Lecture 13: Last time: . we finished talking about series solutions.
. did review of math 114
. defined gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

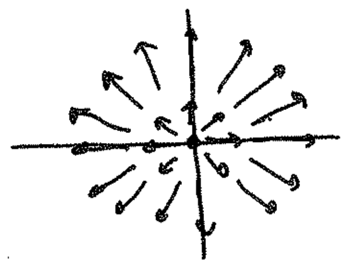
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Vector fields: in two dimensions: $F = P(x,y)\vec{i} + Q(x,y)\vec{j}$

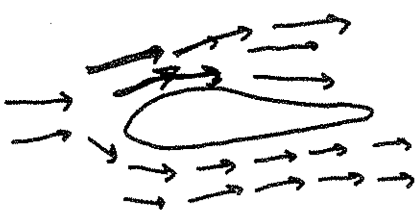
in three $F = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$.

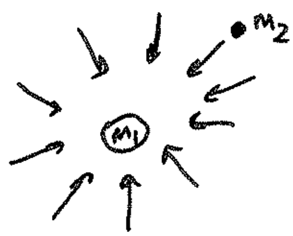
F gives a vector at each point in space.

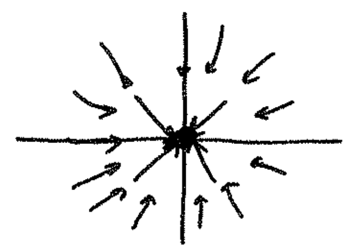
eg: $F(x,y) = x\vec{i} + y\vec{j}$



Vector fields model, for example:
the velocity of each point in a fluid, or some gas.
Or the force field in a physical situation, like gravity acting on a small mass.







We already have a way to get interesting vector fields: the gradient.

For example: If $\phi = \sqrt{x^2+y^2}$, then

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right)$$

we'll come back to this

Recall: cross product of vectors.

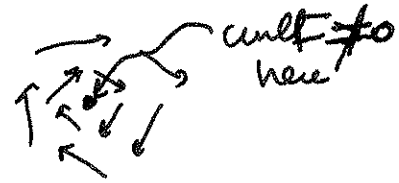
<u>Define</u> the curl of a vector field. $F = P\vec{i} + Q\vec{j} + R\vec{k}$

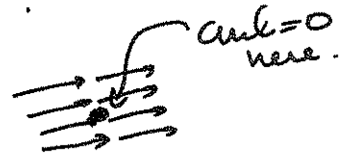
as:

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

What does the curl measure? It measures how much the vector field is curling around a point. We'll see this more when we understand Stokes' theorem.





<u>eg</u>: Let $F = \underbrace{xz}_{P}\vec{i} + \underbrace{yz}_{Q}\vec{j} + \underbrace{xy}_{R}\vec{k}$ $\qquad \operatorname{curl} F = ?$

~~can~~ $\frac{\partial P}{\partial y} = 0 \quad \frac{\partial P}{\partial z} = x$

$\frac{\partial Q}{\partial x} = 0 \quad \frac{\partial Q}{\partial z} = y$

$\frac{\partial R}{\partial x} = y \quad \frac{\partial R}{\partial y} = x$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= (x-y)\vec{i} + (x-y)\vec{j} + (0-0)\vec{k}$$

<u>exercise</u>: Calculate the curl of the vector field:

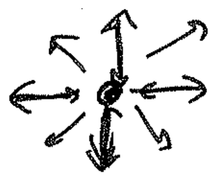
$$F = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right)$$

Divergence: Define the divergence of a vector field F by:

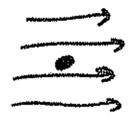
$$\text{DIV}F = \nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Divergence calculates the flux around a small ball (or box) at a point



div > 0

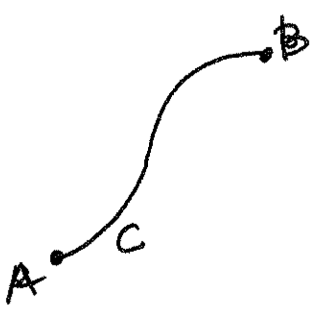


div = 0
no flux around a small ball or box around a point.

eg: let $F = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad$ $\text{div}F = 1 + 1 + 1 = 3$ everywhere!

$F = \vec{i} + \vec{j} + \vec{k}$ constant vector field $\text{div}F = 0$.

Line integrals: are the same as regular integrals, but they are calculated on a ~~line~~ curve in a plane or 3-d space.

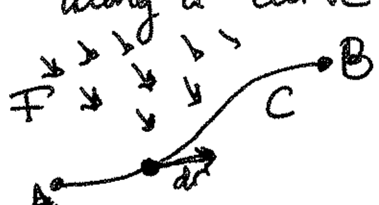


Let C be a curve. $P(x,y)$, $Q(x,y)$ functions.

a line integral along C looks like

$$\int_C P\,dx + Q\,dy$$

why do we write it like this? We want to use line integrals to calculate work. Say a particle is moved in a vector field F along a curve C. Work = Force x distance. Put $dr = (dx, dy)$. At an instant, the particle is moving by dr, but not all the force in F is worked with or against, because F might be at a nonzero angle to dr.



