1

Math 240 Final Exam, Dec 21, 2009 Your name____________________

Problem 1. Solve the system of equations

$$\begin{aligned} x + y + 2z &= 7 \\ x + 2y + 3z &= 9 \\ x + 2y + 4z &= 10 \end{aligned}$$

Answer. x = 4, y = ________, z = ________

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 1 & 2 & 3 & 9 \\ 1 & 2 & 4 & 10 \end{array}\right) \xrightarrow[R_2-R_1]{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2-R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1-R_2]{R_1-2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

2

Problem 4. Diagonalize the matrix A =

$$\begin{matrix} 4 & -3 \\ 1 & 0 \end{matrix}$$

That is, find matrices P and D such that $A = P D P^{-1}$, where D is diagonal.

You must put the following answers in the designated spaces:

(1) Eigenvalues of A in increasing order: 1 and 3

(2) Eigenvectors of A in corresponding order: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(3) Diagonal matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

(4) Matrix $P = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$

$$\begin{matrix} 4-\lambda & -3 \\ 1 & -\lambda \end{matrix} \qquad (4-\lambda)(-\lambda)+3=0$$

$$-4\lambda+\lambda^2+3=0$$

$$\lambda^2-4\lambda+3=0$$

$$(\lambda-3)(\lambda-1)$$

$$\lambda_1=3 \quad \lambda_2=1$$

for $\lambda_1=1$

$$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad v_1=v_2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = k_1$$

for $\lambda_2=3$

$$\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad v_1=3v_2 \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} = k_2$$

3

(1) For which real values of c is the matrix $A = \begin{pmatrix} c & 1 \\ c^2-1 & c \end{pmatrix}$ diagonalizable as a real matrix?

(A) $c = \pm 1$ (B) $-1 < c < 1$ (C) $c \leq -1$
(D) $c \geq 1$ (E) $|c| > 1$ (F) $c = -1$

$\det(A - \lambda I) = \left| \begin{pmatrix} (c-\lambda) & 1 \\ c^2-1 & (c-\lambda) \end{pmatrix} \right|$

$(c-\lambda)^2 - (c^2-1) = 0$

$\cancel{c^2} - 2c\lambda + \lambda^2 - \cancel{c^2} + 1 = 0$

$\lambda^2 - 2c\lambda + 1 = 0$

$\dfrac{2c \pm \sqrt{(2c)^2 - 4}}{2}$

$(2c)^2 - 4 \geq 0$

$(2c)^2 \geq 4$

$4c^2 \geq 4$

$c^2 \geq 1$

$c \geq 1 \quad c \leq -1$

$|c| > 1 \leftarrow$ So

to have distinct real roots we must have $(2c)^2 - 4 > 0$ which makes sure that we are diagonalizable. If $(2c)^2 - 4 = 0$ then you can check separately that you can only find one eigenvector. (but that's not in the options anyway)

$c \overset{?}{=} 1$

$\begin{matrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{matrix}$

$(1-\lambda)^2 = 0$

$\lambda = 1$

$\lambda_1 = 1$

$\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$

$V_2 = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

only 1 value

(4)

Math 240 Final Exam, Dec 21, 2009 Your name____________________

Problem 5. Find the general solution of the differential equation

$$d^4y/dx^4 + 4\, d^3y/dx^3 + 4\, d^2y/dx^2 = 0 .$$

Answer. $y(x) =$ ____________________________________

$$y^{iv} + 4y''' + 4y'' = 0$$

aux:

$$m^4 + 4m^3 + 4m^2 = 0$$

$$m^2(m^2 + 4m + 4) = 0$$

$$m^2(m+2)(m+2) = 0$$

$$m = 0 \text{ (DR)} \qquad m = -2 \text{ (DR)}$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 + c_4 x$$

(5) Math 240 Final Exam, Dec 21, 2009 Your name ____________

Problem 6. Find the function y(x) which satisfies the differential equation

$$x^2 y'' - 5x y' + 8 y = 0$$

and the initial conditions $y(2) = 32$ and $y'(2) = 0$.

