Math 240 Quiz 3

1. Let \( y = \sum_{n=0}^{\infty} c_n x^n \) be a series solution to the differential equation
\[ y'' - xy' - y = 0 \]

Given that \( y(0) = 0 \) and \( y'(0) = 1 \) find \( c_5 \) and \( c_6 \).

\[
y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}
\]

\[
\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0
\]

\[
\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0
\]

Take out the 0th terms from the first and third sum (though we could have done without that too)

\[
2c_2 - c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1) c_{k+2} - (k+1) c_k] x^k = 0
\]

\[
c_2 = \frac{c_0}{2}
\]

\[
C_{k+2} = \frac{c_k}{(k+2)}
\]

\[
y(0) = c_0 = 0
\]

\[
y'(0) = c_1 = 1
\]

So \( c_0 = c_2 = c_4 = c_6 = \ldots = 0 \) and \( c_3 = \frac{c_1}{3} \) and \( c_5 = \frac{c_2}{5} \)

\[
so \quad c_5 = \frac{1}{15}
\]
2. Let $C$ be the part of the circle $x^2 + y^2 = 4$ above the $y = x$ line, traversed counterclockwise, plus the diameter of the same circle on the $y = x$ line traversed from lower left to upper right ($C$ is a closed curve consisting of the sum of those two parts). Let 

$$F = (x^2 + y^2, -2xy)$$

be a vector field. Find

$$\int_C F \cdot dr$$

$$C = C_1 + C_2$$

Parameterize these:

$$C_1: \quad (2\cos t, 2\sin t) \quad t \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

$$C_2: \quad (t, t) \quad t \in \left[-\sqrt{2}, \sqrt{2}\right]$$

$$\int_{C_1} (x^2+y^2)dx + (-2)dy = \int_{\pi/4}^{5\pi/4} 2(2\cos^2 t)dt - 2 \int_{\pi/4}^{5\pi/4} 2\cos t dt$$

$$= -2 \left( \sin \frac{5\pi}{4} - \sin \frac{\pi}{4} \right) - 4 \left( \cos \frac{5\pi}{4} - \cos \frac{\pi}{4} \right)$$

$$= -6 \frac{\sqrt{2}}{2} = -6\sqrt{2}$$

$$\int_{C_2} (x^2+y^2)dx + (-2)dy = \int_{-\sqrt{2}}^{\sqrt{2}} (2t^2 - 2) dt = \left[ \frac{2t^3}{3} - 2t \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{4}{3} \sqrt{2} - 2\sqrt{2} + \frac{4}{3} \sqrt{2} - 2\sqrt{2}$$

$$= \frac{8}{3} \sqrt{2} - 4\sqrt{2}$$

So

$$\int_{C_1 + C_2} F \cdot dr = -6\sqrt{2} + \frac{8}{3} \sqrt{2} - 4\sqrt{2} = -10\sqrt{2} + \frac{8}{3} \sqrt{2}.$$
3. Let $C$ be the closed curve in the previous question. Let 

$$F = (x^2 + y^2, 2xy)$$

be a vector field. Find 

$$\int_C F \cdot dr$$

In this case 

$$\frac{\partial F}{\partial y} = 2y = \frac{\partial F}{\partial x}$$

So 

$$\int_C F \cdot dr = 0$$ since $F$ is defined in a simply connected region.
4. Find the divergence ($\nabla \cdot \mathbf{F}$) and the curl ($\nabla \times \mathbf{F}$) of the following vector field

$$\mathbf{F} = 2xyz \mathbf{i} + x^2z \mathbf{j} + x^2y \mathbf{k}$$

You can do direct calculation by plugging $P, Q, R$ into

$$\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
p & Q & R \\
p & Q & R
\end{vmatrix}$$

and

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

or you can observe that $\mathbf{F} = \nabla \phi$ for $\phi = x^2yz$

Then

$$\nabla \times \nabla \phi = 0$$

and

$$\nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 2yz + 0 + 0 = 2yz$$