Name :

Math 240 Quiz 2

1. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -5 & -1 & -3 \end{pmatrix}$$

a) Find all solutions $\vec{x}$ to the equation $A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

b) Find all solutions $\vec{x}$ to the equation $A\vec{x} = \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$. Do this any way you want, but you can do it without any calculation if I tell you that $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$ and you use part a).

c) What is $\det(A) =$? You can calculate this, but you can also tell the answer without any calculations using part a).

a) Do row operations to solve for $\vec{x}$.

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -5 & -1 & -3 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + 5R_1} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 0 & -6 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & -6 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

So we have $v_1 + 2v_2 = 0$

$3v_2 - v_3 = 0$

So solutions look like

$$\vec{x} = \begin{pmatrix} -2v_2 \\ v_2 \\ 3v_2 \end{pmatrix} = v_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

v_2 can be any number $\in \mathbb{R}$.

b) We know $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$. All other solutions are of the form $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \vec{x}_0$ where $A\vec{x}_0 = 0$. This means all solutions are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + v_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

check that these are solutions: $A\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + v_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}\right) = \underbrace{A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{b}} + \underbrace{v_2 A\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}}_{0} = \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$

c) from part a), we see that nullity $A = 1$ which is equivalent to A being singular!

so $\det A = 0$

2. Find all the eigenvalues and eigenvectors of the matrix.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\big((-1-\lambda)(-1-\lambda) - 0\cdot(-2)\big) - 2\big(6(-1-\lambda) - 0\big)$$
$$+ 1\big(6\cdot(-2) - (-1)(-1-\lambda)\big)$$
$$= (1-\lambda)(\lambda^2 + 2\lambda + 1) + 12(1+\lambda) - 12 - (1+\lambda)$$
$$= -\lambda^3 - \lambda^2 + \lambda + \cancel{1} + 12\lambda + \cancel{12} - \cancel{12} - \cancel{1} - \lambda$$
$$= -\lambda^3 - \lambda^2 + 12\lambda = -\lambda(\lambda^2 + \lambda - 12) = -\lambda(\lambda - 3)(\lambda + 4)$$

so there are three eigenvalues $\lambda = 0$, $\lambda = 3$, $\lambda = -4$

for $\lambda = 0$,

$$A - 0I = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{pmatrix} 13 & 0 & 1 \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so
$$14v_1 + v_3 = 0$$
$$6v_1 - v_2 = 0$$

so $K_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$

for $\lambda = 3$

$$A - 3I = \begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 \cdot 2} \begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -2 & -4 & -8 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ 0 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} -2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -6 & -9 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + 3R_2} \begin{pmatrix} -2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -2 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

so
$$-2v_1 - 2v_3 = 0$$
$$2v_2 + 3v_3 = 0$$

so $K_2 = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$

for $\lambda = -4$

$$A-(-4)I) = \begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \xrightarrow[R_2 \leftarrow 5R_2]{R_3 \leftarrow 5R_3} \begin{pmatrix} 5 & 2 & 1 \\ 30 & 15 & 0 \\ -5 & -10 & 15 \end{pmatrix}$$

$$\xrightarrow[R_2 \leftarrow R_2 - 6R_1]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 5 & 2 & 1 \\ 0 & 3 & -6 \\ 0 & -8 & 16 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2/3]{R_3 \leftarrow R_3/8} \begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 5 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$5v_1 + 5v_3 = 0$ $\qquad v_1 + v_3 = 0$

$\qquad\qquad\qquad\qquad v_1 = -v_3$

$v_2 - 2v_3 = 0$ $\qquad v_2 = 2v_3$

$$K_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

3. Diagonalize the matrix (7pts)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

then find A^{10} (3pts). That is, find a matrix P and a diagonal matrix D such that

$$P^{-1}AP = D$$

Then use this to compute A^{10}.

We first need to find 2 independent eigenvectors and their eigenvalues.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

so $\lambda_1 = 3$ and $\lambda_2 = -1$

<u>for $\lambda_1 = 3$</u>

$$A - 3I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $\lambda_2 = -1$

$$A + I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So:

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ then } P^{-1}AP = D$$

so $PDP^{-1} = A$

$$A^{10} = PD\underbrace{P^{-1}P}_{I}DP^{-1} \cdots PDP^{-1} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{pmatrix}\underbrace{\left(\frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\right)}_{P^{-1}}$$