Answer. $y(x) =$ $y = -2x^4 + 16x^2$

$y = x^m$

$m^2 - m - 5m + 8 = 0$

$m^2 - 6m + 8 = 0$

$(m-4)(m-2) = 0$

$m_1 = 4, m_2 = 2$

$y = c_1 x^4 + c_2 x^2$

$32 = c_1(16) + c_2(4)$

$8 = 4c_1 + c_2$ $\quad c_2 = 8 - 4c_1$

$y' = 4c_1 x^3 + 2c_2 x$

$0 = 4c_1(8) + 2c_2(2)$

$0 = 32c_1 + 4c_2$

$32c_1 + 4(8 - 4c_1) = 0$

$32c_1 + 32 - 16c_1 = 0$

$16c_1 + 32 = 0$

$c_1 = -2$

$c_2 = 16$

6

Math 240 FINAL EXAM Your name ______________________

Problem 2. Find $\det(A^{-1} B A)$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -5 & -2 & 0 \\ 3 & -5 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \\ -1 & -2 & 0 \end{pmatrix}$$

CIRCLE ONE OF THE FOLLOWING ANSWERS.

(a) −2 (b) −1 (c) 0 (d) 1 (e) 2

$\det(A^{-1} B A) = \det(B)$ because $\det(A^{-1}BA) = \det(A)\det(B)\det(A^{-1})$

$= \det(A)\det(A^{-1})\det B$

$= \det(AA^{-1})\det B$

$= \det B$

$2(0 - -4) - 3(0 - -2)$

$8 - 6 = 2$

7

Harder question. But you can do it if you think hard.

(3) Which of the following statements is ***false***:

(A) If A is an 3×3 matrix with real entries, and the only solution of the system $A\overrightarrow{x}=\overrightarrow{0}$ is the zero vector, then A is invertible.

(B) There exists a 2×2 matrix A so that $A\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}4\\8\end{pmatrix}$ and $A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$.

(C) If A and B are diagonalizable 3×3 matrices with the same eigenvectors, then $A+B$ is diagonalizable.

(D) There exists a real 3×3 matrix A which satisfies $A^4=-I_3$.

(E) An $n\times n$ real matrix can have at most n real eigenvalues.

(F) If $\overrightarrow{v}=\begin{pmatrix}3\\0\\-4\end{pmatrix}$, then the matrix $A=\overrightarrow{v}\,\overrightarrow{v}^T$ has three linearly independent eigenvectors.

A) If the only solution to $A\vec{x}=0$ is the zero-vector, the solution space is 0-dimensional. Which means that null $A=0$ so $rk A=3$ and $\det A\neq 0$ and A is invertible

B) If we set $P=\begin{pmatrix}1&1\\2&1\end{pmatrix}$ $D=\begin{pmatrix}4&0\\0&2\end{pmatrix}$ then $A=PDP^{-1}$ (diagonalization)

c) $A=PD_1P^{-1}$ and $B=PD_2P^{-1}$, D_1 and D_2 might be different, but the P's are the same because the eigenvectors are assumed to be the same for A and B. Then we can diagonalize A+B as

$$\begin{aligned}A+B&=PD_1P^{-1}+PD_2P^{-1}\\&=P(D_1P^{-1}+D_2P^{-1})\\&=P\underbrace{(D_1+D_2)}_{\text{diagonal}}P^{-1}\end{aligned}$$

D) If $A^4=-I$ then $\det(A^4)=\det(-I)$

$$(\det(A))^4 = -1 \leftarrow \text{for } I \text{ the } 3\times 3 \text{ identity matrix}$$

can't happen unless $\det A$ is complex.

E) Check that $\begin{pmatrix}3\\0\\4\end{pmatrix}$ is an eigenvector of this matrix. Indeed:

$$\Bigg(\overbrace{\begin{pmatrix}3\\0\\4\end{pmatrix}(3\,0\,4)}^{A}\Bigg)\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}3\\0\\4\end{pmatrix}\left((3\,0\,4)\begin{pmatrix}3\\0\\4\end{pmatrix}\right)=\begin{pmatrix}3\\0\\4\end{pmatrix}\cdot(25)$$

The other two eigenvectors are two independent vectors that satisfy $A\vec{x}=0$. They exist because rank $A=1$, null $A=2$

8

Math 240 FINAL EXAM Your name ____________________

Problem 8. Suppose that the function y(x) satisfies the differential equation $y'' + y' - 6y = 6$ with initial values $y(0) = 1$ and $y'(0) = -1$. Find the value of $y(-1)$.

