Last time: we finished talking about series solutions.
- did review of math 114
- defined gradient \( \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \)
  \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \)

**Vector fields:**
- in two dimensions: \( F = P(x,y) \hat{i} + Q(x,y) \hat{j} \)
- in three \( F = P(x,y,z) \hat{i} + Q(x,y,z) \hat{j} + R(x,y,z) \hat{k} \).

\( F \) gives a vector at each point in space.

**Example:** \( F(x,y) = x \hat{i} + y \hat{j} \)

Vector fields model, for example:
- the velocity of each point in a fluid, or some gas.
- the force field in a physical situation, like gravity acting on a small mass.

We already have a way to get interesting vector fields: the gradient.

**For example:** If \( \phi = \sqrt{x^2 + y^2} \), then

\[ \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = \left( \frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}} \right) \]

we'll come back to this.
Recall: cross product of vectors.

**Define the curl of a vector field.** \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \)

\[
\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}
\]

\[
= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

What does the curl measure? It measures how much the vector field is curling around a point. We'll see this more when we understand Stokes' theorem.

\[
\text{curl} \mathbf{F} \neq 0 \quad \text{here}
\]

\[
\text{curl} \mathbf{F} = 0 \quad \text{here}
\]

**Example:** Let \( \mathbf{F} = \frac{x^2}{P} \mathbf{i} + y^2 \mathbf{j} + xy \mathbf{k} \)

\[
\text{curl} \mathbf{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}
\]

\[
= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

\[
= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

**Exercise:** Calculate the curl of the vector field:

\[
\mathbf{F} = \left( \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right)
\]
**Divergence:** Define the divergence of a vector field \( \mathbf{F} \) by:

\[
\nabla \cdot \mathbf{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (P, Q, R)
\]

\[
= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

Divergence calculates the flux around a small ball at a point. \( \nabla \cdot \mathbf{F} \) or box

\[
div > 0 \quad \implies \quad \text{no flux around a small ball or box around a point.}
\]

Example: let \( \mathbf{F} = x^2 \mathbf{i} + y^3 \mathbf{j} + \mathbf{k} \)

\[\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3 \quad \text{everywhere!}
\]

\( \mathbf{F} = x^2 \mathbf{i} + y^3 \mathbf{j} + \mathbf{k} \) constant vector field \( \nabla \cdot \mathbf{F} = 0 \)

**Line integrals:** are the same as regular integrals, but they are calculated on a line in a plane or 3-d space.

Let \( \mathbf{C} \) be a curve. \( P(x,y), Q(x,y) \) functions. 

a line integral along \( \mathbf{C} \) looks like

\[
\int_{\mathbf{C}} P \, dx + Q \, dy
\]

why do we write it like this? We want to use line integrals to calculate work. Say a particle is moved in a vector field \( \mathbf{F} \) along a curve \( \mathbf{C} \). Work = Force \times \text{distance} \quad \text{Put } \text{d}r = (dx, dy)

At an instant, the particle is moving by \( \text{d}r \), but not all the force \( \mathbf{F} \) is worked with or against, because \( \mathbf{F} \) might be at a nonzero angle to \( \text{d}r \).
For this reason, the work done at that instant is \( F \cdot dr \). We integrate this to find the total amount of work.

\[
\int F \cdot dr = \int (P, Q) \cdot (dx, dy) = \int Pdx + Qdy.
\]

How do we calculate these?

Say the curve is given by \( \gamma(t) = (x(t), y(t)) = (x_1(t), x_2(t)) \) with \( a \leq t \leq b \), then

\[
\int Pdx + Qdy = \int_a^b P(x(t), y(t)) \gamma'(t) \, dt + \int_a^b Q(x(t), y(t)) y'(t) \, dt
\]

Eg: Let's calculate the line integral \( \int_C y \, dx + x \, dy \) for the curve given by \( \gamma(t) = (\cos t, \sin t) \) \( 0 \leq t \leq \pi \) (the half-circle).

\[
\int_C Pdx + Qdy = \int_0^{\pi} P(\cos t, \sin t)(-\sin t) \, dt + \int_0^{\pi} Q(\cos t, \sin t) \cos t \, dt
\]

\[
= \int_0^{\pi} (-\sin^2 t \, dt) + \int_0^{\pi} \cos^2 t \, dt = \int_0^{\pi} \left( \cos^2 t - \sin^2 t \right) \, dt
\]

\[
= \int_0^{\pi} \cos 2t \, dt = \left( \frac{1}{2} \sin 2t \right)_0^{\pi} = 0
\]

If we had calculated with \( C = \) quarter-circle, we would have gotten

\[
\int_C Pdx + Qdy = \left( \frac{1}{2} \sin 2t \right)_0^{\pi/2} = \frac{1}{2}
\]
Similar properties to regular integrals hold for line integrals:

\[ \int_{C_1 + C_2} Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy \]

If \(-C\) is the same curve with reverse orientation, then

\[ \int_{-C} F \cdot dr = -\int_{C} F \cdot dr \]

If \(C\) is a closed curve, then we sometimes put a circle in the integral

\[ \oint_{C} F \cdot dr \]

**Example problem:** Let \(F\) be the vector field \(F = x \mathbf{i} + y \mathbf{j}\). Say I move a particle from \((1,1)\) to \((3,1)\) and then from \((3,1)\) to \((2,2)\). Find the work done.

