Lecture 7: We were looking at consequences of the 3 axioms of probability:

1. \( P(E) \geq 0 \)
2. \( P(S) = 1 \)
3. If \( E_1, E_2, \ldots \) are some mutually exclusive events (i.e. \( E_i \cap E_j = \emptyset \quad \forall i, j \)) then:
   \[ P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i) \]

Consequences:

1. \( P(\emptyset) = 0 \)
2. \( P(E^c) = 1 - P(E) \)
3. If \( E \subseteq F \), then \( P(E) \leq P(F) \)

**Proof:**

\[
F = E \cup (E^c \cap F) = F \cup E
\]

so \( P(F) = P(E) + P(E^c \cap F) \)

\[ \geq 0 \] by (1st axiom

so \( P(F) \geq P(E) \)

4. \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \)

Combine

\( E \cap F \quad F \)
Ex: I am reading two books.

\[ P(\text{I like 1st book}) = 0.5 \]
\[ P(\text{I like 2nd book}) = 0.4 \]
\[ P(\text{I like both books}) = 0.3 \]
\[ P(\text{I don't like either}) = ? \]

Solution: \[ A \cap B \]

\[
P(A) = 0.5 \]
\[
P(\overline{B}) = 0.4 \]
\[
P(\overline{A} \cap \overline{B}) = P(A) + P(\overline{B}) - P(A \cap \overline{B})
\]
\[
= 0.5 + 0.4 - 0.3 = 0.6
\]

3-way version:

\[ P(A \cup B \cup C) \]

\[
= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)
\]

n-way version (Inclusion-Exclusion Principle)

\[
P(\epsilon_1 \cup \epsilon_2 \cup \ldots \cup \epsilon_n) = \sum_{1}^{n} P(\epsilon_i) - \sum_{1}^{n} P(\epsilon_i \epsilon_{i+1}) + \sum_{1}^{n} P(\epsilon_i \epsilon_{i+2} \epsilon_{i+3}) - \ldots + \sum_{1}^{n} (-1)^{n-1} P(\epsilon_1 \epsilon_2 \ldots \epsilon_n)
\]

sum over all pairs \( \{\epsilon_i, \epsilon_j\} \)

+then.
Problems where every element of the sample space is equally likely (uniform distribution)

\[ P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|} \]

**Ex 1:** Two dice are rolled, Prob. that sum is 7?

\[ E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \]

\[ P = \frac{6}{36} = \frac{1}{6} \]

**Ex 2:** Committee of 5 is selected from 6 men and 9 women. What is the probability that 3 men and 2 women are selected?

\[ \binom{6}{3} \cdot \binom{9}{2} = \frac{20 \cdot 36}{1001} \]

**Ex 3:** \( n \) balls, \( k \) are chosen. One ball is special, what is the probability that the special ball is chosen?

\[ \frac{\text{number of ways to choose } k \text{ balls including special ball}}{\binom{n}{k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n} \]

**Ex 4:** 52 cards dealt to 4 players. Prob. that player 1 gets all spades (\( \spadesuit \))

- Number of dealings where player 1 gets all spades?
- Probability that player 1 gets all spades?

\[ \frac{1}{\binom{52}{13}} \]

because player 1 gets 13 random cards only one hand is all spades
Ex5: Same but "one player gets all spades:

\[
\begin{align*}
\text{A:} & \quad \frac{4}{\binom{52}{13}} \quad \text{because it's} \quad P(\text{player 1 gets all spades}) \\
& \quad + P(\text{player 2 gets all spades}) \\
& \quad + \ldots \\
& \quad \text{(mutually exclusive events)}
\end{align*}
\]

Probability as measure of belief:

Say I am writing software for a robot in a grid.

- Robot "thinks" there is 30% chance of finding something to clean at location A, 20% chance at location B.
- 40% at location C, 10% chance at location D.

Should the robot go right or down? (assuming robot wants to clean up at least one square)

We want to compute the prob. that there is something at location A or B. 

\[1 - 0.7 \cdot 0.8 = 0.44.\]

For D: \[1 - 0.6 \cdot 0.9 = 0.46.\]