Series Solutions to differential equations:

Review of power series:

Recall that a power series looks like this:

\[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots \]

(\star)

One way to think about them is to consider them as more and more accurate approximations to a function.

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]  

(with \(0! = 1\))

(\star) is a power series centered at 0. A power series centered at 'a' looks like

\[ \sum_{n=0}^{\infty} c_n (x-a)^n \]

If we have a function \(f\) that has all derivatives at \(a\), then we can consider the approximating power series of \(f\), called the Taylor Series of \(f\):

In principle, we should have

\[ f(x) = \sum \frac{f^{(n)}(a)}{n!} (x-a)^n \]
Other interesting power series:

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \]
\[ \frac{1}{1-x} = 1 + x + x^2 + \ldots \]
\[ \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \]

A power series may not converge for every value of \( x \).
The radius of convergence of a power series is the largest value of \( R \) such that for every \( x \) with \( |x-a| < R \),
the series converges.

**Example:** \( 1 + x + x^2 + \ldots \) has radius of convergence equal to 1.

- We have: \[ \sum_{n=0}^{\infty} c_n (x-a)^n + \sum_{n=0}^{\infty} b_n (x-a)^n = \sum_{n=0}^{\infty} (b_n+c_n) (x-a)^n \]

so it is easy to add power series. But one must be careful
when adding if indices are different.

**Example:** \[ \sum_{n=2}^{\infty} (n)(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \]
the starting point is different

If we put \( k = n+1 \), then the second series becomes

\[ \sum_{k=1}^{\infty} c_{k-1} x^k \]

so we have \[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{k=1}^{\infty} c_k x^k \]

Do the same for the first series by putting \( k = n-2 \)
we get:

\[ \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k \]
the starting points are still different. We can make them the same by removing the first term of the first series.

\[(0+2)(0+1) c_{0+2} x^0 + \sum_{k=1}^{\infty} \frac{(k+2)(k+1)}{k+1} c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k\]

We then take out the \(k=0\) term.

\[
= 2c_2 + \sum_{k=1}^{\infty} \left(\frac{(k+2)(k+1)}{k+1} c_{k+2} + c_{k-1}\right) x^k
\]

Now we know how to manipulate power series. Let's see how we use them to solve differential equations.

**Ex:** \(y'' + xy = 0\)

Say \(y\) is given by a power series \(y = \sum_{n=0}^{\infty} c_n x^n\)

Then \(y' = \sum_{n=1}^{\infty} nc_n x^{n-1}\) and \(y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\)

Plug in:

\[
y'' + xy = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0
\]

\[
= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k
\]

\[
= 2c_2 + \sum_{k=1}^{\infty} \left(\frac{(k+2)(k+1)}{k+1} c_{k+2} + c_{k-1}\right) x^k = 0
\]
This whole power series must be equal to 0. This means that all coefficients must equal 0.

\[ 2c_2 = 0 \quad \text{so} \quad c_2 = 0. \]

\[(k+2)(k+1)c_{k+2} + c_{k-1} = 0 \]

So

\[ c_{k+2} = -\frac{c_{k-1}}{(k+2)(k+1)} \]

This is a relation that holds for every \( k \geq 1 \).

\[ c_2 = c_5 = c_8 = \ldots \]

\[ k=1: \quad c_3 = -\frac{c_0}{3 \cdot 2} \]

\[ k=2: \quad c_4 = -\frac{c_1}{3 \cdot 4} \]

\[ k=3: \quad c_5 = -\frac{c_2}{4 \cdot 5} = 0 \]

\[ k=4: \quad c_6 = -\frac{c_3}{5 \cdot 6} = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0 \]

\[ \vdots \]

You get the idea.

\[ y = c_0 + c_1 x - \frac{c_0}{2 \cdot 3} x^3 - \frac{c_1}{3 \cdot 4} x^4 + 0.1 x^5 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \ldots \]

Regroup:

\[ y = c_0 \left( 1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \ldots \right) + c_1 \left( x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \ldots \right) \]

