Lecture 12:

Last time; we saw how to solve equations of the form

$$y'' + P(x)y' + Q(x)y = 0$$

using series solutions.

We plugged in: $y = \sum_{n=0}^{\infty} c_n x^n$ ← a power series centered at 0

(or we could try $y = \sum c_n (x-x_0)^n$ a power series centered at x_0)

and solved for c_n.

In the end, we could find all the c's in terms of c_0 and c_1 (in our example). These were the two constants for our two solutions y_1 and y_2 which were given as power series.

- Recall: x_0 is an ordinary point of the equation if P and Q are well-defined at x_0. Otherwise, x_0 is singular.
- If x_0 is an ordinary point, then ~~#~~ we know that $y = \sum c_n (x-x_0)^n$ will yield two solutions. This is a theorem.
- If x_0 is a singular point but $(x-x_0)P$ and $(x-x_0)^2 Q$ are well-defined, then we call it a regular singular point. Then it is a theorem that $y = \sum c_n (x-x_0)^{n+r}$ will yield at least one solution (but hopefully two).

eg: $3xy'' + y' - y = 0$

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eg: $3xy'' + y' - y = 0 \rightsquigarrow y'' + \frac{1}{3x}y' - \frac{1}{3x}y = 0$

$x=0$ is a regular singular point.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

plug in:

$$3xy''+y'-y = \sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

we want to take the x^r out.

$$= \sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$k = n-1$

$$= \sum_{k=-1}^{\infty}(k+r+1)(3k+3r+1)c_{k+1} x^{k+r} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= x^{r-1}\, r(3r-2)c_0 + \sum_{k=0}^{\infty}\left((k+r+1)(3k+3r+1)c_{k+1} - c_k\right)x^{k+r}$$

$$= x^r\Big(\underbrace{r(3r-2)c_0}\, x^{-1} + \sum_{k=0}^{\infty}\left((k+r+1)(3k+3r+1)c_{k+1} - c_k\right)x^{k+r}\Big) = 0$$

so $r(3r-2)\cdot c_0 = 0$

So $r=0$ or $3r-2=0$ $\quad r = \frac{2}{3}$

and then we can solve as usual.

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we got $x^r(r(3r-2))c_0x^{-1} + \sum_{k=0}^{\infty}\left((k+r+1)(3k+3r+1)c_{k-1} - c_k\right)x^{k+r}$

notice that our recurrence relation is

$$c_k = \frac{c_{k-1}}{(k+r+1)(3k+3r+1)}$$

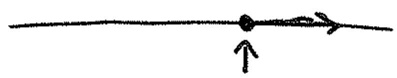
So if c_0 is 0, then our whole solution is 0 so we'd better have $r=0$ or $r=\frac{2}{3}$.

In each of these cases, we can find all the other ~~solutions~~ constants in terms of c_0. So setting $r=0$ and setting $r=\frac{2}{3}$ gives us two solutions.

Vector Calculus:

One-variable calculus can tell us about the motion of an object on a line

$x'(t)$ = velocity $\qquad x''(t)$ acceleration.



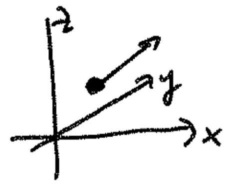
$x(t)$ position of the object at time t.

We do the same in more dimensions.

$$\vec{r}(t) = (x(t), y(t), z(t)) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

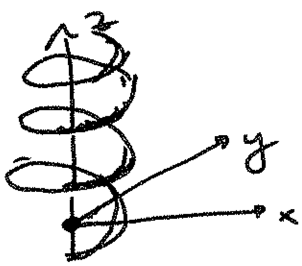
describes the motion of a particle in 3dimensional space.

eg: $\vec{r}(t) = (1+t, 2t, 2+t) = (1,0,2) + t(1,2,1)$



describes the motion of a particle at $(1,0,2)$ at $t=0$ and moving in a straight line in the direction $(1,2,1)$.

eg: $r(t) = (\cos t, \sin t, t)$ draws a circle as it is moving up in the z direction therefore it draws a helix.



If $r(t) = (x(t), y(t), z(t))$ is position then clearly:

$v(t) = r'(t) = (x'(t), y'(t), z'(t))$ is the velocity vector.

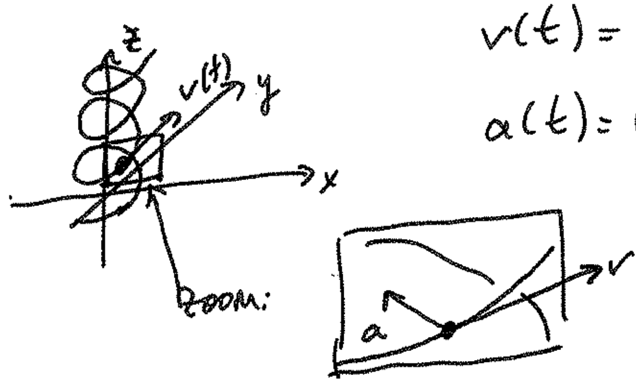
$a(t) = r''(t) = (x''(t), y''(t), z''(t))$ is the acceleration vector.

speed $= \|\vec{v}(t)\|$

eg: Say $r(t) = (2\cos t, 2\sin t, t)$

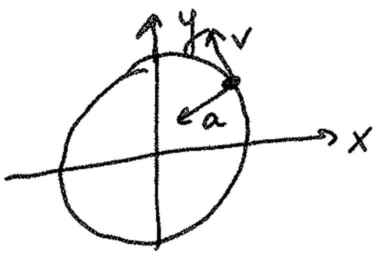
$$v(t) = (-2\sin t, 2\cos t, 1)$$

$$a(t) = (-2\cos t, -2\sin t, 0)$$



- a is pulling inwards.
- v going up and straight (tangent to the curve)

eg: consider $r(t) = (\cos t, \sin t, 0)$



Curvature: define the unit tangent vector of a curve as

$$T = \frac{r'(t)}{\|r'(t)\|}$$

the unit tangent is changing faster as the curve gets more curvy!