CIRCLE ONE OF THE FOLLOWING ANSWERS.

(a) $e^{-3} - e^{2} - 1$ (b) $e^{-3} + e^{2} - 1$ (c) $e^{3} - e^{-2} - 1$ (d) $e^{3} + e^{-2} - 1$ (e) $e^{3} + e^{-2} + 1$

$m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$y = A$

$y' = 0$

$y'' = 0$

$-6A = 6$

$A = -1$

$y = c_1 e^{-3x} + c_2 e^{2x} - 1$

$y(0) = c_1 + c_2 - 1 = 1$

$c_1 + c_2 = 2$

$y' = -3c_1 e^{-3x} + 2c_2 e^{2x}$

$y'(0) = -3c_1 + 2c_2 = -1$

$3c_1 + 3c_2 = 6$

$5c_2 = 5$

$c_2 = 1$

$c_1 = 1$

$y = e^{-3x} + e^{2x} - 1$

$y(-1) = e^{3} + e^{-2} - 1$

(9)

Problem 10. Solve the following system of first order linear differential equations:

$$dx/dt = x + 3y$$
$$dy/dt = 3x + y$$

with initial conditions $x(0) = 0$ and $y(0) = 1$.

You must put your answer here:

$x(t) =$ ______________________________.

$y(t) =$ ______________________________.

$x' = x + 3y$

$y' = 3x + y$

$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \to \begin{matrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{matrix}$

$\cancel{(1-\lambda)}(1-\lambda)^2 - 9 = 0$

$1 - 2\lambda + \lambda^2 - 9 = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2)$

$\lambda_1 = 4 \quad \lambda_2 = -2$

$\lambda_1 = 4 \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

$v_1 = v_2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = k_1$

$\lambda_2 = 2$

$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \quad v_1 = -v_2 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} = k_2$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$

$c_1 e^{4t} + c_2 e^{-2t} = 0$

$x(t) = c_1 e^{4t} + c_2 e^{-2t} = 0$

$c_1 + c_2 = 0 \qquad c_1 = \frac{1}{2}$

$c_1 - c_2 = 1 \qquad c_2 = -\frac{1}{2}$

10

7. If you solve the following system of differential equations

$$\begin{cases} x' = x + 3y \\ y' = 5x + 3y \end{cases}$$

subject to the initial condition $x(0) = 5$ and $y(0) = 3$, then $x(t)$ is given by:

(A) $x = e^{2t} + 4e^{5t}$ (B) $x = 3e^{-2t} + 2e^{6t}$ (C) $x = 2e^{-2t} + 3e^{6t}$

(D) $x = 3e^{2t} + 2e^{5t}$ (E) $x = 6e^{2t} - e^{3t}$

$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$

$\begin{matrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{matrix}$

$(1-\lambda)(3-\lambda) - 15$

$3 - 4\lambda + \lambda^2 - 15 = 0$

$\lambda^2 - 4\lambda - 12 = 0$

$(\lambda-6)(\lambda+2) = 0$

$\lambda = 6, -2$

$K_1 = 6$

$\begin{pmatrix} -5 & 3 \\ 5 & -3 \end{pmatrix}$ $5V_1 = 3V_2$ $\to \begin{pmatrix} 3 \\ 5 \end{pmatrix} = K_1$

$K_2 = -2$

$\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} = K_2$

$X = C_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$

$X(0) = 3C_1 + C_2 = 5$

$Y(0) = 5C_1 - C_2 = 3$

$8C_1 = 8 \quad C_1 = 1$

$C_2 = 2$

$X(t) = 3e^{6t} + 2e^{-2t}$

11

1. For the matrices A^{-1} and B^{-1} below, find $(AB)^{-1}$.

$$A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$$

(A) $\begin{pmatrix} 8 & 12 \\ 12 & 5 \end{pmatrix}$ (B) $\begin{pmatrix} 8 & 3 \\ 12 & 5 \end{pmatrix}$ (C) $\begin{pmatrix} 5 & 6 \\ 6 & 8 \end{pmatrix}$ (D) $\begin{pmatrix} 5 & 8 \\ 8 & 6 \end{pmatrix}$

(E) This can't be done: one of A, B is singular, and AB is undefined.

