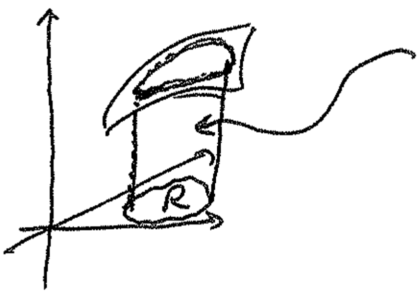
Lecture 15: Last time, we talked in depth about path independence and conservative vector fields. We stressed the physical example of the gravity vector field.
In the end, we recalled how to write and evaluate double integrals.

One important example we did was of a vector field that satisfied $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ but was not conservative in a region containing the origin. It was: $F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$
The reason for this was that F is not defined at the origin so the region containing the origin was not simply connected.

Today:
- quick review of double integrals in polar coordinates.
- Green's theorem,
- surface integrals

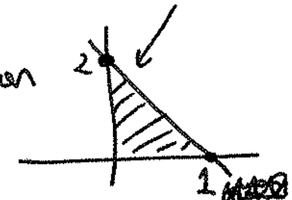
Double integrals:

Say we have a function $z = f(x,y)$ and we want to find the volume under its graph over a region R in the xy plane. The volume is given by:



$$\iint_R f(x,y)\,\underbrace{dx\,dy}_{dA}$$

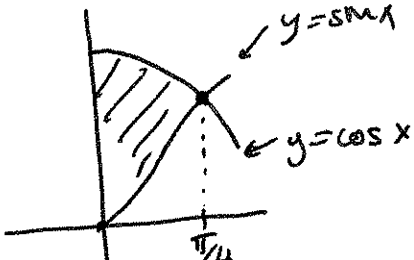
eg: Say we want $\iint_R x^2y\,dA$ where R is the region



we write $\iint_R x^2y = \int_0^1 \int_0^{2-2x} x^2y\,dy\,dx$

equivalently: $\iint_R x^2y = \int_0^2 \int_0^{\frac{2-y}{2}} x^2y\,dx\,dy$ then evaluate one by one.

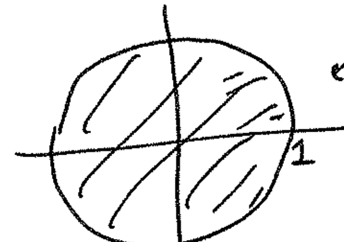
eg:



$$\int_0^{\pi/4}\int_{\sin x}^{\cos x} f(x,y)\,dx\,dy$$

this way is better because we don't have to break it down to two integrals.

eg:

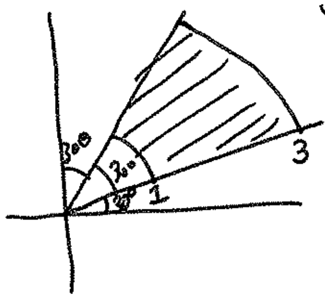


Better in polar. ~~which~~ we'll see next.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dx\,dy$$

Double integrals in polar coordinates:

Say R is given as: It is much easier to do this with polar coordinates.



$$\iint_R f(x,y)\,dx\,dy = \int_{\pi/6}^{2\pi/6}\int_1^3 f(x,y)\,\underline{r}\,dr\,d\theta$$

$$= \int_1^3\int_{\pi/6}^{2\pi/6} f(x,y)\,\underline{r}\,d\theta\,dr$$

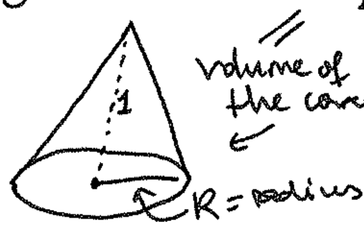
we also have to rewrite f in terms of r and θ.

we add the r for changing from $dx\,dy$ to $dr\,d\theta$. Why?

because a small change in θ gives more area if we are away from the origin!