Curvature: is the length of the derivative of the unit tangent.

$$\left\| \frac{dT}{ds} \right\|$$

parameterized by arc length. why?: because we do not want the curvature artificially increasing when we travel the curve faster than unit speed.

but this is not how we calculate it:

$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt}$ by the chain rule.

so $\frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{\|T'\|}{\|r'\|}$

eg: curvature of the circle of radius R. 

$$r(t) = (R\cos t, R\sin t, 0)$$

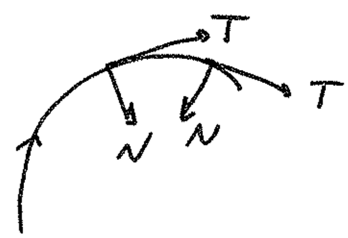
$$T = \frac{r'(t)}{\|r'(t)\|} = \frac{(-R\sin t, R\cos t, 0)}{R} = (-\sin t, \cos t, 0)$$

$$K = \frac{\|T'\|}{\|r'\|} = \frac{1}{R}$$ the curvature is constant along the whole curve (surprise?)

the curvature is bigger if R is smaller.

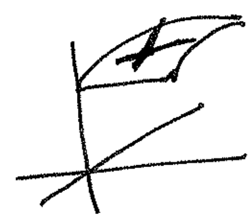
Q: what is the unit normal? answer: $N = \frac{T'}{\|T'\|}$

picture:



Review:

Partial derivatives: Say we are given a function $f(x,y)$ (or $f(x,y,z)$)



$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

measures the rate of change in the x direction

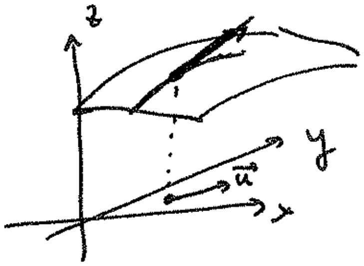
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$ same for the y direction.

eg: let $f(x,y) = 2xy + x^2\sin y$

$\frac{\partial f}{\partial x} = 2y + 2x\sin y$ $\frac{\partial f}{\partial y} = 2x + x^2(\cos y)$ (treating all other variables as constants)

What makes the x direction or the y direction special? nothing it's just a choice. We could define derivatives in any direction.

Say we have a function $z = f(x,y)$ and we want to find the rate of change in the direction of $\vec{u}$.



Let $\vec{u}$ be the unit vector $(\cos\theta, \sin\theta)$

then:

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h}$$

This is a nice definition. But it is much easier to calculate if we use the <u>gradient</u>.

The gradient of $f(x,y)$ is

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \quad \text{or} \quad \nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

We can think of ∇ as an abstract differential operator vector

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \left(\text{or } \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \text{ in two variables}\right)$$

$$\nabla f = \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f\right)$$

What does the gradient measure?.

$\nabla f(x,y)$ gives the direction of steepest ascent at (x,y) i.e the direction you would have to take if you wanted to increase your function as fast as possible.

eg: $f(x,y) = 2xy + x^2\sin y + x$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2y + 2x\sin y + 1 \;,\; 2x + x^2\cos y + 0\right)$$

if $f(xyz) = xyz + x^2 z$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(yz + 2xz \;,\; xz \;,\; xy + x^2\right)$$

Back to directional derivatives:

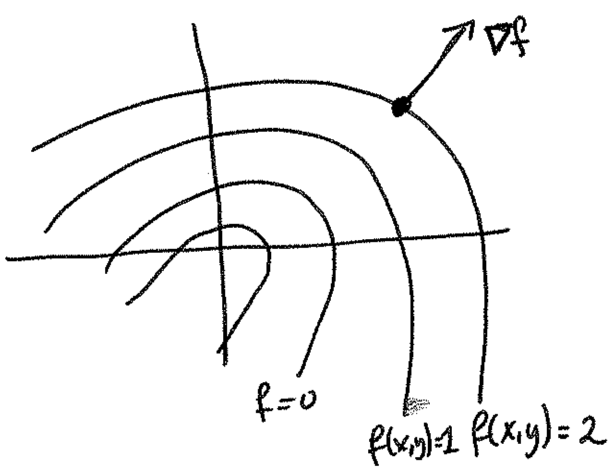
we have (recall: $\vec{u}$ unit direction vector)

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$

From this, we can see that ∇f has not only the direction of the steepest ~~descent~~ ascent!, but also has, as length, the rate of steepest ascent.

eg: calculate: for $f = x^2 + y^2 - xy$ calculate $D_u f(1,1)$ for the $x=y$ direction.

Normal vectors: Let us consider a function $z = f(x,y)$ draw level curves of the function. The gradient of the function gives a normal vector to the level curve because it gives the direction you would get away fastest from that level.



using this: (1) find the equation for the tangent line to the curve $9x^2+4y^2=36$

at the point $\left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$

(2) find the equation for the tangent plane to:

$$x^2+y^2+z^2=5$$

at the point $(2,1,0)$

(look at the book chapter 9 for how to do these)