$(AB)^{-1} = B^{-1}A^{-1}$

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 6 & 8 \end{pmatrix}$$

12

2. Find $y(\pi)$, where y satisfies the differential equation

$$\frac{d^4y}{dx^4} - 16y = 0\,,$$

subject to the initial condition

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -4, \quad y'''(0) = 0\,.$$

As a reminder, $A^4 - 16 = (A^2-4)(A^2+4) = (A-2)(A+2)(A^2+4)$.

(A) 0 (B) 1 (C) $\frac{1}{2}e^{2\pi} + \frac{3}{2}$ (D) $\frac{1}{2}e^{2\pi} + \frac{1}{2}e^{-2\pi} - \frac{1}{2}$ (E) $\frac{1}{2}e^{2\pi} - \frac{1}{2}e^{-2\pi} + \frac{3}{2}$

---

try $y = e^{mx}$. auxiliary eqn: $m^4 - 16 = 0$

$(m-4)(m+4) = 0$

$(m-2)(m+2)(m+4) = 0$

$m_1 = 2 \quad m_2 = -2 \qquad m_{3,4} = \alpha \pm i\beta = 0 \pm 2i$

$y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$

$y' = 2c_1e^{2x} - 2c_2e^{-2x} - 2c_3\sin 2x + 2c_4\cos 2x$

$y'' = 4c_1e^{2x} + 4c_2e^{-2x} - 4c_3\cos 2x - 4c_4\sin 2x$

$y''' = 8c_1e^{2x} - 8c_2e^{-2x} + 8c_3\sin 2x - 8c_4\cos 2x$

$y(0) = 1 \Rightarrow \quad c_1 + c_2 + c_3 = 1$

$y'(0) = 0 \Rightarrow \quad 2c_1 - 2c_2 + 2c_4 = 0$

$y''(0) = -4 \Rightarrow \quad 4c_1 + 4c_2 - 4c_3 = -4$

$y'''(0) = 0 \Rightarrow \quad 8c_1 - 8c_2 - 8c_4 = 0$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 2 & -2 & 0 & 2 & 0 \\ 4 & 4 & -4 & 0 & -4 \\ 8 & -8 & 0 & -8 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 \end{array}\right) \Rightarrow \begin{array}{c} c_3 = 1 \\ c_4 = 0 \\ c_1 = c_2 = 0 \end{array}$$

$y(\pi) = 1$

(13)

14. Compute the determinant

$$\begin{vmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{vmatrix}.$$

(A) -16 (B) -8 (C) 0 (D) 8 (E) 16

$0 + 1(2+6) - 3(2+6)$

$8 - 3(8)$

$8 - 24 = -16$

(14)

(15) Solve the system of first-order linear differential equations with initial value,

$$\frac{dx}{dt} = -4x + 2y, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\frac{dy}{dt} = 2x - 4y,$$

(A) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-6t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$ **(B)** $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

(C) $\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$ **(D)** $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$

(E) $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ **(F)** $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

$X' = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} X$

$\begin{matrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{matrix}$

$(-4-\lambda)^2 - 4$

$16 + 8\lambda + \lambda^2 - 4$

$\lambda^2 + 8\lambda + 12 = 0$

$(\lambda+6)(\lambda+2) = 0$

$\lambda = -6, -2$

$\lambda_1 = -6$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = K_1$

$\lambda_2 = -2$

$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = K_2$

$X = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6x} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2x}$

$x(0) = C_1 + C_2 = 1$

$y(0) = -C_1 + C_2 = 1$

$2C_2 = 0$

$C_2 = 0$

$C_1 = 1$

$X = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$

(15)

15. Solve for x in the system

$$\begin{aligned} x - y - 3z &= 0 \\ x + 3y + 3z &= 2 \\ y + z &= -2 \end{aligned}$$

As a hint:

$$\begin{vmatrix} 1 & -1 & -3 \\ 1 & 3 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

[redacted]

