Name :

Math 240 Quiz 3

1. Let $y = \sum_{n=0}^{\infty} c_n x^n$ be a series solution to the differential equation

$$y'' - xy' - y = 0$$

Given that $y(0) = 0$ and $y'(0) = 1$ find c_5 and c_6.

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

take out the 0th terms from the first and third sum (though we could have done without that too)

$$2c_2 - c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} - (k+1) c_k\right] x^k = 0$$

$$c_2 = \frac{c_0}{2} \qquad y(0) = c_0 = 0$$

$$y'(0) = c_1 = 1$$

$$c_{k+2} = \frac{c_k}{(k+2)}$$

so $c_0 = c_2 = c_4 = c_6 = \ldots = 0$

and $c_3 = \frac{c_1}{3}$ and $c_5 = \frac{c_3}{5}$

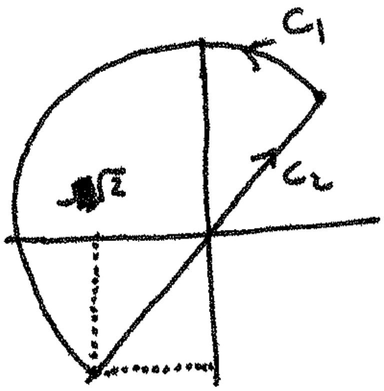
so $c_5 = \frac{1}{15}$

2. Let C be the part of the circle $x^2+y^2=4$ above the $y=x$ line, traversed counterclockwise, plus the diameter of the same circle on the $y=x$ line traversed from lower left to upper right (C is a closed curve consisting of the sum of those two parts). Let

$$\mathbf{F} = (x^2+y^2, -2\text{[redacted]})$$

be a vector field. Find

$$\int_C \mathbf{F}\cdot dr$$



$C = C_1 + C_2$

Parameterize these:

C_1: $(2\cos t, 2\sin t)$ $\quad t\in[\frac{\pi}{4}, \frac{5\pi}{4}]$

C_2: (t,t) $\quad t \in [-\text{[redacted]}\sqrt{2}, \text{[redacted]}\sqrt{2}]$

$$\int_{C_1} (x^2+y^2)dx + (-2)dy = \int_{\pi/4}^{5\pi/4} 1.\text{[redacted]}(2\sin t)dt - 2\int_{\pi/4}^{5\pi/4} 2\cos t\, dt$$

$$= -2\left(\sin\tfrac{5\pi}{4} - \sin\tfrac{\pi}{4}\right) - 4\left(\cos\tfrac{5\pi}{4} - \sin\tfrac{5\pi}{4}\right)$$

$$= -6\,\frac{2\sqrt{2}}{2} = -6\sqrt{2}$$

$$\int_{C_2} (x^2+y^2)dx + (-2)dy = \int_{-\sqrt{2}}^{\sqrt{2}} (2t^2-2)dt = \left(2\frac{t^3}{3} - 2t\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{4}{3}\sqrt{2} - 2\sqrt{2} + \frac{4}{3}\sqrt{2} - 2\sqrt{2}$$

$$= \frac{8}{3}\sqrt{2} - 4\sqrt{2}$$

So $\displaystyle\int_{C_1+C_2} \mathbf{F}\cdot dr = -6\sqrt{2} + \frac{8}{3}\sqrt{2} - 4\sqrt{2} = -10\sqrt{2} + \frac{8}{3}\sqrt{2}.$

3. Let C be the closed curve in the previous question. Let

$$\mathbf{F} = (x^2 + y^2, 2xy)$$

be a vector field. Find

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

In this case $\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$

so $\oint_C F \cdot dr = 0$ since F is defined in a simply connected region.

4. Find the divergence ($\nabla \cdot \mathbf{F}$) and the curl ($\nabla \times \mathbf{F}$) of the following vector field

$$\mathbf{F} = \underbrace{2xyz}_{P}\mathbf{i} + \underbrace{x^2 z}_{Q}\mathbf{j} + \underbrace{x^2 y}_{R}\mathbf{k}$$

You can do direct calculation by plugging P, Q, R into

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

and $\nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

or you can observe that $F = \nabla \phi$ for $\phi = x^2 yz$

then

$$\nabla \times \nabla \phi = 0$$

and

$$\nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 2yz + 0 + 0 = 2yz$$