Lecture 28: Continuing normal random variables.

Last time: \( N(\mu, \sigma^2) \)

Density function:
\[
f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

Properties:
- Total probability is 1.
- \[ \int_{-\infty}^{\infty} f(x) \, dx = 1 \] (book page 188)
- \( \mathbb{E}[X] = \mu \)
  \[
  \mathbb{E}[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \, e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \, dx = \mu
  \]
  - Change of variables
    \[
    u = x - \mu \quad \text{then another change of vars} \quad v = u^2
    \]
  - Key property:
    \[ N(\mu, \sigma^2) = \mu + \sigma N(0, 1) \]
  
So going back to the expectation above
\[ \mathbb{E}[N(\mu, \sigma^2)] = \mu + \sigma \mathbb{E}[N(0, 1)] \] (a little easier)
\[ \text{Var}(\mathcal{N}(\mu, \sigma^2)) = \sigma^2 \] again tricky integral.

\[ \mathcal{N}(0,1) \] standard normal distribution

**Def.** \( \Phi(x) = \text{cumulative distribution function} \)

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \]

because this integral is very hard, there is a table for it in book page 190.

**Ex.** \( X \overset{\text{normal}}{\sim} \) random variable with mean \( \mu = 3 \), std. \( \sigma = 3 \).

find \( P(2 < X < 5) \)

**Key fact!** For \( X \) normal random variable,

\[ F_X(x) = P(X \leq x) = P\left( \frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma} \right) \]

\[ = P\left( \mathcal{N}(0,1) \leq \frac{x - \mu}{\sigma} \right) \]

\[ = \Phi\left( \frac{x - \mu}{\sigma} \right) \]

so we can use \( \Phi \) to get the C.D.F. of any normal random variable.
Solution to problem: \( P(N(3,9) \leq 5) \)

The area we want is between 2 and 5.
\[ F(5) - F(2) \]

\[ F(5) = P(N(3,9) \leq 5) = P\left(\frac{N(3,9) - 3}{3} \leq \frac{5 - 3}{3}\right) = P(N(0,1) \leq \frac{2}{3}) = \Phi\left(\frac{2}{3}\right) \]

Similarly, \( F(2) = \Phi\left(\frac{1}{3}\right) \).

So \( F(5) - F(2) = \Phi\left(\frac{2}{3}\right) - \Phi\left(\frac{1}{3}\right) \).

**Ex:** Binary message 0 or 1 is transmitted on a wire from A to B.