(A) -8 (B) -4 (C) no solution (D) 4 (E) 8

$$\left.\begin{matrix} 1 & -1 & -3 \\ 1 & 3 & 3 \\ 0 & 1 & 1 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 2 \\ -2 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 4 & 6 & 2 \\ 0 & 1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & -7 \end{array}\right)$$

$z = 5$

$y = -7$

$x - y - 3z = 0$

$x = y + 3z = -7 + 15$

$= 8$

16

(2) For the matrix A and the vector of unknowns $\overrightarrow{x}$ given by

$$\begin{array}{c} \begin{array}{cccccc} 1 & 9 & 0 & 2 & 0 & -3 \end{array} \\ A = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 & 3 \\ 2 & 8 & -1 & 1 & 2 & 0 \\ 5 & 5 & -4 & -2 & 8 & 9 \\ \cancel{1} & \cancel{9} & \cancel{0} & \cancel{2} & \cancel{0} & \cancel{-3} \end{pmatrix} \end{array}, \qquad \overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

we have

(A) rank$(A) = 1$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 3 parameters.

(B) rank$(A) = 2$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 2 parameters.

~~(C)~~ rank$(A) = 3$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 3 parameters.

→ ~~(D)~~ rank$(A) = 2$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 4 parameters.

~~(E)~~ rank$(A) = 1$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 5 parameters.

~~(F)~~ rank$(A) = 3$ and the solutins of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 2 parameters.

$$\begin{matrix} 1 & 0 & -1 & -2 & \cancel{0} & \cancel{3} \\ 2 & 10 & -1 & 0 & \cancel{0} & \cancel{0} \\ 5 & 10 & -4 & -6 & \cancel{0} & \cancel{9} \\ 1 & 10 & 0 & 2 & \cancel{0} & \cancel{-3} \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 5 & 1 & 4 & 3 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} \cancel{1} & 0 & 1 & 1 & 0 & 0 \\ \cancel{1} & 1 & 1 & 0 & 0 & 0 \\ \cancel{4} & 1 & 4 & 3 & 0 & 0 \\ \cancel{0} & 1 & 0 & -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 & \cancel{1} & 0 & 0 \\ 0 & 1 & 0 & \cancel{0} & 0 & 0 \\ 0 & 1 & 3 & \cancel{3} & 0 & 0 \\ 0 & 1 & -1 & \cancel{-1} & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 \\ 1 & 3 \\ 1 & -1 \end{matrix}$$

rank$(A) = 2$

Recall: rank = # of linearly independent rows
= # of linearly independent columns
so it's ok to do columen operations to find rank too.

$$\text{Rank} + \underset{4}{\text{nullity}} = n = \underline{\underline{6}} \quad ; \text{ rank} = 2$$

so nullity = 4

17

(4) Consider the matrix

$$B = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

It is known that 2 is an eigenvalue of B. Then

(A) B is diagonalizable with eigenvalues 1, 1, and 2.

(B) B is diagonalizable with eigenvalues 1, 2, and 2.

(C) B is diagonalizable with eigenvalues -1, 1, and 2.

(D) B is not diagonalizable, and has eigenvalues -1, 1, and 2.

(E) B is not diagonalizable, and has eigenvalues 1, 1, and 2.

(F) B is not diagonalizable, and has a unique real eigenvalue 2.

$$\begin{matrix} -\lambda & 0 & -2 \\ 1 & -\lambda & 1 \\ 0 & 1 & 2-\lambda \end{matrix}$$

$$\cancel{-\lambda+2} \qquad 2\lambda^2-\lambda^3$$

$$-1(-\lambda - -2) + (2-\lambda)(\lambda^2)$$

$$\lambda - 2 + 2\lambda^2 - \lambda^3$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$-\lambda^2(\lambda-2) + 1(\lambda-2)$$

$$(-\lambda^2+1)(\lambda-2)$$

$$(1-\lambda)(1+\lambda)(\lambda-2)$$

$$\lambda = 1, -1, 2$$

18

(9) Let $y(x)$ be the general solution of the linear differential equation

$$x^2y'' + 5xy' + 3y = 0,$$

for which $y(1) = 1$ and $y'(1) = -3$. Find $y(1/2)$.