The work done is \(\int_{C} F \cdot dr\) where \(C = C_1 + C_2\).

We need to parameterize these curves first.

- \(C_1\) can be given by \(x = 1 + 2t, y = 1\) for \(t \in [0,1]\).
- \(C_2\) : \(x = 3 - t, y = 1 + t\) for \(t \in [0,1]\).

\[ \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy = \int_{0}^{1} (1+2t)^2 dt + \int_{0}^{1} 1(0) dt \]

\[ \rightarrow \int_{0}^{1} (3+t)(-1)t + \int_{0}^{1} (1+t)^2 dt = (2t + t^2)|_{0}^{1} + \left(1 + 3t + \frac{t^2}{2}\right)|_{0}^{1} + \frac{1}{2}(t + t^2)|_{0}^{1} \]
Independence of path:

Some vector fields are special in the sense that if you calculate a line integral \( \int F \cdot dr \) with them, the answer does not depend on the curve chosen! But depends only on the endpoints. For such an \( F \), we have:

\[
\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr = \int_{C_3} F \cdot dr \quad \ldots
\]

for any curves starting at \( A \) and ending at \( B \). \( F \) is then called conservative.

\[ F = ye^{-x} + x\mathbf{j} \]

You can check that:

\[
\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr = \int_{C_3} F \cdot dr \quad \text{for these curves}
\]

or any other curve with the same endpoints.

Differential of a function: Say I have a function \( \phi : \mathbb{R}^2 \to \mathbb{R} \)

Define \( d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \nabla \phi \cdot dr \)

For \( \phi : \mathbb{R}^3 \to \mathbb{R} \)

\[ d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \nabla \phi \cdot dr \]
One more thing about independence of path:

Say \( \mathbf{F} \) is conservative, then \( \oint_{C} \mathbf{F} \cdot \mathbf{dr} = 0 \) for any closed curve \( C \).

Why? Take a closed curve \( C \). Break it down into \( C = C_1 - C_2 \).

\[
\oint_{C} \mathbf{F} \cdot \mathbf{dr} = \oint_{C_1} \mathbf{F} \cdot \mathbf{dr} - \oint_{C_2} \mathbf{F} \cdot \mathbf{dr} = 0
\]

These two are equal since \( \mathbf{F} \) is conservative!

Say \( \mathbf{F} \) is given by \( \nabla \phi \) for some function \( \phi \). Then \( \mathbf{F} \) is conservative and for \( C = A \to B \),

\[
\oint_{C} \mathbf{F} \cdot \mathbf{dr} = \phi(B) - \phi(A)
\]

Proof: \( \mathbf{F} = \nabla \phi \) means \( \mathbf{F} = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \)

\[
\oint_{C} \mathbf{F} \cdot \mathbf{dr} = \oint_{C} \frac{\partial \phi}{\partial x} \, dx + \oint_{C} \frac{\partial \phi}{\partial y} \, dy + \oint_{C} \frac{\partial \phi}{\partial z} \, dz
\]

If we parameterized the curve into \( x(t), y(t), z(t) \) \( t \in [a, b] \)

Then

\[
= \left[ \left( \frac{\partial \phi}{\partial x} \frac{dx}{dt} \right) + \left( \frac{\partial \phi}{\partial y} \frac{dy}{dt} \right) + \left( \frac{\partial \phi}{\partial z} \frac{dz}{dt} \right) \right] dt
\]

\[
= \phi(y(b)) - \phi(y(a)) = \phi(B) - \phi(A).
\]
Test for path independence.

Let $F$ be defined in a simply connected region $D$.

If $F = (P, Q) = P \, dx + Q \, dy = 0$ if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ for every } (x, y) \text{ in the region}$$

**Ex:** Say we want to calculate $\int_C y \, dx + x \, dy$ along the curve in the picture:

$$F = y \, dx + x \, dy$$

We can check for path independence:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad 1 = 1 \quad \checkmark$$

we can find $\phi$ s.t $F = \nabla \phi \quad \phi = xy$

Then $\int_C y \, dx + x \, dy = \phi(1, 1) - \phi(0, 0) = 1$.

**Ex:** Find, along the same curve $C$, $\int_C 2x \sin y \, dx + x^2 \cos y \, dy$

Path indep check: $\frac{\partial P}{\partial y} = 2x \cos y$

$$\frac{\partial Q}{\partial x} = 2x \cos y$$

$\Rightarrow$ find $\phi = x^2 \sin y$

So $\int_C F \, dr = \phi(1, 1) - \phi(0, 0) = \sin(1)$.

**Ex:** Find the integral of $\int_C 2xe^{x^2} \, y^4 + e^{x^2} \, dy$ along $C$.

We need to find $\phi$, it is a closed curve and $F = (2xe^{x^2}, e^{x^2})$ is conservative. So $\int_C F \, dr = 0$. 

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