Recall: We were thinking about spring-mass systems. The spring constant $k$ is so that if the spring is moved to a distance $x$ from its stable position, it applies a force $kx$.

From $F = ma$,

$$\frac{d^2x}{dt^2} - kx = 0$$

This is the equation that governs the motion of this system. We put in the information of how the system starts by writing initial/boundary conditions.

**Example:** spring and mass are pulled downwards by 1 unit and let go

$x(0) = 1$

$x'(0) = 0$ — no velocity at the beginning.

**Example:** spring is given downward velocity of 2 units at its $x=0$ position.

$x(0) = 0$

$x'(0) = 2$

What if there is more than one spring?

We'll come back to this.
Say there is some resistance in the medium. Then the equation becomes:

\[ \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0 \quad (\lambda > 0) \]

Solutions come from roots to the auxiliary equation:

\[ m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \]

There are three cases:

1. \( \lambda^2 - \omega^2 > 0 \)
   
   Thus, \[ x = c_1 e^{(\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{-(\lambda + \sqrt{\lambda^2 - \omega^2})t} \]
   
   no oscillation!

2. \( \lambda^2 - \omega^2 \geq 0 \)
   
   \[ x = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \]
   
   still no oscillation.

3. \( \lambda^2 - \omega^2 < 0 \)
   
   \[ x = e^{-\lambda t} \left( c_1 \cos(\sqrt{\lambda^2 - \omega^2} t) + c_2 \sin(\sqrt{\lambda^2 - \omega^2} t) \right) \]
   
   oscillatory. If \( \lambda > 0 \), then the graph is like this:
two springs and masses:  

\[ \begin{align*}  
  k_1 & \quad x = 0 \\
  k_2 & \quad y = 0 \\
  m_1, m_2 & 
\end{align*} \]

we'll use \( F = ma \) again to find equations.

\[ \begin{align*}  
  m_1 \frac{d^2x}{dt^2} &= -k_1 x + k_2 (y - x) \\
  m_2 \frac{d^2y}{dt^2} &= -k_2 (y - x) 
\end{align*} \]

so we have two equations that we would need to solve at the same time to find \( y \) and \( x \). We'll do a lot of things like this.

First method: elimination, simultaneous equations.

Ex: \[ \begin{align*}  
  \frac{dx}{dt} &= 3y & \quad \text{Dx - 3y} = 0 \\
  \frac{dy}{dt} &= 2x & \quad \text{2x - Dy} = 0 
\end{align*} \]

multiply the first by 2, apply \( D \) to second to get:

\[ \begin{align*}  
  2Dx - 6y &= 0 \\
  2Dx - D^2y &= 0 \\
  D^2y - 6y &= 0 \\
  y &= c_3 e^{\sqrt{6}t} + c_4 e^{-\sqrt{6}t} 
\end{align*} \]

similarly, cancel the y's:

\[ \begin{align*}  
  D^2x - 3Dy &= 0 \\
  2Dx - 6x &= 0 \\
  2Dx - 3Dy &= 0 \\
  2Dx - D^2y &= 0 \\
  D^2y - 6y &= 0 \\
  y &= c_3 e^{\sqrt{6}t} + c_4 e^{-\sqrt{6}t} \\
  x &= c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} 
\end{align*} \]

but if we just plug these in, we get:

\[ \begin{align*}  
  Dx - 3y &= c_1 \sqrt{6} e^{\sqrt{6}t} + c_2 (-\sqrt{6}) e^{-\sqrt{6}t} - 3c_3 e^{\sqrt{6}t} - 3c_4 e^{-\sqrt{6}t} = 0 
\end{align*} \]

so, if we write these like a vector:

\[ \begin{align*}  
  \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} \\ \frac{\sqrt{6}}{3} c_1 e^{\sqrt{6}t} + \frac{\sqrt{6}}{3} c_2 e^{-\sqrt{6}t} \end{pmatrix} \\
  &= c_1 \left( \frac{1}{\sqrt{6}} \right) e^{\sqrt{6}t} + c_2 \left( \frac{-1}{\sqrt{6}} \right) e^{-\sqrt{6}t} 
\end{align*} \]
Look at the last problem:
\[
\begin{align*}
\frac{dx}{dt} &= 3y \\
\frac{dy}{dt} &= 2x
\end{align*}
\]
we can put it into matrix form.
and write
\[
A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}
\]

then we can write the equation as:
\[
X' = AX
\]

eg:
\[
\begin{align*}
\frac{dx}{dt} &= 3x + 4y \\
\frac{dy}{dt} &= -x + 5y
\end{align*}
\]

