Lecture 16: Last time:

- Green's theorem.

C a positively oriented closed curve bounding a simply connected region R

F a vector field defined on the whole region. Then

$$\oint_C F\cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Say we are given a quantity at each point of the surface S. Then the surface integral calculates the total quantity that we get from S:

- Surface integrals.

Say S is given by $z = f(x,y)$

$$\iint_S G(x,y,z)\,dA$$
$$= \iint_R G(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

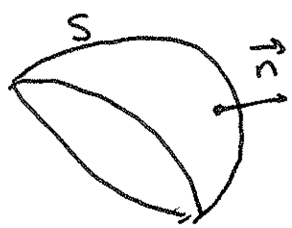
the idea is to express an integral on S by an integral on R. It's not really different than having an integral

$$\int_9^{10} f(t)\,dt \qquad t = x+9$$

and writing it as: $\displaystyle\int_0^1 f(x+9)\,dx$

The difference is that here dx is dt, but dA is not equal to $dx\,dy$ it is $dA = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$

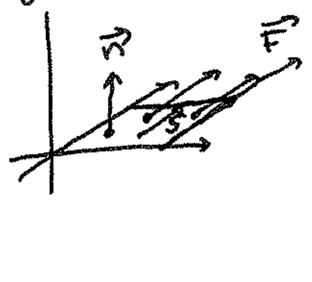
- Flux calculations. The flux of a vector field through a surface S is given by $\displaystyle\iint_S F\cdot\vec{n}\,dA$



here, $\vec{n}$ is the unit normal vector field to S.

simple example:

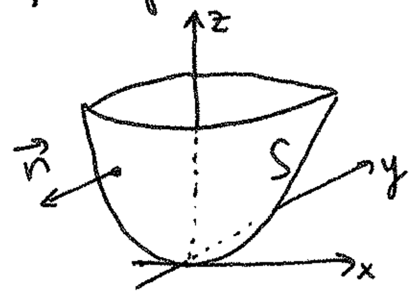
Say S is the square $z=0$, $0\le x\le 1$, $0\le y\le 1$ and $F = x\vec{\imath} + x\vec{k}$



$\vec{n}$, here is $(0,0,1) = \vec{k}$

$$\text{Flux} = \iint_S F\cdot\vec{n} = \int_0^1\int_0^1 x\,dx\,dy = \frac{1}{2}$$

eg: Find the outward flux of the vector field $F = 0\vec{i} + 0\vec{j} + z\vec{k}$.
through the surface given by: $z = x^2 + y^2 \quad 0 \leq z \leq 1$



⚠ we want $\vec{n}$ to be pointing outward.

$$\text{Flux} = \iint_S F \cdot \vec{n}\, dA$$

to find $\vec{n}$, note that S is given as a level curve of the function $z - x^2 - y^2$

so the gradient of this gives a normal vector field.

$$(-2x, -2y, 1)$$

But these vectors point inward! So the normals we want are

$$(2x, 2y, -1)$$

we also need to make this a unit vector.

$$\vec{n} = \frac{(2x, 2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\iint_S F \cdot \vec{n}\, dA = \iint_S (0, 0, z) \cdot \frac{(2x, 2y, -1)}{\sqrt{4x^2+4y^2+1}}\, dA = \iint_S (-z) \frac{1}{\sqrt{4x^2+4y^2+1}}\, dA$$

$z = f(x,y) = x^2 + y^2$ so $\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$

$$= \iint_{R = \text{unit disc}} (-x^2 - y^2) \frac{1}{\sqrt{4x^2+4y^2+1}} \sqrt{1 + (2x)^2 + (2y)^2}\, dx\, dy$$

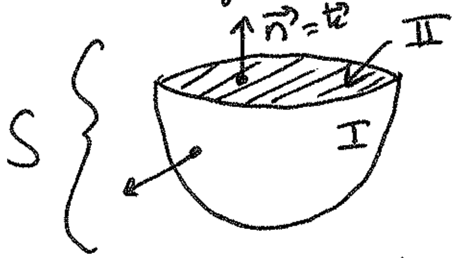
radius = 1

$$= \iint_{\text{unit disc}} (-x^2 - y^2)\, dx\, dy = \int_0^1 \int_0^{2\pi} (-r^2)\, r\, d\theta\, dr = 2\pi \left(-\frac{r^4}{4} \right) \Bigg|_0^1$$

polar coords

$$= -\frac{\pi}{2}$$

eg: Find the flux of the vector field $F = 0\vec{i} + 0\vec{j} + z\vec{k}$ through the same surface, but capped off with a disc on top.



$$\iint_S \vec{F} \cdot \vec{n} = \iint_I F \cdot \vec{n} + \iint_{II} F \cdot \vec{n}$$

we already found $\displaystyle\iint_I F \cdot \vec{n} = -\frac{\pi}{2}$

for II, let's first find the unit normal vector. it is $\vec{k}$

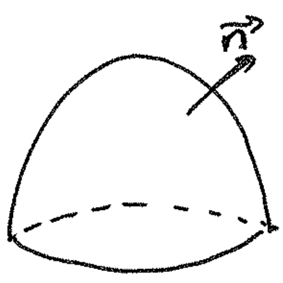
so $$\iint_I F \cdot \vec{n} = \iint_{II} (0,0,z) \cdot (0,0,1)\, dA = \iint_I 1\, dA = \pi$$

$z = 1$ on this surface

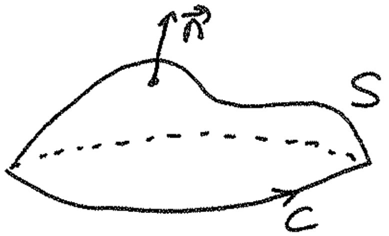
So $$\iint_S F \cdot \vec{n} = \frac{-\pi}{2} + \pi = \frac{\pi}{2}.$$

eg: Let S be given by $z = 9 - x^2 - y^2$ $\quad z \geq 0$

(exercise) Find the outward flux of $F = x\vec{i} + y\vec{j} + z\vec{k}$ through S.



