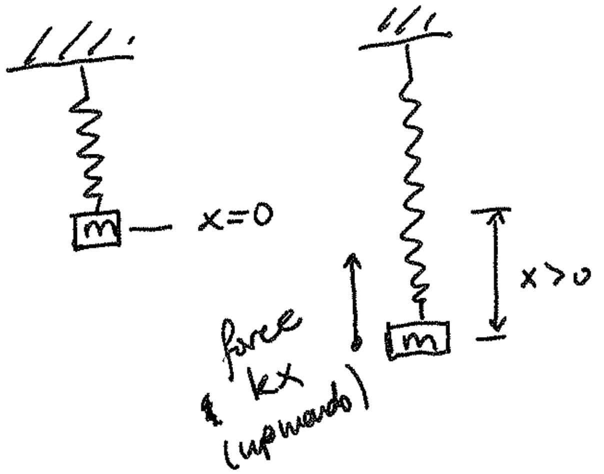
Lecture 10: Last time:
- Cauchy-Euler equation
- more "undetermined coefficients".
- spring-mass systems.

Recall: We were thinking about spring-mass systems. The spring constant k is so that if the spring is moved to a distance x from its stable position, it applies a force kx.



from $\underbrace{F}_{-kx} = m \cdot \underbrace{a}_{\frac{d^2x}{dt^2}}$

we have

$$\frac{d^2x}{dt^2} + kx = 0.$$

this is the equation that governs the motion of this system. We put in the information of how the system starts by writing initial/boundary conditions.

eg: spring and mass are pulled downwards by 1 unit and let go

$x(0) = 1$

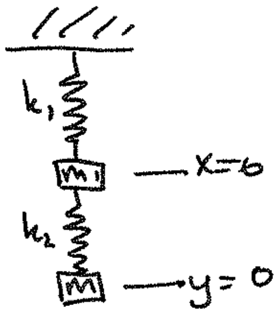
$x'(0) = 0$ ← no velocity at the beginning.

eg: spring is given downward velocity of 2 units at its $x=0$ position.

$x(0) = 0$

$x'(0) = 2$

What if there is more than one spring?



we'll come back to this.

- say there is some resistance in the medium. Then the equation becomes:

$\frac{k}{m}$

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0 \qquad (\lambda > 0)$$

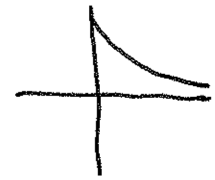
solutions come from roots to the auxiliary equation.

$$m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

there are three cases:

- $\lambda^2 - \omega^2 > 0$

  then $x = c_1 e^{(-\lambda + \sqrt{\lambda^2-\omega^2})t} + c_2 e^{-\lambda - \sqrt{\lambda^2-\omega^2}t}$

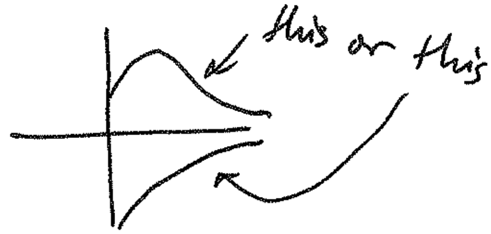


  no oscillation!

- $\lambda^2 - \omega^2 = 0$

  $$x = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

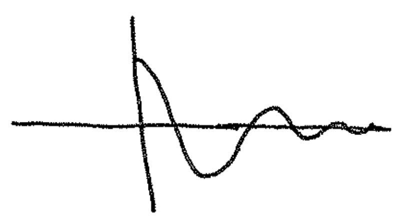
  still no oscillation.



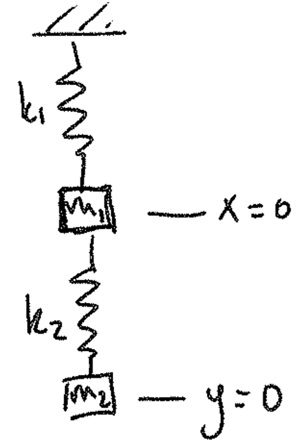
- $\lambda^2 - \omega^2 < 0$

  $$x = e^{-\lambda t}\left(c_1 \cos(\sqrt{\lambda^2-\omega^2}\, t) + c_2 \sin(\sqrt{\lambda^2-\omega^2}\, t)\right).$$

oscillatory. if $\lambda \neq 0$, then the graph is like this:



two springs and masses: we'll use $F = ma$ again to find equations.



$$m_1 \frac{d^2x}{dt^2} = \underbrace{-k_1 x}_{\text{first spring}} + \underbrace{k_2(y-x)}_{\text{second spring}}$$

$$m_2 \frac{d^2y}{dt^2} = -k_2(y-x)$$

so we have two equations that we would need to solve at the same time to find y and x. We'll do a lot of things like this.