For this reason, the work done at that instant is $F \bullet dr$. We integrate this to find the total amount of work.

$$\int_C F \bullet dr = \int_C (P, Q) \bullet (dx, dy) = \int_C P\,dx + Q\,dy.$$

How do we calculate these?

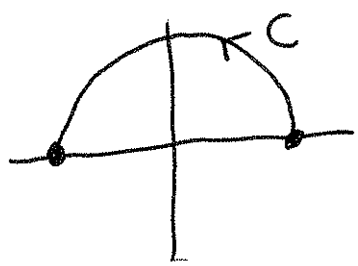
Say the curve is given by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ with $a \leq t \leq b$

then
$$\int_C P\,dx + Q\,dy = \int_a^b P(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t)\,dt + \int_a^b Q(\gamma_1(t), \gamma_2(t))\gamma_2'\,dt$$

eg: Let's calculate the line integral $\int_C \overset{P}{y}\,dx + \overset{Q}{x}\,dy$ for

the curve given by $\gamma(t) = (\underbrace{\cos t}_{\gamma_1}, \underbrace{\sin t}_{\gamma_2})$ $\quad 0 \leq t \leq \pi$

(the half circle)

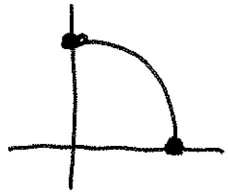


$$\int_C P\,dx + Q\,dy = \int_0^\pi P(\cos t, \sin t)(-\sin t)\,dt + \int_0^\pi Q(\cos t, \sin t)\cos t\,dt$$

$$= \int_0^\pi (-\sin^2 t\,dt) + \int_0^\pi \cos^2 t\,dt = \int_0^\pi \underbrace{(\cos^2 t - \sin^2 t)}_{\cos 2t}\,dt$$

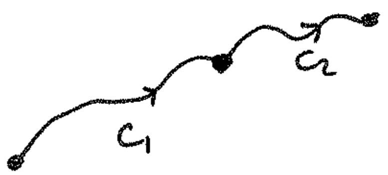
$$= \int_0^\pi \cos 2t\,dt = \left(\frac{1}{2}\sin 2t\right)\Big|_0^\pi = 0$$

If we had calculated with $C =$ quarter-circle, we would have

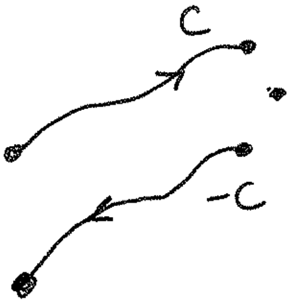


gotten $\int_C P\,dx + Q\,dy = \left(\frac{1}{2}\sin 2t\right)_0^{\pi/2} = \frac{1}{2}$

Similar properties to regular integrals hold for line integrals



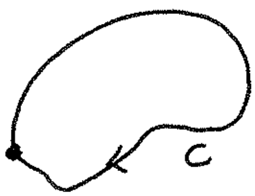
$$\int_{C_1+C_2} P\,dx+Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx+Q\,dy$$



if $-C$ is the same curve with reverse orientation, then

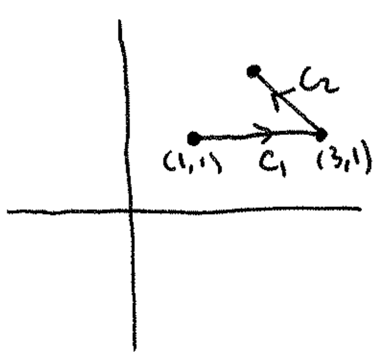
$$\int_{-C} F\cdot dr = -\int_{C} F\cdot dr$$

If C is a closed curve, then we sometimes put a circle in the integral



$$\oint_C F\cdot dr$$

<u>example problem</u>: Let F be the vector field $F = x\vec{i} + y\vec{j}$
say I move a particle from $(1,1)$ to $(3,1)$ and then from $(3,1)$ to $(2,2)$
Find the work done. The work done is $\int_C F\cdot dr$ where $C = C_1 + C_2$.



we need to parameterize these curves first

C_1 can be given by $\{\, x = 1+2t \quad y = 1$ for $t\in[0,1]$

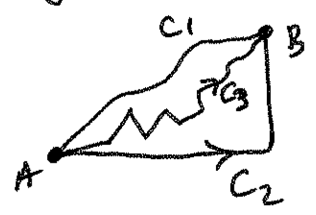
C_2 $\quad x = 3-t \quad y = 1+t \quad t\in[0,1]$

$$\int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr = \int_{C_1} P\,dx+Q\,dy + \int_{C_2} P\,dx+Q\,dy = \int_0^1 (1+2t)\cdot 2\,dt + \int_0^1 1\cdot(0)\,dt \longrightarrow$$

$$\longrightarrow + \int_0^1 (3-t)(-1)t + \int_0^1 (1+t)\cdot 1\,dt = (2t+t^2)\Big|_0^1 + 0 + \left(-3t+\frac{t^2}{2}\right)\Big|_0^1 + \left(t+\frac{t^2}{2}\right)\Big|_0^1$$

