218BC Introduction to Manifolds and Geometry

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Contents

Introduction

This is a continutation of 218A. In 218B, we will discuss vector bundles, differential forms, etc. In 218C, we will discuss de Rham cohomology. References are [Lee13, Spi79, War83].

1 Lecture 1

1.1 Vector bundles

Definition 1.1. A smooth real vector bundle of rank k over a smooth manifold M^n is a topological space E together with a smooth projection

$$\pi : E \to M \tag{1.1}$$

such that

- For $p \in M$, $\pi^{-1}(p)$ is a vector space of dimension k over $\mathbb{R}$.
- There exists local trivializations, that is, there are smooth mappings

  $$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E \tag{1.2}$$

  which maps $p \times \mathbb{R}^k$ linearly onto the fiber $\pi^{-1}(p)$ for every $p \in U_\alpha$.

The transition functions of a bundle are defined as follows.

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \tag{1.3}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{1.4}$$

for $v \in \mathbb{R}^k$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{1.5}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satifsying (1.5), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$. If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle.

Exercise 1.2. If M is a smooth n-dimensional manifold then $\pi : TM \to M$ is a rank n vector bundle. (This was done in 218A). A coordinate system (U_α, x_α), where $x_\alpha : U_\alpha \to \mathbb{R}^n$ gives a trivialization

$$\Phi_\alpha(p, (v^1, \ldots, v^n)) = \sum_{i=1}^n v^i \frac{\partial}{\partial x_\alpha^i}\Big|_p. \tag{1.6}$$

Given another coordinate system (U_β, x_β) with $U_\alpha \cap U_\beta \neq \emptyset$, find the transition functions $\varphi_{\alpha\beta}$.

The tangent bundle has some extra structure which an arbitrary vector bundle does not possess. Recall from 218A that a smooth mapping between smooth manifolds $f : M \to N$ induces a mapping $f_* : TM \to TN$. The following diagram

$$\begin{tikzcd} TM \arrow[r, "f_*"] \arrow[d, "\pi_M"] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{1.7}$$

commutes and f_* restricts to a linear mapping on fibers. Given a smooth mapping $h : N \to X$, consider the composition $h \circ f : M \to X$. The chain rule says that

$$(h \circ f)_* = h_* \circ f_* : TM \to TX. \tag{1.8}$$

Definition 1.3. A bundle mapping between vector bundles E_1 over M and E_2 over N is a mapping $F : E_1 \to E_2$ which maps fibers linearly to fibers and covers a smooth mapping between the base spaces. That is, the diagram

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_M"] & E_2 \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{1.9}$$

commutes.

Definition 1.4. The category of smooth manifolds $\mathbf{Man}^\infty$ has objects as smooth manifolds and morphisms as smooth mappings, where composition of morphisms is just composition of mappings.

Composition of morphisms is obviously associative, i.e.,

$$(\Psi_1 \circ \Psi_2) \circ \Psi_3 = \Psi_1 \circ (\Psi_2 \circ \Psi_3) \tag{1.10}$$

and every manifolds has an identity morphism $id_X : X \to X$, which is obviously smooth, so this is indeed a category.

Definition 1.5. The category **Vect** of smooth vector bundles over smooth manifolds is the collection of all vector bundle (of any rank) over smooth manifolds. The morphisms are the bundle mappings.

We therefore have a functor $\mathcal{T} : \mathbf{Man}^\infty \to \mathbf{Vect}$ where **Vect** is the category of smooth vector bundles over smooth manifolds given by $M \mapsto TM$ and $f : M \to N$ maps to $f_* : TM \to TN$. The mapping $\mathcal{T}$ satisfies $\mathcal{T}(Id_M) = Id_{TM}$ and by (1.8), $\mathcal{T}(f_1 \circ f_2) = \mathcal{T}(f_1) \circ \mathcal{T}(f_2)$, so this is a **covariant** functor, called the tangent functor.

Definition 1.6. For a fixed smooth manifold M, the category $\mathbf{Vect(M)}$ is the collection of smooth vector bundles over M (of any rank). A morphism in this category is a mapping $F : E_1 \to E_2$ covering the identity mapping, that is, the diagram

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_M"] & E_2 \arrow[d, "\pi_M"] \\ M \arrow[r, "id_M"] & M \end{tikzcd} \tag{1.11}$$

We say that bundles E_1 and E_2 over M are isomorphic if there exists an invertible bunble mapping between E_1 and E_2. If E is isomorphic to the trivial bundle over M, $\pi_M : M \times \mathbb{R}^k \to M$ defined by $\pi_M(p, v) = p$, then we say that E is trivial.

We next express the above in coordinates. Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions

$$f_\alpha : U_\alpha \to \mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2}) \tag{1.12}$$

defined by

$$f_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)). \tag{1.13}$$

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi^{E_2}_{\alpha\beta} f_\beta \varphi^{E_1}_{\beta\alpha}, \tag{1.14}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} f_\alpha = f_\beta \varphi^{E_1}_{\beta\alpha}. \tag{1.15}$$

Bundles E_1 and E_2 are equivalent if and only if $\mathrm{rank}(E_1) = \mathrm{rank}(E_2)$ and there exist f_α as above with $\det(f_\alpha) \neq 0$. A vector bundle is *trivial* if and only if there exist functions

$$f_\alpha : U_\alpha \to GL(k, \mathbb{R}) \tag{1.16}$$

such that

$$\varphi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{1.17}$$

Remark 1.7. In the above, for a fixed base space M, we only defined morphisms in the category of vector bundles to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence. More on this later.

1.2 Sections of bundles

Definition 1.8. Let $\pi : E \to M$ be a vector bundle. A section of a bundle is a smooth mapping $s : M \to E$ such that $\pi \circ s = id_M$. The space of sections is denoted by $\Gamma(E)$.

In other words, $s(x) \in E_x$, s maps x to a vector in the fiber over x. In terms of local trivializations we have the following. Let

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha) \tag{1.18}$$

be a local trivialization. Then

$$s_\alpha = \pi_2 \circ \Phi_\alpha^{-1} \circ s : U_\alpha \to \mathbb{R}^k \tag{1.19}$$

is called a local representative of s with respect to Φ_α. On U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\Phi_\beta : U_\beta \times \mathbb{R}^k \to \pi^{-1}(U_\beta). \tag{1.20}$$

Recall that the transition functions of a bundle are

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \tag{1.21}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{1.22}$$

for $v \in \mathbb{R}^k$. Then for any $e_x \in \pi^{-1}(x)$, we have

$$\varphi_{\alpha\beta}(x)(\pi_2 \circ \Phi_\beta^{-1}(e_x)) = \pi_2 \circ \Phi_\alpha^{-1}(e_x). \tag{1.23}$$

Choosing $e_x = s(x)$ we have

$$\varphi_{\alpha\beta}(s)(\pi_2 \circ \Phi_\beta^{-1} \circ s(x)) = \pi_2 \circ \Phi_\alpha^{-1} \circ s(x), \tag{1.24}$$

or simply

$$\varphi_{\alpha\beta} s_\beta = s_\alpha, \text{ on } U_\alpha \cap U_\beta, \tag{1.25}$$

which is the local transformation law for a section.

Conversely, if a bundle $\pi : E \to M$ is given to us in terms of transition functions, then any collection of functions

$$s_\alpha : U_\alpha \to \mathbb{R}^k \tag{1.26}$$

satisfying (1.25) gives a well-defined smooth section $s : M \to E$.

2 Lecture 2

2.1 Pull-back bundles

If M and N are smooth manifolds, and $\pi_N : E \to N$ is a vector bundle over N, then given a smooth mapping $f : M \to N$, define

$$f^*E = \{(p, v) \in M \times E \mid f(p) = \pi_N(v)\}. \tag{2.1}$$

Proposition 2.1. *The pullback f^*E is a vector bundle over M, with projection given by $\pi_1(p, v) = p$, and the fiber f^*E over $p \in M$ is identified with the fiber $E_{f(p)}$, i.e., the following diagram commutes*

$$\begin{tikzcd} f^*(E) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & E \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N. \end{tikzcd} \tag{2.2}$$

Proof. Let $\Phi : U \times \mathbb{R}^k \to \pi_N^{-1}(U)$ be a local trivialization for E. The set $f^{-1}(U)$ is open since f is continuous, and define

$$f^*\Phi : f^{-1}(U) \times \mathbb{R}^k \to \pi_1^{-1}(f^{-1}(U)) \tag{2.3}$$

by

$$f^*\Phi(x, v) = (x, \Phi(f(x), v)). \tag{2.4}$$

The reader can verfiy that this is a local trivialization for f^*E. $\square$

Next we note that sections can be pulled back to sections of the pullback bundle.

Definition 2.2. Let $f : M \to N$ be a smooth mapping between smooth manifolds, and $\pi : E \to N$ be a vector bundle over N. If $\sigma : N \to E$ is a section of E, then $(\sigma \circ f)(x) = (x, \sigma(f(x)))$ is a section of $\pi_1 : f^*E \to M$ and is called the pullback of σ under f.

The fact that this is a section of the pullback bundle is almost obvious, we just need to check that

$$\pi_1(\sigma \circ f)(x) = \pi_1(x, \sigma(f(x))) = x. \tag{2.5}$$

2.2 Push-forward of vector fields

Next, we restrict to tangent bundles. Let $f : M \to N$ be a smooth mapping between smooth manifolds. Then f^*TN is a vector bundle over M. Define

$$(f_*)_B : TM \to f^*TN \tag{2.6}$$

by

$$(f_*)_B(v_p) = (p, f_*v). \tag{2.7}$$

(the subscript B is short for "bundle mapping"). We have the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{(f_*)_B} & f^*TN \\ \big\downarrow{\scriptstyle \pi_M} & & \big\downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{id} & M. \end{array} \tag{2.8}$$

Definition 2.3. If $X \in \Gamma(TM)$, then we can define $f_*X \in \Gamma(f^*TN)$, by

$$f_*X \equiv (f_*)_B \circ X. \tag{2.9}$$

In words: under smooth mappings, vector fields push-forward to sections of the pull-back bundle.

Remark 2.4. Note that for $f : M \to N$, although we can push-forward individual tangent vectors, in general there is *not* a mapping

$$f_* : \Gamma(TM) \to \Gamma(TN). \tag{2.10}$$

For example, f might not even be surjective. This is one reason we had to consider pull-back bundles in the above discussion.

Example 2.5. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve. Then the tangent vector $X = \frac{\partial}{\partial t} \in \Gamma(T\mathbb{R})$, and $\gamma_* X \in \Gamma(\gamma^* T\mathbb{R}^n)$ is a vector field along the curve. Note the curve might have self-intersections, and it could even be a constant path, in which case the pull-back bundle is the trivial bundle.

2.3 Example: The Mobius bundle

Let $B = S^1$ be the base space. Of course, we have the trivial bundle $\pi : S^1 \times \mathbb{R} \to S^1$. Let us define another bundle over S^1. Consider $S^1 = \{v \in \mathbb{R}^2 \mid |v| = 1\}$. Consider $\mathbb{RP}^1 = S^1 / \sim$ where $e^{i\theta} \sim e^{i(\theta+\pi)}$. The quotient space is clearly S^1. Note that a line through the origin is hits the unit circle in exactly 2 opposite points. Therefore we can identify $\mathbb{RP}^1$ with the space of lines through the origin in $\mathbb{R}^2$. Denote the line determined by $p \in \mathbb{RP}^1$ as $[p]$. We then define the following:

$$M = \{(p, v) \in \mathbb{RP}^1 \times \mathbb{R}^2 \mid v \in [p]\}. \tag{2.11}$$

Define a local trivialization as follows. Cover $\mathbb{RP}^1$ by 2 open sets

$$U_1 = \{\ell = re^{i\theta_1} \mid r \in \mathbb{R}, 0 < \theta_1 < \pi\}, \quad U_2 = \{\ell = re^{i\theta_2} \mid r \in \mathbb{R}, \pi/2 < \theta_2 < 3\pi/2\} \tag{2.12}$$

Then define $\Phi_1 : U_1 \times \mathbb{R} \to M$ by

$$\Phi_1(\theta_1, r) = \{[\ell_{\theta_1}], re^{i\theta_1}\}, \tag{2.13}$$

and $\Phi_2 : U_2 \times \mathbb{R} \to M$ by

$$\Phi_2(\theta_2, r) = \{[\ell_{\theta_2}], re^{i\theta_2}\}, \tag{2.14}$$

Next, we determine the overlap mapping

$$\varphi_{12} = \pi_2 \circ \Phi_1^{-1} \circ \Phi_2. \tag{2.15}$$

Note that $U_1 \cap U_2 = V_1 \coprod V_2$, where

$$V_1 = \{0 < \theta_1 < \pi/2\} = \{\pi < \theta_2 < 3\pi/2\}, \tag{2.16}$$
$$V_2 = \{\pi/2 < \theta_1 < \pi\} = \{\pi/2 < \theta_2 < \pi\}. \tag{2.17}$$

We have on V_2, $\varphi_{12} = Id_{\mathbb{R}}$, since $\theta_1 = \theta_2$ there. But on V_1, $\varphi_{12} = -Id_{\mathbb{R}}$, since $\theta_1 = \theta_2 - \pi$ there.

Proposition 2.6. *The bundle $\pi : M \to S^1$ is not trivial.*

Proof. If this bundle were trivial, there would exist a non-zero section. With respect to the above local trivializations, we have

$$s_1 : U_1 \to \mathbb{R}, \quad s_2 : U_2 \to \mathbb{R}, \tag{2.18}$$

with $s_1 = \varphi_{12} s_2$ on $U_1 \cap U_2$. So we have

$$s_1 = \begin{cases} s_2 & \text{on } V_2 \\ -s_2 & \text{on } V_1 \end{cases} \tag{2.19}$$

Since s_2 is not zero on a connected set, it is either positive or negative. But this implies that s_1 is both positive and negative somewhere on U_1. But since U_1 is connected and s_1 is continuous, it would have to be zero somewhere.

□

Proposition 2.7. *Let $f : S^1 \to \mathbb{RP}^1$ be the 2-fold covering mapping. Then $\pi_1 : f^*M \to S^1$ is a trivial bundle.*

Proof. We have

$$f^*M = \{(p, v) \in S^1 \times M \mid \pi(v) = f(p)\}. \tag{2.20}$$

Consider the mapping $\sigma : S^1 \to f^*M$ given by $\sigma : p \mapsto (p, ([p], p))$. Then

$$\pi_1 \sigma(p) = \pi_1(p, [p]) = p, \tag{2.21}$$

so σ is a section of f^*M which is nowhere-zero. We then have a global trivialization of f^*M by $\Phi : S^1 \times \mathbb{R} \to f^*M$ by

$$\Phi(p, r) = (p, r \cdot \sigma). \tag{2.22}$$

□

3 Lecture 3

3.1 Direct sums

If $V_1, \ldots, V_k$ are vector spaces over $\mathbb{R}$, then the direct sum $V_1 \oplus \cdots \oplus V_k$ is the Cartesian product $V_1 \times \cdots \times V_k$ with the following vector space structure:

$$c(v_1, \ldots, v_k) = (cv_1, \ldots, cv_k) \tag{3.1}$$

$$(v_1, \ldots, v_k) + (v_1', \ldots, v_k') = (v_1 + v_1', \ldots, v_k + v_k'), \tag{3.2}$$

for $c \in \mathbb{R}$. The space $V_1 \oplus \cdots \oplus V_k$ satisfies the following "universal" mapping property. For $1 \le i \le k$, let $\iota_i : V_i \to V_1 \oplus \cdots \oplus V_k$ be the inclusion mapping

$$\iota_i : v \mapsto (0, \ldots, \overbrace{v}^{i}, \ldots, 0). \tag{3.3}$$

Let W be any vector space, and $f_i : V_i \to W$ be linear mappings for $1 \le i \le k$. Then there is a *unique* linear map $f : V_1 \oplus \cdots \oplus V_k \to W$ which makes the following diagram

$$\begin{tikzcd} V_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & V_1 \oplus \cdots \oplus V_k \arrow[d, "f"] \\ & W \end{tikzcd} \tag{3.4}$$

commute for $1 \le i \le k$.

This property uniquely characterizes the direct sum. That is, a vector space with the above universal mapping property is isomorphic to the direct sum. Note that obviously

$$\dim_{\mathbb{R}}(V_1 \oplus \cdots \oplus V_k) = \sum_{i=1}^{k} \dim_{\mathbb{R}}(V_i). \tag{3.5}$$

Exercise 3.1. Prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3). \tag{3.6}$$

Definition 3.2. Let $V_i, i \in \mathcal{I}$ be any collection of vector spaces. The Cartesian product $\Pi_{i\in\mathcal{I}} V_i$ is the collection of all functions

$$f : \mathcal{I} \to \cup_{i\in\mathcal{I}} V_i, \tag{3.7}$$

such that $f(i) \in V_i$ for all $i \in \mathcal{I}$. The direct product $\Pi_{i\in\mathcal{I}} V_i$ is the Cartesian product with the vector space structure

$$cf(i) = cf(i) \tag{3.8}$$
$$(f+g)(i) = f(i) + g(i). \tag{3.9}$$

The projection $\pi_i : \Pi_{i\in\mathcal{I}} V_i \to V_i$ is the mapping $\pi_i(f) = f(i)$. The above definition satisfies the following universal property. If V is any vector space and $\phi_i : V \to V_i$ are linear mappings for $i \in \mathcal{I}$, then there is a unique linear mapping $\phi : V \to \Pi_{i\in\mathcal{I}} V_i$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\phi_i"] \arrow[dr, "\phi"'] & V_i \\ & \Pi_{i\in\mathcal{I}} V_i \arrow[u, "\pi_i"'] \end{tikzcd} \tag{3.10}$$

commutes for each $i \in \mathcal{I}$. This property uniquely characterizes the direct product. That is, any vector space with the above universal mapping property is isomorphic to the direct product.

Definition 3.3. Let $V_i, i \in \mathcal{I}$ be any collection of vector spaces. The direct sum $\oplus_{i\in\mathcal{I}} V_i$ is the subspace of the direct product consisting of the functions f such that $f(i) \neq 0$ for only finitely many $i \in \mathcal{I}$.

Remark 3.4. The direct sum satisfies the first universal property (3.4), but not the second (3.10), unless $\mathcal{I}$ is finite. (We leave the proof to the interested reader.)

Definition 3.5. Given vector bundles $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$, the Cartesian product is the bundle $\pi_1 \times \pi_2 : E_1 \times E_2 \to M_1 \times M_2$, defined by $\pi_1 \times \pi_2(e_1, e_2) = (\pi_1(e_1), \pi_2(e_2))$.

Exercise 3.6. (i) Show that this is a vector bundle of rank equal to $\mathrm{rank}(E_1) + \mathrm{rank}(E_2)$ over $M_1 \times M_2$, with fiber over (p_1, p_2) isomorphic to $\pi_1^{-1}(p_1) \oplus \pi_2^{-1}(p_2)$.
(ii) If M_1 and M_2 are smooth manifolds, then $T(M_1 \times M_2)$ is isomorphic to $TM_1 \times TM_2$.

Definition 3.7. Let $\Delta : M \to M \times M$ be the diagonal embedding, that is, $\Delta(p) = (p, p)$. Given vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$, define the direct sum $E_1 \oplus E_2 \equiv \Delta^*(E_1 \times E_2)$.

We can also give a description of the direct sum in terms of trivializations. If $\Phi_1 : U \times \mathbb{R}^{k_1} \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^{k_2} \to \pi_2^{-1}(U)$ are local trivializations then

$$\Phi : U \times (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2}) \to \pi^{-1}(U) \tag{3.11}$$

defined by

$$\Phi(x, (v_1, v_2)) = (\Phi_1(x, v_1), \Phi_2(x, v_2)) \tag{3.12}$$

is a local trivialization for $E_1 \oplus E_2$. Note that the transition functions satisfy

$$\varphi_{\alpha\beta}^{E_1 \oplus E_2} = \varphi_{\alpha\beta}^{E_1} \oplus \varphi_{\alpha\beta}^{E_2} \in GL(k_1 + k_2, \mathbb{R}), \tag{3.13}$$

where this is the "block" matrix

$$\varphi_{\alpha\beta}^{E_1 \oplus E_2}(x)(v, w) = \begin{pmatrix} \varphi_{\alpha\beta}^{E_1}(x) & 0 \\ 0 & \varphi_{\alpha\beta}^{E_2}(x) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \tag{3.14}$$

3.2 Tensor products

Definition 3.8. If A is any set, then the free vector space over A is

$$\mathcal{F}(A) = \oplus_{a \in A} \mathbb{R}. \tag{3.15}$$

This can be thought of as the vector space with basis elements $a \in A$. That is, $\mathcal{F}(A)$ is the set of formal sums

$$\mathcal{F}(A) = \Big\{ \sum_{a \in A} f_a a \mid f_a \neq 0 \text{ for only finitely many } a \in A \Big\} \tag{3.16}$$

with vector space structure

$$c \sum_{a \in A} f_a a = \sum_{a \in A} (c f_a) a \tag{3.17}$$

$$\sum_{a \in A} f_a a + \sum_{a \in A} f'_a a = \sum_{a \in A} (f_a + f'_a) a. \tag{3.18}$$

Definition 3.9. If $V_1, \ldots, V_k$ are vector spaces over $\mathbb{R}$, then the tensor product $V_1 \otimes \cdots \otimes V_k$ is the free real vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$ modulo the subspace spanned by all elements of the form

$$(v_1, \ldots, cv_i, \ldots, v_k) - c(v_1, \ldots, v_i, \ldots, v_k) \tag{3.19}$$

$$(v_1, \ldots, v_i + v_i', \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v_i', \ldots, v_k), \tag{3.20}$$

for $c \in \mathbb{R}$.

The space $V_1 \otimes \cdots \otimes V_k$ satisfies the universal mapping property as follows. Let W be any vector space, and $F : V_1 \times \cdots V_k \to W$ be a multilinear mapping, i.e., F is linear when restricted to each factor, with the other variables held fixed. Then there is a unique *linear* map $\tilde{F} : V_1 \otimes \cdots \otimes V_k$ which makes the following diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "\pi"] \arrow[dr, "F"'] & V_1 \otimes \cdots \otimes V_k \arrow[d, "\tilde{F}"] \\ & W \end{tikzcd} \tag{3.21}$$

commutative, where π is the projection to the quotient space, which we write as

$$\pi(v_1, \ldots, v_k) = v_1 \otimes \cdots \otimes v_k. \tag{3.22}$$

We say that an element in $V_1 \otimes \cdots \otimes V_k$ of the form $v_1 \otimes \cdots \otimes v_k$ is *decomposable.* A general element of $V_1 \otimes \cdots \otimes V_k$ is not decomposable, but can always be written as a finite sum of decomposable elements.

Exercise 3.10. (i) Prove that

$$\dim_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k) = \dim_{\mathbb{R}}(V_1) \cdots \dim_{\mathbb{R}}(V_k). \tag{3.23}$$

(ii) Prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3). \tag{3.24}$$

Definition 3.11. The tensor product of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is the vector bundle $\pi : E_1 \otimes E_2 \to M$ defined by $\pi^{-1}(p) = \pi_1^{-1}(p) \otimes \pi_2^{-1}(p)$. If $\Phi_1 : U \times \mathbb{R}^{k_1} \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^{k_2} \to \pi_1^{-1}(U)$ are local trivializations then consider

$$F : U \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}) \to \pi^{-1}(U) \tag{3.25}$$

defined by

$$F(x, (v_1, v_2)) = \Phi_1(x, v_1) \otimes \Phi_2(x, v_2). \tag{3.26}$$

This is clearly a multilinear mapping on each fiber, so by the universal property of tensor products, there is a unique induced mapping

$$\tilde{F} : U \times (\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}) \to \pi^{-1}(U) \tag{3.27}$$

which, using an isomorphism $\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \cong \mathbb{R}^{k_1 k_2}$, defines a local trivialization for $E_1 \otimes E_2$.

We could have equivalently defined the tensor product in terms of transition functions. To do this, note the following. If $\phi_1 \in GL(k_1, \mathbb{R})$ and $\phi_2 \in GL(k_2, \mathbb{R})$ then define

$$\phi_1 \times \phi_2 : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \tag{3.28}$$

by

$$(\phi_1 \times \phi_2)(v_1, v_2) = \phi_1(v_1) \otimes \phi_2(v_2). \tag{3.29}$$

This is clearly a multilinear mapping, so by the universal property for tensor products, there is a unique induced mapping

$$\phi_1 \otimes \phi_2 : \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \to \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}. \tag{3.30}$$

Given transition functions for E_1

$$\phi^{E_1}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k_1, \mathbb{R}), \tag{3.31}$$

and transition functions for E_2

$$\phi^{E_2}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k_2, \mathbb{R}), \tag{3.32}$$

we define

$$\varphi^{E_1 \otimes E_2}_{\alpha\beta} = \varphi^{E_1}_{\alpha\beta} \otimes \varphi^{E_2}_{\alpha\beta} \in GL(k_1 k_2, \mathbb{R}), \tag{3.33}$$

where we choose some isomorphism $\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \cong \mathbb{R}^{k_1 k_2}$.

3.3 Dual bundles

Definition 3.12. The dual of a vector space V is $V^* = \mathrm{Hom}(V, \mathbb{R})$, which is the space of all linear mappings from V to $\mathbb{R}$.

Remark 3.13. If V is finite-dimensional, we have that $V^* \cong V$, equivalently, $\dim_{\mathbb{R}}(V^*) = \dim_{\mathbb{R}}(V)$. Warning: this is not true over $\mathbb{C}$. In this case, we have $V^* \cong \overline{V}$, more on this later.

Definition 3.14. The dual of a vector bundle $\pi : E \to M$ is the vector bundle $\Pi : E^* \to M$ defined by $\Pi^{-1}(p) = (\pi^{-1}(p))^*$. If $\Phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is a local trivialization then

$$\Phi^* : U \times (\mathbb{R}^k)^* \to \Pi^{-1}(U) \tag{3.34}$$

defined by

$$\Phi^*(x, f)(v_p) = f(\pi_2 \circ \Phi^{-1}(v_p)) \tag{3.35}$$

is a local trivialization for E^*.

Exercise 3.15. Show that the transition functions of E^* are

$$\varphi^{E^*}_{\alpha\beta} = \big((\varphi^E_{\alpha\beta})^{-1}\big)^T = (\varphi^E_{\beta\alpha})^T. \tag{3.36}$$

4 Lecture 4

4.1 Riemannian metrics on real vector bundles

If $\pi : E \to M$ is a real vector bundle, a Riemannian metric on E is a choice of smoothly varying positive definite symmetric inner product on each fiber. That is $g \in \Gamma(E^* \otimes E^*)$ satisfying

$$g(e_1, e_2) = g(e_2, e_1), \tag{4.1}$$

and

$$g(e, e) > 0 \text{ for } e \neq 0. \tag{4.2}$$

Proposition 4.1. *If E is any real vector bundle, then E admits a Riemannian metric.*

Proof. Let

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha) \tag{4.3}$$

be a local trivialization for U_α an open covering of M which is locally finite.. For $x \in U_\alpha$ and $e_1, e_2 \in E_x$, define

$$g_\alpha(e_1, e_2) = \langle \pi_2 \circ \Phi_\alpha^{-1}(e_1), \pi_2 \circ \Phi_\alpha^{-1}(e_2) \rangle, \tag{4.4}$$

where $\langle \cdot, \cdot, \rangle$ denotes the Euclidean inner product on $\mathbb{R}^k$. Next, let χ_α be a partition of unity subordinate to the cover U_α, that is

$$\operatorname{supp}(\chi_\alpha) \subset U_\alpha, \quad 0 \leq \chi_\alpha \leq 1, \text{ and } \sum_\alpha \chi_\alpha = 1. \tag{4.5}$$

Define

$$g(e_1, e_2) = \sum_\alpha \chi_\alpha g_\alpha(e_1, e_2). \tag{4.6}$$

This is clearly symmetric since each g_α is symmetric. It is positive definite since for $e_1 = e_2 = v \in \pi^{-1}(x) \setminus \{0\}$, the right hand side is a finite sum of nonnegative terms, with at least one strictly positive term. □

Corollary 4.2. *For any real vector bundle E, $E^* \cong E$.*

Proof. Choose a Riemannian metric g on E. Then the mapping $\flat : E \to E^*$ defined by

$$\flat(e_1)(e_2) = g(e_1, e_2) \tag{4.7}$$

is an isomorphism on fibers, and covers the identity map. □

Definition 4.3. Given vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ over the same base space M, and assume that $E_1 \subset E_2$. We say that E_1 is a subbundle of E_2, if each fiber $\pi_1^{-1}(x) \subset \pi_2^{-1}(x)$ is a vector subspace.

In bundle terms, existence of a Riemannian metric g implies that there is always a non-zero section of $E^* \otimes E^*$, which says that $E^* \otimes E^*$ always admits a trivial 1-dimensional subbundle $A = c \cdot g$ for $c \in \mathbb{R}$. That is, $\mathrm{span}(g(x))$ defines a 1-dimensional subspace of every fiber, and noting that any 1-dimensional bundle with a non-vanishing section must be a trivial bundle. Of course, the metric gives an isomorphism

$$E^* \otimes E^* \cong E^* \otimes E \cong \mathrm{Hom}(E, E). \tag{4.8}$$

The latter bundle always admits the identity section $I : E \to E$, so $c \cdot I$ for $c \in \mathbb{R}$ defines a 1-dimensional trivial subbundle of $\mathrm{Hom}(E, E)$. The latter choice is canonical, but the sub-bundle A is not.

Remark 4.4. In the special case of a real line bundle $\pi : L \to M$, the bundle $\mathrm{Hom}(L, L)$ must be a trivial line bundle. So $L^* \otimes L^*$ is always a trivial bundle, and a Riemannian metric can simply be viewed as a positive function on M.

Definition 4.5. If $E_1 \subset E_2$ is a subbundle, then the quotient bundle E_2/E_1 is the vector bundle with fiber $\pi_2^{-1}(x)/\pi_1^{-1}(x)$ over x.

Exercise 4.6. Prove that the quotient bundle is a vector bundle. That is, find local trivializations for E_2/E_1.

Note the following corollary.

Corollary 4.7. *If $E_1 \subset E$ is a sub-bundle, then there exists a subbundle $E_2 \subset E$ such that*

$$E \cong E_1 \oplus E_2. \tag{4.9}$$

Furthermore, the quotient bundle $(E/E_1) \cong E_2$.

Proof. Choose a Riemannian metric g on E, and let $E_2 = (E_1)^{\perp}$. Use Gram-Schmidt to construct local trivializations for $(E_1)^{\perp}$ to show this is indeed a subbundle. The rest is just linear algebra. □

Example 4.8. If $f : M^k \to \mathbb{R}^n$ is an embedded (or immersed) submanifold, then define the normal bundle

$$\nu_M = \{(p, v) \in M \times T\mathbb{R}^n \mid v \in T_{f(p)}\mathbb{R}^n, v \perp f_*(T_pM)\}, \tag{4.10}$$

where where use the Euclidean metric on $T\mathbb{R}^n$. This is a bundle of rank $n-k$ over M, since f_* is injective at any point. We then have the decomposition

$$f^*T\mathbb{R}^n = TM \oplus \nu_M. \tag{4.11}$$

For example, take $\iota : S^n \to \mathbb{R}^{n+1}$ to be the standard inclusion. The radial vector field is a nontrivial normal vector field, so we have

$$T\mathbb{R}^{n+1} = TS^n \oplus (S^n \times \mathbb{R}), \tag{4.12}$$

where the latter factor is just the trivial line bundle over S^n. Note this shows that a trivial vector bundle can have a non-trivial sub-bundle (for non-parallelizable spheres).

There is nothing special about $\mathbb{R}^n$ in the above: if $f : M^k \to N^n$ is an immersed submanifold, and g is a Riemannian metric on TN, then we similarly have

$$f^*TN = TM \oplus \nu_M, \tag{4.13}$$

where we use Riemannian metric on N to define the orthogonal complement.

4.2 Reduction of structure group

Definition 4.9. If a bundle $\pi : E \to M$ is equivalent to a bundle which has transition functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to K$, where K is a subgroup of $GL(k, \mathbb{R})$, then we say that the structure group of E can be *reduced* to K.

Another way to state the results from the previous section is as follows.

Proposition 4.10. *We have the following.*

- *A bundle is trivial if and only if its structure group can be reduced to $\{Id\}$.*
- *The structure group of any real vector bundle $\pi : E \to M$ of rank k can be reduced to $O(k)$.*

Proof. If E is trivial, there is a global trivialization $\Phi : M \times \mathbb{R}^k \to E$. Given an open covering $\{U_\alpha\}, \alpha \in \mathcal{I}$ of M, then $\Phi_\alpha = \Phi\Big|_{U_\alpha \times \mathbb{R}_k}$ is a system of local trivializations which has overlap mappings $\phi_{\alpha\beta} = Id$. Conversely, a system of local trivializations which have identity overlap mappings patch together to give a global trivialization.

For the second case, from above E admits a Riemannian metric. By Gram-Schmidt, for any point $x \in M$, there exists a neighborhood U_x and a local basis of sections $\{e_1, \ldots, e_k\}$ which are orthonormal at every point in U_x. Define local trivializations by

$$\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^{k} v^i e_i. \tag{4.14}$$

Then overlaps maps then necessarily satisfy

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k), \tag{4.15}$$

where $O(k)$ is the orthogonal group of $k \times k$ real matrices satisfying $AA^T = I_k$. $\square$

4.3 Real line bundles

Note for a real 1-dimensional line bundle $\pi : L \to M$, we have that the structure group can be reduced to $O(1) = \{\pm 1\}$, Consider the set

$$\tilde{M} = \{v \in L \ |g(v, v) = 1\}. \tag{4.16}$$

Since there are exactly two unit norm vectors in any fiber, we have that $\pi : \tilde{M} \to M$ is a 2-fold covering space. So any real line bundle give an associated 2-fold covering space. Conversely, any 2-fold covering space gives a real line bundle, which is uniquely determined up to equivalence. To see this, note that a 2-fold covering space can be viewed as a fiber bundle with group $\mathbb{Z}_2$, and viewing $\mathbb{Z}_2 = \{\pm 1\} \subset GL(1, \mathbb{R})$, we naturally obtain an associated real line bundle.

Therefore real line bundles over M are in one-to-one correspondence with 2-fold covering spaces of M, up to equivalence. Using some covering space theory, the 2-fold coverings correspond to index 2 subgroups of $\pi_1(M)$, which is

$$\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2). \tag{4.17}$$

Since the abelianization of $\pi_1(M)$ is $H_1(M, \mathbb{Z}_2)$, and from the universal coefficient theorem, we have the isomorphisms

$$\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(M), \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2), \tag{4.18}$$

the first cohomology group with $\mathbb{Z}_2$ coefficients. Consequently, we have proved the following.

Proposition 4.11. *The real line bundles on M up to bundle equivalence, are in one-one correspondence with $H^1(M, \mathbb{Z}_2)$.*

5 Lecture 5

5.1 Čech cohomology with $\mathbb{Z}_2$ coefficients

The above used a lot of topology, so we give another explanation for this isomorphism. Given an open covering $\{U_\alpha\}, \alpha \in \mathcal{I}$, define $C^0(\mathfrak{U}, \mathbb{Z}_2)$ to be the free vector space over $\mathcal{I}$ with $\mathbb{Z}_2$ coefficients. We can think of this as a choice $f_\alpha \in \mathbb{Z}_2$, for each $\alpha \in \mathcal{I}$. Define $C^1(\mathfrak{U}, \mathbb{Z}_2)$ to be the free vector space over $U_\alpha \cap U_\beta$ for nontrivial intersections with $\mathbb{Z}_2$ coefficients, which we can think of as a choice $f_{\alpha\beta} :\in \mathbb{Z}_2$, $\alpha, \beta \in \mathcal{I}$, and $U_\alpha \cap U_\beta \neq \emptyset$. Define $C^2(\mathfrak{U}, \mathbb{Z}_2)$ to be the free vector space over $U_\alpha \cap U_\beta \cap U_\gamma$ with $\mathbb{Z}_2$ coefficients, which we can think of as a choice $f_{\alpha\beta\gamma} \in \mathbb{Z}_2$, $\alpha, \beta, \gamma \in \mathcal{I}$, and $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Finally, define $C^3(\mathfrak{U}, \mathbb{Z}_2)$ to be the free vector space over $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ with $\mathbb{Z}_2$ coefficients, which we can think of as a choice $f_{\alpha\beta\gamma\delta} \in \mathbb{Z}_2$, $\alpha, \beta, \gamma, \delta \in \mathcal{I}$, and $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \neq \emptyset$.

Next, we define linear operators $\delta^0 : C^0(\mathfrak{U}, \mathbb{Z}_2) \to C^1(\mathfrak{U}, \mathbb{Z}_2)$ by $(\delta^0 f)_{\alpha\beta} = f_\beta - f_\alpha$, and $\delta^1 : C^1(\mathfrak{U}, \mathbb{Z}_2) \to C^2(\mathfrak{U}, \mathbb{Z}_2)$ by $(\delta^1 f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}$, and $\delta^2 : C^2(\mathfrak{U}, \mathbb{Z}_2) \to C^3(\mathfrak{U}, \mathbb{Z}_2)$ by $(\delta^2 f)_{\alpha\beta\gamma\delta} = f_{\beta\gamma\delta} - f_{\alpha\gamma\delta} + f_{\alpha\beta\delta} - f_{\alpha\beta\gamma}$. We check that

$$(\delta^1 \circ \delta^0 f)_{\alpha\beta\gamma} = (\delta^1(f_\beta - f_\alpha))_{\alpha\beta\gamma} = f_\gamma - f_\beta - f_\gamma + f_\alpha + f_\beta - f_\alpha = 0, \tag{5.1}$$

and

$$\begin{aligned}(\delta^2 \circ \delta^1 f)_{\alpha\beta\gamma\delta} &= (\delta^1 f)_{\beta\gamma\delta} - (\delta^1 f)_{\alpha\gamma\delta} + (\delta^1 f)_{\alpha\beta\delta} - (\delta^1 f)_{\alpha\beta\gamma} \\ &= f_{\gamma\delta} - f_{\beta\delta} + f_{\beta\gamma} - (f_{\gamma\delta} - f_{\alpha\delta} + f_{\alpha\gamma}) \\ &\quad + f_{\beta\delta} - f_{\alpha\delta} + f_{\alpha\beta} - (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}) = 0.\end{aligned} \tag{5.2}$$

This allows us to define

$$H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathrm{Ker}(\delta^0), \quad H^1(\mathfrak{U}, \mathbb{Z}_2) = \mathrm{Ker}(\delta^1)/\,\mathrm{Image}(\delta^0), \quad H^2(\mathfrak{U}, \mathbb{Z}_2) = \mathrm{Ker}(\delta^2)/\,\mathrm{Image}(\delta^1). \tag{5.3}$$

It is easy to see that if M is connected, then

$$H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.4}$$

Back to real line bundles: after reduction the structure group to $\mathbb{Z}_2$, the transition functions of the bundle are given by

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2. \tag{5.5}$$

Since $\mathbb{Z}_2$ with multiplication is isomorphic to $\mathbb{Z}_2$ with addition, the condition on transition functions

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta}\varphi_{\beta\gamma} \tag{5.6}$$

says that $\varphi_{\alpha\beta}$ form a Čech 1-cocycle, so

$$\varphi_{\alpha\beta} \in \check{H}^1_{\mathfrak{U}}(M, \mathbb{Z}_2). \tag{5.7}$$

The condition that $\varphi_{\alpha\beta}$ be a coboundary is that there exists a 0-cochain $f_\alpha : U_\alpha \to \mathbb{Z}_2$ so that

$$\varphi_{\alpha\beta} = f_\beta f_\alpha^{-1} \tag{5.8}$$

on $U_\alpha \cap U_\beta$ which is exactly the condition for the bundle to the equivalent to a trivial bundle.

However, note there is a slight difference in the definitions because the transition functions in the bundle definition are smooth, so the $\varphi_{\alpha\beta}$ in (5.5) are constant on each components of $U_\alpha \cap U_\beta$, with possibly different values on different components. But a Čech 1-cocycle is the same constant on all components of $U_\alpha \cap U_\beta$.