Our two selections:

\[ \begin{array}{c}
\text{y_1} \\
\text{y_2}
\end{array} \]

You do not need to find a nice function that corresponds to this power series.
Another example: \[ y'' - (1 + x) y = 0 \]

Same method (exercise)

What about? \[ y'' - (1 + x) y = e^x \]

\[ y'' + xy = e^x \]

By plugging in \( y = \sum_{n=0}^{\infty} c_n x^n \), we have again for this side:

\[ 2c_2 + \sum_{k=1}^{\infty} ((k+2)(k+1) c_{k+2} + c_{k-1}) x^k = e^x \]

\[ = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

\[ 2c_2 + \sum_{k=1}^{\infty} ((k+1)(k+2) c_{k+2} + c_{k-1}) x^k \]

\[ - \sum_{k=0}^{\infty} \frac{x^k}{k!} = 0 \]

\[ 2c_2 + 1 + \sum_{k=1}^{\infty} ((k+1)(k+2) c_{k+2} + c_{k-1} - \frac{1}{k!}) x^k = 0 \]

So \( c_2 = \frac{1}{2} \) and \( (k+1)(k+2) c_{k+2} + c_{k-1} - \frac{1}{k!} = 0 \)

Similar recurrence relation!
Consider second degree equations:

\[ y'' + P(x)y' + Q(x)y = 0 \]

If \( P \) and \( Q \) are analytic (their Taylor series converge) at a point \( x_0 \), then the point \( x_0 \) is said to be an ordinary point.

**Theorem:** If \( x_0 \) is an ordinary point of an equation like (1), we can always find two independent solutions of the form \( y_0 = \sum_{n=0}^{\infty} c_n(x-x_0)^n \). These power series converge in an interval given by \( |x-x_0| < R \) where \( R \) is the shortest distance to a singular point of the equation (a singular point is a point that is not a regular point).

This theorem says that we can always find solutions to a second order ODE by considering a power series solution centered at an ordinary point. Swell.

**Ex:** Consider:

\[ (x^2-4)y'' + 3(x-2)y' + 5y = 0 \]

\[ y'' + \frac{3(x-2)}{(x^2-4)^2} y' + \frac{5}{(x^2-4)^2} y = 0 \]

\[ y'' + \frac{3(x-2)}{(x+2)(x-2) (x-2)^2(x+2)^2} y' + \frac{5}{(x^2-4)^2} y = 0 \]

The points \( x_0 = 2 \) and \( x_0 = -2 \) are singular points. All other points are double points.
If $x_0$ is a singular point, and $P(x)(x-x_0)$ and $Q(x)(x-x_0)^2$ are analytic,

then $x_0$ is called a regular singular point.

If $x_0$ is a regular singular point of $y'' + P(x)y' + Q(x)y$,
then there is at least one solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$
Eq: \[ 3xy'' + y' - y = 0 \]

\[ y = \sum_{n=0}^{\infty} c_n x^{n+r} \]

\[ y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \]

\[ y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \]

Plug in:

\[ 3xy'' + y' - y = \sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \]

We want to take the \( x^r \) out.

\[ = \sum_{n=0}^{\infty} (n+r)(3n+3r+2)c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} \]

\[ = \sum_{k=-1}^{\infty} (k+r+1)(3k+3r+1)c_{k+1} x^{k+r} - \sum_{n=0}^{\infty} c_n x^{n+r} \]

\[ = x^{r-1} \left( 3r-2 \right) c_0 + \sum_{k=0}^{\infty} ((k+r+1)(3k+3r+1)c_{k+1} - c_k)x^k \]

\[ = x^r \left( 3r-2 \right) c_0 \cdot x^{-1} + \sum_{k=0}^{\infty} ((k+r+1)(3k+3r+1)c_{k+1} - c_k)x^k \]

So \( r(3r-2) c_0 = 0 \)

So \( r = 0 \) or \( 3r-2 = 0 \) \( r = \frac{2}{3} \)

And then we can solve as usual.