(A) -9 (B) 2 (C) 0

(D) 1/8 (E) 8 (F) 1/16

$m(m-1)+5m+3=0$

$m^2-m+5m+3=0$

$m^2+4m+3=0$

$(m+3)(m+1)=0$

$m=-3,-1$

$y=C_1X^{-3}+C_2X^{-1}$ $\quad$ $y'=-3C_1X^{-4}-C_2X^{-2}$

$y(1)=C_1+C_2=1$ $\quad$ $y'(1)=-3C_1-C_2=-3$

$C_1+C_2=1$

$-2C_1=-2$

$C_1=1$

$C_2=0$

$Y=X^{-3}$

$Y(1/2)=\frac{1}{(1/2)^3}=8$

(19)

18. Select a matrix with eigenvalues 0 and 2, and corresponding eigenvectors $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$.

(A) $\begin{pmatrix}-4 & 2\\-12 & 6\end{pmatrix}$ (B) $\begin{pmatrix}-2 & 1\\-8 & 4\end{pmatrix}$ (C) $\begin{pmatrix}-4 & -12\\2 & 6\end{pmatrix}$ (D) $\begin{pmatrix}0 & 3\\0 & 2\end{pmatrix}$ (E) $\begin{pmatrix}-4 & -8\\3 & 6\end{pmatrix}$

$$\lambda v = A v$$

$\lambda = 0 \quad K_1 = \begin{pmatrix}1\\2\end{pmatrix}$

$\lambda = 2 \quad K_2 = \begin{pmatrix}1\\3\end{pmatrix}$

$$0\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$2\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

$D = \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix} \qquad P = \begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}$

$A = PDP^{-1} \qquad P^{-1} = \begin{pmatrix}3 & -1\\-2 & 1\end{pmatrix}(1)$

$$\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}0+0 & 0+2\\0+0 & 0+6\end{pmatrix}$$

$$= \begin{pmatrix}0 & 2\\0 & 6\end{pmatrix}$$

$$\begin{pmatrix}0 & 2\\0 & 6\end{pmatrix}\begin{pmatrix}3 & -1\\-2 & 1\end{pmatrix} = \begin{pmatrix}0-4 & 2\\-12 & 6\end{pmatrix}$$

$$= \begin{pmatrix}-4 & 2\\-12 & 6\end{pmatrix}$$

19

18. Select a matrix with eigenvalues 0 and 2, and corresponding eigenvectors $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}1\\3\end{pmatrix}$.

Ⓐ $\begin{pmatrix}-4 & 2\\-12 & 6\end{pmatrix}$ Ⓑ $\begin{pmatrix}-2 & 1\\-8 & 4\end{pmatrix}$ Ⓒ $\begin{pmatrix}-4 & -12\\2 & 6\end{pmatrix}$ Ⓓ $\begin{pmatrix}0 & 3\\0 & 2\end{pmatrix}$ Ⓔ $\begin{pmatrix}-4 & -8\\3 & 6\end{pmatrix}$

$Av = \lambda v$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = 0\begin{pmatrix}1\\2\end{pmatrix}$ $\qquad$ $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = 2\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}2\\6\end{pmatrix}$

$a+2b=0$ $\qquad$ $a+3b=2$

$c+2d=0$ $\qquad$ $c+3d=6$

$a+2b=0$
$-a-3b=-2$
$-b=-2$
$b=2 \quad a=-4$

$c+3d=6$
$-c-2d=0$
$d=6$
$c=-12$

$\begin{pmatrix}-4 & 2\\-12 & 6\end{pmatrix}$

20

3. Give the general solution to the differential equation

$$x^2y'' - 2xy' + 2y = 0\,.$$

(A) $y = c_1x^2 + c_2x^3$ (B) $y = c_1 + c_2x^{-2}$ (C) $y = c_1x + c_2x^3$

(D) $y = c_1 + c_2x^2$ (E) $y = c_1x + c_2x^2$

$x^2 y'' - 2xy' + 2y = 0$

$m(m-1) - 2m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1)$

$m_1 = 2 \quad m_2 = 1$

$y = C_1 X^2 + C_2 X$

(21)

Problem 11. Diagonalize the matrix A =

$$\begin{matrix} 2 & -1 \\ 2 & 5 \end{matrix}$$

That is, find matrices P and D such that A = P D P^{-1}, where D is diagonal.

