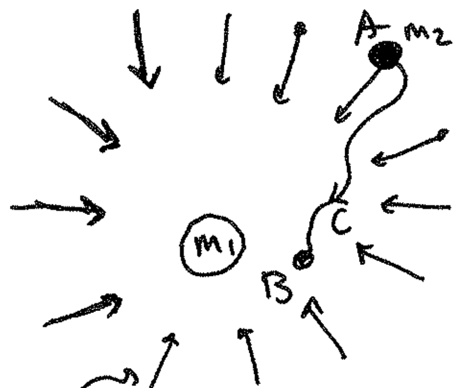
$\nabla \times F$ $\nabla \cdot f$

Lecture 14: . Last time: we talked about vector fields, curl, divergence, line integrals and path independence

Review of vector fields, line integrals and path independence.
Recall the definition of path independence and the test theorem.

Physical picture:



Force field F

m_1 is the frame of reference
(we think of the position of m_1 as $(0,0,0)$ at all times)

Gravity is acting on m_2 as a force field.

If m_2 moves along a curve C, the work done by F is $\int_C F \cdot dr$

If $F = \nabla \phi$ (as is the case for gravity) then ϕ is the 'potential energy'

$$\underbrace{\int_C F \cdot dr}_{\text{"work done" by } F} = \underbrace{\phi(B) - \phi(A)}_{\text{"change in potential energy"}}$$

In the case of gravity, the force has magnitude $G \frac{m_1 m_2}{(\text{distance})^2}$
and is towards m_1 which is at the origin.

$$F = \left(\frac{-x \, G m_1 m_2}{\left(\sqrt{x^2+y^2}\right)^3}, \frac{-y \, G m_1 m_2}{\left(\sqrt{x^2+y^2}\right)^3} \right)$$

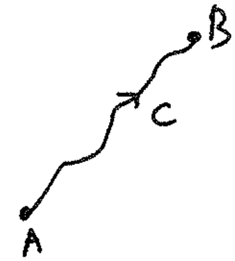
indeed, the force has direction $\left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{x^2+y^2} \right)$.

and $\phi = \frac{1}{\sqrt{x^2+y^2}}$

This is in 2d, you can see what it is in 3d easily from this.

means: no holes in the region

<u>Theorem</u>: Let $F = P\vec{i} + Q\vec{j}$ be defined in a simply connected region R in $\mathbb{R}^2$. Then the following are equivalent:



(1) $\int_C F \cdot dr$ depends only on the endpoints of C

((1') $\oint_C F \cdot dr = 0$ for every closed curve C in R)

(2) There is a function ϕ s.t. $F = \nabla \phi$

and $\int_C F \cdot dr = \phi(B) - \phi(A)$ for every curve C with endpoints A and B

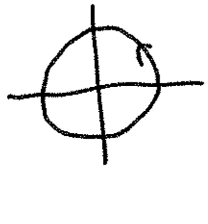
(3) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ for every point in R.

<u>Remark</u>: The first two ~~conditions~~ statements are equivalent whether R is simply connected or not. But the equivalence with the last condition requires that R is simply connected.

Consider $F = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$

check: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

Let C be the unit circle. Parameterize: $x = \cos t$, $y = \sin t$



$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} (-\sin t)(-\sin t)\,dt + \cos t \cos t \,dt$$

$$= \int_0^{2\pi} 1 = 2\pi.$$

the integral is not path independent even though the 3rd condition holds.

The same theorem holds in three dimensions as well.

<u>Theorem</u>: Let F be defined in a simply connected region R.

$F = P\vec{i} + Q\vec{j} + R\vec{k}$ is ~~independent of path~~ conservative

($\int_C F \cdot dr$ is independent of path) if and only if

$$\underbrace{\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

$$\Updownarrow$$

$$\operatorname{curl} F = 0.$$

<u>eg</u> $F = (y + yz)\vec{i} + (x + 3z^3 + xz)\vec{j} + (9yz^2 + xy - 1)\vec{k}$

check that $\operatorname{curl} F = 0$.