Remark 5.1. The Čech cohomology as defined above obviously depends on the open cover. It turns out that if the cover is sufficiently nice, it is independent of the cover. Such a cover is called a "good" cover, which is a covering so that all open sets in the cover and all nontrivial intersections are contractible. We will not prove this right now (maybe later).

Example 5.2 (Case of S^1)**.** If $\mathfrak{U}$ is a covering with 2 intervals so that the intersection is 2 intervals. So

$$C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.9}$$

We conclude that

$$H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(\mathfrak{U}, \mathbb{Z}_2) = 0. \tag{5.10}$$

However, if cover by 3 intervals, so that the intersections are connected intervals, we then have

$$C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 3}, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 3}. \tag{5.11}$$

We conclude that

$$H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.12}$$

Therefore we know all real line bundles on S^1: the trivial bundle and the Mobius bundle.

Example 5.3 (Case of S^2)**.** We can find a good cover of S^2 by slightly enlarging the faces of tetrahedron, call these U_0, U_1, U_2, U_3, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. We see that

$$C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 4}, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 6}, \quad C^2(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 4}. \tag{5.13}$$

We compute that

$$\begin{aligned} \operatorname{Ker} \delta^1 = \{ & f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23} \mid \\ & f_{12} - f_{02} + f_{01} = 0, f_{13} - f_{03} + f_{01} = 0, f_{23} - f_{03} + f_{02} = 0, f_{23} - f_{13} + f_{12} = 0\}. \end{aligned}$$

This shows that a kernel element is determined by f_{01}, f_{02}, f_{03}. We conclude that

$$H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(\mathfrak{U}, \mathbb{Z}_2) = \{0\}, \quad H^2(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.14}$$

Consequently, every real line bundle on S^2 is trivial.

Remark 5.4. Instead of a tetrahedron, one could also use a cube to construct a good cover of S^2. We leave it to the interested student to compute the Čech cohomology of this cover, and verify you get the same answer for the cohomology groups.

Example 5.5 (Case of T^2)**.** We can find a good cover of T^2 by viewing T^2 as a square with opposite sides identified, dividing into 9 squares, and slightly enlarging each square. We see that

$$C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 9}, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 36}, \quad C^2(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 36}, \quad C^3(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 9}. \tag{5.15}$$

Since T^2 is a 2-manifold, it is possible to show that δ_2 is surjective. This implies that $\operatorname{Ker}(\delta_2) = \mathbb{Z}_2^{27}$. Next, by consider a "dual" good cover (where faces become vertices and vice-versa), we can show that $H^2(\mathfrak{U}, \mathbb{Z}_2) = H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2$. Consequently, $\operatorname{Image}(\delta_1) = \mathbb{Z}_2^{26}$, so $\operatorname{Ker}(\delta_1) = \mathbb{Z}_2^{10}$. We already know that $\operatorname{Image}(\delta_0) = \mathbb{Z}_2^8$, so we conclude the following:

$$H^k(T^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = 0, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k = 1 \end{cases}. \tag{5.16}$$

We can write down generators using the following. Let $\pi_i : S^1 \times S^1 \to S^1$ be the projection mappings, and let γ denote the Mobius bundle over S^1. In addition to the trivial bundle, we have $\pi_1^*\gamma$, $\pi_2^*\gamma^1$, and $\pi_1^*\gamma^1 \otimes \pi_2^*\gamma^1$. To get the Čech generator(s), we take a 1-cocycle which is $+1$ along the intersections on a vertical (horizontal) strip of 3 squares, and zero elsewhere.

Remark 5.6. In the previous example, we found there are exactly 4 real line bundles on T^2, up to bundle equivalence. Recall that this equivalence only considers bundle isomorphisms covering the identity mapping. If we had allowed arbitrary diffeomorphisms of the base then pullback under the mapping $f(\theta_1, \theta_2) = (\theta_2, \theta_1)$ would identify $\pi_1^*\gamma$ and $\pi_2^*\gamma^1$. So with this notion of equivalence, there would only be 3 line bundles over T^2.

Example 5.7 (Klein bottle)**.** The Klein bottle can be constructed similar to a torus by identifying opposite sides of a square, but with a twist on one pair of opposite sides. Similar to the above computation for a torus, we can compute that

$$H^k(K^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = 0, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k = 1 \end{cases}. \tag{5.17}$$

Example 5.8. (Tautological bundle on $\mathbb{RP}^n$) Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}),\ c \neq 0. \tag{5.18}$$

Define

$$\gamma_n^1 = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in [x]\} \tag{5.19}$$

We claim that γ_n^1 is a nontrivial 1-dimensional bundle over $\mathbb{RP}^n$. Assume by contradiction that it were the trivial bundle. Then there would exists a nowhere vanishing section $\sigma : \mathbb{RP}^n \to \gamma_n^1$. This is a mapping

$$\sigma : \mathbb{RP}^n \to \mathbb{RP}^n \times \mathbb{R}^{n+1} \tag{5.20}$$

of the form for $x \in S^n$,

$$\sigma([x]) = ([x], c(x) \cdot x) \tag{5.21}$$

For this to be well-defined, we require that $c(x) : S^n \to \mathbb{R}$ is a function satisfying $c(-x) = -c(x)$. Since c must take negative and positive values, by the intermediate value theorem, $c(x_0) = 0$ for some x_0, which is a contradiction.

Example 5.9 (Case of $\mathbb{RP}^2$)**.** We can construct $\mathbb{RP}^2$ by identifying opposite sides of a square, but twisting on both pairs of sides. To find a good cover: divide the square into 9 squares, but shave off the corner squares to become triangles, so that we have an octagon on the boundary. We slightly enlarge the each square and triangle to obtain a good cover. We see that

$$C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 9}, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 32}, \quad C^2(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 32}, \quad C^3(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 8} \tag{5.22}$$

Since $\mathbb{RP}^2$ is a 2-manifold, it is possible to show that δ_2 is surjective. This implies that $\mathrm{Ker}(\delta_2) = \mathbb{Z}_2^{24}$. Next, by consider the "dual" open cover, we can show that $H^2(\mathfrak{U}, \mathbb{Z}_2) = H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2$. Consequently, $\mathrm{Image}(\delta_1) = \mathbb{Z}_2^{23}$, so $\mathrm{Ker}(\delta_1) = \mathbb{Z}_2^9$. We already know that $\mathrm{Image}(\delta_0) = \mathbb{Z}_2^8$, so we conclude the following:

$$H^k(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2, \quad k = 0, 1, 2. \tag{5.23}$$

So there are exactly 2 real line bundles over $\mathbb{RP}^2$: the trivial bundle and the tautological bundle γ_2^1.

6 Lecture 6

6.1 Exterior powers

Let V be a real vector space. The exterior algebra $\Lambda(V)$ is defined as

$$\Lambda(V) = \Big\{ \bigoplus_{k \geq 0} V^{\otimes^k} \Big\} / \mathcal{I} = \bigoplus_{k \geq 0} \Big\{ V^{\otimes^k} / \mathcal{I}_k \Big\} = \bigoplus_{k \geq 0} \Lambda^k V, \tag{6.1}$$

where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $v \otimes v \in V \otimes V$, and $\mathcal{I}_k = V^{\otimes^k} \cap \mathcal{I}$. The wedge product of $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$ is just the multiplication induced by the tensor product in this algebra, that is, lift v and w to $\tilde{v} \in V^{\otimes^p}$, and $\tilde{w} \in V^{\otimes^q}$, and define $v \wedge w = \pi(\tilde{v} \otimes \tilde{w})$, where $\pi : V^{\otimes^{p+q}} \to \Lambda^{p+q}V$ is the projection. This is easily seen to be well-defined. We say that an element in $\Lambda^k(V)$ of the form $v_1 \wedge \cdots \wedge v_k$ is *decomposable.* A general element of $\Lambda^k(V)$ is not decomposable, but can always be written as a sum of decomposable elements.

The space $\Lambda^k(V)$ satisfies the universal mapping property as follows. Let W be any vector space, and let

$$F : \overbrace{V \times \cdots \times V}^{k} \to W \tag{6.2}$$

be an alternating multilinear mapping. That is, F is multilinear and $F(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

$$\begin{tikzcd}
\overbrace{V \times \cdots \times V}^{k} \arrow[r, "\pi"] \arrow[dr, "F"'] & \Lambda^k(V) \arrow[d, "\tilde{F}"] \\
& W
\end{tikzcd}$$

commutative, where π is the projection, which we denote as

$$\pi(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k. \tag{6.3}$$

Exercise 6.1. Prove the following properties of the wedge product.

- Bilinearity: $(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w$, and $(cv) \wedge w = c(v \wedge w)$ for $c \in \mathbb{R}$.
- If $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$, then $v \wedge w = (-1)^{pq} w \wedge v$.
- Associativity $(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3)$.

Exercise 6.2. If $\dim_{\mathbb{R}}(V) = n$, prove that $\Lambda^k(V) = \{0\}$ if $k > n$,

$$\dim(\Lambda^k(V)) = \binom{n}{k} \text{ if } 0 \leq k \leq n, \tag{6.4}$$

and

$$\dim(\Lambda(V)) = 2^n, \tag{6.5}$$

Definition 6.3. For a real vector bundle $\pi : E \to M$, we define $\Pi : \Lambda^p(E) \to M$ by $\Pi^{-1}(x) = \Lambda^p(\pi^{-1}(x))$. If $\Phi : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ is a local trivialization for E, then consider the mapping

$$F : U \times \overbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}^{p} \to \Pi^{-1}(U) \tag{6.6}$$

defined by

$$F(x,(v_1,\dots,v_p)) = \Phi(x,v_1)\wedge\cdots\wedge\Phi(x,v_k) \tag{6.7}$$

This is clearly an alternating multilinear mapping on fibers, so by the universal property, there is an unique induced mapping

$$\tilde{F} : U \times \Lambda^p(\mathbb{R}^k) \to \Pi^{-1}(U) \tag{6.8}$$

which is a local trivialization for $\Lambda^p(E)$.

We can equivalently define the pth exterior power in terms of transition functions. To do this, note that for any linear map $f : \mathbb{R}^k \to \mathbb{R}^k$, there is a naturally induced mapping

$$\Lambda^p f : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \tag{6.9}$$

define as follows. Define

$$F : \overbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}^{p} \to \Lambda^p(\mathbb{R}^k) \tag{6.10}$$

by

$$F(v_1,\dots,v_p) = f(v_1)\wedge\cdots\wedge f(v_p) \tag{6.11}$$

This is clearly an alternating multilinear mapping, so by the universal property, there exists a unique mapping

$$\Lambda^p f = \tilde{F} : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k). \tag{6.12}$$

therefore for any vector bundle E, the pth exterior power $\Lambda^p(E)$ is defined to be the bundle with transition functions

$$\varphi_{\alpha\beta}^{\Lambda^p(E)} = \Lambda^p(\varphi_{\alpha\beta}^E). \tag{6.13}$$

Putting all of these together, we can define the following.

Definition 6.4. For a real vector bundle $\pi : E \to M$, define the exterior algebra bundle $\Lambda(E) = \oplus_{p=0}^k \Lambda^p(E)$.

Note in the above discussion, if we sum together all of the $\Lambda^p f$ mappings, we get an induced mapping between the exterior algebras

$$\Lambda(f) : \Lambda(\mathbb{R}^k) \to \Lambda(\mathbb{R}^k) \tag{6.14}$$

which satisfies

$$\Lambda(f)(\alpha\wedge\beta) = \Lambda(f)(\alpha)\wedge\Lambda(f)(\beta) \tag{6.15}$$

Therefore, the wedge product gives an algebra structure on each fiber of $\Lambda(E)$.

6.2 Differential forms

Definition 6.5. Given a smooth manifold M, a differential k-form on M is smooth section of the kth exterior power of the cotangent bundle, that is, $\omega \in \Gamma(\Lambda^k(T^*M)) \equiv \Omega^k(M)$.

Given a function $f \in C^\infty(M, \mathbb{R})$ we define $df \in \Omega^1(M)$ in two ways. First, viewing vector fields as derivations on smooth functions, we can define

$$df(X) \equiv X(f). \tag{6.16}$$

Alternatively, since $f : M \to \mathbb{R}$, we have $f_* : TM \to T\mathbb{R}$. But there is a natural identification $T_p\mathbb{R} \cong \mathbb{R}$ for any $p \in \mathbb{R}$, so we can view

$$f_* : TU \to \mathbb{R}, \tag{6.17}$$

which is naturally an element in $df \in \Omega^1(U)$.

Exercise 6.6. Verify that these two definitions agree.

Given a coordinate system $x : U \to \mathbb{R}^n$, write the component functions as $x^i : U \to \mathbb{R}$, for $i = 1, \dots, n$. We then have that

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad 1 \le i_1 < i_2 < \cdots < i_k \le n \tag{6.18}$$

are a basis of local sections of $\Lambda^k(T^*U)$. That is, any $\omega \in \Omega^k(U)$ can be written as a linear combination

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{6.19}$$

for some functions $f_{i_1 \dots i_k} \in \Omega^0(U) = C^\infty(U, \mathbb{R})$.

Example 6.7. Let $M = \mathbb{R}^4$, and $f, g \in \Omega^0(\mathbb{R}^4)$, and define

$$\omega_1 = f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \tag{6.20}$$
$$\omega_2 = g(dx^1 \wedge dx^3 - dx^2 \wedge dx^4). \tag{6.21}$$

Then

$$\begin{aligned}
\omega_1 \wedge \omega_1 &= f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\
&= f^2\big(dx^1 \wedge dx^2 \wedge dx^1 \wedge dx^2 + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\
&\quad + dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \wedge dx^3 \wedge dx^4\big) \\
&= f^2\big(0 + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 + 0\big) \\
&= 2f^2 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.
\end{aligned} \tag{6.22}$$

Also,

$$\begin{aligned}
\omega_1 \wedge \omega_2 &= f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge g(dx^1 \wedge dx^3 - dx^2 \wedge dx^4) \\
&= fg\big(dx^1 \wedge dx^2 \wedge dx^1 \wedge dx^3 - dx^1 \wedge dx^2 \wedge dx^2 \wedge dx^4 \\
&\quad + dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^3 - dx^3 \wedge dx^4 \wedge dx^2 \wedge dx^4\big) \\
&= 0.
\end{aligned} \tag{6.23}$$

6.3 Differential forms as multilinear mappings

We could just stick with the above definition of the exterior algebra and prove all results using only this definition. However, it is very useful to think of elements of $\Lambda^k(V^*)$ as alternating multilinear maps on V as follows. One first has to choose a pairing

$$\Lambda^k(V^*) \cong (\Lambda^k(V))^*. \tag{6.24}$$

The pairing we will choose is as follows. If $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $v = v_1 \wedge \cdots \wedge v_k$, then

$$\alpha(v) = \det(\alpha^i(v_j)), \tag{6.25}$$

(note this is not canonical). For example,

$$\alpha^1 \wedge \alpha^2(v_1 \wedge v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{6.26}$$

We would then like to view an element of $(\Lambda^k(V))^*$ as an alternating multilinear mapping from

$$\overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}. \tag{6.27}$$

For this, we specify that if $\alpha \in (\Lambda^k(V))^*$, then

$$\alpha(v_1, \ldots, v_k) \equiv \alpha(v_1 \wedge \cdots \wedge v_k). \tag{6.28}$$

For example

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{6.29}$$

With this convention, if $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ then

$$\alpha \wedge \beta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}). \tag{6.30}$$

This then agrees with the definition of the wedge product given in [Spi79, Chapter 7].

It is convenient to have our 2 definitions of the wedge product because some proofs can be easier using one of the definitions, but harder using the other (for example, associativity of the wedge product).

6.4 Orientability of real bundles

Note that if V is an n-dimensional vector space, then $\Lambda^n V$ is 1-dimensional. So if $L : V \to V$ is a linear transformation then $\Lambda^n L : \Lambda^n V \to \Lambda^n V$ is an endomorphism of a 1-dimensional vector space. Therefore $\Lambda^n L(\omega) = c \cdot \omega$ for some scalar c. So we can make the following definition:

Definition 6.8. For a linear transformation $L : V \to V$, define $\det(L)$ to be the real number so that

$$\Lambda^n L(\omega) = \det(L) \cdot \omega. \tag{6.31}$$

Exercise 6.9. Show that this definition of determinant agrees with the usual linear algebra definition of determinant.

Proposition 6.10. *Let $\pi : E \to M$ be a real vector bundle of rank k. The following are equivalent.*

- *The line bundle $\Lambda^k(E)$ is trivial.*
- *$\Lambda^k(E)$ admits a nowhere zero section.*
- *The double cover $\tilde{M}$ corresponding to $\Lambda^k(E)$ is a trivial 2-fold covering space.*
- *The structure group of E can be reduced to*

$$GL_+(k, \mathbb{R}) \equiv \{A \in GL(k, \mathbb{R}) \mid \det(A) > 0\} \tag{6.32}$$

- *The structure group of E can be reduced to $SO(k)$*

Proof. The proof follows from the above discussion, with the following remarks. If $e_1, \dots, e_k$ is a local basis of sections, we say that $\{e_1, \dots, e_k\}$ is oriented if

$$e_1 \wedge \dots \wedge e_k = f\omega, \tag{6.33}$$

with $f > 0$ and $\omega \in \Lambda^k(E)$ is the nowhere zero section. Restricting to local trivializations arising from oriented local bases of sections will give a reduction of structure group to $GL_+(k, \mathbb{R})$. □

Definition 6.11. We say that a real vector bundle $\pi : E \to M$ is *orientable* if any of the equivalent conditions in Proposition 6.10 are satisfied.

Remark 6.12. If we use the 2-fold covering notion, then we see that if $\pi_1(M) = \{e\}$ then every vector bundle over M is orientable. This is because any covering of a simply connected space is trivial. (Actually, we just need to assume that $H^1(M, \mathbb{Z}_2) = 0$.) Thus, every vector bundle over S^n is orientable for $n \geq 2$.

Example 6.13. Returning to $\mathbb{RP}^n$, since $\mathbb{RP}^n$ is double covered by S^n, we have $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$. Therefore there are exactly 2 real line bundles over $\mathbb{RP}^n$, the trivial bundle and the tautological line bundle. Note that if we put a Riemannian metric on the tautological bundle $\pi : \gamma_n^1 \to \mathbb{RP}^n$, then the total space of the unit sphere bundle is just S^n. But for the trivial bundle over $\mathbb{RP}^n$, the unit sphere bundle is just 2 copies of $\mathbb{RP}^n$.

7 Lecture 7

7.1 Induced mappings

Recall that if $L : V \to W$ is a linear mapping between vector spaces, then there is a mapping, $L^* : W^* \to V^*$ called the *transpose*, defined by the following. If $\omega \in W^*$, and $v \in V$, then

$$(L^*\omega)(v) = \omega(Lv). \tag{7.1}$$

This is called the transpose for the following reason. Let $dim(V) = n$, and $\dim(W) = m$. Let $e_1, \ldots, e_n$ be a basis of V and $f_1, \ldots, f_n$ be a basis of W. Let $e^1, \ldots, e^n$, and $f^1, \ldots, f^n$ denote the dual bases, that is

$$e^i(e_j) = \delta^i_j,\ 1 \leq i, j \leq n \tag{7.2}$$

$$f^i(f_j) = \delta^i_j,\ 1 \leq i, j \leq m. \tag{7.3}$$

We define the quantities L^j_i, $1 \leq i \leq n$, $1 \leq j \leq m$, by

$$Le_i = L^j_i f_j. \tag{7.4}$$

Note that if we write $v \in V$ as $v = v^i e_i$, and $w \in W$ as $w = w^i f_i$, then

$$Lv = L(v^i e_i) = v^i L(e_i) = (v^i L^j_i) f_j. \tag{7.5}$$

So the components of a vector transform like

$$\{v^i\} \mapsto \{L^j_i v^i\}, \tag{7.6}$$

which is the matrix corresponding to the transformation L.

We define the quantities $(L^*)^i_j$, $1 \leq i \leq m,\ 1 \leq j \leq n$, by

$$L^* f^i = (L^*)^i_j e^j \tag{7.7}$$

Plugging in the dual bases, we compute

$$(L^* f^i)(e_k) = (L^*)^i_j e^j(e_k) = (L^*)^i_j \delta^j_k = (L^*)^i_k. \tag{7.8}$$

However, by the definition of the transpose mapping, we have

$$(L^* f^i)(e_k) = f^i(Le_k) = f^i L^j_k f_j = L^j_k f^i(f_j) = L^j_k \delta^i_j = L^i_k \tag{7.9}$$

So if we write $\omega \in V^*$ as $\omega_i e^i$ and $\eta \in W^*$ as $\eta_j f^j$, the components of a dual vector transform like

$$\{\eta_j\} \mapsto \{L^i_j \eta_i\} \tag{7.10}$$

So the matrix corresponding to L^* in the dual basis is indeed the transpose matrix.

The mapping $L^* : W^* \to V^*$ induces a mapping

$$(L^*)^{\times p} : \overbrace{W^* \times \cdots \times W^*}^{p} \to (V^*)^{\otimes p} \tag{7.11}$$

by

$$(L^*)^{\times p}(\alpha^1, \ldots, \alpha^p) \equiv (L^* \alpha^1) \otimes \cdots \otimes (L^* \alpha^p). \tag{7.12}$$

This mapping is a multilinear mapping, so by the universal property of tensor products, this induces a unique mapping

$$(L^*)^{\otimes p} : (W^*)^{\otimes p} \to (V^*)^{\otimes p}. \tag{7.13}$$

By composing with the projection $\pi : (V^*)^{\otimes p} \to \Lambda^p(V^*)$, we obtain an alternating multilinear mapping

$$(L^*)^{\times p} : (W^*)^{\otimes p} \to \Lambda^p(V^*). \tag{7.14}$$

Now by the universal property of exterior products, this induces a mapping

$$\Lambda^p(L^*) : \Lambda^p(W^*) \to \Lambda^p(V^*). \tag{7.15}$$

Note that by taking the direct sum on all p-s, we obtain a mapping between the full exterior algebras

$$\Lambda(L^*) : \Lambda(W^*) \to \Lambda(V^*) \tag{7.16}$$

which is an algebra homomorphism, that is,

$$\Lambda(L^*)(\alpha \wedge \beta) = (\Lambda(L^*)\alpha) \wedge (\Lambda(L^*)\beta). \tag{7.17}$$

7.2 Pull-back of differential forms

Recall that if $f : M \to N$ is a smooth mapping between smooth manifolds, then the mapping $(f_*)_B : TM \to f^*TN$ defined by $(f_*)_B(v_p) = (p, f_*v)$ makes the following diagram commute

$$\begin{array}{ccc} TM & \xrightarrow{(f_*)_B} & f^*TN \\ \Big\downarrow{\scriptstyle \pi_M} & & \Big\downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{\quad id \quad} & M. \end{array} \tag{7.18}$$

Noting that $(f^*(TN))^*$ is naturally isomorphic to $f^*(T^*N)$, let us dualize (7.18) to obtain the commutative diagram

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{\quad id \quad} & M. \end{array} \tag{7.19}$$

Next, by the diagram (21.9) and the above discussion, we obtain bundle mappings

$$\begin{array}{ccc} f^*(\Lambda^p(T^*N)) & \xrightarrow{\Lambda^p(f_B^*)} & \Lambda^p(T^*M) \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{\qquad id \qquad} & M, \end{array} \tag{7.20}$$

Definition 7.1 (Pull-back of a differential form)**.** If $f : M \to N$ is a smooth mapping, and $\omega \in \Lambda^p(T^*N)$, then define $\omega \circ f \in \Gamma(f^*(\Lambda^p(T^*N))$ by $\omega \circ f(p) = (p, \omega_{f(p)})$. Then define

$$f^*\omega \equiv \Lambda^p(f_B^*)(\omega \circ f) \in \Gamma(\Lambda^p(T^*M)). \tag{7.21}$$

Remark 7.2. If we view differential forms as multilinear mappings, for $f : M \to N$, and $\omega \in \Omega^k(N)$, then we have the following "formula". If $p \in M$ and $X_1, \cdots X_k \in T_pM$, then

$$(f^*\omega)(X_1, \cdots, X_k) = \omega_{f(p)}(f_*X_1, \cdots, f_*X_k). \tag{7.22}$$

We could have defined pullback of forms this way, but we would need an extra step to show the pullback of a smooth form is smooth.

For any manifold M, define

$$\Omega(M) = \Gamma(\Lambda(T^*M)) = \Gamma\Big(\bigoplus_{p\geq 0}\Lambda^p(T^*M)\Big) = \bigoplus_{p\geq 0}\Gamma(\Lambda^p(T^*M)) = \bigoplus_{p\geq 0}\Omega^p(M). \tag{7.23}$$

By taking the direct sum of the above mappings on each exterior power, we obtain a mapping

$$f^* : \Omega(N) \to \Omega(M), \tag{7.24}$$

which by (7.17) satisfies

$$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta). \tag{7.25}$$

Proposition 7.3 (The chain rule). *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)^* = f^* \circ h^* : \Omega(M') \to \Omega(M). \tag{7.26}$$

Proof. We have that $f_* : TM \to TN$ is a bundle mapping covering f, and $h_* : TN \to TM'$ is a bundle mapping covering h_*. The above chain rule for the differential says that the bundle mapping $(h \circ f)_* : TM \to TM'$ is given by $(h \circ f)_* = h_* \circ f_*$. Next, we have the mappings $(f_*)_B : TM \to f^*TN$, $(h_*)_B : TN \to h^*TM'$, and $\big((h \circ f)_*\big)_B : TM \to (h \circ f)^*TM'$. The mapping $(h_*)_B$ induces a mapping $(h_*)_B \circ f : f^*TN \to f^*h^*TM'$. Since $(h \circ f)^*TM' = f^*(h^*TM')$, the chain rule implies that $\big((h \circ f)_*\big)_B = \big((h_*)_B \circ f\big) \circ (f_*)_B$. Dualizing and taking the induced mapping on exterior powers then implies the result.

□

Example 7.4. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x^2 + y^2, x^2 - y^2, x^3). \tag{7.27}$$

Denote the coordinates on $\mathbb{R}^3$ as (u, v, w), and let

$$\alpha = wdu \wedge dv - vdu \wedge dw + udv \wedge dw. \tag{7.28}$$

Then

$$f^*\alpha = 4x^4ydx \wedge dy. \tag{7.29}$$

(Details were done in lecture.)

8 Lecture 8

8.1 The exterior derivative

Choose a coordinate system (U, x), and let $\frac{\partial}{\partial x^i}$ denote the coordinate vector field. Recall that viewing vector fields as derivations on germs of functions, this is characterized by

$$\frac{\partial}{\partial x^i}(x^j) = \delta_i^j. \tag{8.1}$$

We then define a local basis of 1-forms dx^i by

$$dx^i\Big(\frac{\partial}{\partial x^i}\Big) = \delta_j^i. \tag{8.2}$$

Note this is just the dual basis, but these are also $d(x^i)$ as defined above in (14.18).

An element $\alpha \in \Omega^p(U)$ can be written as

$$\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{8.3}$$

where the coefficients $\alpha_{i_1 \dots i_p} : U \to \mathbb{R}$ are well-defined functions. Note these coefficients are only defined for strictly increasing sequences $i_1 < \cdots < i_p$. Using our identification of $\Lambda^p(T^*M)$ with $Alt^p(TM)$, the alernating multilinear maps from $TM^{\times p} \to \mathbb{R}$, we have that the coefficient functions are given by

$$\alpha_{i_1 \dots i_p} = \alpha\Big(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_p}}\Big). \tag{8.4}$$

We next define the exterior derivative operator [War83, Theorem 2.20].

Proposition 8.1. *There exists an exterior derivative operator*

$$d : \Omega^p(M) \to \Omega^{p+1}(M) \tag{8.5}$$

which is the unique linear mapping satisfying

- *For $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.*
- *$d^2 = 0$.*
- *If $f \in C^\infty(M, \mathbb{R})$ then df is the differential of f defined above.*

Proof. Note that the differential of a function is given locally by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \tag{8.6}$$

To see this, we have $df = \sum c_i dx^i$, and plugging in the coordinate vector field identifies the coefficient c_i. Since we gave a global definition of df, this is obviously well-defined and independent of the coordinate system. Given a p-form α, write α locally as in (8.3), and then define

$$\begin{aligned} d\alpha &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} d\alpha_{i_1 \dots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^{n} \frac{\partial \alpha_{i_1 \dots i_p}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{8.7}$$

The first "anti-derivation" property is easily verified by computation. The second property holds on functions, because

$$d(df) = d\Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \Big) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0, \tag{8.8}$$

since the Hessian of a smooth function is symmetric.

For existence, we need to check that this definition is independent of the coordinate system. Let d' be the operator defined with respect to another coordinate system $x' : U \to \mathbb{R}^n$. Then

$$\begin{aligned} d'(\alpha) &= d'\Big(\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \Big) \\ &= \sum_{|I|=p} (d'\alpha_{i_1 \dots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &\quad + \sum_{|I|=p} \alpha_{i_1 \dots i_p} \sum_{k} (-1)^{k-1} dx^{i_1} \wedge \cdots \wedge d'(dx^{i_k}) \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{|I|=p} (d\alpha_{i_1 \dots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = d(\alpha), \end{aligned} \tag{8.9}$$

since d and d' agree on functions, and since $d'dx^i = d'd'x^i = 0$.

Then for any p-form α,

$$d(d\alpha) = d\Big(\sum_{|I|=p} (d\alpha_I) \wedge dx^I \Big) = \sum_{|I|=p} (d^2 \alpha_I) \wedge dx^I - d\alpha_I \wedge d(dx^I) = 0. \tag{8.10}$$

Uniqueness will be left as an (optional) exercise. $\square$

An important fact is that d commutes with pull-back.

Proposition 8.2. *If $f : M \to N$ is a smooth mapping, and $\omega \in \Omega^p(N)$, then*

$$f^*(d_N \omega) = d_M(f^* \omega). \tag{8.11}$$

Proof. If ω is a 0-form, which is a function, then $f^*\omega = \omega \circ f$. So by above, we have

$$d(f^*\omega) = d(\omega \circ f) = (\omega \circ f)_*. \tag{8.12}$$

By the chain rule, we then have

$$d(f^*\omega) = \omega_* \circ f_*. \tag{8.13}$$

On the other hand, we have that

$$f^*(d\omega)(X) = d\omega(f_*(X)) = \omega_* \circ f_*(X). \tag{8.14}$$

So the claim is true on functions. Then if ω is a p-form, write

$$\omega = \sum_{|I|=p} \omega_I dx^I. \tag{8.15}$$

Since the pull-back operation is an algebra homomorphism, we have

$$f^*\omega = \sum_{|I|=p} (f^*\omega_I) f^* dx^I = \sum_{|I|=p} (\omega_I \circ f) d(x^I \circ f). \tag{8.16}$$

Then

$$d(f^*\omega) = \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f). \tag{8.17}$$

On the other hand, we have

$$d\omega = \sum_{|I|=p} (d\omega_I) \wedge dx^I, \tag{8.18}$$

so

$$\begin{aligned} f^*(d\omega) = \sum_{|I|=p} f^*(d\omega_I) \wedge f^* dx^I &= \sum_{|I|=p} d(f^*\omega_I) \wedge d(f^* x^I) \\ &= \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f) = d(f^*\omega). \end{aligned} \tag{8.19}$$

□

8.2 Lie derivatives

Given a vector field $X \in \Gamma(TM)$, the Lie derivative of Y with respect to X is

$$\mathcal{L}_X Y = [X, Y], \tag{8.20}$$

where $[X, Y]f = X(Yf) - Y(Xf)$

Proposition 8.3. *For $X, Y \in \Gamma(TM)$, we have $[X, Y] \in \Gamma(TM)$.*

Proof. In a local coordinate system, write

$$X = X^i \frac{\partial}{\partial x^i}, \ Y = Y^i \frac{\partial}{\partial x^i}, \tag{8.21}$$

then

$$\begin{aligned} [X,Y]f &= X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j}\right) - Y^j \frac{\partial}{\partial x^j}\left(X^i \frac{\partial f}{\partial x^i}\right) \\ &= X^i \left(\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right) - Y^j \left(\frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} + X^i \frac{\partial^2 f}{\partial x^j \partial x^i}\right). \end{aligned} \tag{8.22}$$

Since f is smooth, we have equality of the mixed partials, so

$$\begin{aligned} [X,Y]f &= X^i \left(\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j}\right) - Y^j \left(\frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i}\right) \\ &= \left(X^i \frac{\partial Y^l}{\partial x^i} - Y^j \frac{\partial X^l}{\partial x^j}\right)\frac{\partial f}{\partial x^l}. \end{aligned} \tag{8.23}$$

This shows that $[X,Y]$ is a derivation on germs of functions, so is a well-defined vector field. $\square$

Next, for $X, Y \in \Gamma(TM)$, and $\omega \in \Gamma(T^*M)$, define

$$\mathcal{L}_X \omega(Y) = X(\omega(Y)) - \omega(\mathcal{L}_X Y). \tag{8.24}$$

Proposition 8.4. *If $X \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$, then $\mathcal{L}_X \omega \in \Gamma(T^*M)$.*

Proof. Let $f : M \to \mathbb{R}$. Then

$$\begin{aligned} \mathcal{L}_X \omega(fY) &= X(\omega(fY)) - \omega(\mathcal{L}_X(fY)) \\ &= X(f\omega(Y) - \omega([X, fY]) \\ &= (Xf)\omega(Y) + fX(\omega(Y)) - \omega(f[X.Y] - (Xf)Y) \\ &= fX(\omega(Y)) - \omega(f[X,Y]) = f\mathcal{L}_X \omega(Y). \end{aligned} \tag{8.25}$$

Since this expression is linear over C^∞ functions, it is a well-defined tensor. To see this, let $\alpha : \Gamma(TM) \to C^\infty(M)$ be a mapping which is linear over C^∞-functions. It sufices to show that $\alpha(X)(p) = 0$ if $X_p = 0$. This is because if we let X and $\tilde{X}$ be any smooth extensions of X_p, then since $X - \tilde{X}$ vanishes at p

$$\omega(X - \tilde{X})(p) = 0, \tag{8.26}$$

so $\omega(X)(p) = \omega(\tilde{X})(p)$ has a well-defined value, independent of the extension of X_p. To proceed, given a coordinate system around p, choose a cutoff function which is 1 in a coordinate neighborhood of p, and 0 outside. Then

$$X = (\phi X^i)\Big(\phi \frac{\partial}{\partial x^i}\Big) + (1 - \phi^2)X. \tag{8.27}$$

Both terms in the above are smooth vector fields on M, so using linearity,

$$\alpha(X)(p) = (\phi(p)X^i(p))\alpha\Big(\phi\frac{\partial}{\partial x^i}\Big)(p) + (1-\phi^2)(p)\alpha(X)(p) = 0. \tag{8.28}$$

$\square$

9 Lecture 9

9.1 Lie derivative of differential forms

For a functions, we define $\mathcal{L}_X f = Xf$. In the previous lecture, we defined $\mathcal{L}_X \omega$ for $\omega \in \Omega^1(M)$. We now extend the Lie derivative operator to differential forms in $\Omega^p(M)$ for $p > 1$ by

$$\mathcal{L}_X(\omega^1 \wedge \cdots \wedge \omega^p) = \sum_{i=1}^{p} \omega^1 \wedge \cdots \wedge (\mathcal{L}_X \omega^i) \wedge \cdots \wedge \omega^p, \tag{9.1}$$

for $\omega^i \in \Gamma(T^*M)$. Note this is for decomposable forms, but extends to arbitrary forms by linearity. Tensorality is proved similar to the 1-form case. The analog to (8.24) is

$$\begin{aligned}(\mathcal{L}_X \omega)(X_1, \ldots, X_p) = X\Big(&\omega(X_1, \ldots, \ldots, X_p)\Big) \\ &+ \sum_{i=1}^{p} (-1)^i \omega([X, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_p).\end{aligned} \tag{9.2}$$

Proposition 9.1. *For $\omega \in \Omega^p(M)$ and $X \in \Gamma(TM)$, we have $\mathcal{L}_X d\omega = d\mathcal{L}_X \omega$.*

Proof. First, for a function, this is

$$\mathcal{L}_X df = d(Xf). \tag{9.3}$$

To see this, take $Y \in \Gamma(TM)$, and compute

$$\begin{aligned}(\mathcal{L}_X df)(Y) &= X(df(Y)) - df(\mathcal{L}_X Y) \\ &= X(Yf) - (\mathcal{L}_X Y)f \\ &= X(Yf) - X(Yf) + Y(Xf) = Y(Xf)\end{aligned} \tag{9.4}$$

Plugging Y into the right hand side of (9.3) yields

$$d(Xf)(Y) = Y(Xf), \tag{9.5}$$

which proves this for functions.

Next, we consider the case that $p \geq 1$. Since both sides of this equation are tensors, it suffices to prove this in a local coordinate system, and we can assume that ω is of the form $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. We have $d\omega = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, so

$$\mathcal{L}_X d\omega = (\mathcal{L}_X df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} df \wedge dx^{i_1} \wedge \cdots \wedge \mathcal{L}_X dx^{i_k} \wedge \cdots \wedge dx^{i_p} \tag{9.6}$$

Next, we claim that for $\omega \in \Omega^1(M)$, we have

$$\mathcal{L}_X(f\omega) = (Xf)\omega + f\mathcal{L}_X \omega \tag{9.7}$$

To see this, take $Y \in \Gamma(TM)$ and plug into (9.7)

$$\begin{aligned} \mathcal{L}_X(f\omega)(Y) &= X(f\omega(Y)) - f\omega(\mathcal{L}_X Y) \\ &= (Xf)\omega(Y) + fX(\omega(Y)) - f\omega(\mathcal{L}_X Y) \\ &= \big((Xf)\omega + f\mathcal{L}_X\omega\big)(Y). \end{aligned} \tag{9.8}$$

So we then have

$$\begin{aligned} d\mathcal{L}_X\omega &= d\big(\mathcal{L}_X f dx^{i_1} \wedge \cdots \wedge dx^{i_p}\big) \\ &= d\Big(\mathcal{L}_X(f dx^{i_1}) \wedge \cdots \wedge dx^{i_p} + \sum_{k=2}^{p} f dx^{i_1} \wedge \cdots \wedge \mathcal{L}_X dx^{i_k} \wedge \cdots \wedge dx^{i_p}\Big) \\ &= d\Big((Xf)dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} f dx^{i_1} \wedge \cdots \wedge d(Xx^{i_k}) \wedge \cdots \wedge dx^{i_p}\Big) \\ &= d(Xf) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} df \wedge dx^{i_1} \wedge \cdots \wedge d(Xx^{i_k}) \wedge \cdots \wedge dx^{i_p}\Big) \\ &= (\mathcal{L}_X df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} df \wedge dx^{i_1} \wedge \cdots \wedge \mathcal{L}_X dx^{i_k} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{9.9}$$

□

Definition 9.2. Given $\omega \in \Lambda^p(T_x^*M)$, and $X \in T_xM$, define the interior product

$$X \lrcorner : \Lambda^p(T_x^*M) \to \Lambda^{p-1}(T_x^*M) \tag{9.10}$$

by

$$X \lrcorner \alpha(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}). \tag{9.11}$$

Exercise 9.3. Prove that this is equivalent to the following definition. Given $v \in T_xM$, we get a mapping $\iota_v : \Lambda^{p-1}(T_xM) \to \Lambda^p(T_xM)$ by

$$\iota_v(\alpha) = v \wedge \alpha. \tag{9.12}$$

The transpose mapping is

$$\iota_v^* : (\Lambda^p(T_xM))^* \to (\Lambda^{p-1}(T_xM))^* \tag{9.13}$$

Above, we chose an identification of $(\Lambda^k(T_xM))^* \cong \Lambda^k(T_x^*M)$, so using this we get a mapping

$$\iota_v^* : \Lambda^p(T_x^*M) \to \Lambda^{p-1}(T_x^*M). \tag{9.14}$$

Show that $v\lrcorner = (\iota_v)^*$. Also, show that $v\lrcorner$ is an anti-derivation, that is

$$v\lrcorner(\alpha \wedge \beta) = (v\lrcorner\alpha) \wedge \beta + (-1)^p \alpha \wedge (v\lrcorner\beta) \tag{9.15}$$

if $\alpha \in \Lambda^p(T_x^*M)$.