You must put the following answers here:

(1) Eigenvalues of A (smaller one first) are $3 = \lambda_1$ and $4 = \lambda_2$.

(2) The corresponding eigenvectors of A (in the same order) are

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = k_1$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix} = k_2$

(3) The diagonal matrix D = $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

(4) The matrix P = $\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$.

$\begin{matrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{matrix}$

$(2-\lambda)(5-\lambda) - -2$

$10 - 7\lambda + \lambda^2 + 2$

$\lambda^2 - 7\lambda + 12 = 0$

$(\lambda - 4)(\lambda - 3) = 0$

$\lambda = 4, 3$

$\lambda_1 = 3$

$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = k_1$

$\lambda_2 = 4$

$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}$

$2V_1 + V_2 = 0$

$2V_1 = -V_2$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = k_2$

$D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

22

Problem 9. Solve the differential equation $x^2 y'' + 2xy' = 0$ with $y(1) = 1$ and $y'(1) = 1$.

You must put your answer here:

$y(x) =$ ______________________________.

$x^2 y'' + 2xy' = 0$

$m(m-1) + 2m = 0$

$m^2 - \cancel{m} + 2m$

$m^2 + m = 0$

$m(m+1)$

$m_1 = 0 \quad m_2 = -1$.

$y = C_1 x^0 + C_2 x^{-1}$

$y = C_1 + C_2 x^{-1}$

$1 = C_1 + C_2$

$1 = -C_2 \qquad C_2 = -1$

$C_1 = 2$

$y = 2 - \frac{1}{x}$

23

Math 240 FINAL EXAM Your name ______________________

Problem 1. Solve the system of linear equations

$$\begin{aligned} 2x + 3y + z &= 1 \\ 2y + 3z &= -1 \\ x + 3y - z &= -4 \end{aligned}$$

and find the value of y .

CIRCLE ONE OF THE FOLLOWING ANSWERS.

(a) −2 (b) −1 (c) 0 (d) 1 (e) 2

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & 2 & 3 & -1 \\ 1 & 3 & -1 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 3 & -1 \\ 1 & 3 & -1 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & 3 & -3 & -9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & -3 \end{array}\right]$$

Method 1: continue w/ row ops:

$$\xrightarrow{R_3 \times 2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & 2 & -2 & -6 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & -5 & -5 \end{array}\right]$$

$z = 1$

$$\xrightarrow{R_3 / -5} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2 - 3R_3]{R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = 3 \quad y = -2 \quad z = 1$.

METHOD 2:

$2y + 3z = -1$

$y - z = -3$

$y = z + 3$

$2y + 3z = -1$

$2(z-3) + 3z = -1$

$2z - 6 + 3z = -1$

$5z - 6 = -1$

$5z = 5$

$\underline{z = 1}$

$\underline{y = -2}$

$x + 2z = 5$

$x = 3$

(24)

Math 240 Final Exam, Dec 21, 2009 Your name________________

Problem 3. Find a 2×2 matrix A with eigenvalues 1 and 2 and corresponding eigenvectors (3, 1) and (2, 1).

$Av = \lambda v$

Answer. A = $\begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$3a+b=3$ $2a+b=4$

$3c+d=1$ $2c+d=2$

$-3a-b=-3$ $3c+d=1$ $-2+d=2$

$2a+b=4$ $-2c-d=-2$ $d=4$

$-a=+1$ $a=-1$ $c=-1$

$b=6$

Second method: By diagonalization: $A = PDP^{-1}$

where $P = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

25

17. If

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = x\,,$$

then

$$\begin{vmatrix} 2c_2 + a_2 & b_2 & -a_2 \\ 2c_1 + a_1 & b_1 & -a_1 \\ 2c_3 + a_3 & b_3 & -a_3 \end{vmatrix} = ?$$

(A) 0 (B) $-x$ (C) $2x$ (D) $\boxed{-2x}$ (E) $3x$

Switch row → $-X$ Mult row by 2 → $-2X$

Add row → No change Mult row by -1 → $2X$

The second matrix is obtained from the first one by taking transpose (which does not change the det) and doing these operations.