Can we find ϕ s.t. $F = \nabla\phi$

$\frac{\partial \phi}{\partial x} = y + yz$ tells us that $\phi = xy + xyz + g(y,z)$

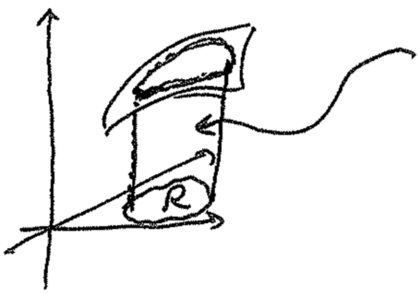
$\frac{\partial \phi}{\partial y} = x + 3z^3 + xz$ says $\phi = xy + xyz + 3yz^3 + g_2(x,z)$

$\frac{\partial \phi}{\partial z} = 9yz^2 + xy - 1$ says $\phi = xyz + 3yz^3 - z + g_3(x,y)$

so $$\phi = xy + xyz + 3yz^3 - z$$

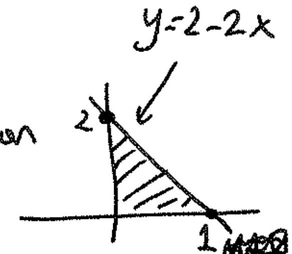
Double integrals:

Say we have a function $z = f(x,y)$ and we want to find the volume under its graph over a region R in the xy plane. The volume is given by:



$$\iint_R f(x,y)\,\underbrace{dx\,dy}_{dA}$$

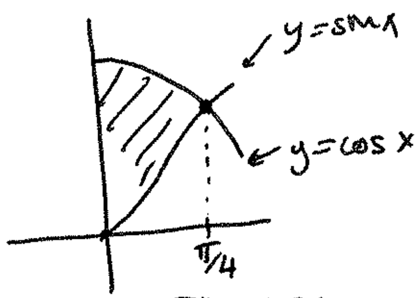
eg: Say we want $\iint_R x^2 y\, dA$ where R is the region



we write $\iint_R x^2 y = \int_0^1 \int_0^{2-2x} x^2 y \, dy\, dx$

equivalently: $\iint_R x^2 y = \int_0^2 \underbrace{\int_0^{\frac{2-y}{2}} x^2 y \, dx}\, dy$ then evaluate one by one.

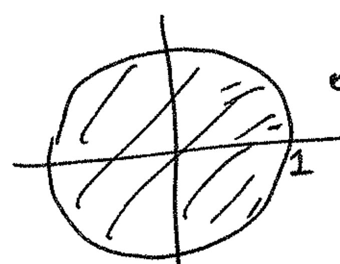
eg:



$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} f(x,y)\,dx\,dy$$

this way is better because we don't have to break it down to two integrals.

eg:

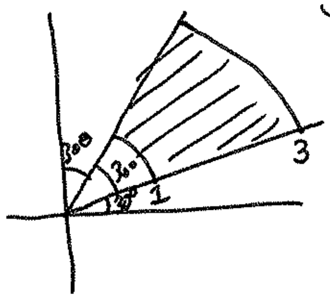


Better in polar. ~~with~~ we'll see next.

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dx\,dy$$

Double integrals in polar coordinates:

Say R is given as: It is much easier to do this with polar coordinates.



$$\iint_R f(x,y)\,dx\,dy = \int_{\pi/6}^{2\pi/6}\int_1^3 f(x,y)\, r\,dr\,d\theta$$

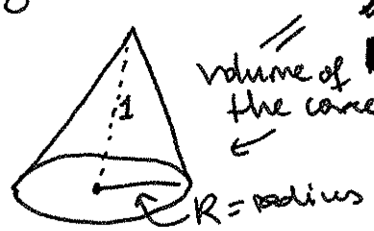
$$= \int_1^3 \int_{\pi/6}^{2\pi/6} f(x,y)\, r\,d\theta\,dr$$

we also have to rewrite f in terms of r and θ.

we add the r for changing from $dx\,dy$ to $dr\,d\theta$. Why?

because a small change in θ gives more area if we are away from the origin!

eg: evaluate: $\displaystyle \iint \left(1 - \frac{\sqrt{x^2+y^2}}{R}\right) dx\,dy$