An important formula is Cartan's formula relating the Lie derivative and the exterior derivative.

Proposition 9.4 (Cartan's magic formula)**.** *If $\omega \in \Omega^p(M)$, then*

$$\mathcal{L}_X\omega = d(X \lrcorner \omega) + X \lrcorner d\omega, \tag{9.16}$$

Proof. Define the operator $H : \Omega^p(M) \to \Omega^p(M)$ by

$$H\omega = d(X \lrcorner \omega) + X \lrcorner d\omega. \tag{9.17}$$

We claim that H is a derivation, that is

$$H(\alpha \wedge \beta) = H(\alpha) \wedge \beta + \alpha \wedge H(\beta). \tag{9.18}$$

To prove this, assume that $\alpha \in \Omega^p(M)$, then

$$\begin{aligned} H(\alpha \wedge \beta) &= d(X \lrcorner(\alpha \wedge \beta) + X \lrcorner d(\alpha \wedge \beta) \\ &= d\big((X \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (X \lrcorner \beta)\big) + X \lrcorner \big(d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta\big) \\ &= d\big((X \lrcorner \alpha)\big) \wedge \beta + (-1)^{p-1}(X \lrcorner \alpha) \wedge d\beta + (-1)^p d\alpha \wedge (X \lrcorner \beta) + (-1)^{2p} \alpha \wedge d(X \lrcorner \beta) \\ &\quad + \big(X \lrcorner d\alpha\big) \wedge \beta + (-1)^{p+1} d\alpha \wedge \big(X \lrcorner \beta\big) + (-1)^p \big(X \lrcorner \alpha\big) \wedge d\beta + (-1)^{2p} \alpha \wedge \big(X \lrcorner d\beta\big) \\ &= H(\alpha) \wedge \beta + \alpha \wedge H(\beta). \end{aligned} \tag{9.19}$$

Then the operators $\mathcal{L}_X$ and H are derivations which commute with d, and agree on functions. So they must be the same operator on forms, since they agree in local coordinates. $\square$

We also have the following formula for the exterior derivative which agrees with the formula for d given in [Spi79, Chapter 7].

Proposition 9.5.

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_p)\Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{9.20}$$

Proof. This formula follows from (9.2) and (9.16), with an induction argument. To see this, if $\omega \in \Omega^1(M)$, then

$$\begin{aligned} d\omega(X_0, X_1) &= (X_0 \lrcorner d\omega)(X_1) = (\mathcal{L}_{X_0}\omega)(X_1) - d(\omega(X_0))(X_1) \\ &= X_0(\omega(X_1)) - \omega([X_0, X_1]) - X_1(\omega(X_0)) \\ &= X_0(\omega(X_1)) - X_1(\omega(X_0)) - \omega([X_0, X_1]). \end{aligned} \tag{9.21}$$

Then we use induction to get the formula for higher degree forms. $\square$

Remark 9.6. The Lie derivative operator can be defined by using the 1-parameter group of diffeomorphisms generated by X via

$$\mathcal{L}_X Y = \frac{d}{dt}(\Phi_{-t})_* Y\Big|_{t=0} \tag{9.22}$$

$$\mathcal{L}_X \omega = \frac{d}{dt}(\Phi_t)^* \omega\Big|_{t=0}, \tag{9.23}$$

but for this, we next first need to discuss the flow of a vector field.

9.2 The flow of a vector field

Given a vector field $X \in \Gamma(TM)$, an integral curve of X is a mapping $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma'(t) = X(\gamma(t))$. Let's look at the case of the real line first, and more general cases next time.

9.3 The real line

Let $M = \mathbb{R}$. Then a vector field $X = f(t)\frac{\partial}{\partial t}$. The differential equation for an integral curve is $\frac{d\gamma}{dt} = f(\gamma)$. We can rewrite this as

$$\frac{d\gamma}{f(\gamma)} = dt, \tag{9.24}$$

which is a separable equation, and we get

$$F(\gamma) = t + C, \tag{9.25}$$

where F is an anti-derivative for $1/f$, and C is a constant of integration. If we plug in $\gamma(0) = t_0$, then $C = F(t_0)$, so we then have

$$\gamma(t) = F^{-1}(t + F(t_0)), \tag{9.26}$$

where F^{-1} is an inverse function to F. Thinking of t_0 as a parameter, we get

$$\Phi(t, t_0) = F^{-1}(t + F(t_0)). \tag{9.27}$$

The standard existence and uniqueness theorem for ODEs says that the domain of Φ is an open set in $\mathbb{R}^2$ containing the t_0-axis.

Example 9.7. We write down a few explicit examples.

- If $f = 1$, then $\Phi(t, t_0) = t + t_0$. For any t_0, the flow is defined for all t.
- If $f = t$, then $\Phi(t, t_0) = t_0 e^t$. For any t_0, the flow is defined for att t
- If $f = t^2$, then $\Phi(t, t_0) = \frac{t_0}{1 - t_0 t}$. The flow is only defined from $(-\infty, t_0^{-1})$ if $t_0 > 0$ and from $(t_0^{-1}, -\infty)$ if $t_0 < 0$. We say this is incomplete.
- If $f = t^2 + 1$, then $\Phi(t, t_0) = \tan(t + \tan^{-1}(t_0))$, also incomplete.

Proposition 9.8. *We have* $\Phi(t + s, t_0) = \Phi(t, \Phi(s, t_0))$

Proof. From the formula (9.27), we have

$$\Phi(t + s, t_0) = F^{-1}(t + s + F(t_0)) = F^{-1}(t + (s + F(t_0)). \tag{9.28}$$

Let us write $s + F(t_0) = F(t_0')$. Then $t_0' = F^{-1}(s + F(t_0))$, so

$$\Phi(t + s, t_0) = F^{-1}(t + F(t_0')) = \Phi(t, t_0') = \Phi(t, \Phi(s, t_0)). \tag{9.29}$$

□

Corollary 9.9. *If the flows at time t and $-t$ are defined for all $x \in \mathbb{R}$, then the mapping $x \mapsto \Phi(t, x)$ is a diffeomorphism.*

Proof. From the previous proposition, we have

$$x = \Phi(t, \Phi(-t, x)) = \Phi(-t, \Phi(t, x)), \tag{9.30}$$

so the mappings $x \mapsto \Phi(t, x)$ and $x \mapsto \Phi(-t, x)$ are inverses of each other. Smoothness follows from the standard ODE existence and uniqueness theorem. □

10 Lecture 10

10.1 The flow of a vector field

Given a vector field $X \in \Gamma(TM)$, an integral curve of X is a mapping $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma'(t) = X(\gamma(t))$. Define $\Phi(t, x) = \gamma(t)$, where $\gamma(t)$ is the integral curve satisfying $\gamma(0) = x$. The fundamental theorem is the following.

Proposition 10.1. *Assume X is smooth. Then through each $x \in M$ there passes a unique integral curve of X. The domain of $\Phi(t, x)$ is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$, and $\Phi : U \to M$ is smooth. We have*

$$\Phi(t + s, x) = \Phi(t, \Phi(s, x)), \tag{10.1}$$

for t, s, x for which the above is defined.

Proof. We discuss the uniqueness, since this is a local property, we assume that M is an open subset of $\mathbb{R}^n$, and the differential equation is

$$y'(t) = X(y(t)), \quad y(0) = x. \tag{10.2}$$

Assume we have 2 solutions y_1 and y_2. Integrating, we obtain

$$y_i(t) = x + \int_0^t X(y_i(s))ds, \tag{10.3}$$

for $i = 1, 2$. We assume that $t > 0$, the argument for $t < 0$ is similar. First, we have

$$\begin{aligned} |y_1(t) - y_2(t)| &= \Big| \int_0^t (X(y_1(s)) - X(y_2(s)))ds \Big| \\ &\leq \int_0^t |X(y_1(s)) - X(y_2(s)))|ds \\ &\leq K \int_0^t |y_1(s) - y_2(s)|ds, \end{aligned} \tag{10.4}$$

using that X, being smooth, is necessarily Lipschitz.

Next, define

$$U(t) = \int_0^t |y_1(s) - y_2(s)| ds. \tag{10.5}$$

Then $U(t) \geq 0$ for $t \geq 0$, and (10.4) says that

$$U'(t) \leq KU(t). \tag{10.6}$$

or $U'(t) - KU(t) \leq 0$. Multiplying by e^{-Kt} yields

$$(e^{-Kt}U(t))' \leq 0. \tag{10.7}$$

Integrating from 0 to t, we obtain

$$e^{-Kt}U(t) \leq 0, \tag{10.8}$$

which says that $U(t) \leq 0$, so $U(t) = 0$ for $t \geq 0$ which implies that $y_1(t) = y_2(t)$ for $t \geq 0$.

Next, we prove (10.1). Note that for x and s fixed, we define $L(t) = \Phi(t + s, x)$. Then

$$\frac{d}{dt}L(t)|_{t=0} = \frac{d}{dt}\Phi(t + s, x)|_{t=0} = \frac{d}{dt}\Phi(t, x)|_{t=s} = X(\Phi(s, x)), \tag{10.9}$$

and $L(0) = \Phi(s, x)$. Define $R(t) = \Phi(t, \Phi(s, x))$. Then

$$\frac{d}{dt}R(t)|_{t=0} = X(\Phi(s, x)), \tag{10.10}$$

and $R(0) = \Phi(s, x)$. So $L(t)$ and $R(t)$ are integral curves of X passing through the same point at $t = 0$, so they are equal by the uniqueness theorem.

Existence of solutions is proved by writing as an integral equation

$$y(t) = x + \int_0^t X(y(s)) ds. \tag{10.11}$$

Let

$$y_0(t) = x, \tag{10.12}$$

$$y_1(t) = x + \int_0^t X(x) ds \tag{10.13}$$

$$y_2(t) = y_0 + \int_0^t X(y_1(s)) ds \tag{10.14}$$

$$\vdots \tag{10.15}$$

$$y_n(t) = y_0 + \int_0^t X(y_{n-1}(s)) ds. \tag{10.16}$$

One then proves this converges to a solution defined on some small interval (details omitted, this is called Picard's iteration method).

Next, we discuss continuous dependence on initial conditions. We have

$$\Phi(t,x) = x + \int_0^t X(\Phi(s,x))ds \tag{10.17}$$

$$\Phi(t,x') = x' + \int_0^t X(\Phi(s,x'))ds. \tag{10.18}$$

Then

$$|\Phi(t,x) - \Phi(t,x')| \leq |x-x'| + K\int_0^t |\Phi(s,x) - \Phi(s,x')|ds \tag{10.19}$$

Letting

$$U(t) = |x-x'| + K\int_0^t |\Phi(s,x) - \Phi(s,x')|ds, \tag{10.20}$$

we have $U(t) \geq 0$ and $U(0) = |x-x'|$. Then

$$U'(t) = K|\Phi(t,x) - \Phi(t,x')| \leq KU(t). \tag{10.21}$$

This implies that $U'(t) - KU(t) \leq 0$. Similar to above, it follows that

$$(e^{-Kt}U(t))' \leq 0. \tag{10.22}$$

Integrating from 0 to t, we obtain

$$e^{-Kt}U(t) \leq |x-x'|. \tag{10.23}$$

This implies that

$$|\Phi(t,x) - \Phi(t,x')| \leq e^{Kt}|x-x'| \tag{10.24}$$

Assuming $t < t'$, we also estimate

$$|\Phi(t,x') - \Phi(t',x')| \leq \int_t^{t'} |X(\Phi(s,x')) - X(\Phi(s,x'))|ds \leq M|t-t'|. \tag{10.25}$$

Finally, we have

$$\begin{aligned} |\Phi(t,x) - \Phi(t',x')| &\leq |\Phi(t,x) - \Phi(t,x')| + |\Phi(t,x') - \Phi(t',x')| \qquad (10.26)\\ &\leq e^{Kt}|x-x'| + M|t-t'| \qquad (10.27)\end{aligned}$$

Given $\epsilon > 0$, choosing

$$\delta = \frac{\epsilon}{2(e^K + M)}, \tag{10.28}$$

proves the continuity. Higher derivatives are estimated in a similar way by differentiating the equation, details are omitted. $\square$

Proposition 10.2. *If M is compact, then the domain of Φ is $\mathbb{R} \times M$. In other words, every vector field on a compact manifold is complete.*

Proof. The previous result says the domain of definition of Φ is an open subset U of $\mathbb{R} \times M$ containing $\{0\} \times M$. Since M is compact, the "Tube Lemma" from basic topology says that U contains $(-\epsilon, \epsilon) \times M$ for some $\epsilon > 0$. Given any $t \in \mathbb{R}$, write $t = k\epsilon + r$, where $0 \leq r < \epsilon$. Writing $\phi_t(x) = \Phi(t, x)$, we define

$$\phi_t(x) = \overbrace{\phi_\epsilon \circ \cdots \circ \phi_\epsilon}^{k} \circ \phi_r(x) \tag{10.29}$$

if $k \geq 0$, and

$$\phi_t(x) = \overbrace{\phi_{-\epsilon} \circ \cdots \circ \phi_{-\epsilon}}^{k} \circ \phi_r(x) \tag{10.30}$$

if $k < 0$. $\square$

Theorem 10.3. *Let X be a vector field on M such that $X(p) \neq 0$. Then there exists a local coordinate system $(x^1, \cdots, x^n)$ around p such that $x_* X = \frac{\partial}{\partial x^1}$.*

Proof. This is local, so we can assume we are in $\mathbb{R}^n$, with coordinates y^i. We can also assume that $X(0) = \frac{\partial}{\partial y^1}|_0$. We then define coordinates $z^i(y^1, \ldots, y^n)$ by

$$(z^1, \ldots, z^n) = \phi_{y^1}(0, y^2, \ldots, y^n) \tag{10.31}$$

We compute that

$$\begin{aligned}
z_*(\frac{\partial}{\partial y^1})f &= \frac{\partial}{\partial y^1}(f \circ z) \\
&= \lim_{h \to 0} \frac{1}{h}(f \circ \phi_{y^1+h}(0, y^2, \ldots, y^n) - f \circ \phi_{y^1}(0, y^2, \ldots, y^n) \\
&= \lim_{h \to 0} \frac{1}{h}(f \circ \phi_h \circ z - f \circ z) \\
&= Xf \circ z.
\end{aligned} \tag{10.32}$$

Clearly, we have

$$z_*(\frac{\partial}{\partial y^i}|_0) = \frac{\partial}{\partial y^i}|_0, \quad i > 1. \tag{10.33}$$

Thus we have that $z_*|0 = Id$. By the inverse function theorem, $x = z^{-1}$ exists in a neighborhood of the origin. Above, we showed that

$$z_*(\frac{\partial}{\partial y^1}) = X \circ z \tag{10.34}$$

so applying x_* to this equation yields

$$\frac{\partial}{\partial y^1} = x_* \circ z_*(\frac{\partial}{\partial y^1}) = x_*(X \circ z) = x_* X \circ z \circ x = x_* X. \tag{10.35}$$

$\square$

11 Lecture 11

11.1 Method of characteristics

The above proof seems magical, so let's give another explanation of the proof of Theorem 10.3. For simplicity, let's just consider the case of $n = 2$, and our vector field is written near the origin as

$$X = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y} \tag{11.1}$$

with $f(0, 0) = 1$ and $g(0, 0) = 0$. We want to find coordinates $u(x, y)$ and $v(x, y)$ so that

$$X = \frac{\partial}{\partial u} = f\Big(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\Big) + g\Big(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\Big) \tag{11.2}$$

$$= \Big(f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y}\Big)\frac{\partial}{\partial u} + \Big(f\frac{\partial v}{\partial x} + g\frac{\partial v}{\partial y}\Big)\frac{\partial}{\partial v} \tag{11.3}$$

So we want to solve the equations

$$f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y} = 1, \quad f\frac{\partial v}{\partial x} + g\frac{\partial v}{\partial y} = 0, \tag{11.4}$$

which are more simply

$$Xu = 1, \quad Xv = 0, \tag{11.5}$$

together with a condition that $u(x, y), v(x, y)$ form a coordinate system near $(0, 0)$. This will be true if the Jacobian determinant

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}(0, 0) \neq 0. \tag{11.6}$$

Remark 11.1. The pair of equations (11.5) is obviously a completely uncoupled system of 2 first order linear PDEs for 2 functions of 2 variables. This is a "determined" system. There is some slight coupling of the "initial conditions" in (11.6).

The method used in Theorem 10.3 is a special case of the "method of characteristics", which we explain next. This method reduces solving a first order linear PDE to solving an ODE. The only "drawback" of this method is that the ODE is nonlinear.

First, consider the equation $Xu = 1$, which is just

$$f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y} = 1. \tag{11.7}$$

Consider the graph of the solution as a hypersurface in $\mathbb{R}^3$:

$$G = \{(x, y, u(x, y)) \mid (x, y) \in U\}. \tag{11.8}$$

A normal vector field to the graph is given by

$$\vec{N} = (u_x, u_y, -1). \tag{11.9}$$

Define the vector field along G by

$$\vec{F} = (f, g, 1). \tag{11.10}$$

Then

$$\vec{F} \cdot \vec{N} = f u_x + g u_y - 1 = 0. \tag{11.11}$$

So the vector field $\vec{F}$ is everywhere tangent to the graph G. Consequently, G is stratified by the integral curves of $\vec{F}$. We then solve the ODE system:

$$\frac{dx}{ds} = f(x,y), \quad \frac{dy}{ds} = g(x,y), \quad \frac{du}{ds} = 1, \tag{11.12}$$

with initial conditions

$$x(0) = 0, \quad y(0) = v, \quad u(0) = 0. \tag{11.13}$$

The last equation gives $u = s + C$, which yields $u = s$. The first 2 equations are just the flow of the vector field X, with initial conditions along the y-axis. The solution is of the form

$$(x(s,v), y(s,v), u(s)) = \big(\Phi(s,(0,v)), s\big) \tag{11.14}$$

so

$$(x(u,v), y(u,v)) = \Phi(u,(0,v)). \tag{11.15}$$

This determines the variables (u,v) implicitly as functions of the variables x and y. This is solvable by the inverse function theorem, provided that

$$\det \begin{pmatrix} \frac{\partial \Phi_1(u,(0,v))}{\partial u} & \frac{\partial \Phi_1(u,(0,v))}{\partial v} \\ \frac{\partial \Phi_2(u,(0,v))}{\partial u} & \frac{\partial \Phi_2(u,(0,v))}{\partial v} \end{pmatrix} (0,0) \neq 0. \tag{11.16}$$

But we have

$$\frac{\partial \Phi_1(u,(0,v))}{\partial u}|_{(0,0)} = \frac{\partial x}{\partial s}|_{(0,0)} = f(0,0) = 1, \tag{11.17}$$

$$\frac{\partial \Phi_2(u,(0,v))}{\partial u}|_{(0,0)} = \frac{\partial y}{\partial s}|_{(0,0)} = g(0,0) = 0, \tag{11.18}$$

$$\frac{\partial \Phi_2(u,(0,v))}{\partial v}|_{(0,0)} = \frac{\partial y}{\partial v}|_{(0,0)} = \lim_{h \to 0} \frac{y(0,h) - y(0,0)}{h} = 1, \tag{11.19}$$

so the determinant is equal to 1 at $(0,0)$, and we can indeed solve for u and v as functions of x and y.

Finally, since

$$\Big(u(x(u,v),y(u,v)),v(x(u,v),y(u,v))\Big) = (u,v), \tag{11.20}$$

by the inverse function theorem,

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11.21}$$

In particular, we have

$$0 = v_x x_u + v_y y_u = f v_x + g v_y. \tag{11.22}$$

Consequently, this v solves the equation $Xv = 0$, and we are done since we already know that (u,v) forms a coordinate system in a neighborhood of the origin.

Remark 11.2. Note that the initial conditions (11.13) specify the solution to vanish along the y-axis, and the parameter u was just a parametrization of the y-axis. We could also get a solution of $Xu = 1$, with initial data specified along a *non-characteristic* curve. That is, a curve passing through $(0,0)$ which is not tangent to X at the origin. Then our initial condition would be

$$x(0) = a(v), \quad y(0) = b(v), \quad u(0) = c(v). \tag{11.23}$$

Here we have parametrized the curve by the parameter v, and we require that $b'(0) \neq 0$. We then have $u = s + c(v)$, so the solution is of the form

$$(x(s,v),y(s,v),u(s)) = \big(\Phi(s,(a(v),b(v))), s + c(v)\big) \tag{11.24}$$

so

$$(x(u,v),y(u,v)) = \Phi(u - c(v),(a(v),b(v))). \tag{11.25}$$

The $c(v)$ term can be eliminated by the transform

$$\tilde{u} = u - c(v), \quad \tilde{v} = v, \tag{11.26}$$

which has Jacobian determinant

$$\det \begin{pmatrix} 1 & -c'(v) \\ 0 & 1 \end{pmatrix} = 1 \neq 0. \tag{11.27}$$

Then the main modification to the above is

$$\begin{aligned} \frac{\partial \Phi_2(u,(a(v),b(v)))}{\partial v}\Big|_{(0,0)} = \frac{\partial y}{\partial v}\Big|_{(0,0)} &= \lim_{h\to 0} \frac{y(a(h),b(h)) - y(0,0)}{h} \\ &= \lim_{h\to 0} \frac{b(h) - 0}{h} = b'(0) \neq 0. \end{aligned} \tag{11.28}$$

Example 11.3. Let's do the above "straightening" procedure for the vector field

$$X = (1+y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{11.29}$$

If we write an integral curve as $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, the flow ODE is

$$\gamma_1' = 1 + \gamma_2, \quad \gamma_2' = \gamma_1 \tag{11.30}$$

Differentiating the first equation yields

$$\gamma_1'' = \gamma_2' = \gamma_1, \tag{11.31}$$

so $\gamma_1 = c_1 \cosh(t) + c_2 \sinh(t)$. The second equation then gives $\gamma_2 = c_1 \sinh(t) + c_2 \cosh(t) + c_3$. The first equation shows that $c_3 = -1$. The initial conditions are $\gamma(0) = (0, y)$, which yields

$$\gamma_1(t) = (y+1)\sinh(t), \quad \gamma_2(t) = -1 + (y+1)\cosh(t) \tag{11.32}$$

So we have

$$(x(u,v), y(u,v)) = \Phi(u,(0,v)) = \big((v+1)\sinh(u), -1 + (v+1)\cosh(u)\big). \tag{11.33}$$

This can be inverted explicitly

$$u(x,y) = \tanh^{-1}\left(\frac{x}{y+1}\right), \quad v(x,y) = -1 + \sqrt{(y+1)^2 - x^2}. \tag{11.34}$$

11.2 Lie derivatives

Now that we understand the flow of a vector field, we give an alternative characterization of the Lie derivative. First, for vector fields. Recall that for $X, Y \in \Gamma(TM)$, we previously defined $L_X Y = [X,Y]$, where $[X,Y]f = X(Yf) - Y(Xf)$. We showed that in a local coordinate system, if

$$X = X^i \frac{\partial}{\partial x^i}, \ Y = Y^i \frac{\partial}{\partial x^i}, \tag{11.35}$$

then

$$[X,Y] = \left(X^i \frac{\partial Y^l}{\partial x^i} - Y^j \frac{\partial X^l}{\partial x^j} \right) \frac{\partial}{\partial x^l}. \tag{11.36}$$

Furthermore, if $\omega \in \Omega^1(M)$, we defined

$$L_X\omega(Y) = X\omega(Y) - \omega(L_X Y). \tag{11.37}$$

Now we will give the new definitions

Definition 11.4. For $X, Y \in \Gamma(TM)$, $f \in \Omega^0(M)$, and $\omega \in \Omega^1(M)$, define

$$L'_X f(p) \equiv \lim_{h\to 0} h^{-1}\Big(f \circ \phi_h(p) - f(p) \Big) \tag{11.38}$$

$$L'_X \omega(p) \equiv \lim_{h\to 0} h^{-1}\Big((\phi_h^*\omega)_p - \omega_p) \Big) \tag{11.39}$$

$$L'_X Y(p) \equiv \lim_{h\to 0} h^{-1}\Big(Y_p - ((\phi_h)_* Y)_p \Big). \tag{11.40}$$

Note that obviously $L'_X f = Xf$.

Proposition 11.5. *The operator L'_X acts as a derivation, that is,*

$$L'_X(fY) = (Xf)Y + fL'_XY \tag{11.41}$$
$$L'_X(f\omega) = (Xf)\omega + fL'_X\omega \tag{11.42}$$
$$L'_X\omega(Y) = X\omega(Y) - \omega(L'_XY). \tag{11.43}$$

Proof. We prove (11.41), the others are proved similarly. We have

$$\begin{aligned}
L'_X(fY) &= \lim_{h\to 0} h^{-1}\Big((fY)_p - ((\phi_h)_* fY)_p\Big)\\
&= \lim_{h\to 0} h^{-1}\Big(f(p)Y_p - (\phi_h)_*(fY)_{\phi_{-h}(p)}\Big)\\
&= \lim_{h\to 0} h^{-1}\Big(f(p)Y_p - f(\phi_{-h}(p))(\phi_h)_*Y_{\phi_{-h}(p)}\Big)\\
&= \lim_{h\to 0} h^{-1}\Big(f(p)Y_p - f(p)(\phi_h)_*Y_{\phi_{-h}(p)} + f(p)(\phi_h)_*Y_{\phi_{-h}(p)} - f(\phi_{-h}(p))(\phi_h)_*Y_{\phi_{-h}(p)}\Big)\\
&= f(p)\lim_{h\to 0} h^{-1}\Big(Y_p - (\phi_h)_*Y_{\phi_{-h}(p)}\Big) + \lim_{h\to 0} h^{-1}\Big(f(p) - f(\phi_{-h}(p))\Big)(\phi_h)_*Y_{\phi_{-h}(p)}\\
&= f(p)L'_XY + \lim_{h\to 0} h^{-1}\Big(f(p) - f(\phi_{-h}(p))\Big)\cdot \lim_{h\to 0}(\phi_h)_*Y_{\phi_{-h}(p)}
\end{aligned} \tag{11.44}$$

Note that by letting $k = -h$, we have

$$\begin{aligned}
\lim_{h\to 0} h^{-1}\Big(f(p) - f(\phi_{-h}(p))\Big) &= \lim_{k\to 0} -k^{-1}\Big(f(p) - f(\phi_k(p))\Big)\\
&= \lim_{k\to 0} k^{-1}\Big(f(\phi_k(p)) - f(p)\Big) = Xf(p).
\end{aligned} \tag{11.45}$$

Finally, we have

$$\begin{aligned}
\lim_{h\to 0}(\phi_h)_*Y_{\phi_{-h}(p)} &= \lim_{h\to 0}\Big((\phi_h)_*Y_{\phi_{-h}(p)} - Y_{\phi_{-h}(p)} + Y_{\phi_{-h}(p)}\Big)\\
&= \lim_{h\to 0}\Big((\phi_h)_* - Id)Y_{\phi_{-h}(p)}\Big) + \lim_{h\to 0} Y_{\phi_{-h}(p)}\Big) = Y(p),
\end{aligned} \tag{11.46}$$

and we are done. $\square$

Exercise 11.6. Prove (11.42) and (11.43).

Proposition 11.7. *For $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^1(M)$, we have*

$$L_XY = L'_XY, \quad L_X\omega = L'_X\omega. \tag{11.47}$$

Proof. From (11.43), we have

$$0 = L'_X\delta^i_j = L'_X(dx^i(\partial_j)) = L'_X(dx^i)(\partial_j) + dx^i(L'_X\partial_j), \tag{11.48}$$

so

$$L'_X\partial_j = -\Big(L'_X(dx^i)(\partial_j)\Big)\partial_i. \tag{11.49}$$

We compute

$$\begin{aligned}
L'_X(dx^i)(\partial_j) &= \lim_{h\to 0} h^{-1}\Big(\phi_h^* dx^i - dx^i\Big)(\partial_j)\\
&= \lim_{h\to 0} h^{-1}\Big((\phi_h^* dx^i)(\partial_j) - dx^i(\partial_j)\Big)\\
&= \lim_{h\to 0} h^{-1}\Big(d(x^i\circ\phi_h)(\partial_j) - dx^i(\partial_j)\Big)\\
&= \lim_{h\to 0} h^{-1}\Big(d(x^i\circ\phi_h)(\partial_j) - dx^i(\partial_j)\Big)\\
&= d\Big(\lim_{h\to 0} h^{-1}\Big(x^i\circ\phi_h - x^i\Big)\Big)(\partial_j)\\
&= d(Xx^i)(\partial_j) = \frac{\partial}{\partial x^j}X^i.
\end{aligned} \tag{11.50}$$

So we have

$$\begin{aligned}
L'_X Y = L'_X(Y^j\partial_j) &= (X(Y^j))\partial_j + Y^j L'_X(\partial_j)\\
&= X(Y^j)\partial_j - Y^j\Big(\frac{\partial}{\partial x^j}X^i\Big)\partial_i\\
&= X^i\Big(\frac{\partial}{\partial x^i}Y^j\Big)\partial_j - Y^j\Big(\frac{\partial}{\partial x^j}X^i\Big)\partial_i\\
&= \Big(X^i\Big(\frac{\partial}{\partial x^i}Y^l\Big) - Y^j\Big(\frac{\partial}{\partial x^j}X^l\Big)\Big)\partial_l,
\end{aligned} \tag{11.51}$$

which agrees with (11.36). This proves $L_X = L'_X$ on vector fields. Since they both satisfy the Leibniz rule on 1-forms,

$$L'_X\omega(Y) = X\omega(Y) - \omega(L'_X Y) \tag{11.52}$$

$$L_X\omega(Y) = X\omega(Y) - \omega(L_X Y), \tag{11.53}$$

they also agree on 1-forms. □

Exercise 11.8. Show that for $\omega\in\Omega^k$ and $k>1$, we have $L_X\omega = L'_X\omega$.

12 Lecture 12

12.1 Frobenius Theorem (local version)

Today we address the following question. Assume we are given vector fields $X_1,\ldots,X_k\in\Gamma(TU)$ which are linearly independent at every point. Then does there exists a coordinate system $(x^1,\ldots,x^n)$ such that $X_i=\partial_i$ for $1\le i\le k$. Before we give the answer, we need a few auxiliary results.

Proposition 12.1. *Let $X \in \Gamma(TM)$ with 1-parameter group ϕ_t. If $\alpha : M \to M$ is a diffeomorphism, then the 1-parameter group of $\alpha_* X$ is given by $\alpha \circ \phi_t \circ \alpha^{-1}$.*

Proof. Recall the formula that

$$(\alpha_* X)_q = (\alpha_*)_{\alpha^{-1}(q)} X_{\alpha^{-1}(q)}, \tag{12.1}$$

for any $q \in M$. Given a point $p \in M$, consider the curve

$$\gamma(t) = \alpha \circ \phi_t \circ \alpha^{-1}(p) = \alpha \circ \Phi(t, \alpha^{-1}(p)). \tag{12.2}$$

This satisfies

$$\begin{aligned}\gamma'(t) = \gamma_*(\partial_t) &= (\alpha_*)_{\phi_t \circ \alpha^{-1}(p)} \circ X_{\phi_t \circ \alpha^{-1}(p)} \\ &= (\alpha_*)_{\alpha^{-1} \circ \alpha \circ \phi_t \circ \alpha^{-1}(p)} \circ X_{\alpha^{-1} \circ \alpha \phi_t \circ \alpha^{-1}(p)} \\ &= (\alpha_* X)_{\alpha \circ \phi_t \circ \alpha^{-1}(p)},\end{aligned} \tag{12.3}$$

using (12.1) with $q = \alpha \circ \phi_t \circ \alpha^{-1}(p)$. Therefore $\gamma(t)$ is an integral curve of $\alpha_* X$, so by uniqueness of integral curves, we are done. $\square$

Proposition 12.2. *Let $X, Y \in \Gamma(TM)$ with 1-parameter groups ϕ_t, ψ_t, respectively. Then $[X, Y] = 0$ if and only if $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ for all s, t.*

Proof. Assume that $[X, Y] = 0$. Then for $q \in M$,

$$0 = \lim_{h \to 0} h^{-1} \Big(Y_q - \big((\phi_h)_* Y \big)_q \Big). \tag{12.4}$$

Next, given any $p \in M$, consider the curve $\gamma : (-\epsilon, \epsilon) \to T_p M$ defined by $c(t) = \big((\phi_t)_* Y \big)_p$. We then compute

$$\begin{aligned}c'(t) &= \lim_{h \to 0} h^{-1} (c(t+h) - c(t)) \\ &= \lim_{h \to 0} h^{-1} \Big(\big((\phi_{t+h})_* Y \big)_p - \big((\phi_t)_* Y \big)_p \Big) \\ &= \lim_{h \to 0} h^{-1} \Big(\big((\phi_t \circ \phi_h)_* Y \big)_p - (\phi_t)_* Y_{\phi_{-t}(p)} \Big) \\ &= \lim_{h \to 0} h^{-1} \Big(\big((\phi_t)_* \circ (\phi_h)_* Y \big)_p - (\phi_t)_* Y_{\phi_{-t}(p)} \Big) \\ &= \lim_{h \to 0} h^{-1} \Big((\phi_t)_* \big((\phi_h)_* Y \big)_{\phi_{-t}(p)} - (\phi_t)_* Y_{\phi_{-t}(p)} \Big) \\ &= (\phi_t)_* \lim_{h \to 0} h^{-1} \Big(\big((\phi_h)_* Y \big)_{\phi_{-t}(p)} - Y_{\phi_{-t}(p)} \Big) = 0,\end{aligned} \tag{12.5}$$

using (12.4) with $q = \phi_{-t}(p)$. Therefore $c(t)$ is constant, and $c(t) = c(0) = Y_p$. This implies that $(\phi_t)_* Y = Y$. By Proposition 12.1, the flow of Y, ψ_s, must be equal to $\phi_t \circ \psi_s \circ \phi_t^{-1}$.

For the converse let $\alpha = \psi_s$, and by assumption $\phi_t = \alpha \circ \phi_t \circ \alpha^{-1}$. By Proposition 12.1, ϕ_t must be the 1-parameter group generated by $\alpha_* X = (\psi_s)_* X$. So we have $(\psi_s)_* X = X$, which obviously implies that $[X, Y] = L'_X Y = 0$. $\square$

The main result is the following.

Theorem 12.3. *Assume we are given vector fields $X_1, \dots, X_k \in \Gamma(TU)$ which are linearly independent in a neighborhood U of $p \in M$ and which satisfy $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Then there exists a local coordinate system $(x^1, \dots, x^n)$ such that $X_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq k$.*

Proof. Without loss of generality, we can assume there is a coordinate system $(t^1, \dots, t^n)$ such that $X_i(0) = \frac{\partial}{\partial t^i}(0)$ for $1 \leq i \leq k$. Call the 1-parameter group of X_i by ϕ^i_t for $1 \leq i \leq k$. Define

$$(t^1(x^1, \dots, x^n), \dots, t^n(x^1, \dots, x^n)) = \phi^1_{x^1}(\phi^2_{x^2}(\cdots(\phi^k_{x^k}(0, \dots, 0, x^{k+1}, \dots, x^n)\cdots))). \quad (12.6)$$

It is easy to see that the Jacobian as 0 is the identity, so by the inverse function theorem we can solve for the $x^i = x^i(t^1, \dots, t^n)$ as functions of the t^i for $1 \leq i \leq n$, in some possibly smaller neighborhood U of p. By (12.2), given any $1 \leq i \leq k$, we can write

$$(t^1(x^1, \dots, x^n), \dots, t^n(x^1, \dots, x^n)) = \phi^i_{x^1}(\phi^1_{x^2}(\cdots(\phi^k_{x^k}(0, \dots, 0, x^{k+1}, \dots, x^n)\cdots))), \quad (12.7)$$

so by the same proof as straightening 1 vector field, we have that $X_i = \frac{\partial}{\partial x^i}$. $\square$

Remark 12.4. We can also phrase the above as follows, which we state for 2 vector fields for simplicity. Given X_1 and X_2 linearly independent vector fields, we require functions $x^1, \dots, x^n$ such that

$$X_1 x^1 = 1, \quad X_1 x^i = 0, i > 1 \quad (12.8)$$

$$X_2 x^2 = 1, \quad X_2 x^i = 0, i \neq 2, \quad (12.9)$$

together with the condition that these form a coordinate system. We know that separately, each line is a determined system. However, together these equations have twice as many equations as unknowns, so this is called an overdetermined system. In general, one does not expect solution to exist for overdetermined systems, unless some extra conditions are satisfied which in this case are exactly the condition that the vector fields commute.

Exercise 12.5. Define the vector fields on $\mathbb{R}^3$ by

$$X_1 = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\partial_x + y\partial_y + z\partial_z) \quad (12.10)$$

$$X_2 = -y\partial_x + x\partial_y \quad (12.11)$$

$$X_3 = \frac{xz}{\sqrt{x^2 + y^2}}\partial_x + \frac{yz}{\sqrt{x^2 + y^2}}\partial_x - \sqrt{x^2 + y^2}\partial_z. \quad (12.12)$$

Show that these vector fields commute, and find a coordinate system which simultaneously straightens X_1, X_2, X_3.

Exercise 12.6. Define the vector fields on $\mathbb{R}^4$ by the following.

$$X_1 = x^1\partial_1 + x^2\partial_2 + x^3\partial_3 + x^4\partial_4 \quad (12.13)$$

$$X_2 = -x^4\partial_1 - x^3\partial_2 + x^2\partial_3 + x^1\partial_4 \quad (12.14)$$

$$X_3 = x^3\partial_1 - x^4\partial_2 - x^1\partial_3 + x^2\partial_4 \quad (12.15)$$

$$X_4 = -x^2\partial_1 + x^1\partial_2 - x^4\partial_3 + x^3\partial_4. \quad (12.16)$$

Show that X_2, X_3, and X_4 commute with X_1, but do not commute with each other. Furthermore, find functions f_1, f_2, f_3, f_4 such that

$$X_1 f_1 = 1, X_1 f_2 = 0, X_2 f_1 = 0, X_2 f_2 = 1, \tag{12.17}$$
$$X_1 f_3 = 0, X_3 f_1 = 0, X_3 f_3 = 1 \tag{12.18}$$
$$X_1 f_4 = 0, X_4 f_1 = 0, X_4 f_4 = 1. \tag{12.19}$$

13 Lecture 13

13.1 Frobenius theorem (geometric version)

We begin with a lemma.

Lemma 13.1. *Let $\phi : M \to N$ be a smooth mapping and $X, Y \in \Gamma(TM)$. Assume that $\phi_* X$ and $\phi_* Y$ are smooth vector fields on N. Then*

$$\phi_*[X, Y] = [\phi_* X, \phi_* Y] \tag{13.1}$$

Proof. Let $f \in C^\infty(N, \mathbb{R})$. Then by definition of the push-forward, we have

$$(\phi_* X) f = X(f \circ \phi). \tag{13.2}$$

Applying (13.2) several times, we then compute

$$\begin{aligned}
[\phi_* X, \phi_* Y] f &= (\phi_* X)\big((\phi_* Y)(f)\big) - (\phi_* Y)\big((\phi_* X)(f)\big) \\
&= X\Big(\big((\phi_* Y)(f)\big) \circ \phi\Big) - Y\Big(\big((\phi_* X)(f)\big) \circ \phi\Big) \\
&= X\big(Y(f \circ \phi)\big) - Y\big(X(f \circ \phi)\big) \\
&= [X, Y](f \circ \phi) = \big(\phi_*[X, Y]\big) f.
\end{aligned} \tag{13.3}$$

□

Definition 13.2. A distribution of rank k, $\Delta \subset TM$ is a sub-bundle of the tangent bundle of rank k. The distribution Δ is said to be *integrable* if for any two local sections $X, Y \in \Gamma(\Delta|U)$, we have $[X, Y] \in \Gamma(\Delta|_U)$.

Definition 13.3. An immersed submanifold $N \subset M$ is called an integral manifold of Δ if $T_p N = \Delta_p$ for all $p \in N$.