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
X' = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} X
\]

Look at the example with \( A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \) that we solved before the solution was:
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{\sqrt{6} t} + c_2 e^{-\sqrt{6} t} \\ \frac{\sqrt{6} c_1}{3} e^{\sqrt{6} t} + \frac{\sqrt{6} c_2}{3} e^{-\sqrt{6} t} \end{pmatrix} = C_1 \left( \frac{1}{\sqrt{3}} \right) e^{\sqrt{6} t} + C_2 \left( \frac{1}{\sqrt{3}} \right) e^{-\sqrt{6} t}
\]

this suggests that the solutions are of the form
\[
X = \begin{pmatrix} x' \\ y' \end{pmatrix} = K e^{\lambda t}
\]
when \( K = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) is some vector.
The equation is $X = AX$. Try $X = ke^{\lambda t}$ as a solution: $X = ke^{\lambda t}$

$\lambda ke^{\lambda t} = A ke^{\lambda t}$

This is satisfied when $AK = \lambda K$. So $K$ is an eigenvector and $\lambda$ is the corresponding eigenvalue of $A$.

E.g.: $X = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} X$

We need to find the eigenvalues and eigenvectors of $A$.

$\det(A - \lambda I) = \det\begin{pmatrix} 2 - \lambda & 3 \\ 2 & 1 - \lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4)$

So $\lambda_1 = -1$ and $\lambda_2 = 4$

For $\lambda_1 = -1$, we want to find $K_1$

$(A + I)K_1 = 0$ \hspace{1cm} \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}K_1 = 0 \hspace{1cm} \text{row ops} \hspace{1cm} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}

So one solution to our equation is: $X_1 = (-1)e^{-t}$

For $\lambda_2 = 4$, $A - 4I = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix} \hspace{1cm} \text{3v}_2 = 2v_1$

$K_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Second solution: $X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$

Final answer: $X = c_1X_1 + c_2X_2 = c_1(-1)e^{-t} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$. 

\[ \]
This is the general strategy.

Now let's all do:
\[
\begin{align*}
\frac{dx}{dt} &= -4x + y + z \\
\frac{dy}{dt} &= x + 5y - z \\
\frac{dz}{dt} &= y - 3z
\end{align*}
\]

What about repeated eigenvalues:

- If we can find all the linearly independent eigenvectors, we can write:
  \[X = c_1 k_1 e^{\lambda_1 t} + c_2 k_2 e^{\lambda_2 t} + \cdots + c_n k_n e^{\lambda_n t}\]
  even if some of the \(\lambda_i\)'s are the same.

What if there is a repeated eigenvalue with only one eigenvector?

\[X_1 = k_1 e^{\lambda_1 t}\] as usual

\[X_2 = k_1 te^{\lambda_1 t} + P e^{\lambda_1 t}\] is the second solution.

Let’s plug into: \[X' = AX\]

\[X_2' = \lambda_1 k_1 e^{\lambda_1 t} + \lambda_1 k_1 te^{\lambda_1 t} + \lambda_1 Pe^{\lambda_1 t}\]

\[X_2' - AX_2 = k_1 e^{\lambda_1 t} + \lambda_1 k_1 te^{\lambda_1 t} + \lambda_1 Pe^{\lambda_1 t} - A(k_1 e^{\lambda_1 t} + P e^{\lambda_1 t})
  = k_1 e^{\lambda_1 t} - (A - \lambda_1 I)P e^{\lambda_1 t} = 0\]

So \[(A - \lambda_1 I)P = k_1\]

We know \(k_1\). This is how to find \(P\).
What if there is a triple eigenvalue with only one eigenvector? 

\[ X_1 = k_1 e^{\lambda_1 t} \]

\[ X_2 = k_1 t e^{\lambda_1 t} + P e^{\lambda_1 t} \text{ as before} \]

with \((A - \lambda I) P = K\)

\[ X_3 = \frac{1}{2} k_1 t^2 e^{\lambda_1 t} + Pte^{\lambda_1 t} + Qe^{\lambda_1 t} \]

with \((A - \lambda I) Q = P\).

you get the idea...

**Ex:** Find all (3) solutions to \( A X = X' \).

\[ A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \]