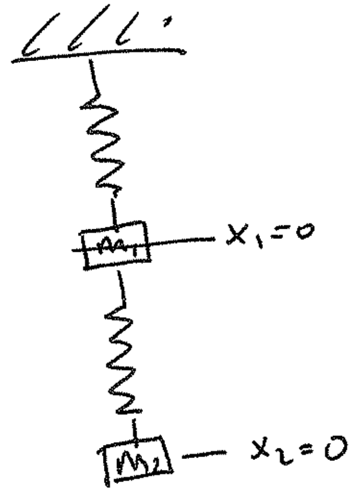
$\lambda^2 - \omega^2 > 0$

$\lambda^2 - \omega^2 < 0$?

$\lambda^2 - \omega^2 = 0$.

what do these correspond to physically? (check book page 155).

What if we have two springs and masses.



now we have two equations :

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 x_2'' = -k_2 (x_2 - x_1)$$

This gives two equations which we must solve together!

eg : $x' + y' + 2y = 0$ these are functions of t.

$x' - 3x - 2y = 0$

write: $Dx + (D+2)y = 0$

$(D-3)x - 2y = 0.$

multiply first eqn by $(D-3)$ and second by D to cancel out the x. and get $(D^2 + D - 6)y = 0.$

similarly: you get: $(D^2 + D - 6)x = 0.$

then we can solve for x and y separately.

eg: $x' - 4x + y'' = t^2$

$x' + x + y' = 0$

solve by elimination. (page 186 of book)