Theorem 13.4. *If Δ is an integrable rank k distribution, then around any point $p \in M$, there exists a local coordinate system (X, U) such that for $q \in U$,*

$$x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n \tag{13.4}$$

is an integral manifold of Δ, for each $(a^{k+1}, \ldots, a^n)$ with $|a^i| < \epsilon$ for $k+1 \le i \le n$.

Proof. For the first part, clearly we can assume that M is an open subset of $\mathbb{R}^n$, and $p = 0$. WLOG, assume that Δ_0 is spanned by

$$\frac{\partial}{\partial t^i}(0), \quad 1 \leq i \leq k. \tag{13.5}$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k factors. Then $\pi_* : \Delta_0 \to \mathbb{R}^k$ is an isomorphism, so by continuity for q sufficiently near p, $\pi_* : \Delta_q \to \mathbb{R}^k$ is injective (because the mapping $q \mapsto \det(\pi_*|_{\Delta_q})$ is continuous, and nonzero at 0). So we can choose $X_i(q) \in \Delta_q$ such that $\pi_* X_i = \frac{\partial}{\partial t^i}$ for $1 \leq i \leq k$. The X_i are smooth vector fields in a neighborhood of the origin in $\mathbb{R}^n$, so by Lemma 13.1 we have

$$\pi_*[X_i, X_j]_q = \left[\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right]_{\pi(q)} = 0. \tag{13.6}$$

Since we assumed that Δ is integrable, $[X_i, X_j]_q \in \Delta_q$, so $[X_i, X_j] = 0$ because π_* is injective. Then we can use Theorem 12.3 to find a coordinate system (x, U) so that $X_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq k$, and we are done.

□

Remark 13.5. One can show that every point $p \in M$ lies on a unique connected maximal integral submanifold. A basic example is lines of some constant slope on a square torus. If the slope is rational, the maximal integral submanifolds are imbedded circles. However, if the slope is irrational, then the maximal integral submanifolds are the real line, since they never close up.

Finally, let's discuss one of the homework problems from a previous lecture. Given $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^1(M)$, we have

$$L'_X\omega(Y) = X\omega(Y) - \omega(L'_X Y), \tag{13.7}$$

where for $p \in M$ we have

$$(L'_X\omega)_p \equiv \lim_{h \to 0} h^{-1}\Big((\phi_h^*\omega)_p - \omega_p)\Big) \tag{13.8}$$

$$(L'_X Y)_p \equiv \lim_{h \to 0} h^{-1}\Big(Y_p - ((\phi_h)_* Y)_p\Big). \tag{13.9}$$

First, note that by definition $L'_X\omega$ is a tensor, i.e., $L'_X\omega \in \Omega^1(M)$. This means that for any vector field $Y \in \Gamma(TM)$, the expression $L'_X\omega(Y)_p$ depends only upon Y_p. For the left hand side, using (11.41), we compute

$$\begin{aligned} X\omega(fY) - \omega(L'_X(fY)) &= X(f\omega(Y)) - \omega\big((Xf)Y + f(L'_X Y)\big) \\ &= Xf \cdot \omega(Y) + fX(\omega(Y)) - Xf \cdot \omega(Y) + f\omega(L'_X Y) \\ &= f\big(X\omega(fY) - \omega(L'_X(fY))\big). \end{aligned} \tag{13.10}$$

So the left hand side is also a tensor since it is linear over C^∞ functions. Since both sides of (13.7) are tensors, it suffices to prove in a coordinate system. Furthermore, without loss of generality, we may assume that $Y = \partial_i$, for $1 \le i \le n$. First, consider the case that $\omega = dx^j$ for some $1 \le j \le n$, and write $X = \sum_k X^k \partial_k$.

The left hand side of (13.7) is

$$L'_X \omega(Y) = L'_X dx^j(\partial_i)_p = \frac{\partial X^j}{\partial x^i}(p), \tag{13.11}$$

which we had already computed above in (11.50). Next, let us compute the right hand side of (13.7),

$$X\omega(Y) - \omega(L'_X Y) = X^k \partial_k dx^j(\partial_i) - dx^j(L'_X \partial_i) = -dx^j(L'_X \partial_i). \tag{13.12}$$

Lemma 13.6. *We have*

$$\phi_*(\partial_i)_p = \sum_i \frac{\partial(x^j \circ \phi)}{\partial x^i}|_p (\partial_j)_{\phi(p)}. \tag{13.13}$$

Proof. We write

$$\phi_*(\partial_i)_p = \sum_j a^j (\partial_j)_{\phi(p)}, \tag{13.14}$$

so by plugging in the coordinate function x^j, we have that

$$a^j = (\phi_*(\partial_i)_p)(x^j) = \frac{\partial(x^j \circ \phi)}{\partial x^i}|_p, \tag{13.15}$$

□

Next, let's compute

$$\begin{aligned}
(L'_X \partial_i)_p &= \lim_{h\to 0} h^{-1}\Big((\partial_i)_p - (\phi_h)_*(\partial_i)_{\phi_{-h}(p)}\Big)\\
&= \lim_{h\to 0} h^{-1}\Big((\partial_i)_p - \sum_j \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_{\phi_{-h}(p)} \partial_j|_p\Big)\\
&= \lim_{h\to 0} h^{-1}\Big(\sum_j \frac{\partial(x^j \circ \phi_0)}{\partial x^i}|_p - \sum_j \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_{\phi_{-h}(p)}\Big)\partial_j|_p\\
&= \lim_{h\to 0} h^{-1}\Big(\sum_j \frac{\partial(x^j \circ \phi_0)}{\partial x^i}|_p - \sum_j \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_p\Big) \qquad (13.16)\\
&\quad + \lim_{h\to 0} h^{-1}\Big(\sum_j \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_p - \sum_j \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_{\phi_{-h}(p)}\Big)\partial_j|_p\\
&= \frac{\partial}{\partial x^i}\Big|_p \lim_{h\to 0} h^{-1}\Big(\sum_j (x^j \circ \phi_0) - \sum_j (x^j \circ \phi_h)\Big)(\partial_j)_p\\
&= -\frac{\partial}{\partial x^i} X(x^j)(\partial_j)_p = -\frac{\partial X^j}{\partial x^i}(\partial_j)_p.
\end{aligned}$$

The interchange of limits is valid since the flow is smooth in both variables. The second limit vanishes because.... This shows that the right hand side is

$$X\omega(Y) - \omega(L'_X Y) = \frac{\partial X^j}{\partial x^i}. \tag{13.17}$$

So we have verified the formula for any vector fields X, Y, but we assumed that $\omega = dx^j$. To finish, from (11.42), we have

$$L'_X(f dx^j) = Xf dx^j + f L'_X dx^j, \tag{13.18}$$

so for any Y,

$$L'_X(f dx^j)(Y) = Xf dx^j(Y) + f(L'_X dx^j)(Y). \tag{13.19}$$

We also have

$$X(f dx^j)(Y) - (f dx^j)(L'_X Y) = (Xf)dx^j(Y) + f\big(X dx^j(Y) - f dx^j(L'_X Y)\big). \tag{13.20}$$

Since the formula is true for $\omega = dx^j$, it is therefore true for $\omega = f dx^j$, so true for any sum $\omega = f_j dx^j$, and we are done.

14 Lecture 14

To begin, let's prove the fact we needed above: for each i and j,

$$\lim_{h \to 0} h^{-1}\Big(\frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_p - \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_{\phi_{-h}(p)}\Big) = 0. \tag{14.1}$$

For this, we need the following lemma.

Lemma 14.1. *If $f : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ is smooth with $f(0, p) = 0$ for all $p \in M$, there there exists a smooth function $g : (-\epsilon, \epsilon) \to \mathbb{R}$ so that $f(t, p) = tg(t, p)$ and $\frac{\partial f}{\partial t}(0, p) = g(0, p)$.*

Proof. Define

$$g(t, p) = \int_0^1 \frac{\partial f}{\partial t}(st, p) ds. \tag{14.2}$$

Then

$$tg(t, p) = \int_0^1 t\Big(\frac{\partial f}{\partial t}\Big)(st, p)ds = \int_0^1 \frac{\partial}{\partial s}\Big(f(st, p)\Big)ds = f(t, p) - f(0, p) = f(t, p), \tag{14.3}$$

and

$$g(0, p) = \lim_{t \to 0} t^{-1} \int_0^1 \frac{\partial}{\partial s}\Big(f(st, p)\Big)ds = \lim_{t \to 0} t^{-1}(f(t, p) - f(0, p)) = \frac{\partial f}{\partial t}(0, p). \tag{14.4}$$

□

Then we apply the lemma to get g_h so that

$$x^j \circ \phi_h - x^j \circ \phi_0 = h g_h, \tag{14.5}$$

with

$$g_0 = \frac{\partial(x^j \circ \phi_h)}{\partial h}|_{h=0} = X x^j = X^j. \tag{14.6}$$

Then we have

$$\begin{aligned}
&\lim_{h\to 0} h^{-1}\Big(\frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_p - \frac{\partial(x^j \circ \phi_h)}{\partial x^i}|_{\phi_{-h}(p)}\Big) \\
&= \lim_{h\to 0} h^{-1}\Big(\frac{\partial(x^j + h g_h)}{\partial x^i}|_p - \frac{\partial(x^j + h g_h)}{\partial x^i}|_{\phi_{-h}(p)}\Big) \\
&= \lim_{h\to 0} h^{-1}\Big(\frac{\partial x^j}{\partial x^i}|_p - \frac{\partial x^j}{\partial x^i}|_{\phi_{-h}(p)}\Big) + \lim_{h\to 0}\Big(\frac{\partial g_h}{\partial x^i}|_p - \frac{\partial g_h}{\partial x^i}|_{\phi_{-h}(p)}\Big) \\
&= \lim_{h\to 0} h^{-1}\Big(\delta^j_i|_p - \delta^j_i|_{\phi_{-h}(p)}\Big) + \frac{\partial X^j}{\partial x^i}|_p - \frac{\partial X^j}{\partial x^i}|_{\phi_0(p)} \\
&= 0 + 0 = 0.
\end{aligned} \tag{14.7}$$

Exercise 14.2. Use the above argument to give a direct proof that $L'_X Y = [X, Y]$ without using coordinates.

14.1 Classical tensor calculus

A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$, and the components of X with respect to a coordinate system $x : U \to \mathbb{R}^n$ are functions $X^i : U \to \mathbb{R}$, $i = 1 \dots n$, defined by

$$X = X^i \frac{\partial}{\partial x^i} \tag{14.8}$$

on U, where $\frac{\partial}{\partial x^i}$ is the ith coordinate partial, which is a vector field on TU. Given another overlapping coordinate system $\tilde{x} : U \to \mathbb{R}^n$, we can write

$$X = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}. \tag{14.9}$$

Proposition 14.3. *The components of a vector field are related by*

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^i} X^i. \tag{14.10}$$

Conversely, any collection of locally defined functions satisfying this relation gives a well-defined vector field $X \in \Gamma(TM)$.

Proof. Since vector fields are derivations on germs of functions, plug in the function $\tilde{x}^j$ to the equality

$$X^i \frac{\partial}{\partial x^i} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}, \tag{14.11}$$

to obtain

$$X^i \frac{\partial}{\partial x^i}(\tilde{x}^j) = \tilde{X}^j. \tag{14.12}$$

□

Similarly, a 1-form is a section of the cotangent bundle, $\omega \in \Gamma(T^*M)$, and the components of ω with respect to a coordinate system $x : U \to \mathbb{R}^n$ are functions $\omega_i : U \to \mathbb{R}$, $i = 1 \dots n$, defined by

$$\omega = \omega_i dx^i \tag{14.13}$$

on U. Given another overlapping coordinate system $\tilde{x} : U \to \mathbb{R}^n$, we can write

$$\omega = \tilde{\omega}_i d\tilde{x}^i. \tag{14.14}$$

Proposition 14.4. *The components of a* 1*-form are related by*

$$\tilde{\omega}_j = \frac{\partial x^i}{\partial \tilde{x}^j}\omega_i. \tag{14.15}$$

Conversely, any collection of locally defined functions satisfying this relation gives a well-defined 1*-form* $\omega \in \Gamma(T^*M)$.

Proof. Plug in the vector field $\frac{\partial}{\partial \tilde{x}^j}$ to the equality

$$\omega_i dx^i = \tilde{\omega}_i d\tilde{x}^i, \tag{14.16}$$

to obtain

$$\omega_i dx^i \left(\frac{\partial}{\partial \tilde{x}^j} \right) = \tilde{\omega}_j. \tag{14.17}$$

But recall the definition of df, where $f : U \to \mathbb{R}$ is a function. We claim that

$$df(X) = X(f). \tag{14.18}$$

To see this, the left hand side is

$$df(X) = \frac{\partial f}{\partial x^i} dx^i \left(X^j \frac{\partial}{\partial x^j} \right) = \frac{\partial f}{\partial x^i} X^i. \tag{14.19}$$

For the right hand side, let $\gamma : (-\epsilon, \epsilon) \to M$ satisfy $\gamma(0) = p$, $\gamma'(0) = X_p$, then

$$X(f) = \frac{d}{dt}(f \circ \gamma)|_{t=0} = \frac{\partial f}{\partial x^i} \frac{d\gamma^i}{dt}_{t=0} = \frac{\partial f}{\partial x^i} X^i_p. \tag{14.20}$$

Then plugging (14.18) into (14.17), we have

$$\tilde{\omega}_j = \omega_i \frac{\partial x^i}{\partial \tilde{x}^j}. \tag{14.21}$$

□

Next, consider a (p, q)-*tensor field*

$$T \in \Gamma\Big((TM)^{\otimes^p} \otimes (T^*M)^{\otimes^q}\Big). \tag{14.22}$$

We can locally write

$$T = T^{j_1 \dots j_p}_{i_1 \dots i_q} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_q}, \tag{14.23}$$

and in another coordinate system

$$T = \tilde{T}^{j_1 \dots j_p}_{i_1 \dots i_q} \frac{\partial}{\partial \tilde{x}^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial \tilde{x}^{j_p}} \otimes d\tilde{x}^{i_1} \otimes \dots \otimes d\tilde{x}^{i_q}. \tag{14.24}$$

The above transformation formulas combine to give the following.

Proposition 14.5. *The components of T satisfy the transformation formulas*

$$\tilde{T}^{j_1 \dots j_p}_{i_1 \dots i_q} = \frac{\partial \tilde{x}^{j_1}}{\partial x^{l_1}} \cdots \frac{\partial \tilde{x}^{j_p}}{\partial x^{l_p}} \frac{\partial x^{k_1}}{\partial \tilde{x}^{i_1}} \cdots \frac{\partial x^{k_q}}{\partial \tilde{x}^{i_q}} T^{l_1 \dots l_p}_{k_1 \dots k_q} \tag{14.25}$$

Conversely, any collection of locallydefined functions satisfying this relation gives a well-defined tensor $T \in \Gamma(TM^{\otimes^p} \otimes T^*M^{\otimes^q})$.

Exercise 14.6. Show that the Kronecker δ symbol, defined by

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{14.26}$$

defines a tensor. That is,

$$T = \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j \tag{14.27}$$

is a well-defined $(1,1)$-tensor $\delta \in \Gamma(TM \otimes T^*M)$. Under the canonical isomorphisms

$$TM \otimes T^*M \cong T^*M \otimes TM \cong Hom(TM, TM), \tag{14.28}$$

identify what is the image of δ.

We note that for an n-form, we can write

$$\omega = \omega_{1 \dots n} dx^1 \wedge \dots \wedge dx^n, \tag{14.29}$$

In another coordinate system, we can write

$$\omega = \tilde{\omega}_{1 \dots n} d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n. \tag{14.30}$$

By Exercise 6.9, these components are related by

$$\tilde{\omega}_{1 \dots n} = \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) \omega_{1 \dots n}, \tag{14.31}$$

which we will use next time to define the integral of an n-form on an orientable manifold. If M is not orientable, we can define a *density* to be a collection of function so that under coordinate changes,

$$\tilde{\omega}_{1 \dots n} = \left| \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) \right| \omega_{1 \dots n}. \tag{14.32}$$

It turns out that these quantities are sections of a trivial 1-dimension line bundle. Their integral is well-defined, even on a non-orientable manifold.

14.2 Coordinate expression for Lie derivatives

We can extend the Lie derivative to any (p, q)-tensor using the formula

$$\mathcal{L}_X(Y \otimes \omega) = \mathcal{L}_X Y \otimes \omega + Y \otimes L_X \omega. \tag{14.33}$$

The follows gives the coordinate expression for the Lie derivative.

Proposition 14.7. *Let T be a tensor of type (p, q). Then*

$$\begin{aligned}(\mathcal{L}_X T)^{j_1 \dots j_p}_{i_1 \dots i_q} &= X^k \partial_k T^{j_1 \dots j_p}_{i_1 \dots i_q} + \partial_{i_1} X^k T^{j_1 \dots j_p}_{k i_2 \dots i_q} + \dots + \partial_{i_q} X^k T^{j_1 \dots j_p}_{i_1 \dots i_{q-1} k} \\ &\quad - \partial_k X^{j_1} T^{k j_2 \dots j_p}_{i_1 \dots i_q} - \dots - \partial_k X^{j_p} T^{j_1 \dots j_{p-1} k}_{i_1 \dots i_q}.\end{aligned} \tag{14.34}$$

Proof. Follows from similar arguments to the above. □

Exercise 14.8. Fill in the details for this proof.

Here is an important point: the expression $\mathcal{L}_X \omega$ is NOT tensorial in the variable X, which is obvious from the above expression, since the right hand side depends on the derivatives of X. To obtain a derivative operator which is tensorial in X will lead us to the concept of a *connection*, which we will cover in the Spring quarter.

15 Lecture 15

15.1 Remarks on flows

We know that if $X \in \Gamma(TM)$ is C^k, then the flow $\Phi(t, x)$ is C^{k+1} in the t-variable, and C^k in both variables. We can use the 1-dimensional Taylor's Theorem to write

$$\phi_t = \phi_0 + \phi_t'|_{t=0} t + \frac{1}{2}\phi_t''|_{t=0} t^2 + \frac{1}{6}\phi_t'''|_{t=0} t^3 + O(t^4), \tag{15.1}$$

as $t \to 0$. The flow equation is

$$\phi_t' = X(\phi_t), \tag{15.2}$$

so $\phi_t'|_{t=0} = X(\phi_0)$. Differentiating again,

$$\phi_t'' = DX(\phi_t)\phi_t' = DX(\phi_t)X(\phi_t), \tag{15.3}$$

so $\phi_t''|_{t=0} = DX(\phi_0)X(\phi_0)$. Differentiating again,

$$\phi_t''' = D^2 X(\phi_t)\phi_t'\phi_t' + DX(\phi_t)DX(\phi_t)\phi_t', \tag{15.4}$$

and evaluating at 0 yields

$$\phi_t'''|_{t=0} = D^2 X(\phi_0)X(\phi_0) \cdot X(\phi_0) + DX(\phi_0) \cdot DX(\phi_0)X(\phi_0). \tag{15.5}$$

So we get the expansion

$$\begin{aligned}\phi_t(p) &= p + tX(p) + \frac{1}{2}t^2 DX(p) \cdot X(p) \\ &\quad + \frac{1}{6}t^3 (D^2 X(p) \cdot X(p) \cdot X(p) + DX(p) \cdot DX(p) \cdot X(p)) + O(t^4),\end{aligned} \tag{15.6}$$

as $t \to 0$.

Remark 15.1. To compute Lie derivatives of tensors, we can actually just use any path of diffeomorphisms ϕ_t such that $\phi_t'|_{t=0} = X$. For example, if we have a Riemannian metric, we could instead use $\phi_t(p) = exp_p(tX_p)$. The equation for a geodesic is

$$(\gamma_t^k)'' = \Gamma_{ij}^k(\gamma_t)(\gamma_t^i)'(\gamma_t^j)', \tag{15.7}$$

with initial conditions $\gamma_0(p) = p$ and $\gamma_0'(p) = X_p$. Imitating the above expansions, we find an expansion like

$$\begin{aligned} \phi_t(p) &= p + tX(p) + \frac{1}{2}t^2\Gamma(p) * X(p) * X(p) \\ &\quad + \frac{1}{6}t^3\Big(\big(\Gamma'(p) + \Gamma^2(p)\big) * X(p) * X(p) * X(p)\Big) + O(t^4) \end{aligned} \tag{15.8}$$

as $t \to 0$. Note this flow has a better property than the vector field flow, the coefficients now only depend upon $X(p)$ and not its derivatives! This flow is what is used to give an infinite-dimensional manifold structure to the diffeomorphism group, not the vector field flow.

Back to the vector field flow, we can obtain an interesting expansion as follows. Fix a k-form $\alpha \in \Omega^k(M)$. Using Taylor's theorem, we can expand

$$\phi_t^*\alpha = \phi_0^*\alpha + \sum_{i=1}^{N} \frac{1}{i!}(\phi_t^*\alpha)^{(i)}|_{t=0}t^i + O(t^{N+1}) \tag{15.9}$$

as $t \to 0$. We can compute these coefficients as follows:

$$\begin{aligned} \frac{d}{dt}\phi_t^*\alpha &= \frac{d}{ds}\phi_{t+s}^*\alpha|_{s=0} = \frac{d}{ds}\phi_{s+t}^*\alpha|_{s=0} \\ &= \frac{d}{ds}(\phi_s \circ \phi_t)^*\alpha|_{s=0} \\ &= \frac{d}{ds}\phi_t^*\phi_s^*\alpha|_{s=0} = \phi_t^*\frac{d}{ds}\phi_s^*\alpha|_{s=0} = \phi_t^*\mathcal{L}_X\alpha. \end{aligned} \tag{15.10}$$

Using this, we obtain

$$(\phi_t^*\alpha)'|_{t=0} = \phi_0^*\mathcal{L}_X\alpha = \mathcal{L}_X\alpha. \tag{15.11}$$

Differentiating again, we obtain

$$(\phi_t^*\alpha)'' = (\phi_t^*\mathcal{L}_X\alpha)' = \phi_t^*\big(\mathcal{L}_X(\mathcal{L}_X\alpha)\big), \tag{15.12}$$

so $(\phi_t^*\alpha)''|_{t=0} = \mathcal{L}_X^2\alpha$. Iterating this computation, we find that

$$(\phi_t^*\alpha)^{(i)}|_{t=0} = \mathcal{L}_X^i\alpha, \tag{15.13}$$

so the above expansion becomes

$$\phi_t^*\alpha = \alpha + \sum_{i=1}^{N} \frac{1}{i!}(\mathcal{L}_X^i\alpha)t^i + O(t^{N+1}) \tag{15.14}$$

as $t \to 0$. Again, we see that the ith coefficient differentiates X i times.

Remark 15.2. If we instead use the Riemannian exponential mapping $\phi_t = exp_p(tX_p)$, we can obtain an expansion

$$\phi_t^*\alpha = \alpha + \sum_{i=1}^{N} \frac{1}{i!}(\nabla_X^i \alpha)t^i + O(t^{N+1}) \tag{15.15}$$

where ∇_X is the covariant derivative operator. Now the ith coefficient only contains $(i-1)$ derivatives of X, so this saves us 1 derivative of X!

16 Lecture 16

16.1 Orientability

Another important fact is that we can integrate top-dimensional differential forms on a compact manifold. But we need to recall orientability. First, an orientation on a n-dimensional vector space V is a choice of ordered basis $(v_1, \ldots, v_n)$ with equivalence relation if 2 ordered bases are related by a change of basis matrix with positive determinant. There are exactly 2 such equivalence classes, and if M is a manifold, the oriented double cover of M denoted by $\pi : \tilde{M} \to M$ is the double covering obtained by replacing a point p with the 2 orientations on T_pM (we choose the topology on $\tilde{M}$ which makes π continuous and open).

Definition 16.1. A manifold M is orientable if any of the following equivalent conditions are satisfied.

- M admits an coordinate atlas (U_α, ϕ_α) such that the overlap maps are orientation-preserving $\phi_\alpha \circ \phi_\beta^{-1}$, that is, the Jacobian $(\phi_\alpha \circ \phi_\beta^{-1})_*$ has positive determinant.
- M admits a nowhere-zero n-form.
- The oriented double cover $\tilde{M} \to M$ is trivial, i.e., it has 2 components.

If M is orientable, the choice of one of the components of $\tilde{M}$ is called an *orientation* on M.

Exercise 16.2. Prove that for any smooth manifold M, the orientable double cover $\tilde{M}$ is always orientable.

Let's illustrate Definition 16.1 with an example:

Example 16.3. We observe that the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is orientable. We have several ways to see this:

- We can cover S^{n-1} by 2 coordinate charts (generalized stereographic projection), with intersection $\mathbb{R}^{n-1} \setminus \{0\}$. By changing the orientation of one of these charts, we can arrange that the overlap mapping is orientation-preserving.

- Let ν be the outer unit normal to S^{n-1}, and define $\omega_{S^{n-1}} = \nu \lrcorner (dx^1 \wedge \cdots \wedge dx^n)$. We claim that ω is non-zero at every point. This form is invariant under rotations, so we only need to check this claim at a point. So let $p = (0, \ldots, 0, 1)$ be the north pole, and $\nu_p = \frac{\partial}{\partial x^n}$. Then

$$\omega_p = dx^1 \wedge \cdots \wedge dx^n \Big(\frac{\partial}{\partial x^n}, \cdot, \cdots, \cdot \Big) = (-1)^{n-1} dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{16.1}$$

  But $T_p S^{n-1} = \text{span}\Big\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}} \Big\}$, so ω_p is clearly non-zero at p, therefore it is everywhere non-zero.

- The bundle $\Lambda^{n-1}(T^* S^{n-1})$ is a real line bundle over S^{n-1}. Choosing any Riemannian metric, the unit sphere bundle can be identified with the oriented double cover. The stucture group can be reduced from $GL(1, \mathbb{R}) = \mathbb{R}_*$ to $\mathrm{O}(1, \mathbb{R}) = \{\pm 1\}$. Covering S^{n-1} by 2 charts as in the previous item, the attaching map is a smooth mapping from $\mathbb{R}^{n-1} \setminus \{0\}$ into ± 1, so must be constant if $n > 2$. Therefore any real line bundle over S^{n-1} is trivial for $n > 2$. For $n = 2$, we know the tangent bundle of S^1 is trivial, so it is clearly orientable. Alternatively, we saw previously that the double covers can be identified with $H^1(S^{n-1}, \mathbb{Z}_2)$, but this vanishes if $n > 2$ (actually, S^{n-1} is simply conneted).

Exercise 16.4. Let $\iota : M^{n-1} \to \mathbb{R}^n$ be an embedded submanifold. Assume that $\mathbb{R}^n \setminus \iota(M^{n-1})$ has exactly 2 components. Prove that M^{n-1} is orientable. (Hint: define a non-zero $(n-1)$-form on M^{n-1} similar to the second bullet point in the previous example).

Exercise 16.5. Prove that real projective space $\mathbb{RP}^n$ is orientable if n is odd, but non-orientable if n is even.

16.2 Integration of differential forms

On an oriented n-dimensional manifold, the integral of $\omega \in \Omega^n(M)$ is defined as follows. Choose an oriented coordinate atlas (U_α, ϕ_α). First, assume that $\omega \in \Omega^n(M)$ has compact support in a single coordinate system U_α. Then

$$(\phi_\alpha)_*(\omega) = f dx^1 \wedge \cdots \wedge dx^n, \tag{16.2}$$

where $f : \phi_\alpha(U_\alpha) \to \mathbb{R}$ has compact support. Define

$$\int_M \omega \equiv \int_{\phi_\alpha(U_\alpha)} f dx^1 \ldots dx^n. \tag{16.3}$$

By the change-of-variables formula for integrals and the formula

$$\tilde{\omega}_{1\ldots n} = \det \left(\frac{\partial x^i}{\partial \tilde{x}^j} \right) \omega_{1\ldots n}, \tag{16.4}$$

fro above (14.31), this definition is independent of coordinate system containing the support of ω.

Next, if M is compact, or if ω has compact support, let χ_α be a partition of unity subordinate to $\{U_\alpha\}$, and define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega. \tag{16.5}$$

Since the sum is finite, this definition is independent of the choice of coordinate atlas and choice of partition of unity. To see this, let U_α and V_β be open covers with subordinate partitions of unity ρ_α, χ_β, respectively. Then

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \sum_\beta \chi_\beta \rho_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \omega = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta \omega, \tag{16.6}$$

since $\rho_\alpha \chi_\beta$ is suported in V_β. Therefore

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \sum_\alpha \rho_\alpha \chi_\beta \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega. \tag{16.7}$$

16.3 Stokes' Theorem

Definition 16.6. A manifold with boundary $M = (M \setminus \partial M) \coprod \partial M$, can be covered by usual manifold coordinate charts in the interior $M \setminus \partial M$, together with coordinate charts near points in ∂M of the the form (U_i, ϕ_i), where $\phi_i : U_i \to H^n$, where

$$H^n = \{(x^1, \dots x^n) \in \mathbb{R}^n \mid x^n \geq 0\}. \tag{16.8}$$

is the upper half space in $\mathbb{R}^n$, such that

$$\phi_i : U_i \cap \partial M \to \mathbb{R}^{n-1} \tag{16.9}$$

is a coordinate chart on ∂M viewed as an $(n-1)$-dimensional smooth manifold.

Integration by parts on manifolds is the following.

Theorem 16.7 (Stokes' Theorem for manifolds with boundary)**.** *Let $(M, \partial M)$ be an oriented manifold with boundary of dimension n. If $\omega \in \Omega^{n-1}(M)$ has compact support, then*

$$\int_{\partial M} \omega = \int_M d\omega, \tag{16.10}$$

where the boundary has the orientation induced from the outer normal, i.e., if $v_i \in T_p(\partial M)$, then the ordered basis $(v_1, \dots v_{n-1})$ is oriented if $(v, v_1, \dots, v_{n-1})$ is positively oriented, for any outward pointing normal vector v.

Proof. We first consider forms compactly supported in a coordinate chart (either an interior chart or a boundary chart). Then just consider an $(n-1)$-form of the form

$$\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \tag{16.11}$$

Note that

$$d\omega = (-1)^{i-1}\partial_i f dx^1 \wedge \cdots \wedge dx^n \tag{16.12}$$

If $i < n$, then ω restricted to the boundary is zero, and

$$\int_{H^n} d\omega = (-1)^{i-1} \int_{H^n} \partial_i f dx^1 \cdots dx^n = 0, \tag{16.13}$$

by Fubini's Theorem and the fundamental theorem of calculus, since f has compact support. If $i = n$, then

$$\begin{aligned}
\int_{H^n} d\omega &= (-1)^{n-1} \int_{H^n} \partial_n f dx^1 \cdots dx^n \\
&= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_n f dx^1 \cdots dx^n \\
&= (-1)^n \int_{\mathbb{R}^{n-1}} \omega(x^1, \ldots, x^n, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{\partial H^n} \omega,
\end{aligned} \tag{16.14}$$

since the outward normal is $-e_n$, so $\{-e_n, e_1, \ldots, e_{n-1}\}$ is oriented, which is equivalent to $(-1)^n$ times $\{e_1, \ldots, e_n\}$. In general ω is a sum of n-terms of the above type, so this proves Stokes' Theorem for $\omega \in \Omega^{n-1}(H^n)$ with compact support.

Next, we choose a partition of unity χ_i subordinate to the cover (U_i, ϕ_i), $\phi_i : U_i \to \mathbb{R}^n$, and write $\omega = \sum_i \chi_i \omega$. Let $\omega_i = \chi_i \omega$. Then for each i in the index set, we have

$$\begin{aligned}
\int_M d\omega_i &= \int_{U_i} d\omega_i = \int_{\phi_i^{-1}(U_i)} (\phi_i^{-1})^*(d\omega_i) = \int_{\phi_i^{-1}(U_i)} d(\phi_i^{-1})^*(\omega_i) \\
&= \int_{H^n} d(\phi_i^{-1})^*(\omega_i) = \int_{\partial H^n} (\phi_i^{-1})^*(\omega_i) = \int_{\partial M} \omega_i,
\end{aligned} \tag{16.15}$$

where the last equality holds since $\phi_i|_{\partial M}$ is a coordinate chart on ∂M as a $(n-1)$-dimensional manifold. Finally, we have

$$\int_M d\omega = \int_M d\Big(\sum_i \omega_i\Big) = \sum_i \int_M d\omega_i = \sum_i \int_{\partial M} \omega_i = \int_{\partial M} \sum_i \omega_i = \int_{\partial M} \omega. \tag{16.16}$$

$\square$

17 Lecture 17

Next, we want to give a generalization of Stokes' Theorem.

17.1 Manifolds with corners

Define

$$\overline{\mathbb{R}}^n_+ = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^i \geq 0, i = 1 \ldots n\}. \tag{17.1}$$

Note that

$$\partial\overline{\mathbb{R}}^n_+ = (\overline{\mathbb{R}}^n_+)_{n-1} \cup (\overline{\mathbb{R}}^n_+)_{n-2} \cup \cdots \cup \{0\} \tag{17.2}$$

where $(\overline{\mathbb{R}}^n_+)_k$ is the subset of $\overline{\mathbb{R}}^n_+$ where exactly $n-k$ of the coordinate functions vanish. Points in $(\overline{\mathbb{R}}^n_+)_k$ for $k < n-1$ are called *corner points*.

A *manifold with corners* M is a Hausdorff, second countable space which is locally homeomorphic to a relatively open subset of $\overline{\mathbb{R}}^n_+$. The set of corner points on M is well-defined, see [Lee13, Proposition 16.20].

Given an $(n-1)$-form ω compactly supported in $\overline{\mathbb{R}}^n_+$, we define

$$\int_{\partial\overline{\mathbb{R}}^n_+} \omega = \int_{(\overline{\mathbb{R}}^n_+)_{n-1}} \omega, \tag{17.3}$$

with orientation induced from the outer normal. Let M be a compact n-manifold with corners. If $\omega \in \Omega^{n-1}(M)$ is supported in a single coordinate chart, we define

$$\int_{\partial M} \omega = \int_{\partial\overline{\mathbb{R}}^n_+} (\phi_i^{-1})^*\omega. \tag{17.4}$$

Finally, if $\omega \in \Omega^{n-1}(M)$, let χ_i be a partition of unity subordinate to an atlas U_i, and define

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \chi_i \omega. \tag{17.5}$$

Theorem 17.1 (Stokes' Theorem on manifolds with corners)**.** *Let M be an oriented manifold with corners, and let $\omega \in \Omega^{n-1}(M)$ have compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega \tag{17.6}$$

Proof. The reduction to a form compactly supported in $\overline{\mathbb{R}}^n_+$ is exactly the same as in the proof of Theorem 16.7. So we only consider the case that $\omega \in \Omega^{n-1}(\overline{\mathbb{R}}^n_+)$ which has compact support. We write

$$\omega = \sum_i \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \tag{17.7}$$

then

$$d\omega = \sum_i (-1)^{i-1} \partial_i \omega_i dx^1 \wedge \cdots \wedge dx^n. \tag{17.8}$$

By Fubini's Theorem and the fundamental theorem of calculus, and since f has compact support, for $R > 0$ sufficiently large, we have

$$
\begin{aligned}
\int_{\overline{\mathbb{R}}^n_+} d\omega &= \sum_i (-1)^{i-1} \int_0^R \cdots \int_0^R \partial_i \omega_i dx^1 \cdots dx^n \\
&= \sum_i (-1)^{i-1} \int_0^R \cdots \int_0^R \partial_i \omega_i dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_i (-1)^{i} \int_0^R \cdots \int_0^R \omega_i(x^1, \dots, 0_i, \dots, x^n) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_i \int_{\overline{\mathbb{R}}^n_+ \cap \{x^i = 0\}} \omega = \int_{\partial \overline{\mathbb{R}}^n_+} \omega.
\end{aligned}
\tag{17.9}
$$

Note that we used here that the outward normal to $\overline{\mathbb{R}}^n_+ \cap \{x^i = 0\}$ is $-e_i$, and $\{-e_i, e_1, \dots, \widehat{e_i}, \dots e_n\}$ is orientation equivalent to $(-1)^i$ times $\{e_1, \dots, e_n\}$. $\square$

17.2 de Rham cohomology

Let M be a smooth manifold of dimension n. Since $d^2 = 0$, we have a "cochain" complex

$$
\cdots \xrightarrow{d^{p-2}} \Omega^{p-1}(M) \xrightarrow{d^{p-1}} \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \xrightarrow{d^{p+1}} \cdots . \tag{17.10}
$$

which terminates at $\Omega^n(M)$, where $n = \dim(M)$. Clearly we have that $\mathrm{Image}(d^{p-1}) \subset \mathrm{Ker}(d^p)$, so we can define the following vector spaces.

Definition 17.2. For $0 \leq p \leq n$, the pth de Rham cohomology group is

$$
H^p_{dR}(M) = \frac{\mathrm{Ker}\{d^p : \Omega^p(M) \to \Omega^{p+1}(M)\}}{\mathrm{Image}\{d^{p-1} : \Omega^{p-1}(M) \to \Omega^p(M)\}}. \tag{17.11}
$$

Example 17.3. If $p = 0$, then $H^0_{dR}(M) = \{f : M \to \mathbb{R} \mid df = 0\}$. Since in local coordinates,

$$
df = \sum_{i=1}^n \frac{\partial f}{\partial x^j} dx^j, \tag{17.12}
$$

it follows that f is constant on connected components of M. Consequently, $\dim(H^0_{dR}(M))$ is equal to the number of components of M.

Example 17.4. Let $M = \mathbb{R}^n \setminus \{0\}$, and consider

$$
\omega_{\mathbb{R}^{n-1}} = \frac{1}{r^n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \in \Omega^{n-1}(M). \tag{17.13}
$$

We compute that $d\omega_{\mathbb{R}^{n-1}} = 0$. We claim that $\omega_{\mathbb{R}^{n-1}}$ cannot be written in the form $\omega_{\mathbb{R}^{n-1}} = d\alpha_{n-2}$ for any $\alpha_{n-2} \in \Omega^{n-2}(M)$. To see this, assume by contradiction that this is true. Let

$\iota : S^{n-1} \to \mathbb{R}^n$ be the inclusion of the unit sphere. We showed above that S^{n-1} is orientable, so we can integrate $(n-1)$-forms on S^{n-1}, once we choose an orientation. Then

$$\int_{S^{n-1}} \iota^* \omega_{\mathbb{R}^{n-1}} = \int_{S^{n-1}} \iota^* d\alpha_{n-2} \int_{S^{n-1}} d\iota^* \alpha_{n-2} = 0, \tag{17.14}$$

by Stokes' Theorem 16.7. However,

$$\iota^* \omega_{\mathbb{R}^{n-1}} = \omega_{S^{n-1}}, \tag{17.15}$$

which we defined above, is non-zero at every point, so the integral must be non-zero. This contradiction proves 2 things:

$$H_{dR}^{n-1}(\mathbb{R}^n \setminus \{0\}) \neq \{0\} \tag{17.16}$$
$$H_{dR}^{n-1}(S^{n-1}) \neq \{0\}. \tag{17.17}$$

Note the latter part of this example proves the following.

Proposition 17.5. *Let M be a compact oriented n-dimensional manifold. Then*

$$H_{dR}^n(M) \neq 0. \tag{17.18}$$

Proof. If M is oriented, then we know there exists a nowhere-zero $\omega \in \Omega^n(M)$ which determines the orientation. In any oriented coordinate system (U, ϕ), we have $\phi_* \omega = f dx^1 \wedge \cdots \wedge dx^n$ where $f > 0$. Therefore we must have

$$\int_M \omega > 0. \tag{17.19}$$

If $\omega = d\alpha_{n-1}$ for $\alpha \in \Omega^{n-1}(M)$, then Stokes' Theorem would say that

$$\int_M \omega = \int_M d\alpha_{n-1} = \int_{\partial M} \alpha_{n-1} = 0, \tag{17.20}$$

since $\partial M = \emptyset$. □

17.3 Simply-connected manifolds

Definition 17.6. We say that M is simply connected if every piecwise smooth path bounds a smoothly embedded disc (embedded as a manifold-with-corners).

