Lecture 9: Hard problems.

(In this lecture, we did the problems at end of notes for lecture 7.)

More examples of uniform distribution problems.

Ex1: \( n \) people are in a room. What is the probability that no two of them have the same birthday. How many people \( n \) makes this probability \( > 50\% \)?

Solution: \[
\frac{\text{# of ways no two birthdays coincide}}{\text{# of ways } n \text{ birthdays can be.}} = \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}
\]

Calculating with computer:

\[\begin{array}{c|c|c}
 n & p & \\
 23 & 0.4927 & \\
 50 & 0.029 & \\
\end{array}\]

Ex2 (Hat) matching problem.

\( n \) people throw their hats, hats get randomly mixed up. What is the probability that nobody gets their own hat?

Solution: \( P(\text{everyone gets their own hat}) = \frac{1}{n!} \)

because it's one permutation \((123 \ldots n)\) among \(n!\) possible permutations of hats.

Let \( E_i = \) the event that \( i \)th person gets their own hat

\[ |\Sigma| = n! \]
In order to compute $P$(nobody gets their own hat) 

$= P(S \setminus (E_1 \cup E_2 \cup \ldots \cup E_n))$

Let us compute $P(E_1 \cup E_2 \cup \ldots \cup E_n)$.

$P(E_1 \cup \ldots \cup E_n) = \sum P(E_i) - \sum P(E_i \cap E_j) + \sum P(E_i \cap E_j \cap E_k) - \ldots$

Can we compute?

$P(E_{i_1}, E_{i_2}, E_{i_3}, \ldots E_{i_k})$

Yes, it's as follows the hats for $i_1, \ldots, i_k$ are fixed, the others can be permuted any way we want.

$P(E_{i_1} \ldots E_{i_k}) = \frac{(n-k)!}{n!} \cdot \frac{\# \text{ of ways to permute remaining hats}}{\text{all hat permutations}}$

$\sum P(E_{i_1} \ldots E_{i_k}) = \frac{n!}{(n-k)!} \cdot \frac{(n-k)!}{n!} \cdot \frac{\# \text{ of ways to permute remaining hats}}{\text{all hat permutations}}$

So

$P(E_1 \cup E_2 \cup \ldots \cup E_n)$

$\approx \frac{1}{e} \approx \frac{1}{e} = e^{-1}$

Taylor series

Remark: The probability doesn't approach 1 as $n \to \infty$!
Ex 3: 10 couples (20 people) are seated (individually) randomly at the table that is round. What is the probability that no husband sits next to his wife?

Solution: We will solve this by computing

\[ 1 - P(\text{some couple ends up sitting next to each other}) \]

How many ways can we sit \( n \) people at a round table anyway?

\[ \frac{n!}{n} \]

divide by \( n \) because rotating everyone doesn't change reality arrangement

\[ \text{i.e. } \frac{5!}{5} = \frac{6!}{6} \]

Let \( E_i \) be the event that the \( i \)th couple sits next to each other. We want \( 1 - P(\bigcup_{i=1}^{10} E_i) \)

\[ P(\bigcup_{i=1}^{10} E_i) = \sum_{i=1}^{10} P(E_i) - \cdots + (-1)^{k+1} \sum_{1 < i_1 < \cdots < i_k} P(E_{i_1} \cap \cdots \cap E_{i_k}) \]

\[ P(E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{(20-k)!}{(20-k)^2} \frac{2}{19!} \]

# of ways to seat 20 people at round table

Think of each couple sitting next to each other as a single item. There are \( k \) couples next to each other and 20 - 2k persons besides those. So 20 - k entities, they can be seated \( (20-k)! \) ways but we can also switch the husband and wife for each of the \( k \) couples (2k)}