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Math 240 Quiz 1

1. Circle true or false (you do not need to justify your answer). A and B are **any** $n \times n$ matrices.

a) The subset $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^{n} \,|\, x_1 + x_2 + \cdots + x_n = 0\}$ is a subspace in $\mathbb{R}^n$.

[✓ True] False

b) The subset $\{(x, y) \in \mathbb{R}^2 \,|\, y = x^3\}$ is a subspace in $\mathbb{R}^2$.

True [✓ False]

c) If $\det(A) = 0$, then the system

$$A \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

has infinitely many solutions.

[✓ True] False

d) $\det(ABA^{-1}) = \det B$ if A is invertible.

[✓ True] False

e) $\det(A + B) = \det A + \det B$

True [✓ False]

2. Find the rank and the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 6 \\ 3 & -1 & 5 \end{pmatrix}$$

row operations:

$$\xrightarrow{R_3 \leftarrow R_3 - 3R_1} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 6 \\ 0 & -4 & 8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & 8 \\ 0 & -4 & 8 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

There are two linearly rows. So this matrix has rank $A = 2$

$\text{rank } A + \text{nullity } A = 3$

so $\text{nullity } A = 1$.

3. Find the inverse of the matrix

$$A = \begin{pmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

two ways to do this.

$$\left(\begin{array}{ccc|ccc} \sin\theta & -\cos\theta & 0 & 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 \sin\theta]{R_1 \leftarrow R_1 \cos\theta} \left(\begin{array}{ccc|cc} \sin\theta\cos\theta & -\cos^2\theta & 0 & \cos\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 & 0 & \sin\theta \\ 0 & 0 & 2 & 0 & 0 \end{array}\right.$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|ccc} \sin\theta\cos\theta & -\cos^2\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 1 & 0 & -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$\cos\theta(1-\cos^2) = \cos\theta\sin^2\theta$

$$\xrightarrow{R_1 \leftarrow R_1 + \cos^2\theta R_2} \left(\begin{array}{ccc|ccc} \sin\theta\cos\theta & 0 & 0 & \boxed{\cos\theta - \cos^3\theta} & \cos^2\theta\sin\theta & 0 \\ 0 & 1 & 0 & -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

now divide:

$$\xrightarrow[R_3 \leftarrow R_3/2]{R_1 \leftarrow R_1/\sin\theta\cos\theta} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 1 & 0 & -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right).$$

$$\underbrace{\begin{matrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1/2 \end{matrix}}_{A^{-1}}$$

The other way is to observe that if we have a matrix of two blocks

$$\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right)$$ then the inverse is $$\left(\begin{array}{c|c} B_1^{-1} & 0 \\ \hline 0 & B_2^{-1} \end{array}\right).$$

4. **Extra credit** No partial credit. Complete answers only.

a) Write down the definition of the *image* of an $m \times n$ matrix A.
b) Find a basis for the image of the matrix A in question 2.
c) Find a basis for the solution space to $A\vec{v} = \vec{0}$ for the same matrix A.

a) A matrix A gives a map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.
The image is the subset of $\mathbb{R}^m$ consisting of elements that come from $\mathbb{R}^n$ via A.
i.e $\operatorname{Im} A = \{ w \in \mathbb{R}^m \mid w = Av \text{ for some } v \in \mathbb{R}^n \}$

b) The image is spanned by the columns of A, but they are not independent. Throw away the last column to get an independent set of vectors that still span the image

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

We know that the basis for the image should have size 2 because rank = 2

c) Our row operations had led us to:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

now continue to get:

$$\xrightarrow{R_2 \leftarrow R_2/4} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the solutions to $Av = 0$ satisfy
$v_1 + v_3 = 0$ so $v_1 = -v_3$
$-v_2 + v_3 = 0$ so $v_2 = v_3$
so all solutions are of the form $\begin{pmatrix} -v_3 \\ v_3 \\ v_3 \end{pmatrix}$

So the solution space is spanned by $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$