eg: evaluate:

$$\iint \left(1 - \frac{\sqrt{x^2+y^2}}{R}\right) dx\,dy$$



volume of the cone

$$= \int_0^R\int_0^{2\pi}\left(1-\frac{r}{R}\right)\underline{r}\,d\theta\,dr$$

$$= \int_0^R \left(\theta r - \frac{r^2\theta}{R}\right)\Big|_0^{2\pi} dr$$

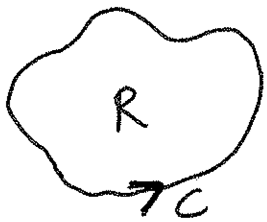
$$= \int_0^R \left(2\pi r - \frac{2\pi r^2}{R}\right) dr$$

$$= \left(2\pi\frac{r^2}{2} - \frac{2\pi r^3}{3R}\right)\Big|_0^R = \pi R^2 - \frac{2\pi}{3}R^2 = \frac{\pi R^2}{3}$$

Green's theorem:

oriented in the positive direction (counterclockwise)

Say F is defined in a simply connected region in $\mathbb{R}^2$. Let C be a closed curve bounding a region R.

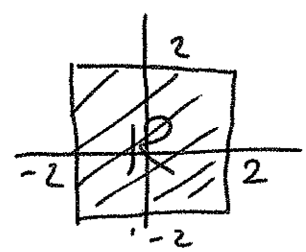


Then:

$$\oint_C F \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

eg: $F = (-16y + \sin x^2)\vec{\imath} + (4e^y + 3x^2)\vec{\jmath}$

R is the region in the picture:

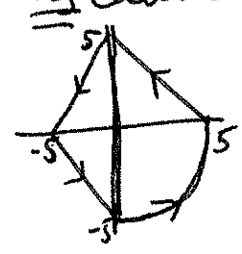


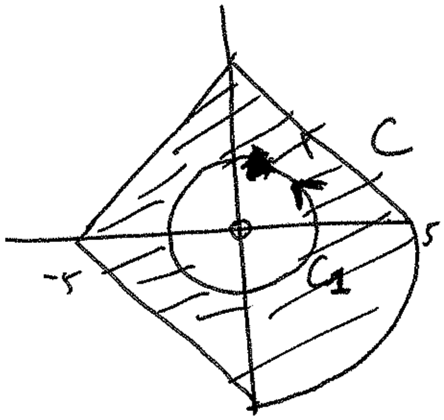
$$\oint_C F \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \int_{-2}^{2}\int_{-2}^{2} (\underbrace{6x + 16}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}})\, dx\,dy = \text{easy}.$$

Observe that Green's theorem says that if F is conservative, then $\oint_C F \cdot dr = 0$ for closed C.

A nice trick: eg Calculate: $\oint F \cdot dr$ for $F = \left(\frac{-y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right)$

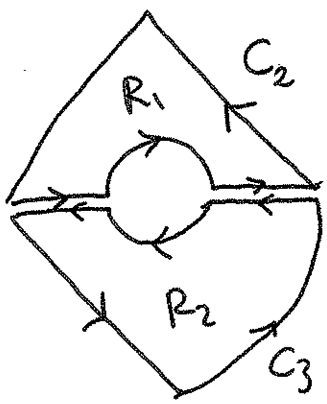
where $C =$ 



We had seen before that, for this F,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{so} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 .$$

But the answer is not 0 because the region is not simply connected. Consider the region in the picture between C and the unit circle. This is still not simply connected. but we can break it down! these two curves are each in a simply connected region where the field F is defined! So we can apply Green's theorem for each of them.



$$\int_{C_2} F\cdot dr = \iint_{R_1} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA \quad \text{and} \quad \int_{C_3} F\cdot dr = \iint_{R_2} \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0} dA$$

But $C_2 + C_3 = C - C_1$

$$\int_{C-C_1} F\cdot dr = \int_{C_2+C_3} F\cdot dr = \iint_{R_1+R_2} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA = 0 .$$

So $\displaystyle\int_{C} F\cdot dr = \int_{C_3} F\cdot dr$

So we can use C_1 to calculate the answer.
(which we had calculated before, and got 2π)

Recall that our theorem for path independence stated the equivalence of three things: For F defined in a simply connected region R.