$=$ volume of the cone

$$= \int_0^R \int_0^{2\pi} \left(1 - \frac{r}{R}\right) r\,d\theta\,dr$$

$$= \int_0^R \left(\theta r - \frac{r^2\theta}{R}\right)\Bigg|_0^{2\pi} dr$$

$$= \int_0^R \left(2\pi r - \frac{2\pi r^2}{R}\right) dr$$

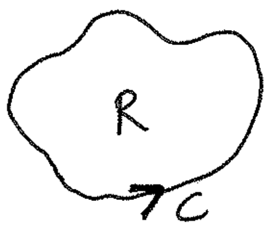
$$= \left(2\pi \frac{r^2}{2} - \frac{2\pi r^3}{3R}\right)\Bigg|_0^R = \pi R^2 - \frac{2\pi}{3}R^2 = \frac{\pi R^2}{3}$$

If we still have time ...

Green's theorem:

oriented in the positive direction (counterclockwise)

Say F is defined in a <u>simply connected</u> region in $\mathbb{R}^2$. Let C be a closed curve bounding a region R.

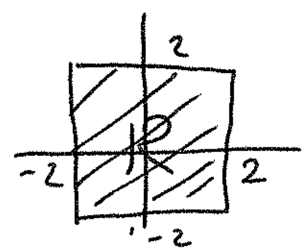


Then:

$$\oint_C F \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

eg: $F = (-16y + \sin x^2)\,\vec{i} + (4e^y + 3x^2)\,\vec{j}$

R is the region in the picture:



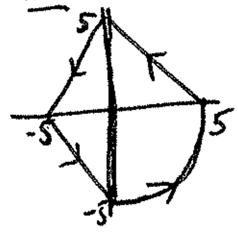
$$\oint_C F \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

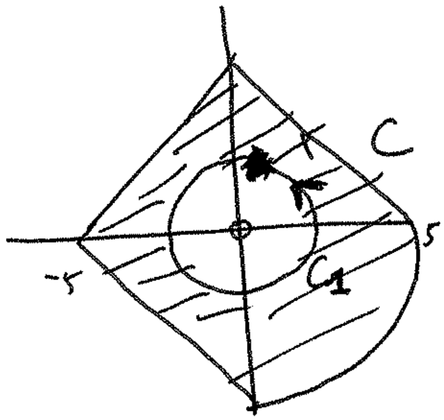
$$= \int_{-2}^{2} \int_{-2}^{2} \underbrace{(6x + 16)}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} \, dx\,dy = \text{easy}.$$

Observe that Green's theorem says that if F is conservative, then $\oint_C F \cdot dr = 0$ for closed C.

A nice trick: eg Calculate: $\oint F \cdot dr$ for $F = \left(\frac{-y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right)$

where $C =$

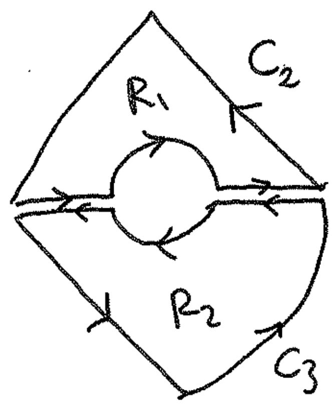




We had seen before that, for this F,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{so} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

But the answer is not 0 because The region is not simply connected. Consider the region in the picture between C and the unit circle. This is still not simply connected. but we can break it down! these two curves are ~~inon~~ each in a simply connected region where the ~~expression~~ field F is defined! So we can apply Green's theorem for each of them.



$$\int_{C_2} F\cdot dr = \iint_{R_1} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA \quad \text{and} \quad \int_{C_3} F\cdot dr = \iint_{R_2} \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0} dA$$

But $C_2 + C_3 = C - C_1$

$$\int_{C-C_1} F\cdot dr = \int_{C_2+C_3} F\cdot dr = \iint_{R_1+R_2} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA = 0.$$

So $\displaystyle\int_{C} F\cdot dr = \int_{C_1} F\cdot dr$

So ~~then~~ we can use C_1 to calculate the answer. (which we had calculated before, and got 2π)