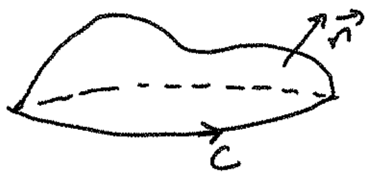
Lecture 17:

Last time: We talked about Stokes' theorem.



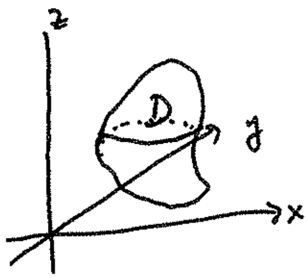
$$\oint_C F\cdot dr = \iint_S (\mathrm{Curl}\,F)\cdot \vec{n}\, dA$$

We saw how:
- Green's theorem is a special case of Stokes' theorem
- You can use Stokes' theorem to change the surface (while keeping the boundary curve the same when integrating $\iint_S (\mathrm{Curl}\,F)\cdot \vec{n}\, dA$.
- The integral $\iint_S \mathrm{Curl}\,F \cdot \vec{n}\, dA = 0$ for surfaces S which are closed (no boundary, eg.)

Today, we are going to talk about the Divergence theorem. But before that:

Triple integrals:

Let D be a region in 3-dimensional space. $f: \mathbb{R}^3 \to \mathbb{R}$ a function.



We can calculate:

$$\iiint_D f(x,y,z)\, dV$$

by turning it into iterated integrals.

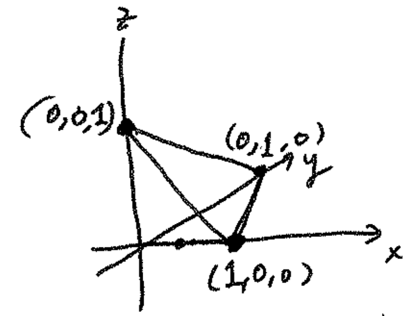
as usual, we have $\iiint_D 1\, dV =$ Volume of D

We could use $\iiint_D f\, dV$ (f = density function) to calculate the total mass of a solid which has varying density. Or we could use triple integrals to calculate the center of mass of such a solid. The x coordinate of the center of mass would be $\dfrac{\iiint_D f\cdot x\, dV}{\text{mass of } D}$

eg: Let's find the result of the integral.

$$\iiint_D xy \, dx\,dy\,dz$$ where D is the region with $x+y+z \leq 1$

and $x \geq 0,\ y \geq 0,\ z \geq 0$.



We would write:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy \, dz\,dy\,dx$$

and evaluate the integrals one by one.

as x goes from 0 to 1

for each x, y goes from 0 to 1-x

for each x and y, z goes from 0 to 1-x-y

$$= \int_0^1\int_0^{1-x} (xyz)\Big|_0^{1-x-y} dy\,dx = \int_0^1\int_0^{1-x} xy(1-x-y)\,dy\,dx = \int_0^1\int_0^{1-x} xy - x^2y - xy^2 \, dy\,dx$$

$$= \int_0^1 \left(\frac{xy^2}{2} - \frac{x^2y^2}{2} - x\frac{y^3}{3}\right)\Big|_0^{1-x} dx = \int_0^1 \frac{x(1-x)^2}{2} - \frac{x^2(1-x)^2}{2} - x\frac{(1-x)^3}{3}\, dx$$

$$= \int_0^1 (1-x)^2\left(\frac{x}{2} - \frac{x^2}{2} - \frac{x}{3}\right)dx = \frac{1}{2}\int_0^1 (x^2-2x+1)\left(\frac{x}{3} - x^2\right)dx$$

$$= \int_0^1 \left(-x^4 + \left(2+\frac{1}{3}\right)x^3 + \ldots\right.$$ you get the idea.

We could have written the integral as dxdydz or dydxdz etc provided that we redo the boundaries.

Cylindrical coordinates: (same as polar coordinates but with a z too)

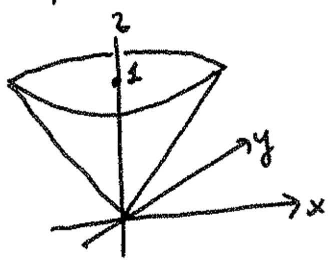
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

eg. Let's find the integral $\iiint_D (x^2+y^2)\, dV$

where D is the region with

$z^2 \geq x^2+y^2$ and $0 \leq z \leq 1$

this is a cone.