- First method: elimination,

eg: $\frac{dx}{dt} = 3y$, $\frac{dy}{dt} = 2x$ $\rightsquigarrow$ $Dx - 3y = 0$, $2x - Dy = 0$

multiply the first by 2, apply D to second to get:

$$\begin{array}{r} 2Dx - 6y = 0 \\ -\;\; 2Dx - D^2y = 0 \\ \hline D^2y - 6y = 0 \end{array}$$

$$y = c_3 e^{\sqrt{6}t} + c_4 e^{-\sqrt{6}t}$$

similarly, cancel the y's:

$$D^2x - 3Dy = 0$$
$$6x - 3Dy = 0$$

$$D^2x - 6x = 0$$
$$x = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$$

but if we just plug these in, we get:

$$Dx - 3y = c_1\sqrt{6}e^{\sqrt{6}t} + c_2(-\sqrt{6})e^{-\sqrt{6}t} - 3c_3 e^{\sqrt{6}t} - 3c_4 e^{-\sqrt{6}t} = 0$$

so $\sqrt{6}c_1 - 3c_3 = 0$ so $c_3 = \frac{\sqrt{6}c_1}{3}$ and $c_4 = \frac{-\sqrt{6}}{3}c_2$

so, if we write these like a vector:

so $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} \\ \frac{\sqrt{6}c_1}{3}e^{\sqrt{6}t} + \frac{-\sqrt{6}}{3}c_2 e^{-\sqrt{6}t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{\sqrt{6}}{3} \end{pmatrix} e^{\sqrt{6}t} + c_2 \begin{pmatrix} 1 \\ -\frac{\sqrt{6}}{3} \end{pmatrix} e^{-\sqrt{6}t}$$

Look at the last problem:

$$\frac{dx}{dt} = 3y$$
$$\frac{dy}{dt} = 2x$$

we can put it into matrix form.

call $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

then $X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

the solution was:
$x =$

and write $A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$

then we can write the equation as:

$$X' = AX$$

eg: $\frac{dx}{dt} = 3x + 4y$
$\frac{dy}{dt} = -x + 5y$

$$\longleftrightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$X' = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} X$$

Look at the example with $A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$ that we solved before

the solution was

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} \\ \frac{\sqrt{6}c_1}{3} e^{\sqrt{6}t} + \frac{-\sqrt{6}}{3} c_2 e^{-\sqrt{6}t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{\sqrt{6}}{3} \end{pmatrix} e^{\sqrt{6}t} + c_2 \begin{pmatrix} 1 \\ \frac{-\sqrt{6}}{3} \end{pmatrix} e^{-\sqrt{6}t}$$

this suggests that the solutions are of the form

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = K e^{\lambda t}$$

where $K = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is some vector.

the equation is $X' = AX$.

try $Ke^{\lambda t}$ as a solution $\quad X = Ke^{\lambda t}$

$$X' = \lambda K e^{\lambda t}$$

$$\lambda K e^{\lambda t} = A K e^{\lambda t}$$

this is satisfied when $AK = \lambda K$. So K is an eigenvector and λ is the corresponding eigenvalue of A.

<u>eg</u>: $X' = \underbrace{\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}}_{A} X$ $\qquad$ we need to find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0$$

so $\lambda_1 = -1$ and $\lambda_2 = 4$

for $\lambda_1 = -1$, we want to find K_1

$$(A+I)K_1 = 0 \qquad \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} K_1 = 0 \quad \xrightarrow{\text{row ops}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

so $K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so one solution to our equation is:

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{(-1)t}$$

for $\lambda_2 = 4$, $A - 4I = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix}$ $\quad 3v_2 = 2v_1$

$$K_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

second solution: $X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$

final answer: $X = c_1 X_1 + c_2 X_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$.

This is the general strategy.

Now let's all do:
$$\frac{dx}{dt} = -4x + y + z$$
$$\frac{dy}{dt} = x + 5y - z$$
$$\frac{dz}{dt} = y - 3z$$

What about repeated eigenvalues:

- If we can find all the linearly independent eigenvectors, we are fine $X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \dots + c_n K_n e^{\lambda_n t}$ even if some of the λ_i's are the same.

What if there is a repeated eigenvalue with only one eigenvector?

$X_1 = K_1 e^{\lambda_1 t}$ as usual

$X_2 = K_1 t e^{\lambda_1 t} + P e^{\lambda_1 t}$ is the second solution.

Let's plug into: $X' = AX$

$$X_2' = K_1 e^{\lambda_1 t} + \lambda_1 K_1 t e^{\lambda_1 t} + \lambda_1 P e^{\lambda_1 t}$$

$$X_2' - AX_2 = K_1 e^{\lambda_1 t} + \cancel{\lambda_1 K_1 t e^{\lambda_1 t}} + \lambda_1 P e^{\lambda_1 t} - \cancel{\underbrace{AK_1}_{\lambda_1 K_1} t e^{\lambda_1 t}} - AP e^{\lambda_1 t}$$

$$= K_1 e^{\lambda_1 t} - (A - \lambda I) P e^{\lambda_1 t} = 0$$

so $(A - \lambda I) P = K_1$

we know K_1. This is how to find P.

What if there is a triple eigenvalue with only one eigenvector?

$$X_1 = K_1 e^{\lambda_1 t}$$

$$X_2 = K_1 t e^{\lambda_1 t} + P e^{\lambda_1 t} \quad \text{as before}$$

with $(A - \lambda I) P = K$

$$X_3 = \frac{1}{2} K_1 t^2 e^{\lambda_1 t} + P t e^{\lambda_1 t} + Q e^{\lambda_1 t}$$

with $(A - \lambda I) Q = P$.

you get the idea...

eg: Find all (3) solutions to $AX = X'$.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}.$$