Independence of path:

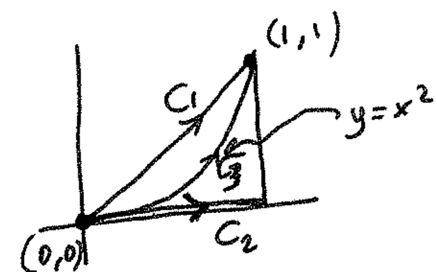
Some vector fields are special in the sense that if you calculate a line integral $\int_C F\cdot dr$ with them, the answer does not depend on the curve chosen! But depends only on the end-points. For such an F, we have:



$$\int_{C_1} F\,dr = \int_{C_2} F\,dr = \int_{C_3} F.dr \quad \ldots$$

for any curves starting at A and ending at B. F is then called <u>conservative</u>.

eg: $F = y\vec{i} + x\vec{j}$



you can check that:

$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr = \int_{C_3} F\cdot dr$$ for ~~all~~ these curves (or any other curve with the same endpoints)

d of a function: Say I have a function $\phi : \mathbb{R}^2 \to \mathbb{R}$

define $$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \nabla\phi\cdot dr$$

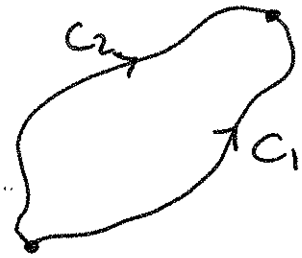
for $\phi : \mathbb{R}^3 \to \mathbb{R}$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \nabla\phi\cdot dr$$

One more thing about independence of path:

Say F is conservative, then $\oint_C F \cdot dr = 0$ for any closed curve C.

<u>why?</u>: take a closed curve C. Break it down into $C = C_1 - C_2$



$C = C_1 - C_2$.

$$\int_C F \cdot dr = \underbrace{\int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr}_{\text{these two are equal since } F \text{ is conservative!}} = 0$$

Say F is given by $\nabla \phi$ for some function ϕ. Then F is conservative and for $C =$

$$\int_C F \cdot dr = \phi(B) - \phi(A)$$

This is just like the fundamental theorem of calculus! $\int_C d\phi = \phi(B) - \phi(A)$

<u>proof</u>: $F = \nabla \phi$ means $F = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$

$$\int_C F \cdot dr = \int_C \frac{\partial \phi}{\partial x} dx + \int_C \frac{\partial \phi}{\partial y} dy + \int_C \frac{\partial \phi}{\partial z} dz$$

if we parameterized the curve into $x(t), y(t), z(t)$ $t \in [a,b]$ then

$$= \int_a^b \underbrace{\left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \right)}_{\frac{d\phi(\gamma)}{dt}} dt$$

$$= \phi(\gamma(b)) - \phi(\gamma(a)) = \phi(B) - \phi(A).$$

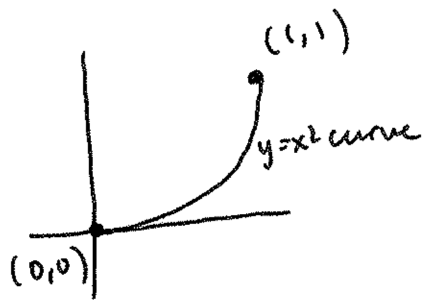
Test for path independence.

Let F be defined in a simply connected region.

$F = (P, Q) = P\vec{\imath} + Q\vec{\jmath} = 0$ if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{for every } (x,y) \text{ in the region}$$

eg: Say we want to calculate $\int y\,dx + x\,dy$ along the curve in the picture



$F = y\vec{\imath} + x\vec{\jmath}$

we can check for path independence:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$1 = 1 \quad \checkmark$$

we can find ϕ s.t $F = \nabla\phi$ $\qquad \phi = xy$

then $\int_C y\,dx + x\,dy = \phi(1,1) - \phi(0,0) = 1$.

eg: Find, along the same curve C, $\int_C 2x\sin y\,dx + x^2\cos y\,dy$

path indep check: $\frac{\partial P}{\partial y} = 2x\cos y$ $\qquad$ find $\phi = x^2\sin y$

$\frac{\partial Q}{\partial x} = 2x\cos y$

so $\int_C F\cdot dr = \phi(1,1) - \phi(0,0) = \sin(1)$

eg: Find the integral of $\oint_C 2xe^{x^2}y\,dx + e^{x^2}dy$ along 

no need to find ϕ, it is a closed curve and $F = (2xe^{x^2}y, e^{x^2})$ is conservative so answer $= 0$.