In regular dx, dy, dz we would write the integral as follows:

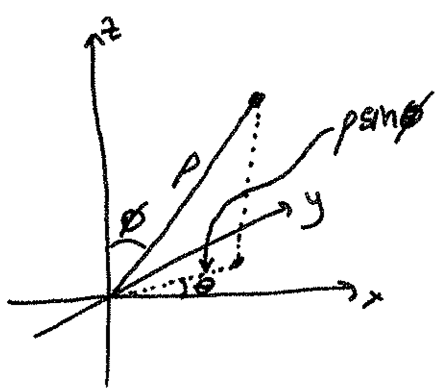
$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)\, dz\, dy\, dx$$

In cylindrical coordinates, it's much nicer:

$$\int_0^{2\pi}\int_0^1\int_r^1 r^2 \cdot \underline{\underline{r}}\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \left(\frac{r^4}{4}\right)\Big|_r^1 dr\, d\theta = \dots$$

$dxdydz = r\, dz\, dr\, d\theta$.

Spherical coordinates:



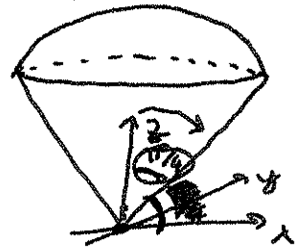
$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$.

and we have $dxdydz = \boxed{\rho^2\sin\phi}\, d\rho\, d\phi\, d\theta$

eg: Let's find the volume of the region given by

$$x^2+y^2+z^2 \leq 1$$

and $\quad z^2 \geq x^2+y^2$



to find the volume, we need: $\iiint_D 1\, dV$

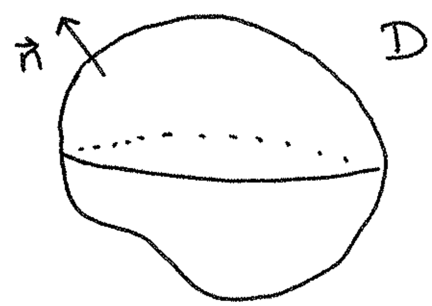
In spherical coordinates, we can write this integral as:

$$\underbrace{\int_0^{2\pi}}_{\theta} \underbrace{\int_0^{\pi/4}}_{\phi} \underbrace{\int_0^{1}}_{\rho} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \left(\tfrac{1}{3}\right) \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (-\cos\phi) \Big|_0^{\pi/4} d\theta = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

We'll get to do more examples of these as we study the:

<u>Divergence theorem:</u> Let D be a region in space, bounded by a closed surface S. $\vec{n}$ outward pointing unit normal vector field.



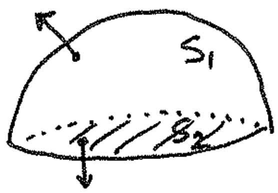
Then:

$$\boxed{\iint_S F \cdot n \, dA = \iiint_D \operatorname{div} F \, dV}$$

<u>Remark:</u> We had seen before that the divergence of a vector field at a point (x,y,z) gives the flux through a small cube around that point. If we add the fluxes through all those cubes, the faces that overlap have fluxes that cancel out. So we get the flux through a boundary. So if we add (integrate out) the fluxes at points, we get the flux through S.

Let's apply it:

eg: Let D be the region between $x^2+y^2+z^2=9$ and the xy plane with $z \geq 0$. D is bounded by the surface which is the union of the top hemisphere and the disc of radius 3 at the origin on the xy plane.



$S = S_1 + S_2$

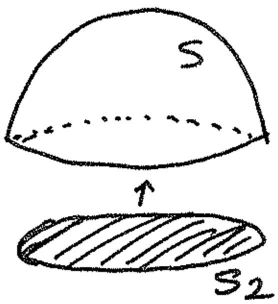
Let's find the outward flux of the vector field $F = 2x\vec{i} + y\vec{j} + (z-1)\vec{k}$.

$$\iint_S F \cdot \vec{n}\, dA = \iiint_D \operatorname{div} F\, dV = \iiint_D 4\, dV = 4\begin{pmatrix}\text{volume of} \\ \text{this half} \\ \text{sphere}\end{pmatrix} = 4 \cdot \frac{2}{3}\pi (3^3) = 72\pi$$

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2+1+1 = 4$$

<u>Same example, in a different way</u>: ~~For~~ Let S be the top part of the sphere $x^2+y^2+z^2=9$ $z \geq 0$.

Let $F = (2x, y, z-1)$. Find $\iint_S F \cdot \vec{n}\, dA$



S is not bounding a region because it is not a closed surface. To use the divergence theorem, we need to close it off by capping it at the bottom.

then we have $\iint_S F \cdot \vec{n} + \iint_{S_2} F \cdot \vec{n} = \iiint \operatorname{div} V = 72\pi$

what is $\vec{n}$ on S_2, it is $(0,0,-1)$ because it has to be outward pointing.

$$\text{so } \iint_S F \cdot \vec{n} = \iiint_D \operatorname{div} V - \iint_{S_2} F \cdot \vec{n} = 72\pi - \iint_{S_2} (2x, y, z-1) \cdot (0,0,1)\, dA$$

$$= 72\pi - \iint (z-1)\, dA = 72\pi - (-1 \cdot 9\pi)$$