Proposition 17.7. *If M is simply connected then $H_{dR}^1(M) \cong \{0\}$.*

Proof. Let $\omega \in \Omega^1(M)$ with $d\omega = 0$, and fix a point $p \in M$. For any $q \in M$, choose a smoothly embedded path $\gamma : [0,1] \to M$ with $\gamma(0) = 0$ and $\gamma(1) = q$. Then γ is a 1-chain, and define

$$f(q) = \int_\gamma \omega = \int_0^1 \gamma^* \omega. \tag{17.21}$$

This yields a well-defined function $f : M \to \mathbb{R}$. To see this, if $\tilde{\gamma}$ is another smoothly embedded path from p to q, then define

$$\overline{\gamma} = \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \tilde{\gamma}(2-2t) & t \in [1/2, 1] \end{cases} \tag{17.22}$$

Then $\overline{\gamma}$ is a piecwise smooth path, and by assumption, $\overline{\gamma}$ is the boundary of an embedded disc $\iota : D \to M$, as a manifold-with-corners. By Stokes' Theorem for chains, we have

$$\int_{\overline{\gamma}} \omega = \int_D \iota^* d\omega = 0, \tag{17.23}$$

which implies that f is well-defined.

To see that $df = \omega$, without loss of generality, we can assume that we are in a domain in $\mathbb{R}^n$, and write $\omega = \omega_i dx^i$. Then

$$df(\partial_i) = \partial_i f = \frac{d}{ds}\Big(\int_{\gamma_s} \gamma_s^* \omega \Big)\Big|_{s=0}, \tag{17.24}$$

where $\gamma_s(t) = \gamma(t) + ste_i$ is the path $\gamma_s : [0,1] \to M$ satisfying $\gamma_s(0) = p$ and $\gamma_s(t) = q + se_i$. Then

$$\begin{aligned} \gamma_s^* \omega &= \omega_j(\gamma_s(t)) d(x^j \circ \gamma_s(t)) \\ &= \omega_j(\gamma_s(t)) d(\gamma^j(t) + st\delta_i^j) \\ &= \omega_j(\gamma_s(t))(\gamma^j)'(t)dt + \omega_i(\gamma_s(t))sdt. \end{aligned} \tag{17.25}$$

Then we have

$$\begin{aligned} \frac{d}{ds}\Big(\int_{\gamma_s} \gamma_s^* \omega \Big)\Big|_{s=0} &= \frac{d}{ds}\Big(\int_0^1 \Big(\omega_j(\gamma_s(t))(\gamma^j)'(t)dt + \omega_i(\gamma_s(t))s \Big) dt \Big)\Big|_{s=0} \\ &= \int_0^1 \frac{d}{ds}\Big(\omega_j(\gamma_s(t))(\gamma^j)'(t) + \omega_i(\gamma_s(t))s \Big)\Big|_{s=0} dt \\ &= \int_0^1 \frac{d}{ds}\Big(\omega_j(\gamma_s(t))(\gamma^j)'(t) \Big)\Big|_{s=0} dt + \int_0^1 \frac{d}{ds}\Big(\omega_i(\gamma_s(t))s \Big)\Big|_{s=0} dt \\ &= \int_0^1 \Big(\frac{\partial}{\partial x^k} \omega_j)(\gamma_s(t)) \frac{d}{ds}(\gamma_s^k(t))\Big|_{s=0} (\gamma^j)'(t)dt + \int_0^1 \omega_i(\gamma(t))dt. \end{aligned} \tag{17.26}$$

Since $d\omega = 0$, we have

$$0 = d(\omega_j dx^j) = \frac{\partial \omega_j}{\partial x^k} dx^k \wedge dx^j, \tag{17.27}$$

which implies that

$$\frac{\partial \omega_j}{\partial x^k} = \frac{\partial \omega_k}{\partial x^j}. \tag{17.28}$$

Then we have

$$\begin{aligned}
\frac{d}{ds}\Big(\int_{\gamma_s}\gamma_s^*\omega\Big)\Big|_{s=0} &= \int_0^1\Big(\frac{\partial}{\partial x^j}\omega_k)(\gamma(t))t\delta_i^k(\gamma^j)'(t)dt + \int_0^1 \omega_i(\gamma(t))dt \\
&= \int_0^1\Big(\frac{\partial}{\partial x^j}\omega_i)(\gamma(t))t(\gamma^j)'(t)dt + \int_0^1 \omega_i(\gamma(t))dt \\
&= \int_0^1 t\frac{d}{dt}(\omega_i(\gamma(t)))dt + \int_0^1 \omega_i(\gamma(t))dt \\
&= t\omega_i(\gamma(t))\Big|_0^1 - \int_0^1 \omega_i(\gamma(t))dt + \int_0^1 \omega_i(\gamma(t))dt \\
&= \omega_i(q) = \omega_q\Big(\frac{\partial}{\partial x^i}\Big),
\end{aligned} \tag{17.29}$$

which proves that $df = \omega$. $\square$

Example 17.8. Consider $S^n \subset \mathbb{R}^{n+1}$. A piecewise smooth loop $\gamma : [0,1] \to S^n$ must miss a point p. So we can view $\gamma : [0,1] \to S^n \setminus \{p\} = \mathbb{R}^n$. Every piecwise smooth loop in $\mathbb{R}^n$ bounds a disc embedded in $\mathbb{R}^n$ if $n > 1$, so we conclude that S^n is simply-connected, and thus $H^1(S^n, \mathbb{R}) = \{0\}$ for $n \geq 2$.

18 Lecture 18

18.1 Finite group quotients

Let M be a smooth manifold, and Γ be a finite group. A left action of Γ on M is a smooth mapping

$$A : \Gamma \times M \to M \tag{18.1}$$

satisfying

$$A(g_1 g_2, p) = A(g_1, A(g_2, p)) \tag{18.2}$$
$$A(e, p) = p \text{ for all} p \in M. \tag{18.3}$$

For each $g \in \Gamma$, then mapping $A_g : M \to M$ is a diffeomorphism since it has inverse $A_{g^{-1}}$. Also, $A(e,p) = p$ for all $p \in M$, where e is the identity element of Γ.

Definition 18.1. The action A is *free* if $A(g,p) = p$ for some $p \in M$ implies that $g = e$.

We define the quotient space M/Γ as the set of equivalence classes $[p]$ where the equivalence relation is $p_1 \sim p_2$ if there exist $g \in \Gamma$ such that $A(g, p_1) = p_2$.

Proposition 18.2. *If the action is free, then the quotient space M/Γ is a manifold. Furthermore, $\pi : M \to M/\Gamma$ is a covering space of order $|\Gamma|$ with deck transformation group Γ.*

Proof. We will leave the details of this as an optional exercise. $\square$

Proposition 18.3. *Let Γ be a free action of a finite group on a simply-connected manifold M. Then $H^1_{dR}(M/\Gamma) = \{0\}$.*

Proof. Let $\omega \in \Omega^1(M/\Gamma)$ such that $d\omega = 0$, and let $\pi : M \to M/\Gamma$ denote the projection mapping. Then $\pi^*\omega \in \Omega$ satisfies $d\pi^*\omega = \pi^* d\omega = 0$. From Proposition 17.7, there exists $f : M \to \mathbb{R}$ such that $\pi^*\omega = df$. New we average over the group action, that is, define

$$\tilde{f} = \frac{1}{\Gamma} \sum_{g \in \Gamma} f \circ A_g. \tag{18.4}$$

Then clearly $\tilde{f}$ descends to a function $\tilde{f} : M/\Gamma \to \mathbb{R}$. We have that

$$A_g^* \pi^* \omega = (\pi \circ A_g)^* \omega = \pi^* \omega. \tag{18.5}$$

Then $\pi^*\omega = df$ implies

$$|\Gamma| \pi^* \omega = \sum_{g \in \Gamma} A_g^* \pi^* \omega = \sum_{g \in \Gamma} A_g^* df = \sum_{g \in \Gamma} d(A_g^* f) = d\Big(\sum_{g \in \Gamma} f \circ A_g \Big), \tag{18.6}$$

which is equivalent to

$$\pi^* \omega = d\tilde{f} \tag{18.7}$$

Now this equation is invariant under the action of Γ, which proves that $\omega = d\tilde{f}$ downstairs, so $H^1_{dR}(M) = \{0\}$. $\square$

Corollary 18.4. *We have that $H^1_{dR}(\mathbb{RP}^n) = \{0\}$ for $n \geq 2$.*

18.2 Diffeomorphism invariance

Note that

$$H^*_{dR}(M) \equiv \bigoplus_{p=0}^{n} H^p_{dR}(M) \tag{18.8}$$

has an algebra structure induced by the wedge product. To see this, for $[\alpha] \in H^p_{dR}(M)$ and $[\beta] \in H^q_{dR}(M)$, represented by $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, we have that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0, \tag{18.9}$$

so we define

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]. \tag{18.10}$$

To see that this is well-defined, we have

$$(\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d\gamma \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta), \tag{18.11}$$

since β is closed, so

$$[(\alpha + d\gamma) \wedge \beta] = [\alpha \wedge \beta]. \tag{18.12}$$

Well-definedness in the other factor is similar, or just use the skew-symmetry property of the wedge product. Therefore we have

$$\wedge : H^p_{dR}(M) \otimes H^q_{dR}(M) \to H^{p+q}_{dR}(M). \tag{18.13}$$

Note that from Proposition 8.1, we have

$$[\alpha] \wedge [\beta] = (-1)^{pq}[\beta] \wedge [\alpha]. \tag{18.14}$$

Next, let $f : X \to Y$ be a smooth mapping between smooth manifolds. As discussed before, we have a pullback operation on differential forms, $f^* : \Omega^*(Y) \to \Omega^*(X)$, which makes the following diagram commute

$$\begin{array}{ccc} \Omega^p(Y) & \xrightarrow{d^p_Y} & \Omega^{p+1}(Y) \\ \Big\downarrow{\scriptstyle (f^*)^p} & & \Big\downarrow{\scriptstyle (f^*)^{p+1}} \\ \Omega^p(X) & \xrightarrow{d^p_X} & \Omega^{p+1}(X). \end{array} \tag{18.15}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The de Rham cohomology algebra is a diffeomorphism invariant.

Corollary 18.5. *If $f : X \to Y$ then there are induced mappings*

$$(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X). \tag{18.16}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{18.17}$$

Consequently, if X and Y are diffeomorphic, then $H^p_{dR}(X) \cong H^p_{dR}(Y)$ for every $p \geq 0$, and moreover, the cohomology algebras are isomorphic $H^*_{dR}(X) \cong H^*_{dR}(Y)$.

Proof. We first note that any smooth mapping $f : X \to Y$ induces a well-defined mapping on cohomology $(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X)$ by the following. If $[\alpha] \in H^p_{dR}(Y)$ is represented by a form α, such that $d^p_Y \alpha = 0$, then we have

$$d^p_X (f^*)^p \alpha = (f^*)^{p+1} d^p_Y \alpha = (f^*)^{p+1} 0 = 0, \tag{18.18}$$

so we can define $f^*[\alpha] = [f^*\alpha]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$(f^*)^p(\alpha + d^{p-1}_Y \beta) = (f^*)^p \alpha + (f^*)^p d^{p-1}_Y \beta = (f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1} \beta, \tag{18.19}$$

so $[(f^*)^p(\alpha + d^p_Y \beta)] = [(f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1} \beta] = [(f^*)^p \alpha]$.

If f is a diffeomorphism, then f^{-1} exists and is smooth, so we have

$$f \circ f^{-1} = id_Y, \quad f^{-1} \circ f = id_X, \tag{18.20}$$

and from Proposition 7.3, the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^*_{dR}(X)}, \quad (f^{-1})^* \circ f^* = id_{H^*_{dR}(Y)}, \tag{18.21}$$

Finally, since $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$, together these mappings form an algebra homomorphism on cohomology algebras, which will be an algebra isomorphism if X and Y are diffeomorphic. □

19 Lecture 19

Last time, we showed that de Rham cohomology is a diffeomorphism invariant. Today, we will show that it is moreover a homotopy invariant.

19.1 The Poincaré Lemma

Let M be a smooth manifold, possibly noncompact, and let $N = M \times [0,1]$, which is an $(n+1)$-dimensional manifold with boundary. Let $\pi : N \to M$ be the projection $\pi(x,t) = x$. Also, let $\iota_t : M \to M \times [0,1]$ be the inclusion $\iota_t(x) = (x,t)$. Define a mapping

$$I^k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M) \tag{19.1}$$

by the following. Since

$$T_{(p,t)}N = \mathrm{Ker}(\tilde{\pi}_*)_{(p,t)} \oplus \mathrm{Ker}(\pi_*)_{(p,t)} \cong T_pM \oplus T_t[0,1] \tag{19.2}$$

where $\tilde{\pi}(x,t) = t$, any k-form on N can be uniquely written as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)dt \wedge (\pi^*\phi_{k-1}), \tag{19.3}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0(N)$. The mapping I^k is then defined by

$$I^k(\omega) = \Big(\int_0^1 f(x,t)dt \Big) \phi_{k-1}. \tag{19.4}$$

Proposition 19.1. *For $\omega \in \Omega^k(N)$, we have*

$$(\iota_1)^*\omega - (\iota_0)^*\omega = d_M I^k \omega + I^{k+1} d_N \omega. \tag{19.5}$$

Proof. Writing ω in the form (19.3), since $\iota_t^* dt = 0$, and $\pi \circ \iota_t = id_M$, the left hand side of (19.5) is

$$\begin{aligned}(\iota_1)^*\omega - (\iota_0)^*\omega &= (\iota_1)^* h(x,t)\pi^*\phi_k - (\iota_0)^* h(x,t)\pi^*\phi_k \\ &= (h(x,1) - h(x,0))\phi_k.\end{aligned} \tag{19.6}$$

Next, assume that ω is just of the form

$$\omega = h(x,t)\pi^* \phi_k. \tag{19.7}$$

Then, choosing a local coordinate system $\{x^i\}$ on M near any point, we have

$$d_N \omega = \Big(\sum_{i=1}^{m} \frac{\partial h}{\partial x^i} dx^i + \frac{\partial h}{\partial t} dt \Big) \wedge \pi^* \phi_k + h(x,t) \pi^* d_M \phi_k. \tag{19.8}$$

By the definition of I^k, we have $I^k \omega = 0$, so obviously

$$d_M I^k \omega = 0, \tag{19.9}$$

and

$$\begin{aligned} I^{k+1} d_N \omega &= I^{k+1} \Big\{ \Big(\sum_{i=1}^{n} \frac{\partial h}{\partial x^i} dx^i + \frac{\partial h}{\partial t} dt \Big) \wedge \pi^* \phi_k + h(x,t) \pi^* d_M \phi_k \Big\} \\ &= I^{k+1} \Big\{ \frac{\partial h}{\partial t} dt \wedge \pi^* \phi_k \Big\} = \Big(\int_0^1 \frac{\partial h}{\partial t} dt \Big) \phi_k = (h(x,1) - h(x,0)) \phi_k. \end{aligned} \tag{19.10}$$

So the proposition holds for forms of this type.

Next, assume that ω is just of the form

$$\omega = f(x,t) dt \wedge (\pi^* \phi_{k-1}). \tag{19.11}$$

From (19.6) above, we have

$$(\iota_1)^* \omega - (\iota_0)^* \omega = 0. \tag{19.12}$$

Note that

$$\begin{aligned} d_N \omega &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dt \wedge (\pi^* \phi_{k-1}) - f(x,t) dt \wedge \pi^* (d_M \phi_{k-1}) \\ &= - \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dt \wedge \pi^* \Big(dx^i \wedge \phi_{k-1} \Big) - f dt \wedge \pi^* \Big(d_M \phi_{k-1} \Big). \end{aligned} \tag{19.13}$$

So by definition of I^{k+1} and (19.13), we have

$$I^{k+1} d_N \omega = - \Big(\sum_{i=1}^{n} \int_0^1 \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} - \Big(\int_0^1 f dt \Big) d_M \phi_{k-1}. \tag{19.14}$$

Next, by defintion of I^k,

$$\begin{aligned} d_M I^k \omega &= d_M \Big\{ \Big(\int_0^1 f(x,t) dt \Big) \phi_{k-1} \Big\} \\ &= \Big(\int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} + \Big(\int_0^1 f dt \Big) d_M \phi_{k-1}. \end{aligned} \tag{19.15}$$

Consequently, on forms of this type, we have

$$d_M I^k \omega + I^{k+1} d_N \omega = 0. \tag{19.16}$$

So the proposition is true for forms of the second type. By linearity, the proposition holds for all forms, and we are done. □

Remark 19.2. Note that we used a coordinate system in the above proof. This is OK since these are local expressions of global quantities, so the local identity therefore implies the global identity.

19.2 Homotopy invariance of de Rham cohomology

Definition 19.3. Let X and Y be smooth manifolds. Smooth mappings $f, g : X \to Y$ are said to be smoothly homotopic if there exists a smooth mapping $F : X \times [0,1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Proposition 19.4. *Let X and Y be smooth manifolds. If $f, g : X \to Y$ are smoothly homotopic then*

$$f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X) \tag{19.17}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x, t)$, and note that

$$(\iota_t)^* : \Omega^*(X \times [0,1]) \to \Omega^*(X). \tag{19.18}$$

In Proposition 19.1, we constructed

$$I^k : \Omega^k(X \times [0,1]) \to \Omega^{k-1}(X) \tag{19.19}$$

satisfying

$$(\iota_1)^* - (\iota_0)^* = I^{k+1} d_{X\times[0,1]} + d_X I^k. \tag{19.20}$$

This clearly implies that as mappings on de Rham cohomology

$$(\iota_0)^* = (\iota_1)^* : H^k_{dR}(X \times [0,1]) \to H^k_{dR}(X). \tag{19.21}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, we have

$$f^* = (\iota_0)^* \circ F^*, \quad g^* = (\iota_1)^* \circ F^*, \tag{19.22}$$

therefore $f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X)$. □

Definition 19.5. Smooth manifolds X and Y have the same smooth homotopy type if there exist smooth mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is smoothly homotopic to Id_X and $f \circ g$ is smoothly homotopic to id_Y.

Corollary 19.6. *If X and Y have the same smooth homotopy type, then $H^*_{dR}(X) \cong H^*_{dR}(Y)$.*

Proof. From Proposition 33.2, we have

$$f^* \circ g^* = Id_{H^*_{dR}(X)} \tag{19.23}$$
$$g^* \circ f^* = Id_{H^*_{dR}(Y)}, \tag{19.24}$$

so f^* and g^* are isomorphisms. □

Some special cases of this are the following.

Definition 19.7. A smooth manifold X is smoothly contractible if X has the same smooth homotopy type as a point.

Corollary 19.8. *If X is smoothly contractible, then*

$$H^k_{dR}(X) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{19.25}$$

Proof. By definition, there is a mapping $f : X \to \{p\}$ and a mapping $g : \{p\} \to X$ so that $g \circ f$ is homotopic to Id_X. This is equivalent to the existence of $H : X \times [0,1] \to X$ so that $H(x,1) = x$ for all $x \in X$ and $H(x,0) = x_0$ where $x_0 = g(p)$. We already know that $H^0_{dR}(X) = \mathbb{R}$, so let $k \geq 1$, and $\omega \in \Omega^k(X)$ such that $d_X\omega = 0$. Plugging in $H^*\omega$ into the Poincaré Lemma yields

$$\begin{aligned}(\iota_1)^*H^*\omega - (\iota_0)^*H^*\omega &= (H \circ \iota_1)^*\omega - (H \circ \iota_0)^*\omega \\ &= I^{k+1}d_{X\times[0,1]}H^*\omega + d_X I^k H^*\omega = d_X I^k H^*\omega,\end{aligned} \tag{19.26}$$

because ω is closed and d commutes with pullback. However, $H \circ \iota_1 = Id_X$ and $H \circ \iota_0$ is a constant map, therefore we have

$$\omega = d_X I^k H^*\omega, \tag{19.27}$$

so ω is exact. □

Exercise 19.9. A domain $A \subset \mathbb{R}^n$ is star-shaped if there exists a $p \in A$ such that for any $x \in A$, the line segment between p and x is contained in A. In this case, let $H : A\times[0,1] \to \mathbb{R}^n$ be the mapping $H(x,t) = tx + (1-t)p$. This shows that A is (smoothly) contractible to a point, so A has the same de Rham cohomology groups as a point. Show that the Poincaré Lemma gives the explicit formula as follows. Without loss of generality, we can assume that $p = \{0\}$. Writing $\omega \in \Omega^k(A)$ for $k \geq 1$ with $d\omega = 0$ as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{19.28}$$

then

$$\gamma \equiv \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \Big(\int_0^1 t^{k-1}\omega_{i_1\cdots i_k}(tx)dt \Big) x^{i_\alpha} dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_k} \tag{19.29}$$

is an explicit $(k-1)$-form solving $d\gamma = \omega$. (Hint: $\gamma = I^k H^*\omega$.)

Definition 19.10. A submanifold $i : A \hookrightarrow X$ is a smooth deformation retraction of X if there exists a smooth mapping $r : X \to X$ such that

$$r \circ i = id_A, \tag{19.30}$$

and $i \circ r$ is smoothly homotopic to Id_X.

Corollary 19.11. *If A is a smooth deformation retraction of X then*

$$H^k_{dR}(A) \cong H^k_{dR}(X), \tag{19.31}$$

for all $k \geq 0$.

Example 19.12. Consider $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \subset \mathbb{R}^n$ given by $r(x) = x/|x|$. The mapping $F(x,t) = (1-t)x + t(x/|x|)$ is a smooth homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, so S^{n-1} is a smooth deformation retraction of $\mathbb{R}^n \setminus \{0\}$ and we therefore have

$$H^k_{dR}(S^{n-1}) = H^k_{dR}(\mathbb{R}^n \setminus \{0\}). \tag{19.32}$$

From earlier, we know that this vector space is non-zero when $k = n-1$, and our next task is to prove that this cohomology group is 1-dimensional (and determine all of the other groups as well).

20 Lecture 20

20.1 Poincaré Lemma for cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega^{p-1}_c(M) \xrightarrow{d} \Omega^p_c(M) \xrightarrow{d} \Omega^{p+1}_c(M) \xrightarrow{d} \cdots, \tag{20.1}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Lemma 20.1. *Let M be a differentiable n-manifold, then for $k \geq 1$,*

$$H^k_{c,dR}(M \times \mathbb{R}) \cong H^{k-1}_{c,dR}(M). \tag{20.2}$$

Proof. First, we define a mapping "integration over the fiber" by

$$\pi_* : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M) \tag{20.3}$$

by the following. Any k-form on $N = M \times \mathbb{R}$ can be written as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{20.4}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0_c(M \times \mathbb{R})$. Define

$$\pi_*(\omega) = \Big(\int_{-\infty}^{\infty} f(x,t) dt \Big) \phi_{k-1}, \tag{20.5}$$

noting that the integral is defined because ω is assumed to have compact support, and this form has compact support since f has compact support.

We claim that

$$d_M \circ \pi_* = \pi_* \circ d_{M \times \mathbb{R}}, \tag{20.6}$$

To see this, the left hand side of (20.6) is

$$\begin{aligned} d_M \circ \pi_* \omega &= d_M \Big(\Big(\int_{-\infty}^{\infty} f(x,t) dt \Big) \phi_{k-1} \Big) \\ &= \Big(\int_{-\infty}^{\infty} \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(x,t) dt \Big) d_M \phi_{k-1}. \end{aligned} \tag{20.7}$$

The right hand side of (20.6) is

$$\begin{aligned} \pi_* \circ d_N \omega &= \pi_* \Big(\frac{\partial h}{\partial t} dt \wedge \pi^* \phi_k + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge \pi^* \phi_{k-1} \wedge dt + f(x,t) \pi^* (d_M \phi_{k-1}) \wedge dt \Big) \\ &= \pi_* \Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge \pi^* \phi_{k-1} \wedge dt + f(x,t) \pi^* (d_M \phi_{k-1}) \wedge dt \Big) \\ &= \Big(\int_{-\infty}^{\infty} \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(x,t) dt) \Big) d_M \phi_{k-1}, \end{aligned} \tag{20.8}$$

since the term involving h is zero because h has compact support, and using the fundamental theorem of calculus. Therefore π_* induces a mapping

$$\pi_* : H^k_{c,dR}(M \times \mathbb{R}) \to H^{k-1}_{c,dR}(M). \tag{20.9}$$

Next, we choose $e \in \Omega^1_c(\mathbb{R})$ with $\int_R e = 1$, and define

$$e_* : \Omega^k_c(M) \to \Omega^{k+1}_c(M \times \mathbb{R}) \tag{20.10}$$

by

$$e_*(\omega) = (\pi^* \omega) \wedge e. \tag{20.11}$$

We have that

$$d_{M \times \mathbb{R}} \circ e_* = e_* \circ d_M, \tag{20.12}$$

because

$$d_N \circ e_*(\omega) = d_N(\pi^* \omega \wedge e) = (d_N \pi^* \omega) \wedge e = \pi^*(d_M \omega) \wedge e = e_* \circ d_M(\omega). \tag{20.13}$$

Therefore e_* induces a mapping

$$e_* : H^k_{c,dR}(M) \to H^{k+1}_{c,dR}(M \times \mathbb{R}). \tag{20.14}$$

Let us write $e = \chi dt$, then

$$\pi_* \circ e_*(\omega) = \pi_*\Big(\chi(t)(\pi^*\omega) \wedge dt\Big) = \Big(\int_{-\infty}^{\infty} \chi(t)dt\Big)\omega = \omega \tag{20.15}$$

Therefore, we have $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$, so $\pi_* \circ e_* = 1$ on $H^k_{c,dR}(M)$.

We next claim that $e_* \circ \pi_* = 1$ on $H^k_{c,dR}(M \times \mathbb{R})$. To see this, writing $\omega \in \Omega^k_c(N)$ as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{20.16}$$

define a mapping

$$K : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R}) \tag{20.17}$$

by

$$K(\omega) = \pi^*\phi_{k-1}\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big). \tag{20.18}$$

Note that the right hand side is indeed a $(k-1)$-form on $M \times \mathbb{R}$ with compact support, which is clear if t is sufficiently large. We claim that if $\omega \in \Omega^k_c(M \times \mathbb{R})$ then

$$(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{20.19}$$

which we will separately verify for forms of type $\omega = h(x,t)\pi^*\phi_k$, and for forms of type $\omega = f(x,t)\pi^*\phi_{k-1} \wedge dt$.

For forms of the first type, we obviously have

$$(1 - e_*\pi_*)h(x,t)\pi^*\phi_k = h(x,t)\pi^*\phi_k. \tag{20.20}$$

On the other hand, since K is zero on forms of this type,

$$\begin{aligned}
(dK - Kd)(h(x,t)\pi^*\phi_k) &= -K\Big(\Big(\frac{\partial h}{\partial x}\Big)dx \wedge \pi^*\phi_k + \Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k + h(x,t)\pi^* d\phi_k\Big)\\
&= -K\Big(\Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k\Big)\\
&= (-1)^{k-1}K\Big(\Big(\frac{\partial h}{\partial t}\Big)(\pi^*\phi_k) \wedge dt\Big)\\
&= (-1)^{k-1}\pi^*\phi_k\Big(\int_{-\infty}^{t} \frac{\partial h}{\partial t}ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} \frac{\partial h}{\partial t}ds\Big)\\
&= (-1)^{k-1}(\pi^*\phi_k)h(x,t).
\end{aligned} \tag{20.21}$$

For forms of the second type $\omega = f(x,t)\pi^*\phi_{k-1} \wedge dt$, we have

$$
\begin{aligned}
(1 - e_*\pi_*) f(x,t)\pi^*\phi_{k-1} \wedge dt &= f(x,t)\pi^*\phi_{k-1} \wedge dt - \Big(\int_{-\infty}^{\infty} f(x,t)dt\Big)(\pi^*\phi_{k-1}) \wedge e \\
&= \pi^*\phi_{k-1} \wedge \Big(f(x,t)dt - \Big(\int_{-\infty}^{\infty} f(x,t)dt\Big)e\Big) \\
&= \Big(f(x,t) - \Big(\int_{-\infty}^{\infty} f(x,t)dt\Big)\chi(t)\Big)\pi^*\phi_{k-1} \wedge dt
\end{aligned}
\tag{20.22}
$$

Next,

$$
\begin{aligned}
d_N K\omega &= d_N\Big(\pi^*\phi_{k-1}\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big)\Big) \\
&= \pi^*(d_M\phi_{k-1})\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big) \\
&\quad + (-1)^{k-1}\pi^*\phi_{k-1}\sum_{i=1}^{n}\Big(\int_{-\infty}^{t}\frac{\partial f}{\partial x^i}(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}(x,s)ds\Big) \wedge dx^i \\
&\quad + (-1)^{k-1}\pi^*\phi_{k-1}\Big(f(x,t)dt - e\int_{-\infty}^{\infty} f(x,s)ds\Big).
\end{aligned}
\tag{20.23}
$$

We compute

$$
\begin{aligned}
K d_N\omega &= K\Big(\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i \wedge \pi^*\phi_{k-1} + f(x,t)\pi^* d_M\phi_{k-1}\Big) \wedge dt \\
&= \sum_{i=1}^{n} K\Big(\frac{\partial f}{\partial x^i}dx^i \wedge \pi^*\phi_{k-1} \wedge dt\Big) + K\Big(f(x,t)\pi^* d_M\phi_{k-1} \wedge dt\Big) \\
&= \sum_{i=1}^{n}\pi^*(dx^i \wedge \phi_{k-1})\Big(\int_{-\infty}^{t}\frac{\partial f}{\partial x^i}(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}(x,s)ds\Big) \\
&\quad + \pi^*(d_M\phi_{k-1})\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big).
\end{aligned}
\tag{20.24}
$$

Adding together (20.23) and (20.24) and using (20.22), we obtain

$$
(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{20.25}
$$

which finishes the proof of the claim.

The claim implies that $e_* \circ \pi_* = 1$ as a mapping on $H^k_{c,dR}(M \times \mathbb{R})$, and the Poincaré Lemma for compactly supported cohomology follows. $\square$

Corollary 20.2. *We have*

$$
H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{20.26}
$$

and a generator for $H^n_{c,dR}(\mathbb{R}^n)$ is given by any compactly supported n-form μ with $\int_{\mathbb{R}^n}\mu = 1$.

Proof. We start with $M = \{p\}$ a single point. From above, we have an isomorphism

$$\mathbb{R} \cong H^0_{c,dR}(\{p\}) \cong H^1_{c,dR}(\mathbb{R}). \tag{20.27}$$

Furthermore, since the isomorphism is given by e_*, the proof shows that a generator of the left hand side is $\chi(x^1)dx^1$. Next, we have

$$H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}, \tag{20.28}$$

and a generator of the left hand side is $\chi(x^1)dx^1 \wedge \chi(x^2)dx^2$. In general, a generator is given by

$$\chi(x^1) \cdots \chi(x^n) dx^1 \wedge \cdots \wedge dx^n. \tag{20.29}$$

Next, we use the fact that π_* is an isomorphism. The isomorphism

$$H^1_{c,dR}(\mathbb{R}) \cong H^0_{c,dR}(\{p\}) \cong \mathbb{R} \tag{20.30}$$

is given by

$$\phi_1 \mapsto \int_{\mathbb{R}} \phi_1 dx^1. \tag{20.31}$$

Then the isomorphism

$$H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}, \tag{20.32}$$

is given by

$$f(x^1, x^2) dx^1 \wedge dx^2 \mapsto \left(\int_{\mathbb{R}} f(x^1, x^2) dx^2 \right) dx^1. \tag{20.33}$$

Composing these isomorphisms and using Fubini's Theorem, we get

$$f(x^1, x^2) dx^1 \wedge dx^2 \mapsto \int_{\mathbb{R}^2} f(x^1, x^2) dx^1 \wedge dx^2. \tag{20.34}$$

In general, the isomorphism is given by

$$f(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n \mapsto \int_{\mathbb{R}^n} f(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n. \tag{20.35}$$

□

Remark 20.3. This shows that $H^*_{c,dR}(M)$ is not a homotopy invariant, since (20.26) is not the same as the cohomology of a point. But of course, $H^*_{c,dR}(M)$ *is* a diffeomorphism invariant.

21 Lecture 21

21.1 Exact sequences of cochain complexes

Definition 21.1. A sequence of vector spaces A, B, C, with linear mappings $\alpha : A \to B$, $\beta : B \to C$

$$0 \xrightarrow{0} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{0} 0 \tag{21.1}$$

is called *exact* if the kernel of each mapping is equal to the image of the previous mapping. That is $Ker(\alpha) = \{0\}$ if and only if α is injective. Next, $Ker(\beta) = Im(\alpha)$. Finally, $Im(\beta) = C$, if and only if β is surjective.

Let C_i be a co-complex of vector spaces for $i = 1, 2, 3$.

$$\cdots \xrightarrow{d_i^{p-2}} C_i^{p-1} \xrightarrow{d_i^{p-1}} C_i^p \xrightarrow{d_i^p} C_i^{p+1} \xrightarrow{d_i^{p+1}} \cdots . \tag{21.2}$$

with $d^2 = 0$. A morphism from C_i to C_j are mappings $\alpha^k : C_i^k \to C_j^k$ such that the following diagram commutes for every p

$$\begin{array}{ccc} C_i^p & \xrightarrow{d_i^p} & C_i^{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ C_j^p & \xrightarrow{d_j^p} & C_j^{p+1} \end{array} \tag{21.3}$$

For co-complexes C_1, C_2, C_3, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of co-complexes is exact if

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{21.4}$$

if the sequence

$$0 \xrightarrow{0} C_1^p \xrightarrow{\alpha^p} C_2^p \xrightarrow{\beta^p} C_3^p \xrightarrow{0} 0 \tag{21.5}$$

is exact for every p.

Lemma 21.2 (The zig-zag lemma for cochain complexes)**.** *If*

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{21.6}$$

is a short exact sequence of co-complexes, then there exist connecting homomorphisms

$$\delta^p : H^p(C_3) \to H^{p+1}(C_1) \tag{21.7}$$

for every p such that the sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\delta^p} H^{p+1}(C_1) \longrightarrow \cdots \tag{21.8}$$

is exact.

Proof. We look at the huge commutative diagram

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_1^{p-1} \arrow[r, "\alpha^{p-1}"] \arrow[d, "d_1^{p-1}"] & C_2^{p-1} \arrow[r, "\beta^{p-1}"] \arrow[d, "d_2^{p-1}"] & C_3^{p-1} \arrow[r] \arrow[d, "d_3^{p-1}"] & 0 \\
0 \arrow[r] & C_1^{p} \arrow[r, "\alpha^{p}"] \arrow[d, "d_1^{p}"] & C_2^{p} \arrow[r, "\beta^{p}"] \arrow[d, "d_2^{p}"] & C_3^{p} \arrow[r] \arrow[d, "d_3^{p}"] & 0 \\
0 \arrow[r] & C_1^{p+1} \arrow[r, "\alpha^{p+1}"] \arrow[d] & C_2^{p+1} \arrow[r, "\beta^{p+1}"] \arrow[d] & C_3^{p+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
\tag{21.9}
$$

which has all horizontal rows exact.

To define the connecting homomorphism, take $c_3^p \in C_3^p$ with $d_3^p c_3^p = 0$. By exactness of the middle row, β_p is surjective, so $c_3^p = \beta^p(c_2^p)$ for some $c_2^p \in C_2^p$. Then since the diagram commutes, we have

$$\beta^{p+1} d_2^p c_2^p = d_3^p \beta^p c_2^p = d_3^p c_3^p = 0. \tag{21.10}$$

By exactness of the bottow row, we have $d_2^p c_2^p = \alpha^{p+1} c_1^{p+1}$ for some $c_1^{p+1} \in C_1^{p+1}$. Since C_1 is a co-complex, and by commutativity of the diagram, we have

$$0 = d_2^{p+1} d_2^p c_2^p = d_2^{p+1} \alpha^{p+1} c_1^{p+1} = \alpha^{p+2} d_1^{p+1} c_1^{p+1}, \tag{21.11}$$

which implies that $d_1^{p+1} c_1^{p+1} = 0$, since α^{p+2} is injective. So we define $\delta^p(c_3^p) = [c_1^{p+1}]$, the homology class of c_1^{p+1} in $H^{p+1}(C_1)$.

To prove this mapping is well-defined, assume that we started with $c_p^3 \in C_p^3$ which was of the form $c_p^3 = d_3^{p-1} c_3^{p-1}$. Then we can write $c_3^{p-1} = \beta^{p-1} c_2^{p-1}$, and the element $\tilde{c}_2^p = d_2^{p-1} c_2^{p-1}$ satisfies $\beta^p(\tilde{c}_2^p) = c_3^p$. But this element is exact, so the next step clearly gives zero. Independence of the choice of c_2^p is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing. □

Exercise 21.3. Prove that the sequence (21.8) is exact.

21.2 Mayer-Vietoris for de Rham cohomology

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{\beta^p} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{\alpha^p} \Omega^p(U \cap V) \longrightarrow 0 \tag{21.12}$$

where

$$\beta^p(\omega) = \big((i_{U \hookrightarrow M})^* \omega, (i_{V \hookrightarrow M})^* \omega\big). \tag{21.13}$$

and

$$\alpha^p(\omega_U, \omega_V) = (i_{U\cap V \hookrightarrow U})^*\omega_U - (i_{U\cap V\hookrightarrow V})^*\omega_V \tag{21.14}$$

To see this, β^p is obviously injective. For exactness at the middle step, obviously $\alpha^p\beta^p\omega = 0$. If $\beta^p(\omega_U, \omega_V) = 0$, then $\omega_U = \omega_V$ on $U \cap V$, so then (ω_U, ω_V) is a well-defined global form on M.

To show that α is onto, let $\omega \in \Omega^p(U\cap V)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \alpha(\phi_V\omega, -\phi_U\omega)$.

By the zig-zag lemma for cohomology, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(U\cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U\cap V) \xrightarrow{\delta^p} \cdots \tag{21.15}$$

22 Lecture 22

22.1 General remarks on Mayer-Vietoris

Let us review the definition of the mapping $\delta^p : H^p(U\cap V) \to H^{p+1}(M)$. Given a cohomology class $[\omega] \in H^p_{dR}(U \cap V)$, represented by $\omega \in \Omega^p(U \cap V)$ with $d\omega = 0$, we first write $\omega = \alpha^p(\phi_V\omega, -\phi_U\omega)$, then we apply the exterior derivative to get

$$(d(\phi_V\omega), -d(\phi_U\omega)) = (d\phi_V \wedge \omega, -d\phi_U \wedge \omega) \in \Omega^p(U) \oplus \Omega^p(V). \tag{22.1}$$

Note that on $U \cap V$, we have $(\phi_U + \phi_V)\omega = \omega$, so applying d to this equation, we have that $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$ on $U \cap V$, so together these define a global form

$$\delta^p\omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{22.2}$$

and we take the cohomology class of this form.

Remark 22.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

Corollary 22.2. *If a smooth manifold M has a finite covering of open sets such that each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has finite-dimensional de Rham cohomology, then M has finite-dimensional de Rham cohomology.*

Proof. Note that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{22.3}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{22.4}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

Now we look at the following portion of the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\alpha^{p-1}} H^{p-1}_{dR}(U\cap V) \xrightarrow{\delta^{p-1}} H^p_{dR}(U\cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} \cdots \tag{22.5}$$

Using induction on the number of open sets in the covering, the corollary follows. □

Remark 22.3. If M is compact, there always exists such a covering. More on this later.

The following lemma will be extremely useful.