(1) $\int_C F\,dr$ is independent of path

(1') $\oint_C F\cdot dr = 0$ for every closed curve C.

(2) There is a ϕ s.t. $F = D\phi$ and $\int_C F\cdot dr = \phi(B) - \phi(A)$

(3) $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

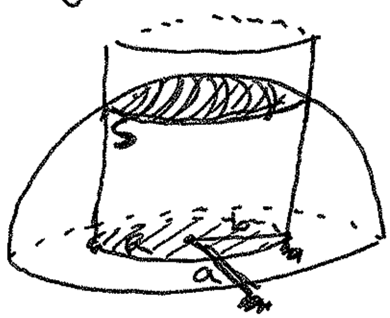
$$\begin{array}{c}(1')\\ \Updownarrow\end{array}$$

We had seen why (1) and (2) were equivalent. It is also easy to see why (2) $\Rightarrow$ (3) (by the commutativity of partial derivatives) We had not said why (3) $\Rightarrow$ (1) or (3) $\Rightarrow$ (2). Green's theorem gives this immediately. Since for a closed curve C,

$$\oint_C F\cdot dr = \iint_{\substack{\text{Region}\\ \text{bounded}\\ \text{by } C}} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\overset{\shortparallel}{0} \text{ if (3) is true.}} dx\,dy = 0 .$$

Surface integrals:

Say we want to find the area of ~~a~~ the portion of a sphere $x^2+y^2+z^2=a^2$ that is inside a cylinder $x^2+y^2=a^2$



we can do this by integrating 1 over that surface. But how do we do that?

$$\iint_S 1\, dA = ?$$

area element on the surface.

We can calculate this integral by changing it to an integral on the disc in the $z=0$ plane lying under it.

$$\iint_S 1\, dA = \iint_R 1 \underbrace{\sqrt{1+(f_x)^2+(f_y)^2}}\, dx\,dy$$

we need to change the area element because dA is different from $dxdy$!!!

the surface is given by $z=f(x,y)=\sqrt{a^2-x^2-y^2}$

$$f_x=\frac{\partial f}{\partial x}=\frac{-x}{\sqrt{a^2-x^2-y^2}} \qquad \frac{\partial f}{\partial y}=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

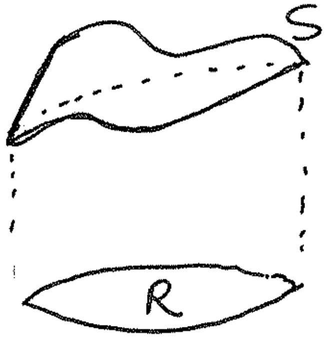
$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}=\sqrt{\frac{a^2}{a^2-x^2-y^2}}=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

don't forget the r.

$$\text{so} \quad \iint_S 1\, dA = \iint_{\substack{\text{disc}\\ \text{of radius}\\ b}} \frac{a}{\sqrt{a^2-x^2-y^2}}\, dx\,dy = a\int_0^{2\pi}\int_0^b \frac{1}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$\ldots = 2\pi a\left(a-\sqrt{a^2-b^2}\right)$$

We can use the same method to calculate the integral of any function on any surface. Say $z=f(x,y)$ gives the surface.



$$\iint_S G(x,y,z)\,dA = \iint_R G(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy$$

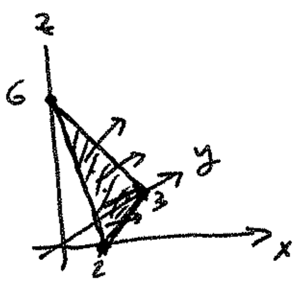
$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}$$

adjustment to change dA into $dxdy$.

<u>exercise</u>: find $\iint_S x\,dA$ for the surface in the example below.

Recall $\vec{n}$ always represents unit normal vector to a surface.

