Series Solutions to differential equations:

Review of power series:

Recall that a power series looks like this:

$$(*) \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

One way to think about them is to consider them as more and more accurate approximations to a function.

eg: $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (\text{with } 0! = 1)$$

$(*)$ is a power series centered at 0. A power series centered at 'a' looks like

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

If we have a function f that has all derivatives at a, then we can consider the approximating power series of f, called the Taylor Series of f:

In principle, we should have

$$f(x) = \sum \frac{f^{(n)}(a)}{n!} (x-a)^n$$

other interesting power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

A power series may not converge for every value of x. The radius of convergence of a power series is the largest value of R such that for every x with $|x-a| < R$, the series converges.

eg: $1+x+x^2+\cdots$ has radius of convergence equal to 1.

- We have: $$\sum_{n=0}^{\infty} c_n (x-a)^n + \sum_{n=0}^{\infty} b_n (x-a)^n = \sum_{n=0}^{\infty} (b_n + c_n)(x-a)^n$$

so it is easy to add power series. But one must be careful when adding if indices are different.

eg: $$\sum_{n=2}^{\infty} (n)(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

the starting point is different

the powers of x are different.

if we put $k = n+1$, then the second series becomes

$$\sum_{k=1}^{\infty} c_{k-1} x^k$$

so we have $$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{k=1}^{\infty} c_{k-1} x^k$$

do the same for the first series by putting $k = n-2$

we get: $$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k$$

the starting points are still different. we can make them the same by removing the first term of the first series.

$$(0+2)(0+1)\, c_{0+2}\, x^0 + \underbrace{\sum_{k=1}^{\infty} (k+2)(k+1)\, c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1}\, x^k}_{\text{we can now combine these two}}$$

we took out the k=0 term

$$= 2.c_2 + \sum_{k=1}^{\infty} \left((k+2)(k+1)\, c_{k+2} + c_{k-1} \right) x^k$$

Now we know how to manipulate power series; let's see how we use them to solve differential equations.

<u>eg</u>: $y'' + xy = 0$

Say y is given by a power series $y = \sum_{n=0}^{\infty} c_n x^n$

then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2}$

Plug in:

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \cancel{x} \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)\, c_{k+2}\, x^k + \sum_{k=1}^{\infty} c_{k-1}\, x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} \left((k+2)(k+1)\, c_{k+2} + c_{k-1} \right) x^k \doteq 0$$

this whole power series must be equal to 0. This means that all coefficients must equal 0.

$$2c_2 = 0 \qquad \text{so} \qquad c_2 = 0.$$

$$(k+2)(k+1)c_{k+2} + c_{k-1} = 0$$

$$\text{so} \quad \left[c_{k+2} = -\frac{c_{k-1}}{(k+2)(k+1)} \right]$$

this is a relation that holds for every $k \geq 1$.

this means $0 = c_2 = c_5 = c_8 = \cdots$

we can find as many of the c's as we would like. (in terms of c_0 and c_1)

$\underline{k=1}$: $c_3 = -\dfrac{c_0}{3.2}$

$\underline{k=2}$: $c_4 = -\dfrac{c_1}{3.4}$

$\underline{k=3}$ $c_5 = -\dfrac{c_2}{4.5} = 0$

$\underline{k=4}$ $c_6 = -\dfrac{c_3}{5 \cdot 6} = \dfrac{1}{2.3.5 \cdot 6} c_0$

$\vdots$

you get the idea.

$$\text{so } y = c_0 + c_1 x - \frac{c_0}{2.3} x^3 - \frac{c_1}{3.4} x^4 + 0.x^5 + \frac{c_0}{2.3.5.6} x^6 + \frac{c_1}{3.4.6.7} x^7 + \cdots$$

regroup:

$$y = \underbrace{c_0 \left(1 - \frac{1}{2.3} x^3 + \frac{1}{2.3.5.6} x^6 - \cdots \right)}_{y_1} + \underbrace{c_1 \left(x - \frac{1}{3.4} x^4 + \frac{1}{3.4.6.7} x^7 - \cdots \right)}_{y_2}$$

our two solutions.

you do $\underline{\underline{\text{not}}}$ need to find a nice function that corresponds to this power series.

another example: $y'' - (1+x)y = 0$ same method (exercise)

what about? $\cancel{y'' - (1+x)y} = e^x$

$$y'' + xy = e^x$$

by plugging in $y = \sum_{n=0}^{\infty} c_n x^n$, we have again

for this side

$$2c_2 + \sum_{k=1}^{\infty} \left((k+2)(k+1)c_{k+2} + c_{k-1}\right)x^k = e^x$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$2c_2 + \sum_{k=1}^{\infty} \left((k+1)(k+2)c_{k+2} + c_{k-1}\right)x^k - \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}} = 0$$

$$2c_2 - 1 + \sum_{k=1}^{\infty} \left((k+1)(k+2)c_{k+2} + c_{k-1} - \frac{1}{k!}\right)x^k = 0$$

so $c_2 = \frac{1}{2}$ and $\underbrace{(k+1)(k+2)c_{k+2} + c_{k-1} - \frac{1}{k!} = 0}_{\text{similar recurrence relation!}}$

Consider second degree equations:

$$(*) \qquad y'' + P(x)y' + Q(x)y = 0$$

If P and Q are analytic (they are well defined and their Taylor series converge) at a point x_0, then the point x_0 is said to be an ordinary point.

<u>Theorem</u>: If x_0 is an ordinary point of an equation like $(*)$, we can always find two independent solutions of the form $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$. These power series converge in an interval given by $|x-x_0| < R$ where R is the shortest distance to a singular point of the equation. (a singular point is a point that is not a regular point)

This theorem says that we can always find solutions to a second order ODE by considering a power series solution centered at an ordinary point. Swell.

eg: consider: $(x^2-4)^2 y'' + 3(x-2)y' + 5y = 0$

$$y'' + \frac{3(x-2)}{(x^2-4)^2}y' + \frac{5}{(x^2-4)^2}y = 0$$

$$y'' + \frac{3}{(x+2)^2(x-2)}y' + \frac{5}{(x-2)^2(x+2)^2}y = 0$$

the points $x_0 = 2$ and $x_0 = -2$ are singular points.
All other points are double points.

If x_0 is a singular point, and $P(x)(x-x_0)$ and
$Q(x).(x-x_0)^2$ are analytic

then x_0 is called a regular singular point.

If x_0 is a regular singular point of $y''+P(x)y'+Q(x)$
then there is at least one solution of the form

$$y=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$

eg: $3xy'' + y' - y = 0 \quad \rightsquigarrow \quad y'' + \frac{1}{3x}y' - \frac{1}{3x}y = 0$

$x=0$ is a regular singular point.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

plug in:

$$3xy''+y'-y = \sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

we want to take the x^r out.

$$= \sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$k = n-1$

$$= \sum_{k=-1}^{\infty}(k+r+1)(3k+3r+1)c_{k+1} x^{k+r} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= x^{r-1}\, r(3r-2)c_0 + \sum_{k=0}^{\infty}\left((k+r+1)(3k+3r+1)c_{k+1} - c_k\right)x^{k+r}$$

$$= x^r\Big(\underbrace{r(3r-2)c_0 x^{-1}} + \sum_{k=0}^{\infty}\left((k+r+1)(3k+3r+1)c_{k+1} - c_k\right)x^k\Big)$$

$$= 0$$

so $r(3r-2)\cdot c_0 = 0$

so $r=0$ or $3r-2=0 \quad r = \frac{2}{3}$

and then we can solve as usual.