More precisely:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{\text{transpose}} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \xrightarrow[(-1)]{\text{switch row 1 and row 2}} \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\xrightarrow[(-1)]{\text{switch col 1 and col 3}} \begin{pmatrix} c_2 & b_2 & a_2 \\ c_1 & b_1 & a_1 \\ c_3 & b_3 & a_3 \end{pmatrix} \xrightarrow[2]{\text{multiply col 1 by 2}} \begin{pmatrix} 2c_2 & b_2 & a_2 \\ 2c_1 & b_1 & a_1 \\ 2c_3 & b_3 & a_3 \end{pmatrix}$$

$$\xrightarrow[\text{no effect on det}]{\text{add col 3 to col 1}} \begin{pmatrix} 2c_2+a_2 & b_2 & a_2 \\ 2c_1+a_1 & b_1 & a_1 \\ 2c_3+a_3 & b_3 & a_3 \end{pmatrix} \xrightarrow[(-1)]{\text{multiply col 3 by } (-1)} \begin{pmatrix} 2c_2+a_2 & b_2 & -a_2 \\ 2c_1+a_1 & b_1 & -a_1 \\ 2c_3+a_3 & b_3 & -a_3 \end{pmatrix}$$

total effect of these operations: $(-1)(-1)(-1)\cdot 2 = \underline{\underline{-2}}$

26

13. A certain forced undamped oscillator is modelled by the differential equation

$$m\frac{d^2x}{dt^2} + 18x = 4\cos 3t\,.$$

What mass $m > 0$ corresponds to resonance, that is, $x(t)$ is unbounded as $t \to \infty$?

(A) $m = 1$ (B) $m = 2$ (C) $m = 3$ (D) $m = 4$ (E) $m = 5$

$m\frac{d^2y}{dx^2} + 18y = 4\cos(3x)$ $y(x)$ unbounded as $x \to \infty$

z=mass (the m in the question)

$z(m^2) + 18 = 0$

$zm^2 + 18 = 0$

$zm^2 = -18$

$m^2 = -18/z$

$m = \pm\sqrt{18/z}\, i$

$y = \underline{c_1\cos\left(\sqrt{18/z}\, x\right) + c_2\sin\left(\sqrt{18/z}\, x\right)}$

these are the solutions to the homogeneous version.

$\sqrt{18/z} = 3$

$18/z = 9$

$18/9 = z$ $z = 2$

$\boxed{\text{mass} = 2}$

(*) $y = A\cos 3x + B\sin 3x$

$y' = -3A\sin 3x + 3B\cos 3x$

$y'' = (-9zA\cos 3x - 9zB\sin 3x)m$

$+ 18A\cos(3x) + 18B\sin(3x) = 4\cos 3t$

$(18 - 9z)A = 4$ $(-9 + 18)B = 0$

$A = \frac{4}{18 - 9z}$ $B = 0$

$c_1\cos\left(\sqrt{18/z}\, x\right) + c_2\sin\left(\sqrt{18/z}\, x\right) + \frac{4}{18-9z}\cos(3x)$

$z = 2$

(**) $y = Ax\cos(3x) + Bx\sin(3x)$

$y' = A\cos(3x) + -3Ax\sin(3x) +$
$B\sin(3x) + 3Bx\cos(3x)$

$y'' = -3A\,\cancel{\sin}(3x) - 3A\,\cancel{\sin}(3x) - 9Ax\cos(3x)$
$+ 3B\sin(3x) + 3B\cos(3x) - 9Bx\sin(3x)$

$z(-6A + 3B)\sin(3x) + z3B\cos(3x)$
$-z9Ax\cos(3x) - z9Bx\sin(3x)$
$+18Ax\cos(3x) + 18Bx\sin(3x)$

...

The point is that, when $m \neq 2$ ($z \neq 2$) the solution is y_p as in (*) above (which is bounded) but when $\sqrt{\frac{18}{z}} = 3$, $A\cos x + B\sin x$ is not a solution because it is a solution to the homogeneous version. So, as usual we try $y_p = At\cos 3t + Bt\sin 3t$ which is not bounded! $\longrightarrow$ whatever A and B are, Y_p is unbounded

This is when the bridge collapses!