Lemma 22.4. *If*

$$0 \longrightarrow V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \longrightarrow V_{k-1} \xrightarrow{\alpha_{k-1}} V_k \longrightarrow 0. \tag{22.6}$$

is exact, then

$$0 = \dim(V_1) - \dim(V_2) + \dim(V_3) + \cdots + (-1)^{k-1}\dim(V_k). \tag{22.7}$$

Proof. We use induction. If $k = 1$, then we have $0 \to V_1 \to 0$, which obviously implies that $\dim(V_1) = 0$. Assume the theorem is true up to $k - 1$. The mapping $\alpha_2 : V_2 \to V_3$ has kernel given by $\alpha_1(V_1)$. So there is an induced mapping $\alpha_2 : V_2/\alpha_1(V_1) \to V_2$. If $\alpha_2(v_2 + \alpha_1(V_1)) = \alpha_2(v_2) = 0$, then $v_2 = \alpha_1(w_1)$, so the induced mapping is injective. We therefore have an exact sequence

$$0 \longrightarrow V_2/\alpha_1(V_1) \xrightarrow{\alpha_2} V_3 \xrightarrow{\alpha_3} \cdots \longrightarrow V_{k-1} \xrightarrow{\alpha_{k-1}} V_k \longrightarrow 0. \tag{22.8}$$

This is an exact seqence of length $k - 1$, so by induction

$$\begin{aligned} 0 &= \dim\Big(V_2/\alpha_1(V_1)\Big) - \dim(V_3) + \cdots \\ &= -\dim(V_1) + \dim(V_2) - \dim(V_3) + \cdots , \end{aligned} \tag{22.9}$$

and we are done. $\square$

22.2 Spheres

We consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

Corollary 22.5. *We have*

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases} \tag{22.10}$$

Proof. Cover with 2 open sets U, V, with $U \cong \mathbb{R}^n \cong V$ and $U \cap V \cong S^{n-1}$. First, consider the case of S^1. In this case, the Mayer-Vietoris sequence is

$$\begin{tikzcd} 0 \arrow[r] & H^0_{dR}(S^1) \arrow[r, "\beta^0"] & H^0_{dR}(U) \oplus H^0_{dR}(V) \arrow[r, "\alpha^0"] & H^0_{dR}(U \cap V) \arrow[dll, "\delta^0"] & \\ & H^1_{dR}(S^1) \arrow[r, "\beta^1"] & H^1_{dR}(U) \oplus H^1_{dR}(V) \arrow[r, "\alpha^1"] & H^1_{dR}(U \cap V) \arrow[r] & 0. \end{tikzcd} \tag{22.11}$$

But $U \cap V$ is contractible to 2 points, so this is

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "\beta^0"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\alpha^0"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, "\delta^0"] \\ & H^1_{dR}(S^1) \arrow[r, "\beta^1"] & 0. & \end{tikzcd} \tag{22.12}$$

Lemma 22.4 then says that $H^1_{dR}(S^1) \cong \mathbb{R}$.

Next, for $n > 1$, look at the beginning of the Mayer-Vietoris sequence

$$0 \longrightarrow H^0_{dR}(S^n) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots \tag{22.13}$$

But now $U \cap V$ is connected, so this is

$$0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \xrightarrow{\delta^0} \cdots . \tag{22.14}$$

Since β is injective, the kernel of α^0 is 1-dimensional. But α^0 has a 2-dimensional domain, so the image of α^0 is 1-dimensional, that is α^0 is surjective. So we can move to the next level and get

$$0 \longrightarrow H^1_{dR}(S^n) \xrightarrow{\beta^0} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^0} H^1_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots , \tag{22.15}$$

Since U and V are contractible, this says that $H^1_{dR}(S^n) = \{0\}$ for $n \geq 2$.

Next, we look at the upper portion of the Mayer-Vietoris sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-2}_{dR}(U) \oplus H^{n-2}_{dR}(V) \arrow[r, "\alpha^{n-2}"] & H^{n-2}_{dR}(S^{n-1}) \arrow[dll, "\delta^{n-2}"'] \\
H^{n-1}_{dR}(S^n) \arrow[r, "\beta^p"] & 0 \arrow[r, "\alpha^p"] & H^{n-1}_{dR}(S^{n-1}) \arrow[dll, "\delta^{n-1}"'] \\
H^n_{dR}(S^n) \arrow[r, "\beta^{p+1}"] & 0. &
\end{tikzcd} \tag{22.16}$$

This yields

$$H^n_{dR}(S^n) \cong H^{n-1}_{dR}(S^{n-1}) \cong \mathbb{R}, \tag{22.17}$$

and

$$H^k_{dR}(S^n) \cong H^{k-1}_{dR}(S^{n-1}) = \{0\}, \tag{22.18}$$

for $2 \leq k \leq n-1$, so this finishes the proof. $\square$

22.3 The 2-torus

Next we consider the 2-dimensional torus $T^2 = S^1 \times S^1$.

Corollary 22.6. *We have*

$$H^2_{dR}(T^2) = \begin{cases} \mathbb{R} & k = 0 \text{ or } 2 \\ \mathbb{R}^2 & k = 1 \end{cases} . \tag{22.19}$$

Proof. Writing the first S^1-factor as the union of two intervals, we can cover T^2 by $U = \mathbb{R}\times S^1$ and $V = \mathbb{R} \times S^1$ and such that the intersection is diffeomorphic to $U \cap V$ is $\mathbb{R} \times S^1 \coprod \mathbb{R} \times S^1$. The full Mayer-Vietoris sequence is

$$\begin{tikzcd}
0 \arrow[r] & H^0(T^2) \arrow[r, "\beta^0"] & H^0(U) \oplus H^0(V) \arrow[r, "\alpha^0"] & H^0(U \cap V) \arrow[dll, "\delta^0"] \\
& H^1(T^2) \arrow[r, "\beta^1"] & H^1(U) \oplus H^1(V) \arrow[r, "\alpha^1"] & H^1(U \cap V) \arrow[dll, "\delta^1"] \\
& H^2(T^2) \arrow[r, "\beta^2"] & H^2(U) \oplus H^2(V) \arrow[r, "\alpha^2"] & H^2(U \cap V) \arrow[r, "\delta^2"] & 0.
\end{tikzcd} \tag{22.20}$$

Using the Poincaré Lemma and Corollary 22.5, we know this is

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\beta^0"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\alpha^0"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, "\delta^0"] \\
& H^1(T^2) \arrow[r, "\beta^1"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\alpha^1"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, "\delta^1"] \\
& H^2(T^2) \arrow[r, "\beta^2"] & 0 \arrow[r, "\alpha^2"] & 0 \arrow[r, "\delta^2"] & 0.
\end{tikzcd} \tag{22.21}$$

If we try and use Lemma 22.4, we just get that $\dim(H^1(T^2)) = \dim(H^2(T^2)) + 1$, so we really need to look at the mappings. For this, note that the α^0 mapping has a 1-dimensional image, because the difference on the intersection is the same on both components of $U \cap V$. We also claim that α^1 has a 1-dimensional image. To see this, note that $d\theta$ is a generator for $H^1(S^1)$. From the proof of homotopy invariance of de Rham cohomology, a generator of $H^1(U) = H^1(S^1 \times \mathbb{R})$ is $\pi^* d\theta$, where π is projection onto the S^1 factor. Since the generator is independent of the $\mathbb{R}$ coordinate, the difference is the same on both components of the intersection, which proves the claim. So we can split off 2 exact sequences.

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta^0} H^1(T^2) \xrightarrow{\beta^1} \mathbb{R}^2 \xrightarrow{\alpha^1} \mathbb{R} \longrightarrow 0, \tag{22.22}$$

and

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha^1} \mathbb{R}^2 \xrightarrow{\delta^1} H^2(T^2) \longrightarrow 0. \tag{22.23}$$

Using Lemma 22.4, the first sequence implies that $b^1 = \dim(H^1(T^2)) = 2$ and the second sequence implies that $b^2 = \dim(H^2(T^2)) = 1$. $\square$

Remark 22.7. The above proof also shows that $H^1(T^2)$ is spanned by $d\theta_1$ and $d\theta_2$, where θ_i is the angular coordinate on the ith factor. Also, $H^2(T^2)$ is spanned by $d\theta_1 \wedge d\theta_2$.

23 Lecture 23

23.1 Top degree cohomology

We can now determine the top degree compactly supported cohomology of any manifold.

Theorem 23.1. *Let M be a connected smooth n-dimensional manifold. Then*

$$H^n_{c,dR}(M) = \begin{cases} \mathbb{R} & M \text{ is orientable} \\ 0 & M \text{ is non-orientable} \end{cases}. \tag{23.1}$$

In particular, if M is compact, then

$$H^n_{dR}(M) = \begin{cases} \mathbb{R} & M \text{ is orientable} \\ 0 & M \text{ is non-orientable} \end{cases} \tag{23.2}$$

Proof. Assume that M is orientable. From the Poincaré Lemma for cohomology with compact support, we know that $H^n_c(\mathbb{R}^n) = \mathbb{R}$. Cover M by open sets U_i diffeomorphic to $\mathbb{R}^n$. Given $\omega \in \Omega^n_c(M)$, write

$$\omega = \sum_i \omega_i = \sum_i \chi_i \omega, \tag{23.3}$$

where χ_i is a partition of unity subordinate to $\{U_i\}$. Since the result is true for $\mathbb{R}^n$, we know that $\omega_i = d\alpha_i + c_i\beta_i$, where β_i is a bump form, which we can take to be supported near any point $p_i \in U_i$. So we can assume that ω is cohomologous to a sum of bump forms. We claim that any 2 bump forms are cohomologous to a multiple of each other. To see this, given $p, p' \in M$, we can connect by a sequence of coordinate patches V_i and sequence of points chosen as follows. Let $p_0 = p$, contained in U_0. Then choose $p_1 \in V_1 \cap V_2$, $p_2 \in V_2 \cap V_3$, etc. This shows that $H^n_c(M)$ is at most 1-dimensional. By Stokes' theorem, we already knew that $H^n_c(M)$ was a least 1-dimensional. So M orientable implies $\dim(H^n_c(M)) = 1$.

If M is non-orientable, then there must be a path $\gamma : S^1 \to M$ such that $\Lambda^n(\gamma^* TM)$ is the non-orientable bundle on S^1. This is because the orientation double cover $\pi : \tilde{M} \to M$ is a non-trivial double covering, so there must be a closed path γ which lifts to an open path in $\tilde{M}$ connecting different points in a single fiber. Consequently, there exists a sequence of coordinate neighborhoods $V_1, \dots, V_N$ with $V_1 = V_N$ and such that there are an odd number of orientation-reversing transition functions. The above argument then shows that β_1, a bump form supported in V_1, is cohomologous to $-\beta_1$, and therefore must be trivial in cohomology. $\square$

Remark 23.2. We have not yet determined $H^n(M)$ where M is a non-compact manifold, we will return to this later.

23.2 Real projective 2-space

Using the above, we can determine the de Rham cohomology of $\mathbb{RP}^2$.

Proposition 23.3. *We have*

$$H^k(\mathbb{RP}^2) = \begin{cases} \mathbb{R} & k = 1 \\ 0 & k = 0, 2 \end{cases} \tag{23.4}$$

Proof. We can consider $\mathbb{RP}^2$ as a disc D^2 with the boundary identified by the antipodal map. Cover by $U = D^2$, and V a tubular neighborhood of the boundary $\mathbb{RP}^1 = S^1$. The intersection is a annulus so is homotopic to S^1. The Mayer-Vietoris sequence is

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\beta^0"] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "\alpha^0"] & \mathbb{R}\oplus\mathbb{R} \arrow[dll, rounded corners, "\delta^0"] & \\
& H^1(\mathbb{RP}^2) \arrow[r, "\beta^1"] & \mathbb{R} \arrow[r, "\alpha^1"] & \mathbb{R} \arrow[dll, rounded corners, "\delta^1"] & \\
& H^2(\mathbb{RP}^2) \arrow[r, "\beta^2"] & 0 \arrow[r, "\alpha^2"] & 0 \arrow[r, "\delta^2"] & 0.
\end{tikzcd} \tag{23.5}$$

Since $\mathbb{RP}^2$ is non-orientable, we know that $H^2(\mathbb{RP}^2) = \{0\}$, so we conclude that $H^1(\mathbb{RP}^2) = \{0\}$. $\square$

Remark 23.4. We could also prove this directly without using Theorem 23.1: it is not hard to show that the mapping α^1 is multiplication by 2, so is an isomorphism since $\mathbb{R}$ has no torsion.

23.3 Connected sum of surfaces

By "surface" we will mean a 2-dimensional smooth manifold. Next, we define the connected sum operation. Given surfaces M_1, M_2 choose points $p_i \in M_i$, $i = 1, 2$. Let U_i be a neighborhood of p_1 which is diffeomorphic to $B_0(2)$, a ball centered at the origin of radius 2 in $\mathbb{R}^2 = \mathbb{C}$. Let $\Psi_i : B_2(0) \to M_i$ be a diffeomorphism such $\Psi_i(B_2(0)) = U_i$ and $\Psi_i(0) = p_i$. Let $V_i = M_i \setminus \Psi_i(B_{1/2}(0))$.

Definition 23.5. We define $M_1 \# M_2 = V_1 \coprod V_2 / \sim$ where the equivalence relation is $z \sim w$ if $z \in \Psi_1(B_2(0) \setminus B_{1/2}(0))$ and $w \in \Psi_2(B_2(0) \setminus B_{1/2}(0))$ satisfy

$$\Psi_1^{-1}(z) \sim (\Psi_2^{-1}(w))^{-1}, \tag{23.6}$$

where the right hand side means the inverse as a complex number.

Remark 23.6. Instead of the inverse as a complex number, we could have used the mapping $z \mapsto z/|z|^2 = (\overline{z})^{-1}$. Since every surface admit an orientation-reversing diffeomorphism, the resulting connected sums are diffeomorphic. However, this is not true in higher dimensions in which case $M \# N$ and $M \# \overline{N}$ are not necessarily diffeomorphic.

Remark 23.7. We have the following properties of the connected sum:

(i) $M_1 \# M_2$ is a surface which is independent of the choice of base points. This is because given any surface M, and $p_1, p_2 \in M$, there exists a diffeomorphism $f : M \to M$ with $f(p_1) = p_2$.

(ii) $M \# S^2$ is diffeomorphic to M.

(iii) If M_1 and M_2 are orientable, then so is $M_1 \# M_2$. This is because the map $z \mapsto z^{-1}$ is orientation-preserving.

(iv) If either M_1 or M_2 is non-orientable, then $M_1\#M_2$ is non-orientable.

We leave the detailed proofs of these properties as an exercise for the interested student.

24 Lecture 24

24.1 Cohomology of compact surfaces

By "surface" we will mean a 2-dimensional smooth manifold. We will compute the de Rham cohomology of any compact surface. Recall that we have already computed the de Rham cohomology of S^2, T^2, and $\mathbb{RP}^2$. Using the connected sum operation, we can obtain all other surface from these building blocks.

Theorem 24.1 (Classification of compact surfaces)**.** *If M is an orientable compact surface then M is diffeomorphic to*

$$S^2 \quad or \quad \overbrace{T^2\#\cdots\#T^2}^{k} \equiv k\#T^2, \tag{24.1}$$

for some $k \in \mathbb{Z}_+$. If M is a non-orientable compact surface then M is diffeomorphic

$$\overbrace{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}^{k} \equiv k\#\mathbb{RP}^2 \tag{24.2}$$

for some $k \in \mathbb{Z}_+$.

We will not prove this, but will determine the de Rham cohomology groups of each of these cases. This will show that these examples are pairwise non-diffeomorphic. A useful definition is the following.

Definition 24.2. For a compact surface M, the Euler characteristic is

$$\chi(M) \equiv \dim(H^0(M)) - \dim(H^1(M)) + \dim(H^2(M)) \equiv b^0 - b^1 + b^2. \tag{24.3}$$

Theorem 24.3. *We have*

$$H^k_{dR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & k = 1 \end{cases}, \tag{24.4}$$

and $\chi(S^2) = 2$. If $M = g\#T^2$, then

$$H^k_{dR}(M) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^{2g} & k = 1 \end{cases}, \tag{24.5}$$

and $\chi(M) = 2 - 2g$. If $M = (g+1)\#\mathbb{RP}^2$ then

$$H^k_{dR}(M) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^{g} & k = 1 \\ 0 & k = 2 \end{cases}, \tag{24.6}$$

and $\chi(M) = 1 - g$.

Proof. We already know this is true for S^2 and $\mathbb{RP}^2$, and T^2. We claim that the Euler characteristic of a connect sum of surfaces is given by

$$\chi(M_1\# M_2) = \chi(M_1) + \chi(M_2) - 2. \tag{24.7}$$

Given any decomposition of $M = U \cup V$ into the union of 2 open sets, we have the Mayer-Vietoris sequence,

$$\begin{tikzcd}
0 \arrow[r] & H^0(M) \arrow[r] & H^0_{dR}(U) \oplus H^0_{dR}(V) \arrow[r] & H^0_{dR}(U \cap V) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^1_{dR}(M) \arrow[r] & H^1_{dR}(U) \oplus H^1_{dR}(V) \arrow[r] & H^1_{dR}(U \cap V) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^2_{dR}(M) \arrow[r] & H^2_{dR}(U) \oplus H^2_{dR}(V) \arrow[r] & H^1_{dR}(U \cap V) \arrow[r] & 0.
\end{tikzcd} \tag{24.8}$$

Lemma 22.4 implies that

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V). \tag{24.9}$$

Given a surface M and $p \in M$, cover M by open sets $U = M \setminus \{p\}$, V a small ball around p, and $U \cap V \sim S^1$. Applying (24.9),

$$\chi(M \setminus \{p\}) = \chi(M) - 1. \tag{24.10}$$

Next, we cover $M_1\# M_2$ by $U \sim M_1 \setminus \{p_1\}$, $V \sim M_2 \setminus \{p_2\}$ and such that $U \cap V$ retracts onto S^1. Applying (24.9) with (24.10) yields

$$\chi(M_1\# M_2) = \chi(M_1) - 1 + \chi(M_2) - 1 - \chi(S^1) = \chi(M_1) + \chi(M_2) - 2. \tag{24.11}$$

In the orientable case, the proposition implies that

$$\chi\left(g\# T^2\right) = 2 - 2g. \tag{24.12}$$

In the nonorientable case, the proposition implies that

$$\chi\big((g+1)\#\mathbb{RP}^2\big) = 1 - g. \tag{24.13}$$

and the dimension of the middle de Rham homology group follows since we know the dimension of the top de Rham cohomology group. $\square$

Remark 24.4. We have the miscellaneous facts about surfaces, which follow from the classification (but can also be proved directly):

- The orientable double cover of $M = (g+1)\#\mathbb{RP}^2$ is $\tilde{M} = g\# T^2$. This follows since $\chi(\tilde{M}) = 2\chi(M)$, a fact which we will prove below.
- The Klein bottle K is diffeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2$, where K is a cylinder with the 2 boundary circles identified by a twist (without a twist, we would obtain a torus). This contains a Mobius strip, so is non-orientable. The orientation double cover of K is T^2, so $\chi(T^2) = 0 = 2\chi(K)$ implies that $\chi(K) = 0$.

- We have $T^2\#\mathbb{RP}^2$ is diffeomorphic to $3\#\mathbb{RP}^2$, since both are non-orientable, with Euler characteristic equal to -1.
- Any compact nonorientable surface is diffeomorphic to a compact orientable surface connect sum with an $\mathbb{RP}^2$ or Klein bottle. This follows from the classification and the previous two items.

24.2 Embeddings of surfaces

Any compact orientable surface embeds in $\mathbb{R}^3$. This is clear since the torus embeds as a surface of revolution, and we can perform the connect sum of disjoint embedded tori by removing a disc from each and attaching an embedded cylinder. It turns out that a compact non-orientable surface cannot be embedded in $\mathbb{R}^3$, but can be embedded in $\mathbb{R}^4$. To see the latter, we will first show that $\mathbb{RP}^2$ can be embedded in $\mathbb{R}^4$. We define a mapping from $\mathbb{R}^3$ to $\mathbb{R}^6$ by

$$\phi : (x, y, z) \mapsto (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz) \tag{24.14}$$

When restricted to $S^2 \subset \mathbb{R}^3$, this mapping is invariant under the antipodal map, so we get a mapping

$$\phi : \mathbb{RP}^2 \to \mathbb{R}^6 \tag{24.15}$$

which is easily checked to be an embedding. The image of ϕ lies in the subset of $\mathbb{R}^6$:

$$V = \Big\{(x_1, \dots, x_6) \in \mathbb{R}^6 \mid x_1 + x_2 + x_3 = 1, \sum_{i=1}^{6} x_i^2 = 1\Big\} \tag{24.16}$$

This is $S^5 \subset \mathbb{R}^6$ intersected with a hyperplane, so is diffeomorphic to S^4. So we have an embedding of

$$\phi : \mathbb{RP}^2 \to S^4, \tag{24.17}$$

which is called the Veronsese $\mathbb{RP}^2$ in S^4. Since $S^4 \setminus \{p\}$ is diffeomorphic to $\mathbb{R}^4$ under stereographic projection, by removing a point not in the image of ϕ, we find the claimed embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

To get an embedding of the connect sum $\mathbb{RP}^2\#\mathbb{RP}^2$, just take 2 Veronese $\mathbb{RP}^2$-s which are disjoint, remove discs from each, and attach together with an embedded cylinder. Higher order connect sums of $\mathbb{RP}^2$-s follow from the same argument.

24.3 Triangulations of smooth manifolds

Define the standard p-simplex to be

$$\Delta^p = \Big\{(t_0, \dots, t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\Big\}. \tag{24.18}$$

For $0 \leq i \leq p$, the ith face of Δ^p is the $(p-1)$-simplex

$$\Delta_i^p : \Delta^{p-1} \to \Delta^p \tag{24.19}$$

defined by

$$(t_0, \ldots, t_{p-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots t_{p-1}). \tag{24.20}$$

More generally, for $k < p-1$, a k-face of Δ^p is a simplex obtained from Δ_p obtained by setting $p-k$ of the coordinates equal to 0.

Definition 24.5. If M is a smooth compact n-dimensional manifold, a triangulation of M is collection of diffeomorphisms

$$c_i^n : \Delta^n \to M \tag{24.21}$$

for $i = 1 \ldots N$ whose images cover M and such that if

$$c_i^n(\Delta^p) \cap c_j^n(\Delta^p) \neq \emptyset, \tag{24.22}$$

for $i \neq j$, then the intersection is exactly a k-face of both simplices for $0 < k \leq p-1$.

We will refer to image of c_i^n as an n-simplex of the triangulation, and the image of any k-face of a simplex will be called a k-simplex of the triangulation.

Remark 24.6. Using his embedding theorem, Whitney proved that every smooth manifold M admits a triangulation. The basic idea is to embed M into $\mathbb{R}^n$. Taking a very fine cubical lattice in general position, he constructs a simpicial complex in a tubular neighborhood of M which projects to a triangulation on M. We will not give details of this, but refer the interested student to Cairns' 1961 paper "A simple triangulation method for smooth manifolds", which is just 2 pages!

25 Lecture 25

25.1 Euler's polyhedral formula

If M^n is a smooth compact n-dimensional manifold with a triangulation, then let α_k be the number of k-simplices in a triangulation. For surfaces, these are also notated as $V = \alpha_0, E = \alpha_1, F = \alpha_2$ since V is the number of vertices, E is the number of edges, and F is the number of faces in the triangulation. We also define Euler characteristic

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b^i(M), \tag{25.1}$$

where $b^i(M) = \dim(H^i(M))$ is called the ith Betti number of M.

Theorem 25.1. *If a smooth compact surface M admits a triangulation, then*

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2 = V - E + F. \tag{25.2}$$

Consequently, the sum $V - E + F$ is independent of the triangulation.

Proof. Clearly, we can assume that M is connected. Let U be the union of small balls around the barycenters (center of mass) of the 2-simplices p_i. Let $V_1 = M \setminus \cup_{i=1}^{\alpha_2}\{p_i\}$. Then $U \cap V$ is homotopic to the disjoint union of α_2 copies of S^1.

Applying the Mayer-Vietoris sequence to U and V yields that

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^{\alpha_2} \oplus H^0_{dR}(V_1) \arrow[r] & \mathbb{R}^{\alpha_2} \arrow[dlll, rounded corners] \\
& H^1_{dR}(M) \arrow[r] & H^1_{dR}(V_1) \arrow[r] & \mathbb{R}^{\alpha_2} \arrow[dll, rounded corners] \\
& H^2_{dR}(M) \arrow[r] & H^2_{dR}(V_1) \arrow[r] & 0.
\end{tikzcd} \tag{25.3}$$

Lemma 22.4 implies that

$$0 = 1 - \alpha_2 - b^0(V_1) + \alpha_2 - b^1(M) + b^1(V_1) - \alpha_2 + b^2(M) - b^2(V_1), \tag{25.4}$$

or

$$\chi(M) = \chi(V_1) + \alpha_2. \tag{25.5}$$

We next apply the Mayer-Vietoris sequence on V_1, with a new U and V. For this, if 2 2-simplices intersect along a 1-face, then we can connect the barycenters by a curve which intersects the 1-face in the barycenter of the 1-face.

Then let U be the disjoint union of sets diffeomorphic to balls which are slight "fattenings" of slight shrinkings of the curves (so that they are disjoint near the endpoints). Let V_0 be the complement in V_1 of the union of the curves joining the barycenters of the faces. Then $V_1 = U \cup V_0$, U is the union of α_1 balls, and the set V_0 deformation retracts onto the set of 0-faces. Also, the intersection $U \cap V_0$ consists of $2\alpha_1$ sets diffeomorphic to balls, since each curve cuts the fattenings into 2 pieces, so we have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^{\alpha_1} \oplus \mathbb{R}^{\alpha_0} \arrow[r] & \mathbb{R}^{2\alpha_1} \arrow[dll, rounded corners] \\
& H^1_{dR}(V_1) \arrow[r] & 0
\end{tikzcd} \tag{25.6}$$

Lemma 22.4 implies that

$$\chi(V_1) = \alpha_0 - \alpha_1. \tag{25.7}$$

Combining with (25.5), we have

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2. \tag{25.8}$$

$\square$

Remark 25.2. In dimension n, it is true that

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \alpha_k, \tag{25.9}$$

but we leave the details as an exercise for the interested student. (The above proof extends to the higher-dimensional case with a litte extra work; see [Spi79, Chapter 11].)

Corollary 25.3. *If a finite group Γ acts freely on a compact manifold M, then*

$$\chi(M) = |\Gamma| \cdot \chi(M/\Gamma). \tag{25.10}$$

In particular, if $\pi : \tilde{M} \to M$ is a d-fold covering space, then

$$\chi(\tilde{M}) = d \cdot \chi(M). \tag{25.11}$$

Proof. If M/Γ has a triangulation with α_k k-simplices, then we can pull-back the triangulation to M, which is a triangulation with $\tilde{\alpha}_k = |\Gamma|\alpha_k$ k-simplices. □

Remark 25.4. This is a special case of a much more general formula. If $\pi : E \to B$ is any orientable fiber bundle with fiber F then $\chi(E) = \chi(F) \cdot \chi(B)$. This follows from the Serre spectral sequence.

25.2 The Riemann-Hurwitz formula

Definition 25.5. Let M and N be compact surfaces. We say that $f : M \to N$ is a degree d branched covering if there exist $S = \{p_1, \dots p_k\} \in N$ such that f is is a d-fold covering space away from S, $f^{-1}(p_i)$ is finite, and near and $q \in f^{-1}(p_i)$, f is diffeomorphically conjugate to $z \mapsto z^{d_q}$, for some integer d_q.

Note that the sum of the branching degrees $d_{i,j}$ for $q_{i,j} \in f^{-1}(p_i)$ must satisfy $\sum_j d_{i,j} = d$.

Theorem 25.6 (Riemann-Hurwitz formula)**.** *If $f : M \to N$ is a degree d branched covering, then*

$$\chi(M) = d \cdot \chi(N) - \sum_{p \in M} (d_p - 1), \tag{25.12}$$

where d_p is the local branching degree at p.

Proof. Consider a triangulation of N which has vertices at all of the critical values of f. This lifts to a triangulation of M which has d times the number of faces, and d times the number of edges. Then number of vertices is d times the number of vertices which are not at branching points. At a branching point, the number of vertices is reduced by d_p. □

26 Lecture 26

26.1 Riemann surfaces

Definition 26.1. A *Riemann surface* is a smooth surface with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

Example 26.2. Consider $S^2 = \mathbb{C} \cup \{\infty\}$. We can cover S^2 by 2 copies of $\mathbb{C}$, with overlap mapping $z \mapsto z^{-1}$, which is holomorphic. Thus S^2 is a Riemann surface.

Example 26.3. View T^2 as $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$, where the group is generated by translations along a lattice L. Since translations are holomorphic, we see that T^2 is a Riemann surface.

Since nonsingular holomorphic mappings are necessarily orientation preserving, it follows that Riemann surfaces must be orientable. Actually, any orientable smooth surface is a Riemann surface, but we will not prove this right now. The main source of branched coverings arises from holomorphic mappings between Riemann surfaces.

Proposition 26.4. *Assuming that M and N are compact Riemann surfaces. Then any nonconstant holomorphic mapping $f : M \to N$ is a branched covering.*

Proof. This follows because the set of critical points of f must be finite: in local holomorphic coordinates, a critical point satisfies $\frac{\partial}{\partial z} f = 0$. Since f is nonconstant, this equation can only have finitely many zeroes. Away from the critical values, f must be a covering space. Near a critical point, a nonconstant holomorphic function has a zero of some finite order, and is holomorphically conjugate to $z \mapsto z^q$. $\square$

Example 26.5. For $q \in \mathbb{N}$, consider the mapping $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^q$. We can extend this to a meromorphic function $f : S^2 \to S^2$ by mapping ∞ to ∞. There are precisely 2 branch points 0 and ∞, both of order q. The Riemann-Hurwitz formula yields

$$\chi(S^2) = q\chi(S^2) - 2(q-1) = 2q - 2q + 2 = 2, \tag{26.1}$$

which is correct!

26.2 Hypersurfaces in $\mathbb{CP}^2$

Complex projective n-space is defined to be the space of lines through the origin in $\mathbb{C}^{n+1}$. This is equivalent to $\mathbb{C}^{n+1}/\sim$, where $\sim$ is the equivalence relation

$$(z^0, \dots, z^n) \sim (w^0, \dots, w^n) \tag{26.2}$$

if there exists $\lambda \in \mathbb{C}^*$ so that $z^j = \lambda w^j$ for $j = 1 \dots n$. The equivalence class of $(z^0, \dots, z^n)$ will be denoted by $[z^0 : \cdots : z^n]$. Letting $U_j = \{[z^0 : \cdots : z^n] | z^j \neq 0\}$, $\mathbb{CP}^n$ is covered by $(n+1)$ coordinate charts $\phi_j : U_j \to \mathbb{C}^n$ defined by

$$\phi_j : [z^0 : \cdots : z^n] \mapsto \Big(\frac{z^0}{z_j}, \dots, \frac{z^{j-1}}{z_j}, \frac{z^{j+1}}{z_j}, \dots, \frac{z^n}{z_j} \Big), \tag{26.3}$$

with inverse given by

$$\phi_j^{-1} : (w^1, \dots, w^n) \mapsto [w^1 : \dots w^{j-1} : 1 : w^j : \cdots : w^n]. \tag{26.4}$$

Remark 26.6. It is easy to see that $\mathbb{CP}^1$ is diffeomorphic to S^2. For $n > 1$, note the overlap maps are holomorphic functions of several variables, which gives $\mathbb{CP}^n$ the structure of a complex manifold.

We will now restrict to $\mathbb{CP}^2$, and consider the homogeneous degree d polynomial

$$f_d = z_0^d + z_1^d + z_2^d. \tag{26.5}$$

Although f_d is not a well-defined function from $\mathbb{CP}^2$ to $\mathbb{C}$, the subset

$$V_d = \{p \in \mathbb{CP}^2 : f_d(p) = 0\} \tag{26.6}$$

is a well-defined subset of $\mathbb{CP}^2$.

Proposition 26.7. *The subset V_d is a compact orientable surface and has genus*

$$g = \frac{(d-1)(d-2)}{2}. \tag{26.7}$$

Proof. The subset V is closed, and since $\mathbb{CP}^2$ is compact, V is compact. To see that V is a submanifold, consider the (complex) partial derivatives

$$\frac{\partial}{\partial z^i} f_d = d z_i^{d-1}. \tag{26.8}$$

The set where $\frac{\partial}{\partial z^i} f_d = 0$ is the subset $\{[z_0, z_1, z_2] \in \mathbb{CP}^n \mid z_i = 0\}$. These 3 subsets have no common zero. This means that at each point on V_d, the complex gradient is non-zero. By the implicit function theorem, this implies that V_d is a smooth Riemann surface.

Since f_d is a holomorphic polynomial, V_d is a Riemann surface. Since holomorphic maps are orientation-preserving, V is orientable. To find the genus, let $\pi : \mathbb{CP}^2 \setminus [1,0,0] \to \mathbb{CP}^1$ by $[z_0, z_1, z_2] \mapsto [z_1, z_2]$. Since the point $[1,0,0]$ is not on $V = V_d$, the restriction of π to V_d gives a holomorphic mapping $\pi : V_d \to \mathbb{CP}^1$. Since f is a degree d polynomial, this mapping has degree d. The branch points are the subset

$$\{[z_1, z_2] \in \mathbb{CP}^1 \mid z_1^d + z_2^d = 0\} = \{[1, \zeta_d]\}, \tag{26.9}$$

where ζ_d is a dth root of -1. There are exactly d of these, so there are d branch points of order d. The Riemann-Hurwitz formula then gives

$$\chi(V_d) = d\chi(\mathbb{CP}^1) - d(d-1) = -d^2 + 3d. \tag{26.10}$$

But we know that $\chi(V_d) = 2 - 2g$, and solving for g yields (26.7). ◻

The first few values of this, starting with $d = 1$ are

$$0, 0, 1, 3, 5, 10, 15, 21... \tag{26.11}$$

The degree 1 case is a line, so is obviously $\mathbb{CP}^1$. The degree 2 case is also an S^2.

The degree 3 case is a torus; this is called an elliptic curve. There is actually a holomorphic function $f : V_3 \to \mathbb{CP}^1$ which is of degree 2, and has 4 branch points of order 2. To see this, we can make a linear change of variables so that V_3 is equivalent to the Riemann surface defined by

$$V_3' = \{[z_0, z_1, z_2] \mid z_2 z_1^2 = z_0^3 - 432 z_2^3\}. \tag{26.12}$$

Then the mapping $\pi : [z_0, z_1, z_2] \mapsto [z_0, z_2]$ exhibits V_3' as such a branched covering. There is a slight problem since the point $[0, 1, 0] \in V_3'$, but it is easy to check that π can be extended holomorphically to all of V_3' with a double branched point there. Note this checks out with Riemann-Hurwitz:

$$0 = \chi(T^2) = 2\chi(S^2) - \sum_{i=1}^{4}(2-1) = 0. \tag{26.13}$$

27 Lecture 27

27.1 Degree of a smooth mapping

Definition 27.1. A mapping $f : X \to Y$ between topological spaces is proper if the inverse image of any compact set is compact.

Exercise 27.2. Prove the following statements about proper mappings.

(i) Let X and Y be metric spaces. For a sequence of points $x_i \in X$, we say that $\lim_{i\to\infty} x_i = \infty$ if given any compact subset $K \subset X$, then there exists an integer N so that $x_i \in X \setminus K$ for $i > N$. Show that $f : X \to Y$ is proper iff for any sequence $x_i \in X$ such that $\lim_{i\to\infty} x_i = \infty$, then $\lim_{i\to\infty} f(x_i) = \infty$.

(ii) If Y is a manifold and $f : X \to Y$ is proper and continuous, then f is a closed mapping, that is, f maps closed sets to closed sets.

Let $f : M \to N$ be a proper smooth mapping between n-dimensional connected and oriented smooth manifolds. Since f is proper, $f^* : \Omega^n_c(N) \to \Omega^n_c(M)$, and therefore there is an induced mapping $f^* : H^n_{c,dR}(N) \to \Omega^n_{c,dR}(M)$. From Theorem 23.1, we know that $H^n_{c,dR}(M) \cong \mathbb{R}$, with isomorphism given by $[\omega] \mapsto \int_M \omega$, and similarly for N. Therefore, we can make the following definition.

Definition 27.3. The degree of f is the real number $\deg(f)$ so that

$$\int_M f^*\omega = \deg(f) \int_N \omega \tag{27.1}$$

for all $\omega \in \Omega^n_c(N)$.

Proposition 27.4. *If $f : M \to N$ and $g : N \to \tilde{M}$ are both proper then $g \circ f : N \to \tilde{M}$ is proper and*

$$\deg(g \circ f) = \deg(g) \circ \deg(f) \tag{27.2}$$

Proof. The composition of proper maps is obviously proper. Given $\omega \in \Omega^n_c(\tilde{M})$ then

$$\int_M (g \circ f)^* \omega = \int_M f^* g^* \omega = \deg(f) \int_N g^* \omega = \deg(f) \deg(g) \int_{\tilde{M}} \omega. \tag{27.3}$$

□

Proposition 27.5. *Let $f : M \to N$ be a diffeomorphism. Then* $\deg(f) = 1$ *is f is orientation preserving, and* $\deg(f) = -1$ *if f is orientation reversing.*

Proof. This follows from the change of variables formula, the definition of the integral is clearly invariant under diffeomorphisms, but only up to sign. □

Proposition 27.6. *If M and N are compact, and $f : M \to N$ is smoothly homotopic to $g : M \to N$, then* $\deg(f) = \deg(g)$.

Proof. If M and N are compact, we know that $H^n_{c,dR}(M) = H^n_{dR}(M)$ and $H^n_{c,dR}(N) = H^n_{dR}(N)$. By the Proposition 33.2 $f^* = g^* : H^n_{c,dR}(N) \to H^n_{c,dR}(M)$. □

Remark 27.7. This is not true in the noncompact case. The functions z and z^2 as mappings from $\mathbb{C}$ to itself are properly homotopic, yet have different degrees.

Proposition 27.8. *If $f : M \to N$ is proper and not surjective, then* $\deg(f) = 0$.

Proof. If f is not surjective, then there exists $q \in N$ which is not in the image of f. Furthermore, since f is a closed mapping, there exists a neighborhood U of q which contains no points in the image of f. Let χ be an n-form supported in U with $\int_U \chi = 1$. But $f^*\chi \equiv 0$, so $\int_M f^*\chi = 0$, and thus $\deg(f) = 0$. □

Proposition 27.9. *If $f : M \to N$ is proper, then* $\deg(f) \in \mathbb{Z}$. *Furthermore, if $q \in N$ be a regular value of f, and let $f^{-1}(q) = \{p_1, \ldots, p_k\}$. Then*

$$\deg(f) = \sum_i \operatorname{sgn}(f_*|_{p_i}), \tag{27.4}$$

where

$$\operatorname{sgn}(f_*|_{p_i}) = \det(f_*|_{p_i}) / |\det(f_*|_{p_i})|, \tag{27.5}$$

Proof. By Sard's Theorem, there exists a regular value q. Since f is proper, $f^{-1}(q)$ is compact, and since it consists of isolated points (by the inverse function theorem), it must be a finite set. We can choose a neighborhood U of q so that $f^{-1}(U) = U_1 \coprod \cdots \coprod U_k$ and such that $f : U_i \to U$ is a diffeomorphism. Let $\chi \in \Omega^n_c(U)$ satisfy $\int_U \chi = 1$. Then $f^*\chi$ is supported in $U_1 \cup \cdots \cup U_k$, and we have

$$\int_M f^*\chi = \sum_{i=1}^k \int_{U_i} f^*\chi = \sum_{i=1}^k \operatorname{sgn}(f_*|_{p_i}) \int_U \chi = \sum_{i=1}^k \operatorname{sgn}(f_*|_{p_i}). \tag{27.6}$$

□

27.2 Applications of degree

In this subsection, we will give several applications of the degree theory developed above.

Corollary 27.10 (Fundamental theorem of algebra)**.** *Any nonconstant polynomial in $\mathbb{C}$ has a zero.*

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Recall that $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$ where $(z^1, z^2) \sim \lambda(z^1, z^2)$ for $\lambda \neq 0$. It is easy to see that we can extend $P(x) : S^2 \to S^2$, as a holomorphic map (it is a meromorphic function on $\mathbb{C}$ with a single pole at infinity of order n). Since P is holomorphic, it is orientation preserving. Since P is a polynomial, the set of critical values is a finite set. Since P is non-constant, P attains some value $q \in S^2$ which is not critical. Since P is orientation preserving at all regular points, the degree must then be non-zero (in fact, the degree is n). Proposition 27.8 then implies that P is surjective. □

Corollary 27.11. *There does not exist any non-zero vector field on S^n for n even.*

Proof. Let $A : S^n \to S^n$ be the antipodal map. We first claim that $\deg(A) = -1$ for n even. Clearly A is a diffeomorphism, so we just need to check if it is orientation preserving or not. The standard orientation of S^n is given by

$$\omega = \left(x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \cdots \wedge dx^{n+1}). \tag{27.7}$$

Clearly

$$A^*(\omega) = (-1)^{n+1}\omega. \tag{27.8}$$

So if n is even, we have $\deg(A) = -1$.

