Lecture 24:

Other random variables less popular than binomial or Poisson.

Geometric r.v.

Suppose independent trials are performed, each successful with probability \( p \), until a success occurs.

\[ X = \text{"\# of trials performed"} \]

\[ P(X = n) = (1-p)^{n-1} \ p \quad n=1,2,... \]

Let's look at the total probability:

\[ \sum_{n=1}^{\infty} P(X = n) = \sum_{n=1}^{\infty} (1-p)^{n-1} \ p = p \sum_{j=0}^{\infty} (1-p)^j = p \cdot \frac{1}{1-(1-p)} = 1. \]

Geometric series

\[ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \]

**Ex.** I decided that I'm going to play the lotto (winning probability \( p \)) every week once, until I win and then retire. What is the expected number of times I would play? (expected number of weeks until retirement)

Before we solve, let's find the expected number in general:

\[ \mathbb{E}[X] = ? \]
Expectation of geometric r.v. 

\[ E[X] = \sum_{i=1}^{\infty} i q^{i-1} p \] 

\[ = \sum_{i=1}^{\infty} \frac{i p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{1-p} \] 

Let \( q = 1-p \).

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

\[ \sum_{n=1}^{\infty} n x^{n-1} = -1 \cdot \frac{(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \]

Number of (expected) weeks till retirement: \( \frac{1}{p} \approx 13,000,000 \) weeks

We can do the series for the variance too and get:

\[ \text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1-p}{p^2} \]

Negative Binomial Random variable:

Independent trials are performed until we get \( r \) successes.

\( X = \# \text{ of trials until we get } r \text{ successes} \)

\[ P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r} \quad n = r, r+1, \ldots \]
What is the probability of having $r$ successes before $m$ failures? Each success is with probability $p$.

**Solution:** $r$ successes must occur before $(r+m)$th trial because otherwise I am guaranteed to have $m$ failures. So the total probability is:

$$
\sum_{n=r}^{r+m-1} \Pr(X=n) = \sum_{n=r}^{r+m-1} \binom{n-1}{r-1} p^r (1-p)^{n-r}
$$

**Expected value:** we can do the sum as usual, but we can also be very clever:

Observe that, if $Y =$ # of trials until one success (geometric),

then $X = Y + Y + \cdots + Y$

$r$ times.

So $E[X] = E[Y] + \cdots + E[Y] = \frac{r}{p}$

**Hypergeometric r.v.:**

I have a box with $n$ balls, $m$ white, $n-m$ black.

$X =$ the number of white balls I would get if I randomly select $r$ balls (without replacement)
\[ P(X = i) = \frac{{m \choose i} \cdot {N-m \choose r-i}}{{N \choose r}} \]

\[ \mathbb{E}[X] = \frac{nm}{N} \quad (\text{Book page 154}) \]