<u>eg</u>: Let $\mathbf{F}(x,y,z) = z\vec{j} + z\vec{k}$ represent the flow of some liquid. Find the flux $\iint_S \mathbf{F}\cdot\vec{n}$ of F through the surface S given by the portion of the plane $z = 6-3x-2y$ lying in the first octant $(x>0, y>0, z>0)$



How do we find $\vec{n}$. The surface is given as

$$z+3x+2y-6=0$$

If we take the gradient of this, and plug in any point on the surface, we will get a normal vector. (since this is a plane, all the normal vectors are the same)

$\vec{n} \approx (3,2,1)$ but this is not $\vec{n}$, because $\vec{n}$ has to be the unit normal vector, so we need to divide this vector by its length.

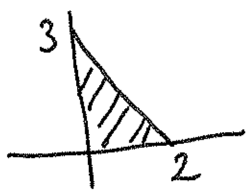
$$\vec{n} = \left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$$

So flux $= \iint_S F \cdot n\,dA = \iint_S (0, z, z) \cdot \left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right) dA$

$$= \frac{1}{\sqrt{14}} \iint_S \underbrace{3z}_{G(x,y,z)}\, dA$$

to evaluate, we evaluate over the region on the xy plane

$z = 6 - 3x - 2y = f(x,y)$ $\quad \frac{\partial f}{\partial x} = -3 \quad \frac{\partial f}{\partial y} = -2$



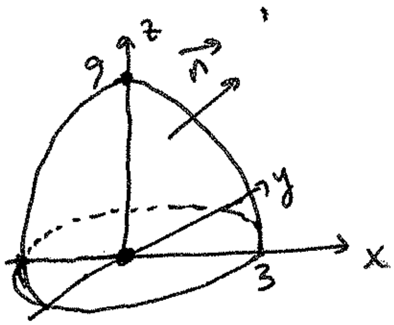
$$\frac{1}{\sqrt{14}} \iint_S 3z\,dA = \frac{1}{\sqrt{14}} \int_0^2 \int_0^{\frac{6-3x}{2}} 3(6-3x-2y) \underbrace{\sqrt{1+(-3)^2+(-2)^2}}_{\sqrt{14}}\,dy\,dx$$

$$= 3 \int_0^2 \int_0^{\frac{6-3x}{2}} (6-3x-2y)\,dy\,dx$$

$$\ldots = 18$$

<u>one more example</u>: $F = z\vec{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ above the xy plane.

Find the flux. Flux is $\iint_S F \cdot \vec{n}$, we first need $\vec{n}$



the surface is given by

$$z + x^2 + y^2 - 9 = 0$$

take the gradient:

$$(2x, 2y, 1)$$

normalize: $\vec{n} = \left(\frac{2x}{\sqrt{1+4x^2+4y^2}}, \frac{2y}{\sqrt{1+4x^2+4y^2}}, \frac{1}{\sqrt{1+4x^2+4y^2}} \right)$

$$\vec{F} \cdot \vec{n} = (0, 0, z) \cdot \vec{n} = \frac{z}{\sqrt{1+4x^2+4y^2}}$$

$$z = 9 - x^2 - y^2$$
$$\frac{\partial f}{\partial x} = -2x \quad \frac{\partial f}{\partial y} = -2y$$

$$\iint_S F \cdot \vec{n}\, dA = \iint_S \frac{z}{\sqrt{1+4x^2+4y^2}}\, dA$$

$$= \iint_{\substack{\text{disc}\\ \text{of radius } 3}} \frac{z}{\sqrt{1+4x^2+4y^2}} \sqrt{1+4x^2+4y^2}\, dx\,dy$$

$$= \iint_{\substack{\text{disc}\\ \text{of radius } 3}} z\, dx\,dy = \iint_{\substack{\text{disc}\\ \text{of radius } 3}} (9-x^2-y^2)\, dx\,dy$$

polar coords

$$= \int_0^{2\pi}\int_0^{3} (9-r^2)\,\underline{\underline{r}}\, dr\,d\theta = \int_0^{2\pi} \left(9r - \frac{r^3}{3}\right)\Big|_0^3 dr\,d\theta$$

$$= 2\pi(27-9) = 36\pi.$$