But if X is a non-zero vector field on S^n, let $\gamma_p(t)$ be the portion of the great circle such that $\gamma_p(0) = p$, $\gamma_p(1) = -p$ and such that $\gamma_p'(0)$ points in the direction of X_p. Then $H(p,t) = \gamma_p(t)$ is a homotopy between Id and A. This is a contradiction since $\deg(Id) = 1$. □

Remark 27.12. Odd-dimensional spheres always have a non-zero vector field:

$$X = (-x_2, x_1, -x_4, x_3, \cdots) \tag{27.9}$$

We also have the following:

Proposition 27.13. *If $f : S^n \to S^n$ is smooth and* $\deg(f) \neq (-1)^{n+1}$*, then f has a fixed point.*

Proof. If no fixed point, then for $p \in S^n$, the line segment from $f(p)$ to $-p$ does not hit the origin. Then we can define

$$H(p,t) = \frac{(1-t)f(p) - tp}{|(1-t)f(p) - tp|}, \tag{27.10}$$

which is a homotopy between f and the antipodal map. □

Next, we have

Proposition 27.14. *Let M be a smooth n-dimensional oriented compact manifold with connected boundary ∂M. Let N be a compact connected oriented $(n-1)$ manifold. Let $g : \partial M \to N$ be a smooth mapping which extends to a smooth mapping $G : M \to N$. Then* $\deg(g) = 0$.

Proof. Let ω be any smooth $(n-1)$ form on N such that $\int_N \omega = 1$. Obviously $d_N\omega = 0$ since N is of dimension $n-1$. Then by Stokes' Theorem

$$\deg(g) = \int_{\partial M} g^*\omega = \int_{\partial M} G^*\omega = \int_M d_M(G^*\omega) = \int_M G^*(d_N\omega) = 0. \tag{27.11}$$

□

Corollary 27.15 (Brouwer fixed point theorem)**.** *If $f : \overline{B^n} \to \overline{B^n}$ is smooth, then f has a fixed point.*

Proof. Assume by contradiction that f has no fixed point. Then define $G : \overline{B^n} \to S^{n-1}$ by

$$G(x) = \frac{x - f(x)}{|x - f(x)|}. \tag{27.12}$$

Letting $g = G|_{S^{n-1}}$, by the previous proposition, we have $\deg(g) = 0$. However, define $H : S^{n-1} \times [0,1] \to S^{n-1}$ by

$$H(x,t) = \frac{x - tf(x)}{|x - tf(x)|}. \tag{27.13}$$

Clearly, the denominator never vanishes, so H is a smooth homotopy between g and the identity map. Since $\deg(Id) = 1$, this is a contradiction. □

27.3 Top degree cohomology

We prove the following, which will complete our understanding of the top degree de Rham cohomology of any smooth manifold.

Proposition 27.16. *If M is a non-compact smooth manifold of dimension n, then*

$$H^n_{dR}(M) \cong \{0\}. \tag{27.14}$$

Proof. Assume that $\omega \in \Omega^n$ has compact support in a coordinate neighborhood $U = U_1$. We then find a "ray" of coordinate neighborhoods, that is, U_i such that $U_i \cap U_{i+1} \neq \emptyset$ and is connected, but the sequence leaves any compact subset of M. Choose $\omega_i \in \Omega^n_c(U_i \cap U_{i+1})$ which generates $H^n_c(U_i)$. Then

$$\begin{aligned} \omega &= d\eta_1 + c_1\omega_1 \\ &= d\eta_1 + c_1(c_2\omega_2 + d\eta_2) \\ &= d\eta_1 + c_1 d\eta_2 + c_1c_2(d\eta_3 + c_3\omega_3) = \cdots . \end{aligned} \tag{27.15}$$

So we have

$$\begin{aligned}\omega &= d\eta_1 + c_1 d\eta_2 + c_1c_2 d\eta_3 + c_1c_2c_3 d\eta_4 + \cdots \\ &= d\Big(\eta_1 + c_1 d\eta_2 + c_1c_2 d\eta_3 + c_1c_2c_3 d\eta_4 + \cdots\Big).\end{aligned} \tag{27.16}$$

Since the open sets U_i leave any compact subset, this is a finite sum at any point, so this shows that $\omega = d\eta$, where $\eta \in \Omega^{n-1}(M)$.

Next, given any ω, we can find a sequence of coordinate neighborhoods U_i such that $\{U_i\}$ covers M and which is locally finite, and which leaves any compact subset. Let ϕ_i be a partition of unity subordinate to $\{U_i\}$. Then by the above, we can write $\omega_i = \phi_i \omega = d\eta_i$, where the support of η_i is contained in $U_i \cup U_{i+1} \cup \cdots$. Finally,

$$\omega = \sum \omega_i = \sum d\eta_i = d(\sum \eta_i), \tag{27.17}$$

and the form $\eta = \sum \eta_i$ makes sense since the U_i-s leave any compact subset. □

Let us summarize our results on the top-dimensional cohomology of any connected smooth n-dimensional manifold.

Theorem 27.17. *Let M be a connected smooth n-dimensional manifold. Then*

$$H^n_{dR}(M) = \begin{cases} \mathbb{R} & M \text{ compact and orientable} \\ 0 & M \text{ otherwise} \end{cases} \tag{27.18}$$

and

$$H^n_{c,dR}(M) = \begin{cases} \mathbb{R} & M \text{ orientable} \\ 0 & M \text{ otherwise} \end{cases}. \tag{27.19}$$

28 Lecture 28

28.1 Real projective spaces

Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}), \ c \neq 0. \tag{28.1}$$

Since every line through the origin hits the unit sphere in exactly two points, we can desribe $\mathbb{RP}^n$ as a quotient space. That is,

$$\mathbb{RP}^n = S^n / \mathbb{Z}_2, \tag{28.2}$$

where $\mathbb{Z}_2$ acts by $p \mapsto A(p) = -p$. Let $\pi : S^n \to \mathbb{RP}^n$ denote the projection mapping.

Proposition 28.1. *$\mathbb{RP}^n$ is orientable if n is odd, and non-orientable if n is even.*

Proof. In the last lecture, we saw that

$$\omega = \left(x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \cdots \wedge dx^{n+1}) \tag{28.3}$$

is a nowhere-vanishing n-form on $S^n \subset \mathbb{R}^{n+1}$. Clearly,

$$A^*\omega = (-1)^{n+1}\omega. \tag{28.4}$$

So if n is odd, ω is invariant under the above $\mathbb{Z}_2$ action and thus descends to be nowhere-zero n-form on $\mathbb{RP}^n$.

In n is even, then $A^*\omega = -\omega$. This says that A is orientation-reversing. If $\mathbb{RP}^n$ were orientable, then it would have a non-zero n-form $\omega \in \Omega^n(\mathbb{RP}^n)$, and the pull back form $\pi^*\omega$ would be a non-zero n-form on S^n which is invariant under A:

$$A^*\pi^*\omega = (\pi \circ A)^*\omega = \pi^*\omega, \tag{28.5}$$

since $\pi \circ A = \pi$. This says that A is orientation-preserving, which is a contradiction. $\square$

We next compute the de Rham cohomology of $\mathbb{RP}^n$.

Theorem 28.2. *We have*

$$H^k_{dR}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k < n \\ \mathbb{R} & k = n \text{ odd} \\ 0 & k = n \text{ even} \end{cases} \tag{28.6}$$

Proof. Since $A : S^n \to S^n$ satisfies $A^2 = Id_{S^n}$. For each $0 \leq k \leq n$, we have that

$$\Omega^k(S^n) = \Omega^k_+(S^n) \oplus \Omega^k_-(S^n) \tag{28.7}$$

where

$$\Omega^k_\pm(S^n) = \{\omega \in \Omega^k(S^n) \mid A^*\omega = \pm\omega\}, \tag{28.8}$$

because we can write

$$\omega = \frac{1}{2}(\omega + A^*\omega) + \frac{1}{2}(\omega - A^*\omega) \tag{28.9}$$

We claim that

$$\pi^* : \Omega^k(\mathbb{RP}^n) \to \Omega^k_+(S^n) \subset \Omega^k(S^n), \tag{28.10}$$

and is an isomorphism. Just as above $\pi \circ A = \pi$ implies that $A^*\pi^*\omega = \pi^*\omega$, so clearly the image of the pull-back lies in the space of invariant forms. Next, we need to show that if $\omega \in \Omega^k_+(S^n)$, then ω is the pull-back of a form $\alpha \in \Omega^k(\mathbb{RP}^n)$. That is, if $A^*\omega = \omega$, then for $p \in S^n$, and $X_1, \ldots, X_k \in T_pS^n$,

$$\omega_p(X_1, \ldots, X_k) = (\pi^*\alpha)_p(X_1, \ldots, X_k) = \alpha_{\pi(p)}(\pi_*X_1, \ldots, \pi_*X_k). \tag{28.11}$$

Let us use this equation to define $\alpha_{\pi(p)}$. We need to prove this is well-defined. Given $[p] \in \mathbb{RP}^n$, there are exactly 2 preimages p and $A(p) = -p$. The mappings $(\pi_*)_p : T_pS^n \to T_{[p]}\mathbb{RP}^n$, $(\pi_*)_{A(p)} : T_{A(p)}S^n \to T_{[p]}\mathbb{RP}^n$, and $(A_*)_p : T_pS^n \to T_{A(p)}S^n$ are isomorphisms. Given $[p] \in \mathbb{RP}^n$ and $Y_1, \dots Y_k \in T_{[p]}\mathbb{RP}^n$, there are exactly 2 choices:

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_p((\pi_*)_p^{-1}Y_1, \dots, (\pi_*)_p^{-1}Y_k), \tag{28.12}$$

or

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A(p)}((\pi_*)_{A(p)}^{-1}Y_1, \dots, (\pi_*)_{A(p)}^{-1}Y_k). \tag{28.13}$$

Since $\pi \circ A = \pi$, we have

$$(\pi_*)_{A(p)}(A_*)_p = (\pi_*)_p. \tag{28.14}$$

Since all of the mappings are isomorphisms, this implies that

$$(\pi_*)_{A(p)}^{-1} = (A_*)_p(\pi_*)_p^{-1}, \tag{28.15}$$

so (28.13) can be rewritten as

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A(p)}((A_*)_p(\pi_*)_p^{-1}Y_1, \dots, (A_*)_p(\pi_*)_p^{-1}Y_k). \tag{28.16}$$

The condition that ω is invariant under A, $A^*\omega = \omega$ says that

$$(A^*\omega)_p(X_1, \dots, X_k) = \omega_{A(p)}(A_*X_1, \dots, A_*X_k) \tag{28.17}$$

Choosing $X_i = (\pi_*)_p^{-1}Y_i$, we see that (28.12) = (28.13), therefore α is well-defined.

We next note that if $A^*\omega = \omega$ then

$$A^*d\omega = dA^*\omega = d\omega, \tag{28.18}$$

so the exterior derivative maps

$$d : \Omega^k_+(S^n) \to \Omega^{k+1}_+(S^n). \tag{28.19}$$

We therefore have the commutative diagram

$$\begin{CD}
\cdots @>d>> \Omega^{k-1}(\mathbb{RP}^n) @>d>> \Omega^k(\mathbb{RP}^n) @>d>> \Omega^{k+1}(\mathbb{RP}^n) @>d>> \cdots \\
@. @VV{\pi^*}V @VV{\pi^*}V @VV{\pi^*}V @. \\
\cdots @>d>> \Omega^{k-1}_+(S^n) @>d>> \Omega^k_+(S^n) @>d>> \Omega^{k+1}_+(S^n) @>d>> \cdots
\end{CD} \tag{28.20}$$

Since π^* is an isomorphism, we have

$$H^k_{dR}(\mathbb{RP}^n) = H^k(\Omega^*_+(S^n)) \tag{28.21}$$

We next note that if $A^*\omega = -\omega$ then

$$A^*d\omega = dA^*\omega = -d\omega, \tag{28.22}$$

so the exterior derivative maps

$$d : \Omega^k_-(S^n) \to \Omega^{k+1}_-(S^n). \tag{28.23}$$

We can also decompose the de Rham complex on S^n by

$$\Omega^{k-1}_+(S^n) \oplus \Omega^{k-1}_-(S^n) \xrightarrow{d \oplus d} \Omega^k_+(S^n) \oplus \Omega^k_-(S^n) \xrightarrow{d \oplus d} \Omega^{k+1}_+(S^n) \oplus \Omega^{k+1}_-(S^n). \tag{28.24}$$

This implies that

$$H^k_{dR}(S^n) = H^k(\Omega^*_+(S^n)) \oplus H^k(\Omega^*_-(S^n)). \tag{28.25}$$

Next, note that since A is a diffeomorphism satisfying $A^2 = Id_{S^n}$, we have that A induces a mapping on cohomology

$$A^* : H^k_{dR}(S^n) \to H^k_{dR}(S^n), \tag{28.26}$$

which also satisfies $(A^*)^2 = Id_{H^k(S^n)}$. Consequently, we can decompose

$$H^k_{dR}(S^n) = H^k_+(S^n) \oplus H^k_-(S^n), \tag{28.27}$$

where $H^k_+(S^n), H^k_-(S^n)$ are the invariant and anti-invariant cohomology classes, respectively. Equivalently, these are the $+1$ and -1 eigenspaces of A^*. We next claim that

$$H^k(\Omega^*_\pm(S^n)) = H^k_\pm(S^n). \tag{28.28}$$

This follows because we have two decompositions

$$\begin{aligned} H^k_{dR}(S^n) &= H^k(\Omega^*_+(S^n)) \oplus H^k(\Omega^*_-(S^n)) \\ &= H^k_+(S^n) \oplus H^k_-(S^n), \end{aligned} \tag{28.29}$$

the first factors are the $+1$ eigenspace, and the second factors are the -1 eigenspace, so they must be equal.

To finish the proof, we clearly have $H^k_{dR}(\mathbb{RP}^n) = \{0\}$ for $0 < k < n$. For $k = n$, we know that $A^* = (-1)^{n+1}$ acting on $H^n_{dR}(S^n)$, so we have that $H^n_{dR}(\mathbb{RP}^n) = \mathbb{R}$ if n is odd, and $H^n_{dR}(\mathbb{RP}^n) = \{0\}$ if n is even.

□

28.2 Finite group quotients

Let M be a smooth manifold, and Γ be a finite group acting freely on M. That is, we have is a smooth mapping

$$A : \Gamma \times M \to M \tag{28.30}$$

satisfying

$$A(g_1 g_2, p) = A(g_1, A(g_2, p)) \tag{28.31}$$

and $A(e, p) = p$ for all $p \in M$, where e is the identity element of Γ. The action A is *free* if $A(g, p) = p$ for some $p \in M$ implies that $g = e$. Let $A_g : M \to M$ denote the diffeomorphism $A_g(p) = A(g, p)$.

Recall that the quotient space M/Γ is a manifold. Furthermore, $\pi : M \to M/\Gamma$ is a covering space of order $|\Gamma|$ with deck transformation group Γ.

Definition 28.3. The space of invariant k-forms

$$\Omega^k_+(M) = \{\omega \in \Omega^k(M) \mid A_g^*\omega = \omega \text{ for all } g \in \Gamma\}. \tag{28.32}$$

Proposition 28.4. *The mapping $\pi^* : \Omega^k(M/\Gamma) \to \Omega^k_+(M)$ is an isomorphism.*

Proof. For each $g \in \Gamma$, we have $\pi \circ A_g = \pi$ which implies that $A^*\pi^*\omega = \pi^*\omega$, so clearly the image of the pull-back lies in the space of invariant forms. Next, we need to show that if $\omega \in \Omega^k_+(M)$, then ω is the pull-back of a form $\alpha \in \Omega^k(M/\Gamma)$. That is, if $A_g^*\omega = \omega$ for all $g \in \Gamma$, then for $p \in M$, and $X_1, \dots, X_k \in T_pM$,

$$\omega_p(X_1, \dots, X_k) = (\pi^*\alpha)_p(X_1, \dots, X_k) = \alpha_{\pi(p)}(\pi_*X_1, \dots, \pi_*X_k). \tag{28.33}$$

Let p be any preimage of $[p]$ under the projection π. The mapping $(\pi_*)_p : T_pM \to T_{[p]}(M/\Gamma)$ is an isomorphism. Given $Y_1, \dots Y_k \in T_{[p]}(M/\Gamma)$, we define

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_p((\pi_*)_p^{-1}Y_1, \dots, (\pi_*)_p^{-1}Y_k). \tag{28.34}$$

We need to show this is well-defined. Let $\tilde{p}$ be any other preimage. Then there exists $g \in \Gamma$ such that $\tilde{p} = A_g p$. Using $A_g p$ instead of p in the definition yields

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A_g(p)}((\pi_*)_{A_g(p)}^{-1}Y_1, \dots, (\pi_*)_{A_g(p)}^{-1}Y_k). \tag{28.35}$$

Since $\pi \circ A_g = \pi$, we have

$$(\pi_*)_{A_g(p)}((A_g)_*)_p = (\pi_*)_p. \tag{28.36}$$

Since all of these mappings are isomorphisms, this implies that

$$(\pi_*)_{A_g(p)}^{-1} = ((A_g)_*)_p(\pi_*)_p^{-1}, \tag{28.37}$$

so (28.35) can be rewritten as

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A_g(p)}(((A_g)_*)_p(\pi_*)_p^{-1}Y_1, \dots, (A_*)_p(\pi_*)_p^{-1}Y_k). \tag{28.38}$$

The condition that ω is invariant under A_g, $A_g^*\omega = \omega$ says that

$$\omega_p(X_1, \dots, X_k) = (A_g^*\omega)_p(X_1, \dots, X_k) = \omega_{A_g(p)}((A_g)_*X_1, \dots, (A_g)_*X_k). \tag{28.39}$$

Choosing $X_i = (\pi_*)_p^{-1}Y_i$, we see that (28.34) = (28.35), therefore α is well-defined. $\square$

Proposition 28.5. *We have*

$$H^k_{dR}(M/\Gamma) = H^k(\Omega^*_+(M)) \tag{28.40}$$

Proof. If $A^*\omega = \omega$ then

$$A^*d\omega = dA^*\omega = d\omega, \tag{28.41}$$

so the exterior derivative maps

$$d : \Omega^k_+(M) \to \Omega^{k+1}_+(M). \tag{28.42}$$

We therefore have the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(M/\Gamma) & \xrightarrow{d} & \Omega^k(M/\Gamma) & \xrightarrow{d} & \Omega^{k+1}(M/\Gamma) & \xrightarrow{d} & \cdots \\ & & \big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \pi^*} & & \\ \cdots & \xrightarrow{d} & \Omega^{k-1}_+(M) & \xrightarrow{d} & \Omega^k_+(M) & \xrightarrow{d} & \Omega^{k+1}_+(M) & \xrightarrow{d} & \cdots \end{array} \tag{28.43}$$

Since π^* is an isomorphism, this finishes the proof. □

Next, we have the following

Proposition 28.6. *The induced mapping*

$$\pi^* : H^k_{dR}(M/\Gamma) \to H^k_{dR}(M) \tag{28.44}$$

is injective.

Proof. We have that $\Omega^*_+(M) \subset \Omega^*(M)$ is a subcocomplex. This induces a mapping

$$H^k(\Omega^*_+(M)) \to H^k_{dR}(M) \tag{28.45}$$

by the following. Take an equivalence class $[\omega] \in H^k(\Omega^*_+(M))$ represented by $\omega \in \Omega^k_+(M)$, and map this to the cohomology class $[\omega] \in H^k_{dR}(M)$. This is well-defined, since if $\omega = d\alpha$ where $\alpha \in \Omega^{k-1}_+(M)$ then obviously $[\omega] = 0$ in $H^k_{dR}(M)$ also.

By the previous proposition, we just need to show that the mapping (28.45) is an injection. For this, we need to show that if $\omega \in \Omega^k_+(M)$ satisfies $\omega = d\alpha$ for $\alpha \in \Omega^{k-1}(M)$, then $\omega = d\beta$, where $\beta \in \Omega^{k-1}_+(M)$. For this, simply define

$$\beta = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} A_g^*\alpha. \tag{28.46}$$

For any $g' \in G$, this satisfies

$$A^*_{g'}\beta = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} A^*_{g'}A^*_g\alpha = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} A^*_{gg'}\alpha = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} A^*_g\alpha = \beta \tag{28.47}$$

so $\beta \in \Omega^{k-1}_+(M)$. Then

$$d\beta = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} dA^*_g\alpha = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} A^*_g d\alpha = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} d\alpha = d\alpha. \tag{28.48}$$

□

Definition 28.7. The kth Betti number of M is

$$b^k(M) = \dim(H^k_{dR}(M)). \tag{28.49}$$

We can phrase the above result as follows.

Theorem 28.8. *If $\pi : M \to M/\Gamma$ is as above, then*

$$b^k(\tilde{M}) \geq b^k(M). \tag{28.50}$$

Example 28.9. Recall we have $\pi : T^2 \to K$ as the orientation double cover of the Klein bottle. The inequalities $1 = b^2(T^2) > b^2(K) = 0$ and $2 = b^1(T^2) > b^1(K) = 1$ show that (28.50) can be strict.

Example 28.10 (Lens spaces)**.** Choose relatively prime integers $1 \leq q < p$. Consider $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}. \tag{28.51}$$

Let $\Gamma = \mathbb{Z}/p\mathbb{Z}$ act on S^3 generated by

$$(z_1, z_2) \mapsto (\zeta_p z_1, \zeta_p^q z_2), \tag{28.52}$$

where ζ_p is a primitive pth root of unity. It is easy to see this is a free action, so $S^3/\Gamma \equiv L(p, q)$ is a smooth 3-manifold. The inequalities (28.50) show that $b^k(S^3/\Gamma) = b^k(S^3)$, so the de Rham theory is unable to distinguish these spaces. However, since $\pi_1(S^3/\Gamma) = \Gamma$, if $p \neq p'$ then $L(q, p)$ cannot be homeomorphic to $L(q', p')$, so the fundamental group *can* distinguish these. An interesting question is: when exactly are $L(q, p)$ and $L(q', p)$ diffeomorphic?

28.3 Compactly supported cohomology

If M is noncompact, the mapping $\pi : M \to M/\Gamma$ is proper. Therefore we have

$$\pi^* : \Omega_c^k(M/\Gamma) \to \Omega_{c,+}^k(M). \tag{28.53}$$

The above arguments holds verbatim for compactly supported cohomology, so we have:

Proposition 28.11. *The mapping $\pi^* : \Omega_c^k(M/\Gamma) \to \Omega_{c,+}^k(M)$ is an isomorphism, and*

$$H_{c,dR}^k(M/\Gamma) = H^k(\Omega_{c,+}^*(M)). \tag{28.54}$$

Furthermore, the induced mapping

$$\pi^* : H_{c,dR}^k(M/\Gamma) \to H_{c,dR}^k(M) \tag{28.55}$$

is injective.

As an application, we can give another proof of the following.

Theorem 28.12. *If M is a smooth manifold of dimension n which is non-orientable and connected then $H_{c,dR}^n(M) = \{0\}$.*

Proof. Recall the construction of the orientable double cover $\pi : \tilde{M} \to M$: the bundle $\Lambda^n(M)$ is a real line bundle. Endow this bundle with a Riemannian metric, and then $\tilde{M}$ is the unit sphere bundle. Since M is non-orientable, $\tilde{M}$ is connected. The mapping $A : \omega_p \mapsto -\omega_p$ is clearly a free $\mathbb{Z}_2$-action on $\tilde{M}$, and $M = \tilde{M}/\mathbb{Z}_2$.

We claim that $\tilde{M}$ is orientable. Given $p \in M$, there are precisely 2 preimages $\tilde{p}$ and $A\tilde{p}$ under π. The point $\tilde{p} = \omega_p$ is, by definition, a non-zero n-form on T_pM, so determines an orientation on T_pM. The mapping $\pi_* : T_{\tilde{p}}\tilde{M} \to T_pM$ is an isomorphism, so we give $T_{\tilde{p}}\tilde{M}$ the induced orientation. Similarly, we give $T_{A\tilde{p}}$ the induced orientation. This clearly gives a smooth orientation on $\tilde{M}$, called the *tautological* orientation. Note that the mapping A is orientation-reversing (otherwise, the quotient space would also be orientable).

For compactly supported cohomology, we have

$$\pi^* : H^k_{c,dR}(M/\Gamma) \to H^k_{c,dR}(M) \tag{28.56}$$

is injective. Given $\omega \in \Omega^n_c(M)$, we have $\tilde{\omega} = \pi^*\omega$ satisfies $A^*\tilde{\omega} = \tilde{\omega}$. But

$$\int_{\tilde{M}} \tilde{\omega} = -\int_{\tilde{M}} A^*\tilde{\omega} = -\int_{\tilde{M}} A^*\tilde{\omega}, \tag{28.57}$$

since A is orientation-reversing. Consequently,

$$\int_{\tilde{M}} \tilde{\omega} = 0. \tag{28.58}$$

By the first part of the proof of Theorem 23.1, this implies that $[\tilde{\omega}] = 0 \in H^n_{c,dR}(\tilde{M})$. But since π^* is injective, this implies that $[\omega] = 0 \in H^n_{c,dR}(M)$. $\square$

29 Lecture 29

29.1 Mayer-Vietoris for cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega^{p-1}_c(M) \xrightarrow{d} \Omega^p_c(M) \xrightarrow{d} \Omega^{p+1}_c(M) \xrightarrow{d} \cdots, \tag{29.1}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Write $M = U \cup V$ as the union of two open sets in M. Note that if $U_1 \subset U_2$ and $\omega \in \Omega^k_c(U_1)$ then ω extends to be a compactly supported form in U_2. Letting $\iota : U_1 \hookrightarrow U_2$ denote the inclusion mapping, we denote by $i_*\omega$ this extension map on forms. We claim that the following sequence is exact:

$$0 \longrightarrow \Omega^p_c(U \cap V) \xrightarrow{\tilde{\alpha}^p} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{\tilde{\beta}^p} \Omega^p_c(U \cup V) \longrightarrow 0 \tag{29.2}$$

where

$$\tilde{\alpha}^p(\omega_{U\cap V}) = \big((i_{U\cap V \hookrightarrow U})_*\omega_{U\cap V}, -(i_{U\cap V\hookrightarrow V})_*\omega_{U\cap V}\big) \tag{29.3}$$

and

$$\tilde{\beta}^p(\omega_U, \omega_V) = (i_{U\hookrightarrow M})_*\omega_U + (i_{V\hookrightarrow M})_*\omega_V. \tag{29.4}$$

To see this, $\tilde{\alpha}^p$ is obviously injective. For exactness at the middle step, obviously $\tilde{\beta}^p\tilde{\alpha}^p\omega = 0$. If $\tilde{\beta}^p(\omega_U, \omega_V) = 0$, then $\omega_U = -\omega_V$. This implies that the support of both forms is contained in $U \cap V$, and since they are equal there, take $\omega_{U\cap V} = \omega_U$, and then $(\omega_U, \omega_V) = \tilde{\alpha}^p(\omega_U)$.

To show that $\tilde{\beta}$ is onto, let $\omega \in \Omega^p_c(M)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$.

Consequently, from the ziz-zag Lemma, we obtain a long exact sequence

$$\cdots \xrightarrow{\tilde{\delta}^{p-1}} H^p_{c,dR}(U\cap V) \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U\cup V) \xrightarrow{\tilde{\delta}^p} \cdots \tag{29.5}$$

Let us review the definition of the mapping $\tilde{\delta}^p$. Given a cohomology class $[\omega] \in H^p_{c,dR}(U\cup V)$, represented by $\omega \in \Omega^p_c(U\cup V)$ with $d\omega = 0$, we first write $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$, then we apply the exterior derivative to get

$$(d(\phi_U\omega), d(\phi_V\omega)) = (d\phi_U \wedge \omega, d\phi_V \wedge \omega) \in \Omega^p_c(U) \oplus \Omega^p_c(V) \tag{29.6}$$

Either of these elements is supported in $U \cap V$ and then since $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$,

$$\tilde{\delta}^p\omega = [d\phi_U \wedge \omega] = [-d\phi_V \wedge \omega] \in H^{p+1}_{c,dR}(U \cap V). \tag{29.7}$$

Remark 29.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

29.2 Good covers

Recall the Poincaré lemma for compactly supported cohomology, Lemma 20.1, showed that

$$H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{29.8}$$

Remark 29.2. This shows that $H^*_{c,dR}(M)$ is not a homotopy invariant, since (20.26) is not the same as the cohomology of a point. But of course, $H^*_{c,dR}(M)$ *is* a diffeomorphism invariant.

We have the following definition.

Definition 29.3. We say that a manifold M has a good cover U_i each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has the same de Rham cohomology as $\mathbb{R}^n$, and the same compactly supported de Rham cohomology as $\mathbb{R}^n$.

Recall we proved ealier that if M has a finite good cover, then the de Rham cohomology is finite-dimensional. We next extend this to compactly supported cohomology.

Corollary 29.4. *If M has a finite good cover, then the compactly supported de Rham cohomology is finite-dimensional.*

Proof. Recall that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{29.9}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{29.10}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

We prove the corollary using induction on the number of open sets in a finite good cover. To see this, let k be the number of sets in a good cover. For $k = 1$, we know the corollary is true. Assume the corollary is true up to k, and let $\{U_1, \dots, U_{k+1}\}$ be a good cover of a manifold M. Let $U = U_1 \cup \cdots \cup U_k$, and let $V = U_{k+1}$. Then U and V have good covers with fewer that $k+1$ open sets, so their compactly supported de Rham cohomology is finite-dimensional. Also, $U_1 \cap U_{k+1}, \dots, U_k \cap U_{k+1}$ is a good cover of $U \cap V$, so the theorem is true for $U \cap V$ as well.

Now we look at the following portion of the compactly supported Mayer-Vietoris sequence

$$\cdots \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} H^{p+1}_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^{p+1}} \cdots \tag{29.11}$$

The above observation then implies that $H^p_{c,dR}(U \cup V)$ is finite-dimensional. □

Next, we need the following technical lemma.

Lemma 29.5. *If U is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{c,dR}(U) \cong H^k_{c,dR}(\mathbb{R}^n)$ for all $0 \leq k \leq n$. Furthermore, an isomorphism of $H^n_{c,dR}(U)$ and $\mathbb{R}$ is given by integration.*

Proof. The proof is not too difficult, but we only give an outline. The main idea is to first show it is true for a star-shaped open set U whose boundary is a smooth graph over a small sphere around the star point; it is easy to show such a set is diffeomorphic to the unit ball in $\mathbb{R}^n$ by a diffeomorphism which is maps lines through the star point to lines through the origin. Then one shows that an arbitary star-shaped open set U can be approximated from the inside by star-shaped open sets U_i with smooth boundary such that $U_i \subset U_{i+1}$ and $U = \cup_i U_i$. Then if $\omega \in \Omega^k(U)$ has compact support K, then there exists i such that $K \subset U_i$, which reduces to the case with smooth boundary.

For $k = n$, we already determined the top degree compactly supported cohomology of any oriented manifold; the isomorphism with $\mathbb{R}$ is given by integration; see Theorem 23.1. □

Remark 29.6. It is actually true that U is diffeomorphic to $\mathbb{R}^n$, but this is more difficult to show, and we do not need such a strong result.

Corollary 29.7. *If M is compact, then M admits a finite good cover.*

Proof. Using a Riemannian metric, there exists a covering of M by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map at any point, a geodesically convex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$. This is contractible, so from the Poincaré Lemma, it has the same de Rham cohomology as $\mathbb{R}^n$. Lemma 29.5 tells us that it also has the same compactly supported de Rham cohomology as $\mathbb{R}^n$, so we are done. □

29.3 The five lemma

We next discuss some homological algebra, which we will use next time to prove Poincaré duality and the Künneth formula.

Lemma 29.8 (The five lemma)**.** *Assume the diagram*

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{\alpha_1} & V_2 & \xrightarrow{\alpha_2} & V_3 & \xrightarrow{\alpha_3} & V_4 & \xrightarrow{\alpha_4} & V_5 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\ W_1 & \xrightarrow{\beta_1} & W_2 & \xrightarrow{\beta_2} & W_3 & \xrightarrow{\beta_3} & W_4 & \xrightarrow{\beta_4} & W_5 \end{array} \tag{29.12}$$

commutes, and has exact rows. If $\phi_1, \phi_2, \phi_4, \phi_5$ are isomorphisms, then ϕ_3 is also an isomorphism.

Proof. Injectivity of ϕ_3: If $\phi_3(v_3) = 0$, then $\beta_3(\phi_3(v_3) = 0 = \phi_4\alpha_3(v_3)$. Since ϕ_4 is injective, $\alpha_3(v_3) = 0$. By exactness, $v_3 = \alpha_2(v_2)$. Then $\phi_3\alpha_2(v_2) = 0 = \beta_2\phi_2(v_2)$. By exactness, $\phi_2(v_2) = \beta_1(w_1)$. By surjectivity of ϕ_1, $w_1 = \phi_1(v_1)$. Then

$$\phi_2(v_2) = \beta_1\phi_1(v_1) = \phi_2\alpha_1(v_1), \tag{29.13}$$

but since ϕ_2 is injective, this implies that $v_2 = \alpha_1(v_1)$. Finally, $v_2 = \alpha_2(v_2) = \alpha_2\alpha_1(v_1) = 0$, by exactness.

The proof of surjectivity is similar, and left to the student. □

Exercise 29.9. Prove the surjectivity of ϕ_3.

30 Lecture 30

30.1 Some more algebra

Another useful lemma is the following.

Lemma 30.1. *If the sequence*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{30.1}$$

is exact at W_2, then the dual sequence

$$W_3^* \xrightarrow{\beta^*} W_2^* \xrightarrow{\alpha^*} W_1^* \tag{30.2}$$

is exact at W_2^*.

Proof. First, if $w_3^* \in W_3^*$, and $w_1 \in W_1$, then

$$\alpha^*(\beta^* w_3^*)(w_1) = (\beta^* w_3^*)(\alpha(w_1)) = w_3^*(\beta\alpha(w_1)) = 0, \tag{30.3}$$

since $\beta \circ \alpha = 0$ by assumption. This proves that $Im(\beta^*) \subset Ker(\alpha^*)$. For the other direction, if $w_2^* \in Ker(\alpha^*)$, then for all $w_1 \in W_1$, $\alpha^*(w_2^*)(w_1) = w_2^*(\alpha(w_1))$. So the element $0 =$

$w_2^* \circ \alpha \in W_1^*$. We want to find $w_3^* \in W_3^*$ such that $w_2^* = \beta^* w_3^*$. For all $w_2 \in W_2$, this is $w_2^*(w_2) = w_3^* \beta w_2$, which is just $w_2^* = w_3^* \circ \beta$. So if $w_3 \in W_3$ is of the form $\beta(w_2)$ then define

$$w_3^*(w_3) \equiv w_2^*(w_2). \tag{30.4}$$

If $w_3 = \beta(w_2')$, then $\beta(w_2 - w_2') = 0$, so $w_2 - w_2' = \alpha(w_1)$. Then

$$w_2^*(w_2 - w_2') = w_2^*(\alpha(w_1)) = (w_2^*\alpha)(w_1) = 0. \tag{30.5}$$

So we have defined w_3^* on the subspace $Im(\beta) \subset W_3$. To extend to a linear mapping on all of W_3, just take any subspace so that $W_3 = Im(\beta) \oplus W$, and define w_3^* to vanish on W. Then the condition $w_2^* = w_3^* \circ \beta$ is obviously satisifed. $\square$

Next, another lemma.

Lemma 30.2. *Let B and C be vector spaces. Then*

$$(B \oplus C)^* \cong B^* \oplus C^* \tag{30.6}$$

Proof. Let $\iota_B : B \to B \oplus C$ and $\iota_C : C \to B \oplus C$ denote the inclusion mappings. Define $f : (B \oplus C)^* \to B^* \oplus C^*$ by

$$f(m^*) = (\iota_B^* m^*, \iota_C^* m^*). \tag{30.7}$$

Define $g : B^* \oplus C^* \to (B \oplus C)^*$ by

$$g(b^*, c^*)(b, c) = b^*(b) + c^*(c). \tag{30.8}$$

Then

$$\pi_{B^*}(f \circ g)(b^*, c^*)(b) = \iota_B^* g(b^*, c^*)(b) = g(b^*, c^*)(b, 0) = b^*(b). \tag{30.9}$$

Similarly,

$$\pi_{C^*}(f \circ g)(b^*, c^*)(c) = \iota_C^* g(b^*, c^*)(c) = g(b^*, c^*)(0, c) = c^*(c). \tag{30.10}$$

This implies that $f \circ g = Id_{B^* \oplus C^*}$. Next,

$$\begin{aligned} g \circ f(m^*)(b, c) &= g(\iota_B^* m^*, \iota_C^* m^*)(b, c) = \iota_B^* m^*(b) + \iota_C^* m^*(c) \\ &= m^*(\iota_B(b)) + m^*(\iota_C(c)) = m^*(b, 0) + m^*(0, c) = m^*(b, c), \end{aligned} \tag{30.11}$$

so $g \circ f = Id_{(B \oplus C)^*}$. $\square$

30.2 Poincaré duality

If M is any oriented manifold of dimension n, then we have a pairing

$$\Omega^k(M) \times \Omega_c^{n-k}(M) \to \mathbb{R}, \tag{30.12}$$

given by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta. \tag{30.13}$$

By Stokes' Theorem, this mapping descends to cohomology, and since this mapping is bilinear, we obtain a pairing

$$PD : H^k_{dR}(M) \otimes H^{n-k}_c(M) \to \mathbb{R}. \tag{30.14}$$

In the case M^n has the same de Rham cohomology and the same compactly supported de Rham cohomology as $\mathbb{R}^m$, then we have $H^k_{c,dR}(M) \cong H^{n-k}_{dR}(M)$. Furthermore, we have an isomorphism

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{30.15}$$

given by $PD(\alpha)(\beta) = \int_M \alpha \wedge \beta$. This follows from Theorem 23.1.

Theorem 30.3. *If M^n is orientable and has a finite good cover, then*

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{30.16}$$

is an isomorphism for all $0 \leq k \leq n$.

Proof. Let $m = n - k$, and consider the diagram

$$\begin{tikzcd}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \arrow[r, "\alpha^{k-1}"] \arrow[d, "PD\oplus PD"] & H^{k-1}_{dR}(U \cap V) \arrow[r, "\delta^{k-1}"] \arrow[d, "PD"] & H^k_{dR}(U \cup V) \arrow[r, "\beta^k"] \arrow[d, "PD"] & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[r, "\alpha^k"] \arrow[d, "PD\oplus PD"] & H^k_{dR}(U \cap V) \arrow[d, "PD"] \\
(H^{m+1}_{c,dR}(U) \oplus H^{m+1}_{d,dR}(V))^* \arrow[r, "(\tilde{\alpha}^{m+1})^*"] & H^{m+1}_{c,dR}(U \cap V)^* \arrow[r, "(\tilde{\delta}^m)^*"] & H^m_{c,dR}(U \cup V)^* \arrow[r, "(\tilde{\beta}^m)^*"] & (H^m_{c,dR}(U) \oplus H^m_{c,dR}(V))^* \arrow[r, "(\tilde{\alpha}^m)^*"] & H^m_{c,dR}(U \cap V)^*
\end{tikzcd} \tag{30.17}$$

The top horizontal row is exact since it is the usual Mayer-Vietoris sequence. The bottom horizontal row is exact since is the dual exact sequence of the Mayer-Vietoris sequence with compact support. We next claim that this diagram commutes up to sign, so by changing some of the vertical maps to their negatives if necessary, we obtain a commutative diagram.