Stokes' theorem: This is in $\mathbb{R}^3$. Say we have a curve bounding a surface S. Then:



$$\oint_C F \cdot dr = \iint_S (\text{curl} F) \cdot \vec{n} \, dS$$

(the orientation of C and $\vec{n}$ have to satisfy the right hand rule)

Interpretation: we had said that curl measures the amount of swirl a vector field does at a point.

If F is similar to dr for the curve C then there should be a lot of swirling inside the curve.

Let's apply it in an example:

eg: Let C be the triangle whose vertices are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

Let $F = (2z+x)\vec{i} + (y-z)\vec{j} + (x+y)\vec{k}$

find $\oint_C F \cdot dr$.

By Stokes' theorem: $\oint_C F \cdot dr = \iint_S (\text{curl} F) \cdot \vec{n} \, dA$.

$$\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (1+1)\vec{i} + \vec{j}(2-1) + \vec{k}(0-0)$$

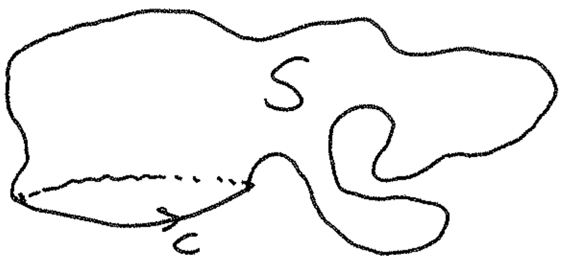
$$= (2, 1, 0) \quad \text{constant vector field.}$$

$$\vec{n} = (1,1,1) \cdot \frac{1}{\sqrt{3}}$$

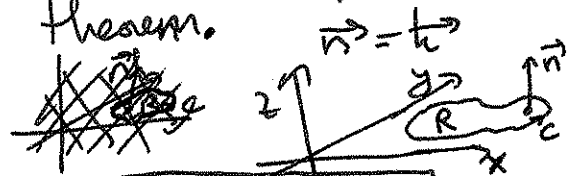
check!

$$\iint_S \text{curl} F \cdot \vec{n} \, dA = \iint_S (2,1,0) \cdot (1,1,1) \frac{1}{\sqrt{3}} \, dA = \iint_S \frac{3}{\sqrt{3}} \, dA = \sqrt{3} \iint_S 1 \, dA = \frac{\sqrt{3}}{2}$$

<u>Remarks</u>: (1) The shape of the surface can be as weird as we want



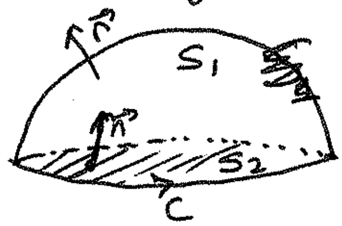
(2) Green's theorem is a special case of Stokes' theorem. $\vec{n} = \vec{k}$



(3) Since $\iint_S \text{Curl}F \cdot \vec{n}\, dA = \oint_C F \cdot dr$, the surface doesn't really matter as long as it bounds the same curve.

$\text{Curl}F \cdot \vec{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Indeed:



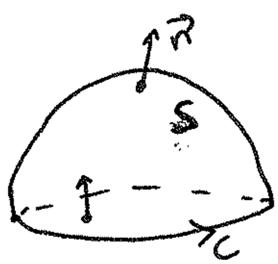
$$\iint_{S_1} \text{Curl}F \cdot \vec{n} = \oint_C F \cdot dr$$

$$\iint_{S_2} \text{Curl}F \cdot \vec{n} = \oint_C F \cdot dr$$

So $\iint_{S_1} \text{Curl}F \cdot \vec{n} = \iint_{S_2} \text{Curl}F \cdot \vec{n}$.

<u>eg</u>: Let $F = 0\vec{i} + 0\vec{j} + xyz\vec{k}$.

S: $z = 1 - x^2 - y^2 \quad z \geq 0$ Find: $\iint_S \text{Curl}F \cdot \vec{n}\, dA$



$$\iint_S \text{Curl}F \cdot \vec{n} = \oint_C F \cdot dr = \iint_{\substack{\text{unit disc} \\ \text{on xy plane}}} \text{Curl}F \cdot \vec{n}$$

here, $\vec{n} = (0,0,1) = \vec{k}$

so we can calculate more easily (but first find Curl$F \cdot \vec{n}$

since $\vec{n} = (0,0,1)$, we only need the $\vec{k}$ component of Curl$F$ which is 0 since P and Q are 0. so $\iint_S \text{Curl}F \cdot \vec{n} = 0$.