Consider the square

$$\begin{tikzcd}
H^{k-1}_{dR}(U \cap V) \arrow[r, "\delta^{k-1}"] \arrow[d, "PD"] & H^k_{dR}(U \cup V) \arrow[d, "PD"] \\
H^{m+1}_{c,dR}(U \cap V)^* \arrow[r, "(\tilde{\delta}^m)^*"] & H^m_{c,dR}(U \cup V)^*
\end{tikzcd} \tag{30.18}$$

For the mapping

$$PD \circ \delta^{k-1} : H^{k-1}_{dR}(U \cap V) \to H^m_{c,dR}(U \cup V)^* \tag{30.19}$$

let's take an element $[\omega] \in H^{k-1}_{dR}(U \cap V)$, and an element $[\tau] \in H^m_{c,dR}(U \cup V)$. Then

$$(PD \circ \delta^{k-1}[\omega])[\tau] = PD(\delta^{k-1}[\omega])[\tau] = \int_M (\delta^{k-1}\omega) \wedge \tau. \tag{30.20}$$

Recall from our discussion of the Mayer-Vietoris sequence that

$$\delta^{k-1}\omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{30.21}$$

This form is supported in $U \cap V$, so we have

$$(PD \circ \delta^{k-1}[\omega])[\tau] = \int_{U\cap V} (\delta^{k-1}\omega) \wedge \tau = \int_{U\cap V} (-d\phi_U \wedge \omega) \wedge \tau. \tag{30.22}$$

Next, we look at the mapping

$$(\tilde{\delta}^m)^* \circ PD : H^{k-1}_{dR}(U \cap V) \to H^m_{c,dR}(U \cup V)^*. \tag{30.23}$$

We then have

$$\big((\tilde{\delta}^m)^* \circ PD[\omega]\big)[\tau] = PD[\omega](\tilde{\delta}^m[\tau]) = \int_{U\cap V} \omega \wedge \tilde{\delta}^m \tau. \tag{30.24}$$

Recall from our discussion of the compactly supported Mayer-Vietoris sequence that

$$\tilde{\delta}^m \tau = [d\phi_U \wedge \tau] = [-d\phi_V \wedge \tau] \in H^{m+1}_{c,dR}(U \cap V). \tag{30.25}$$

So we have

$$\big((\tilde{\delta}^m)^* \circ PD[\omega]\big)[\tau] = \int_{U\cap V} \omega \wedge d\phi_U \wedge \tau = (-1)^{k-1} \int_{U\cap V} d\phi_U \wedge \omega \wedge \tau. \tag{30.26}$$

So we see that

$$(\tilde{\delta}^m)^* \circ PD = (-1)^k PD \circ \delta^{k-1} \tag{30.27}$$

Next, we look at the square

$$\begin{array}{ccc} H^k_{dR}(U \cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U) \oplus H^k_{dR}(V) \\ \Big\downarrow{\scriptstyle PD} & & \Big\downarrow{\scriptstyle PD \oplus PD} \\ H^m_{c,dR}(U \cup V)^* & \xrightarrow{(\tilde{\beta}^m)^*} & \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^* \end{array} \tag{30.28}$$

For the mapping

$$(PD \oplus PD) \circ \beta^k : H^k_{dR}(U \cup V) \to \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^*, \tag{30.29}$$

choose $[\omega] \in H^k_{dR}(U \cup V)$, $[\tau_1] \in H^m_{c,dR}(U)$ and $[\tau_2] \in H^m_{c,dR}(V)$, and we have

$$\begin{aligned} \big((PD \oplus PD) \circ \beta^k[\omega]\big)([\tau_1], [\tau_2]) &= \big(PD_U \circ \beta^k_U[\omega]\big)([\tau_1]) + PD_V \circ \beta^k_V[\omega]\big)([\tau_2]) \\ &= \int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2. \end{aligned} \tag{30.30}$$

Next, we look at the mapping

$$(\tilde{\beta}^m)^* \circ PD : H^k_{dR}(U \cup V) \to \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^*, \tag{30.31}$$

for which we have

$$\begin{aligned}\Big((\tilde{\beta}^m)^* \circ PD[\omega]\Big)([\tau_1],[\tau_2]) &= PD[\omega](\tilde{\beta}^m(([\tau_1],[\tau_2])) = PD[\omega](\tau_1+\tau_2)\\ &= \int_M \omega\wedge(\tau_1+\tau_2) = \int_M \omega\wedge\tau_1 + \int_M \omega\wedge\tau_2\\ &= \int_U \omega\wedge\tau_1 + \int_V \omega\wedge\tau_2,\end{aligned} \tag{30.32}$$

since τ_1 has compact support on U and τ_2 has compact support on V. So we have that

$$(PD \oplus PD)\circ\beta^k = (\tilde{\beta}^m)^* \circ PD. \tag{30.33}$$

We leave the remaining α square(s) as an exercise.

By the five lemma, if the outer 4 vertical maps are isomorphisms, then so is the central vertical map. The proof is completed by induction on the number of open sets in the good cover, since we know it is true for $\mathbb{R}^n$ from the previous lecture. □

Exercise 30.4. Show that the α squares commute up to sign.

31 Lecture 31

31.1 Some consequences of Poincaré duality

An immediate corollary of Poincaré duality is the following.

Corollary 31.1. *If M^n is a connected and orientable n-manifold with a finite good cover then $H^k_{dR}(M)$ and $H^{n-k}_{c,dR}(M)$ have the same dimension. If M is moreover compact, then $H^k_{dR}(M)$ and $H^{n-k}_{dR}(M)$ have the same dimension.*

We also have the following corollary.

Corollary 31.2. *If M^n is a compact odd-dimensional manifold, then the Euler characteristic $\chi(M)=0$.*

Proof. If n is odd and M is orientable, then by Poincaré duality,

$$\begin{aligned}\chi(M) = \sum_{i=0}^{n}(-1)^i b^i(M) &= \sum_{i=0}^{[n/2]}(-1)^i b^i(M) + \sum_{i=[n/2]+1}^{n}(-1)^i b^i(M)\\ &= \sum_{i=0}^{[n/2]}(-1)^i b^i(M) + \sum_{j=0}^{[n/2]}(-1)^{[n/2]+1+j} b^{[n/2]+1+j}(M)\\ &= \sum_{i=0}^{[n/2]}(-1)^i b^i(M) - \sum_{j=0}^{[n/2]}(-1)^j b^j(M) = 0.\end{aligned} \tag{31.1}$$

If M is not orientable, then let $\tilde{M}$ be the orientable double cover. Then $0=\chi(\tilde{M}) = 2\chi(M)$ implies that $\chi(M)=0$. □

Remark 31.3. By the Poincaré-Hopf Theorem, this implies that every odd-dimensional manifold admits a nowhere-vanishing vector field.

31.2 The intersection form

Let M^n be a compact even-dimensional oriented manifold, and write $n = 2m$. Poincaré duality says that

$$PD : H^m_{dR}(M) \otimes H^m(M) \to \mathbb{R} \tag{31.2}$$

is a non-degenerate pairing. We have that

$$PD(\alpha, \beta) = \int_M \alpha \wedge \beta = (-1)^{m^2} \int_M \beta \wedge \alpha = (-1)^{m^2} PD(\beta, \alpha), \tag{31.3}$$

therefore if m is even, PD is symmetric, while if m is odd, PD is skew-symmetric.

Corollary 31.4. *If M^n is a compact oriented manifold of dimension $n = 4k+2$, then $\chi(M)$ is even.*

Proof. Using Poincaré duality, we have that

$$\begin{aligned}
\chi(M) = \sum_{i=0}^{4k+2}(-1)^i b^i(M) &= \sum_{i=0}^{2k+1}(-1)^i b^i(M) + \sum_{i=2k+2}^{4k+2}(-1)^i b^i(M) \\
&= -b^{2k+1}(M) + \sum_{i=0}^{2k}(-1)^i b^i(M) + \sum_{j=0}^{2k}(-1)^{2k+2+j} b^{2k+2+j}(M) \\
&= -b^{2k+1}(M) + 2\sum_{i=0}^{2k}(-1)^i b^i(M).
\end{aligned} \tag{31.4}$$

So the claim is equivalent to $b^{2k+1}(M)$ being even. However, as observed above, the intersection form is a non-degenerate skew-symmetric form on $H^{2k+1}(M)$, so it must be even dimensional. ☐

In case $n = 4k$, then PD is a nondegenerate symmetric bilinear form on $H^{2k}(M)$. By Sylvester's Theorem, we can make the following definition.

Definition 31.5. If M^{4k} is a compact oriented manifold of dimension a multiple of 4, let $b^{2k}_+(M)$ denote the number of positive eigenvalues and $b^{2k}_-(M)$ denote the number of negative eigenvalues of PD on $H^{2k}(M)$. The *signature* of M^{4k} is

$$\sigma(M^{4k}) \equiv b^{2k}_+(M) - b^{2k}_-(M). \tag{31.5}$$

Since PD is non-degenerate, it cannot have a zero eigenvalue, so we must have

$$b^{2k}(M) = b^{2k}_+(M) + b^{2k}_-(M). \tag{31.6}$$

Note that $b^{2k}_+(M), b^{2k}_-(M)$, and $\sigma(M)$ are oriented diffeomorphism invariants of a compact oriented manifold M^{4k} of dimension a multiple of four.

31.3 Künneth formula

We begin with an algbraic lemma.

Lemma 31.6. *If the sequence of vector spaces*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{31.7}$$

is exact at W_2, and V is any finite dimensional vector space, then the sequence

$$W_1 \otimes V \xrightarrow{\alpha\otimes 1_V} W_2 \otimes V \xrightarrow{\beta\otimes 1_V} W_3 \otimes V \tag{31.8}$$

is exact at $W_2 \otimes V$.

Proof. Clearly, $(\beta \otimes 1_V) \circ (\alpha \otimes 1_V) = 0$, which implies that $\mathrm{Image}(\alpha \otimes 1_V) \subset \mathrm{Ker}(\beta \otimes 1_V)$. For the reverse inclusion, choose a basis $e_i, 1 \leq i \leq \dim(V)$, of V. Any element $v \in W_2 \otimes V$ may be written as a linear combination

$$v = \sum_{i=1}^{\dim(V)} b_i \otimes e_i, \tag{31.9}$$

where $b_i \in W_2$. If $\beta \otimes 1_V(v) = 0$, then

$$0 = \sum_{i=1}^{\dim(V)} \beta(b_i) \otimes e_i. \tag{31.10}$$

The only way this element can vanish in the tensor product is that $\beta(b_i) = 0$ for all i. By exactness of the original sequence, this implies that $b_i = \alpha(a_i)$ for elements $a_i \in W_1$. Then

$$v = \sum_{i=1}^{\dim(V)} \alpha(a_i) \otimes e_i = (\alpha \otimes 1_V)\Big(\sum_{i=1}^{\dim(V)} a_i \otimes e_i\Big). \tag{31.11}$$

□

Let M and N be smooth manifolds. Let $\pi : M \times N \to M$ denote the projection onto the first factor, and $\rho : M \times N \to N$ be projection onto the second factor. There is a mapping from

$$K : \Omega^p(M) \times \Omega^q(N) \to \Omega^{p+q}(M \times N) \tag{31.12}$$

given by $K : (\omega, \phi) \mapsto \pi^*\omega \wedge \rho^*\phi$. Since K is bilinear, there is an induced mapping

$$K : \Omega^p(M) \otimes \Omega^q(N) \to \Omega^{p+q}(M \times N) \tag{31.13}$$

Note that if $d\omega = 0$ and $d\phi = 0$ then

$$\begin{aligned} K((\omega + d\alpha), \phi) &= \pi^*(\omega + d\alpha) \wedge \rho^*\phi \\ &= \pi^*\omega \wedge \rho^*\phi + d\pi^*\alpha \wedge \rho^*\phi \\ &= \pi^*\omega \wedge \rho^*\phi + d(\pi^*\alpha \wedge \rho^*\phi). \end{aligned} \tag{31.14}$$

Consequently, there is an induced mapping

$$K : H^p_{dR}(M) \otimes H^q_{dR}(N) \to H^{p+q}(M \times N). \tag{31.15}$$

By taking direct sums, we obtain a mapping

$$\psi : \bigoplus_{p+q=k} H^p(M) \otimes H^q(N) \to H^k(M \times N). \tag{31.16}$$

The next theorem says that Ψ is an isomorphism for each k.

Theorem 31.7 (Künneth formula)**.** *For any $k \in \mathbb{Z}, k \geq 0$, we have*

$$H^k_{dR}(M \times N) \cong \bigoplus_{p+q=k} H^p_{dR}(M) \otimes H^q_{dR}(N). \tag{31.17}$$

Proof. If $M = U \cup V$, then consider the Mayer-Vietoris sequence on M:

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{31.18}$$

In the category of vector spaces, tensor products preserve exact sequences, so we have an exact sequence

$$\cdots \longrightarrow H^p(U \cup V) \otimes H^{k-p}(N) \longrightarrow (H^p(U) \otimes H^{k-p}(N)) \oplus (H^p(V) \otimes H^{k-p}(N)) \longrightarrow H^p(U \cap V) \otimes H^{k-p}(N) \longrightarrow \cdots \tag{31.19}$$

Next, take the direct sum on p from 0 to k, and we have a long exact sequence. Consider the following diagram.

$$\begin{array}{ccccc} \oplus_{p=0}^k H^p(U \cup V) \otimes H^{k-p}(N) & \longrightarrow & \oplus_{p=0}^k (H^p(U) \otimes H^{k-p}(N)) \oplus (H^p(V) \otimes H^{k-p}(N)) & \longrightarrow & \oplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(N) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ H^k((U \cup V) \times N) & \longrightarrow & H^k(U \times N) \oplus H^k(V \times N) & \longrightarrow & H^k((U \cap V) \times N), \end{array} \tag{31.20}$$

where the lower row is the Mayer-Vietoris sequence with respect to the open cover $\{U \times N, V \times N\}$ of $M \times N$. This is straightforward to check commutativity. If we continue the diagram to the right, we see the following square

$$\begin{array}{ccc} \oplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(N) & \xrightarrow{\delta_1} & \oplus_{p=0}^k H^{p+1}(U \cup V) \otimes H^{k-p}(N) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ H^k((U \cap V) \times F) & \xrightarrow{\delta_2} & H^{k+1}((U \cup V) \times N). \end{array} \tag{31.21}$$

If $\omega \otimes \phi \in H^p(U \cap V) \otimes H^{k-p}(N)$. Then

$$\psi\delta_1(\omega \otimes \phi) = \psi((\delta\omega) \otimes \phi) = \pi^*(\delta\omega) \wedge \rho^*\phi \tag{31.22}$$

$$\delta_2\psi(\omega \otimes \phi) = \delta_2(\pi^*\omega \wedge \rho^*\phi). \tag{31.23}$$

Recall the definition of δ: if ρ_U, ρ_V is a partition of unity with respect to the covering $\{U, V\}$ of M, then

$$\delta(\omega) = \begin{cases} d(\rho_V\omega) & \text{in } U \\ -d(\rho_U\omega) & \text{in } V. \end{cases} \tag{31.24}$$

Note also that $\pi^*\rho_U, \pi^*\rho_V$ is a partition of unity with respect to the open covering $\{U \times N, V \times N\}$ of $M \times N$. Therefore

$$\delta_2(\gamma) = \begin{cases} d(\pi^*\rho_V\gamma) & \text{in } U \times N \\ -d(\pi^*\rho_U\gamma) & \text{in } V \times N. \end{cases} \tag{31.25}$$

On the set $U \times N$, we have

$$\begin{aligned} \psi\delta_1(\omega \otimes \phi) &= \pi^*(d(\rho_V\omega)) \wedge \rho^*\phi \\ &= d(\pi^*\rho_V) \wedge \pi^*\omega \wedge \rho^*\phi, \end{aligned} \tag{31.26}$$

and

$$\begin{aligned} \delta_2\psi(\omega \otimes \phi) &= \delta_2(\pi^*\omega \wedge \rho^*\phi) \\ &= d\big(\pi^*\rho_V(\pi^*\omega \wedge \rho^*\phi)\big) \\ &= d(\pi^*\rho_V) \wedge \pi^*\omega \wedge \rho^*\phi. \end{aligned} \tag{31.27}$$

If $M = \mathbb{R}^n$, this is the Poincaré Lemma, see Proposition 19.1. The result then follows by the five lemma and the usual argument of induction on the number of sets in a good cover of M. □

There is also a Künneth formula for cohomology with compact support.

Theorem 31.8. *Let M and N be orientable. Then for any $k \in \mathbb{Z}, k \geq 0$, we have*

$$H^k_{c,dR}(M \times N) \cong \bigoplus_{p+q=k} H^p_{c,dR}(M) \otimes H^q_{c,dR}(N). \tag{31.28}$$

Proof. If M and N are orientable, then $M \times N$ is orientable. The result then follows from the Künneth formula for ordinary de Rham cohomology, and Poincaré duality. □

Remark 31.9. The above result is true without any orientability assumption. For this, use the Mayer-Vietoris sequence for compactly supported cohomology, and imitate the above proof of Künneth for ordinary de Rham cohomology.

We end this lecture with some corollaries. The first is the cohomology of higher-dimensional tori.

Corollary 31.10. *Let*

$$T^n = \overbrace{S^1 \times \cdots \times S^1}^{n}, \tag{31.29}$$

then

$$\dim(H^k(T^n)) = \binom{n}{k}. \tag{31.30}$$

The next corollary is the cohomology of the product of spheres.

Corollary 31.11. *Let $m, n \in \mathbb{Z}_+$, then*

$$H^k_{dR}(S^n \times S^m) = \begin{cases} \mathbb{R} & k = 0, m+n \\ \mathbb{R} & k = m \text{ or } n \text{ if } m \neq n \\ \mathbb{R}^2 & k = m \text{ if } m = n \\ 0 & \text{otherwise} \end{cases}. \tag{31.31}$$

In a special case of the above, we furthermore have the following.

Corollary 31.12. *Let $m \in \mathbb{Z}_+$. Then*

$$b^{2m}_{\pm}(S^{2m} \times S^{2m}) = 1 \tag{31.32}$$

and $\sigma(S^{2m} \times S^{2m}) = 0$.

32 Lecture 32

32.1 Stokes' Theorem on chains

Define the standard n-simplex to be

$$\Delta^p = \Big\{(t_0, \dots, t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\Big\}. \tag{32.1}$$

We orient Δ_p with respect to the normal $\hat{n} = (1, \dots, 1)$. I.e., $(v_1, \dots, v_p) \in T_x\Delta^p$ is oriented if $(\hat{n}, v_1, \dots, v_p)$ is oriented equivalent to $(e_0, \dots, e_p)$ in $\mathbb{R}^{p+1}$. The ith face of Δ^p is the $(p-1)$-simplex

$$\Delta^p_i : \Delta^{p-1} \to \Delta^p \tag{32.2}$$

defined by

$$(t_0, \dots, t_{p-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{p-1}). \tag{32.3}$$

For a topological space X, a continuous mapping

$$c : \Delta^p \to X. \tag{32.4}$$

is called a singular p-simplex. If X is a smooth manifold, and c is smooth, then we say that c is a smooth singular p-simplex. In the smooth case, if $\omega \in \Omega^p(X)$, and c is a smooth singular p-simplex, define

$$\int_c \omega = \int_{\Delta^p} c^*\omega. \tag{32.5}$$

Definition 32.1. The pth singular chain group $C_p(X, \mathbb{R})$ is the free vector space over $\mathbb{R}$ generated by a singular p-simplices. The smooth pth singular chain group $C^\infty_p(X, \mathbb{R})$ is the free vector space over $\mathbb{R}$ generated by smooth singular p-simplices.

A singular p-chain is a finite linear combination

$$c = \sum_{i=1}^{N} a_i c_i, \tag{32.6}$$

where $a_i \in \mathbb{R}$ and c_i are singular p-simplices. In the smooth case, for $\omega \in \Omega^p(M)$, define

$$\int_c \omega = \sum_{i=1}^{N} a_i \int_{c_i} \omega. \tag{32.7}$$

Define the boundary operator

$$\partial : C_p(X, G) \to C_{p-1}(X, G) \tag{32.8}$$

by the following given a singular p-simplex $c : \Delta^p \to X$, let

$$\partial c = \sum_{i=0}^{p} (-1)^i c \circ \Delta_i^p, \tag{32.9}$$

and extend to all chains by linearity.

Theorem 32.2 (Stokes' Theorem on chains)**.** *Let M be a compact oriented manifold. If $\omega \in \Omega^{p-1}(M)$, then for any chain $c \in C_p^{\infty}(X, \mathbb{R})$,*

$$\int_{\partial c} \omega = \int_c d\omega. \tag{32.10}$$

Proof. The standard n-simplex is a manifold with corners, and the sign in the definition of the boundary operator gives the correct orientation on each face. □

32.2 Singular homology

Somewhat in analogy with $d^2 = 0$, we have the following.

Proposition 32.3. *We have $\partial^2 = 0$.*

Since $\partial^2 = 0$, we have a *chain complex*

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1}(X, \mathbb{R}) \xrightarrow{\partial_{p+1}} C_p(X, \mathbb{R}) \xrightarrow{\partial_p} C_{p-1}(X, \mathbb{R}) \xrightarrow{\partial_{p-1}} \cdots . \tag{32.11}$$

Define the pth singular homology group by

$$H_p(X, \mathbb{R}) = \frac{Ker\{\partial_p : C_p(X, \mathbb{R}) \to C_{p-1}(X, \mathbb{R})\}}{Im\{\partial_{p+1} : C_{p+1}(X, \mathbb{R}) \to C_p(X, \mathbb{R})\}} \tag{32.12}$$

We next consider the functorality of homology. If $f : X \to Y$ is a continuous mapping between topological spaces, then we can push forward chains by the following. For a simplex in X, $c : \Delta^p \to X$, we define $(f_*)_p c = f \circ c$, and extend to chains by linearity. This yields mappings

$$(f_*)_p : C_p(X, \mathbb{R}) \to C_p(Y, \mathbb{R}), \tag{32.13}$$

for $p = 0, 1, 2, \ldots$.

The following says that the collections of mappings $(f_*)_p$ are a *morphism* of chain complexes.

Proposition 32.4. *The following diagram*

$$\begin{tikzcd} C_{p+1}(X,\mathbb{R}) \arrow[r, "\partial^X_{p+1}"] \arrow[d, "(f_*)_{p+1}"] & C_p(X,\mathbb{R}) \arrow[d, "(f_*)_p"] \\ C_{p+1}(Y,\mathbb{R}) \arrow[r, "\partial^Y_{p+1}"] & C_p(Y,\mathbb{R}) \end{tikzcd} \tag{32.14}$$

commutes.

Corollary 32.5. *If $f : X \to Y$ then there are induced mappings*

$$(f_*)_p : H_p(X, \mathbb{R}) \to H_p(Y, \mathbb{R}). \tag{32.15}$$

If $g : Y \to Z$, then

$$((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p. \tag{32.16}$$

Consequently, if X and Y are homeomorphic, then $H_p(X, \mathbb{R}) \cong H_p(Y, \mathbb{R})$ for every $p \geq 0$.

32.3 Homotopy invariance of homology

Proposition 32.6. *If $f, g : X \to Y$ are continuously homotopic then*

$$H_k f = H_k g : H_k(X, \mathbb{R}) \to H_k(Y, \mathbb{R}) \tag{32.17}$$

Proof. To prove this, one defines an operator

$$S_p : C^0_p(X, \mathbb{R}) \to C^0_{p+1}(X \times [0, 1], \mathbb{R}) \tag{32.18}$$

such that

$$(\iota_1)_* - (\iota_0)_* = \partial_{p+1} S_p + S_{p-1} \partial_p, \tag{32.19}$$

where $\iota_t : X \to X \times [0, 1]$ is the inclusion $\iota_t(x) = (x, t)$. In other words, S is a chain homotopy between the morphisms $(\iota_0)_*$ and $(\iota_1)_*$ from the singular chain complex on X and the singular chain complex on $X \times [0, 1]$. We only need to define S_p for for singular p-simplices, and extend to all chain by linearity. This is called the “prism” operator.

We will divide $\Delta^n \times [0, 1]$ into $(n + 1)$ $(n + 1)$-simplices. For $i = 0, \ldots, n$, define

$$p^n_i : \Delta^{n+1} \to \Delta^n \times [0, 1] \tag{32.20}$$

by

$$(t_0, \ldots, t_{n+1}) \mapsto ((t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), t_{i+1} + \cdots + t_{n+1}) \tag{32.21}$$

We will view $\Delta^n \times [0,1] \subset \mathbb{R}^{n+2}$, so will henceforth omit the inner parenthesis. Define $S_n : C_n(X, \mathbb{R}) \to C_{n+1}(X \times [0,1], \mathbb{R})$ by

$$S_n(c_n) = \sum_{i=0}^{n} (-1)^i (c_n \times id) \circ p_i^n, \tag{32.22}$$

and extend to all chains by linearity. A long calculation, which is omitted, shows that (32.19) is satisfied. Homotopy invariance follows like we did for de Rham cohomology. □

32.4 Mayer-Vietoris for singular chains

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow C_p(U \cap V) \xrightarrow{\alpha_p} C_p(U) \oplus C_p(V) \xrightarrow{\beta_p} C_p(U) + C_p(V) \longrightarrow 0 \tag{32.23}$$

where

$$\alpha(c_p) = \big((i_{U \cap V \hookrightarrow U})_* c_p, (i_{U \cap V \hookrightarrow V})_* c_p \big) \tag{32.24}$$

and

$$\beta(a_p, b_p) = (i_{U \hookrightarrow M})_* a_p - (i_{V \hookrightarrow M})_* b_p. \tag{32.25}$$

It is not hard to see this sequence is exact. Furthermore, by a barycentric subdivision argument, the homology $H_*(C_p(U) + C_p(V))$ is isomorphic to $H_*(U \cup V)$. (Roughly, keep subdividing simplices until their images are contained in U or V.) Consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\partial_{p+1}} H_p(U \cap V) \xrightarrow{\alpha_p} H_p(U) \oplus H_p(V) \xrightarrow{\beta_p} H_p(U \cup V) \xrightarrow{\partial_p} \cdots \tag{32.26}$$

33 Lecture 33

33.1 Singular cohomology

To define singular cohomology, let $C^p(X, \mathbb{R})$ denote the singular cochains, which are dual to singular chains, i.e.,

$$C^p(X, \mathbb{R}) = Hom(C_p(X, \mathbb{R}), \mathbb{R}), \tag{33.1}$$

and let $\delta^p : C^p(X, \mathbb{R}) \to C^{p+1}(X, \mathbb{R})$ denote the dual to the boundary operator $\partial_{p+1} : C_{p+1}(X, \mathbb{R}) \to C_p(X, \mathbb{R})$, defined as follows. For $c^p \in C^p(X, \mathbb{R})$ and $c_{p+1} \in C_p(X, \mathbb{R})$,

$$(\delta^p c^p)(c_{p+1}) = c^p(\partial_{p+1} c_{p+1}). \tag{33.2}$$

Since $\partial_p \circ \partial_{p+1} = 0$, we have $\delta^{p+1} \circ \delta^p = 0$, so we have a *cochain complex*

$$\cdots \xrightarrow{\delta^{p-2}} C^{p-1}(X, \mathbb{R}) \xrightarrow{\delta^{p-1}} C^p(X, \mathbb{R}) \xrightarrow{\delta^p} C^{p+1}(X, \mathbb{R}) \xrightarrow{\delta^{p+1}} \cdots . \tag{33.3}$$

Define the pth singular cohomology group by

$$H^p(X, \mathbb{R}) = \frac{Ker\{\delta^p : C^p(X, \mathbb{R}) \to C^{p+1}(X, \mathbb{R})\}}{Im\{\delta^{p-1} : C^{p-1}(X, \mathbb{R}) \to C^p(X, \mathbb{R})\}}. \tag{33.4}$$

Next, let $f : X \to Y$ be a smooth mapping between topological spaces. The mapping on chains $(f_*)_p : C_p(X, \mathbb{R}) \to C_p(Y, \mathbb{R})$ induces the dual mapping on cochains

$$f^* : C^p(Y, \mathbb{R}) \to C^p(X, \mathbb{R}) \tag{33.5}$$

by the following. For $c^p \in C^p(Y, \mathbb{R})$ and $c_p \in C_p(X, \mathbb{R})$, define

$$(f^* c^p)(c_p) = c^p(f_* c_p) \tag{33.6}$$

Dualizing the diagram (32.14), we have the following commutative diagram

$$\begin{array}{ccc} C^p(Y, \mathbb{R}) & \xrightarrow{\delta^p_Y} & C^{p+1}(Y, \mathbb{R}) \\ \Big\downarrow{\scriptstyle (f^*)^p} & & \Big\downarrow{\scriptstyle (f^*)^{p+1}} \\ C^p(X, \mathbb{R}) & \xrightarrow{\delta^p_X} & C^{p+1}(X, \mathbb{R}). \end{array} \tag{33.7}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The singular cohomology spaces are a topological invariant.

Corollary 33.1. *If $f : X \to Y$ is continuous, then there are induced mappings*

$$(f^*)^p : H^p(Y, \mathbb{R}) \to H^p(X, \mathbb{R}). \tag{33.8}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{33.9}$$

Consequently, if X and Y are homeomorphic, then $H^p(X, \mathbb{R}) \cong H^p(Y, \mathbb{R})$ for every $p \geq 0$.

Proof. Exactly the same as the proof of Corollary 18.5, with d_X, d_Y replaced by δ_X, δ_Y. $\square$

33.2 Homotopy invariance of singular cohomology

Proposition 33.2. *If $f, g : X \to Y$ are continuously homotopic then*

$$H^k f = H^k g : H^k(Y, \mathbb{R}) \to H^k(X, \mathbb{R}) \tag{33.10}$$

This folows by dualizing the prism operator to obtain a co-chain homotopy.

33.3 Mayer-Vietoris for singular co-chains

Write $M = U \cup V$ as the union of two open sets in M. Dualizing the Mayer-Vietoris chain sequence, the following sequence is exact:

$$0 \longrightarrow (C_p(U) + C_p(V))^* \xrightarrow{\beta^p} C^p(U) \oplus C^p(V) \xrightarrow{\alpha^p} C^p(U \cap V) \longrightarrow 0 \tag{33.11}$$

where $\beta^p = (\beta_p)^*$ and $\alpha^p = (\alpha_p)^*$.

By the barycentric subdivision argument, one can show that

$$H^p\Big((C_p(U) + C_p(V))^*\Big) \cong H^p(U \cup V, \mathbb{R}), \tag{33.12}$$

consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(U \cup V) \xrightarrow{\beta^p} H^p(U) \oplus H^p(V) \xrightarrow{\alpha^p} H^p(U \cap V) \xrightarrow{\delta^p} \cdots \tag{33.13}$$

33.4 Smooth singular cohomology

Next, we restrict to the category of smooth manifolds. For a smooth manifold space X, a smooth mapping

$$c : \Delta^p \to X. \tag{33.14}$$

is called a smooth singular p-simplex.

Definition 33.3. The smooth pth singular chain group $C_p^\infty(X, \mathbb{R})$ is the free vector space over $\mathbb{R}$ generated by smooth singular p-simplices.

We note that the usual boundary operator maps

$$\partial_{p+1} : C_{p+1}^\infty(X; \mathbb{R}) \to C_p^\infty(X; \mathbb{R}) \tag{33.15}$$

To define smooth singular cohomology, let $C^p_\infty(X; \mathbb{R})$ denote the smooth singular cochains, which are dual to smooth singular chains, i.e.,

$$C^p_\infty(X; \mathbb{R}) = Hom(C_p^\infty(X; \mathbb{R}), \mathbb{R}), \tag{33.16}$$

and let $\delta^p : C^p_\infty(X; \mathbb{R}) \to C^{p+1}_\infty(X; \mathbb{R})$ denote the dual to the boundary operator defined as before. For $c^p \in C^p_\infty(X; \mathbb{R})$ and $c_{p+1} \in C_p^\infty(X; \mathbb{R})$,

$$(\delta^p c^p)(c_{p+1}) = c^p(\partial_{p+1} c_{p+1}). \tag{33.17}$$

Since $\partial_p \circ \partial_{p+1} = 0$, we have $\delta^{p+1} \circ \delta^p = 0$, so we have a *cochain complex*

$$\cdots \xrightarrow{\delta^{p-2}} C^{p-1}_\infty(X; \mathbb{R}) \xrightarrow{\delta^{p-1}} C^p_\infty(X; \mathbb{R}) \xrightarrow{\delta^p} C^{p+1}_\infty(X; \mathbb{R}) \xrightarrow{\delta^{p+1}} \cdots . \tag{33.18}$$

Define the pth smooth singular cohomology group by

$$H^p_\infty(X; \mathbb{R}) = \frac{Ker\{\delta^p : C^p_\infty(X; \mathbb{R}) \to C^{p+1}_\infty(X; \mathbb{R})\}}{Im\{\delta^{p-1} : C^{p-1}_\infty(X; \mathbb{R}) \to C^p_\infty(X; \mathbb{R})\}}. \tag{33.19}$$

The smooth singular cohomology satisfies the same axioms as the topological singular cohomology.

Proposition 33.4 (Smooth homotopy invariance)**.** *If $f : X \to Y$ is smoothly homotopic to $g : X \to Y$ then $H^p f = H^p g : H^p_\infty(Y;\mathbb{R}) \to H^p_\infty(X;\mathbb{R})$. Consequently, if X is homotopy equivalent to Y then $H^p_\infty(X;\mathbb{R}) \cong H^p_\infty(Y;\mathbb{R})$ for all $p \geq 0$.*

Proposition 33.5 (Mayer-Vietoris)**.** *If $X = U \cup V$ where U and V are open, then there is a long exact sequence*

$$\cdots \xrightarrow{\delta^{p-1}} H^p_\infty(U \cup V;\mathbb{R}) \xrightarrow{\beta^*_p} H^p_\infty(U;\mathbb{R}) \oplus H^p_\infty(V;\mathbb{R}) \xrightarrow{\alpha^*_p} H^p_\infty(U \cap V;\mathbb{R}) \xrightarrow{\delta^p} \cdots \tag{33.20}$$

Proposition 33.6. *If M is a smooth manifold which has a finite good cover, then*

$$H^p_\infty(M;\mathbb{R}) \cong H^p(M;\mathbb{R}) \tag{33.21}$$

Proof. Both cohomology theories agree on contractible spaces, and both satisfy a Mayer-Vietoris sequence. There is an obvious chain mapping from smooth co-chains to topological co-chains. By the same argument as before using the five lemma and induction on the number of elements in a good cover, the cohomology groups are isomorphic in any degree. □

33.5 de Rham's Theorem

We are now in a position to state the theorem of de Rham relating de Rham cohomology and singular cohomology with real coefficients of a smooth manifold M. Morever, we can write the explicit mapping. Consider the following diagram:

$$\begin{tikzcd} \cdots \arrow[r, "d^{p-2}"] & \Omega^{p-1}(M) \arrow[r, "d^{p-1}"] \arrow[d, "\mathcal{F}^{p-1}"] & \Omega^p(M) \arrow[r, "d^p"] \arrow[d, "\mathcal{F}^p"] & \Omega^{p+1}(M) \arrow[r, "d^{p+1}"] \arrow[d, "\mathcal{F}^{p+1}"] & \cdots \\ \cdots \arrow[r, "\delta^{p-2}"] & C^{p-1}_\infty(M;\mathbb{R}) \arrow[r, "\delta^{p-1}"] & C^p_\infty(M;\mathbb{R}) \arrow[r, "\delta^p"] & C^{p+1}_\infty(M;\mathbb{R}) \arrow[r, "\delta^{p+1}"] & \cdots, \end{tikzcd} \tag{33.22}$$

where the vertical maps are defined as follows. If $\omega \in \Omega^p(M)$, and c_p is a smooth p-chain, then let

$$(\mathcal{F}^p \omega)(c_p) = \int_{c_p} \omega. \tag{33.23}$$

Proposition 33.7. *The diagram* (33.22) *commutes. Consequently, there are induced mappings*

$$H^p\mathcal{F}^p : H^p_{dR}(M) \to H^p_\infty(M;\mathbb{R}). \tag{33.24}$$

Proof. Commutativity says that

$$\delta^p \mathcal{F}^p = \mathcal{F}^{p+1} d^p \tag{33.25}$$

Given $\omega \in \Omega^p(M)$, and $(p+1)$-chain c_{p+1}, the left hand side of (33.23) evaluates to

$$\delta^p(\mathcal{F}^p(\omega))(c_{p+1}) = \mathcal{F}^p(\omega)(\partial_{p+1} c_{p+1}) = \int_{\partial_{p+1} c_{p+1}} \omega. \tag{33.26}$$

The right hand side of (33.23) evaluates to

$$\mathcal{F}^{p+1}d^p\omega(c_{p+1}) = \int_{c_{p+1}} d^p\omega, \tag{33.27}$$

These are equal by Theorem 32.2, Stokes' Theorem on chains.

Consequently, $\mathcal{F}^*$ is a morphism of co-chain complexes, so there are well-defined induced maps on the cohomology groups.

□

We can now prove the main result.

Theorem 33.8 (de Rham)**.** *If M has a finite good cover then the mappings*

$$\mathcal{F}^p : H^p_{dR}(M) \to H^p_\infty(M;\mathbb{R}), \tag{33.28}$$

are isomorphisms for all $p \geq 0$.

Proof. Note we have the following morphism of short exact sequences of co-complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^p(U\cup V) & \xrightarrow{\beta^p} & \Omega^p(U)\oplus\Omega^p(V) & \xrightarrow{\alpha^p} & \Omega^p(U\cap V) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\mathcal{F}^p} & & \big\downarrow{\scriptstyle\mathcal{F}^p\oplus\mathcal{F}^p} & & \big\downarrow{\scriptstyle\mathcal{F}^p} & & \\
0 & \longrightarrow & (C^\infty_p(U;\mathbb{R})+C^\infty_p(V;\mathbb{R}))^* & \xrightarrow{\beta^*_p} & C^p_\infty(U;\mathbb{R})\oplus C^p(V;\mathbb{R}) & \xrightarrow{\alpha^*_p} & C^p_\infty(U\cap V;\mathbb{R})^* & \longrightarrow & 0.
\end{array} \tag{33.29}$$

The above diagram is easily seen to commute at every square. It follows that the following diagram of associated Mayer-Vietoris exact sequences commutes at every square

$$\begin{array}{ccccccccc}
H^{k-1}_{dR}(U)\oplus H^{k-1}_{dR}(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_{dR}(U\cap V) & \xrightarrow{\delta^k} & H^k_{dR}(U\cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U)\oplus H^k_{dR}(V) & \xrightarrow{\alpha^k} & H^k_{dR}(U\cap V) \\
\big\downarrow{\scriptstyle\mathcal{F}^{k-1}\oplus\mathcal{F}^{k-1}} & & \big\downarrow{\scriptstyle\mathcal{F}^{k-1}} & & \big\downarrow{\scriptstyle\mathcal{F}^k} & & \big\downarrow{\scriptstyle\mathcal{F}^k\oplus\mathcal{F}^k} & & \big\downarrow{\scriptstyle\mathcal{F}^k} \\
H^{k-1}_s(U)\oplus H^{k-1}_s(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_s(U\cap V) & \xrightarrow{\delta^k} & H^k_s(U\cup V) & \xrightarrow{\beta^k} & H^k_s(U)\oplus H^k_s(V) & \xrightarrow{\alpha^k} & H^k_s(U\cap V)
\end{array} \tag{33.30}$$

If there is only 1 element in the covering, then we are done by the homotopy invariance of both theories. By the Five Lemma, if the result is true for U, V and $U\cap V$, then it is also true for $U\cup V$. By induction on the number of elements in a finite good cover, the theorem is then true for any manifold which admits a finite good cover. □

From Proposition 33.6 above, we have

Theorem 33.9. *If X is a smooth manifold with a finite good cover, then*

$$H^p_{dR}(X) \cong H^p(X;\mathbb{R}). \tag{33.31}$$

Consequently, the de Rham cohomology groups are a topological invariant. That is if smooth manifolds X and Y are homeomorphic, then $H^p_{dR}(X)\cong H^p_{dR}(Y)$.

Remark 33.10. There do exist examples of homeomorphic but non-diffeomorphic smooth manifolds! But de Rham cohomology will never be able to tell these apart – for this one needs more refined invariants.

References

[Lee13] John M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013.

[Spi79] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I*, second ed., Publish or Perish, Inc., Wilmington, Del., 1979. MR 532830

[War83] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition.

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