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Introduction

This is a continuation of 218A. In 218B, we will discuss vector bundles, differential forms, etc. References are [Lee13, Spi79, War83].

1 Lecture 1

1.1 Vector bundles

Definition 1.1. A smooth real vector bundle of rank \( k \) over a smooth manifold \( M^n \) is a topological space \( E \) together with a smooth projection

\[
\pi : E \to M
\]

such that

- For \( p \in M \), \( \pi^{-1}(p) \) is a vector space of dimension \( k \) over \( \mathbb{R} \).
- There exists local trivializations, that is, there are smooth mappings

\[
\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E
\]

which maps \( p \times \mathbb{R}^k \) linearly onto the fiber \( \pi^{-1}(p) \) for every \( p \in U_\alpha \).

The transition functions of a bundle are defined as follows.

\[
\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})
\]

defined by

\[
\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)),
\]

for \( v \in \mathbb{R}^k \).

On a triple intersection \( U_\alpha \cap U_\beta \cap U_\gamma \), we have the identity

\[
\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}.
\]

Conversely, given a covering \( U_\alpha \) of \( M \) and transition functions \( \varphi_{\alpha\beta} \) satisfying (1.5), there is a vector bundle \( \pi : E \to M \) with transition functions given by \( \varphi_{\alpha\beta} \). If the transitions function \( \varphi_{\alpha\beta} \) are \( C^\infty \), then we say that \( E \) is a smooth vector bundle.

Exercise 1.2. If \( M \) is a smooth \( n \)-dimensional manifold then \( \pi : TM \to M \) is a rank \( n \) vector bundle. (This was done in 218A). A coordinate system \( (U_\alpha, x_\alpha) \), where \( x_\alpha : U_\alpha \to \mathbb{R}^n \) gives a trivialization

\[
\Phi_\alpha(p, (v^1, \ldots, v^n)) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i_\alpha} \bigg|_p.
\]

Given another coordinate system \( (U_\beta, x_\beta) \) with \( U_\alpha \cap U_\beta \neq \emptyset \), find the transition functions \( \varphi_{\alpha\beta} \).
The tangent bundle has some extra structure which an arbitrary vector bundle does not possess. Recall from 218A that a smooth mapping between smooth manifolds \( f : M \to N \) induces a mapping \( f_* : TM \to TN \). The following diagram
\[
\begin{array}{ccc}
TM & \xrightarrow{f_*} & TN \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M & \xrightarrow{f} & N
\end{array}
\] (1.7)
commutes and \( f_* \) restricts to a linear mapping on fibers. Given a smooth mapping \( h : N \to X \), consider the composition \( h \circ f : M \to X \). The chain rule says that
\[
(h \circ f)_* = h_* \circ f_* : TM \to TX.
\] (1.8)

**Definition 1.3.** A bundle mapping between vector bundles \( E_1 \) over \( M \) and \( E_2 \) over \( N \) is a mapping \( F : E_1 \to E_2 \) which maps fibers linearly to fibers and covers a smooth mapping between the base spaces. That is, the diagram
\[
\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_2 \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M & \xrightarrow{f} & N
\end{array}
\] (1.9)
commutes.

**Definition 1.4.** The category of smooth manifolds \( \text{Man}^\infty \) has objects as smooth manifolds and morphisms as smooth mappings, where composition of morphisms is just composition of mappings.

Composition of morphisms is obviously associative, i.e.,
\[
(\Psi_1 \circ \Psi_2) \circ \Psi_3 = \Psi_1 \circ (\Psi_2 \circ \Psi_3)
\] (1.10)
and every manifolds has an identity morphism \( \text{id}_X : X \to X \), which is obviously smooth, so this is indeed a category.

**Definition 1.5.** The category \( \text{Vect} \) of smooth vector bundles over smooth manifolds is the collection of all vector bundle (of any rank) over smooth manifolds. The morphisms are the bundle mappings.

We therefore have a functor \( \mathcal{T} : \text{Man}^\infty \to \text{Vect} \) where \( \text{Vect} \) is the category of smooth vector bundles over smooth manifolds given by \( M \mapsto TM \) and \( f : M \to N \) maps to \( f_* : TM \to TN \). The mapping \( \mathcal{T} \) satisfies \( \mathcal{T}(\text{Id}_M) = \text{Id}_{TM} \) and by (1.8), \( \mathcal{T}(f_1 \circ f_2) = \mathcal{T}(f_1) \circ \mathcal{T}(f_2) \), so this is a **covariant** functor, called the tangent functor.

**Definition 1.6.** For a fixed smooth manifold \( M \), the category \( \text{Vect}(M) \) is the collection of smooth vector bundles over \( M \) (of any rank). A morphism in this category is a mapping \( F : E_1 \to E_2 \) covering the identity mapping, that is, the diagram
\[
\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_2 \\
\downarrow{\pi_M} & & \downarrow{\pi_M} \\
M & \xrightarrow{\text{id}_M} & M
\end{array}
\] (1.11)
We say that bundles $E_1$ and $E_2$ over $M$ are isomorphic if there exists an invertible bundle mapping between $E_1$ and $E_2$. If $E$ is isomorphic to the trivial bundle over $M$, $\pi_M : M \times \mathbb{R}^k \rightarrow M$ defined by $\pi_M(p, v) = p$, then we say that $E$ is trivial.

We next express the above in coordinates. Assume we have a covering $U_\alpha$ of $M$ such that $E_1$ has trivializations $\Phi_\alpha$ and $E_2$ has trivializations $\Psi_\alpha$. Then any vector bundle mapping gives locally defined functions

$$f_\alpha : U_\alpha \rightarrow \text{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})$$

defined by

$$f_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)).$$

(1.13)

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi_{\alpha \beta}^E f_\beta \varphi_{\beta \alpha}^E,$$

(1.14)

equivalently,

$$\varphi_{\alpha \beta}^E f_\alpha = f_\beta \varphi_{\beta \alpha}^E.$$

(1.15)

Bundles $E_1$ and $E_2$ are equivalent if and only if $\text{rank}(E_1) = \text{rank}(E_2)$ and there exist $f_\alpha$ as above with $\det(f_\alpha) \neq 0$. A vector bundle is trivial if and only if there exist functions

$$f_\alpha : U_\alpha \rightarrow GL(k, \mathbb{R})$$

(1.16)

such that

$$\varphi_{\beta \alpha} = f_\beta f_\alpha^{-1}.$$

(1.17)

**Remark 1.7.** In the above, for a fixed base space $M$, we only defined morphisms in the category of vector bundles to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence. More on this later.

### 1.2 Sections of bundles

**Definition 1.8.** Let $\pi : E \rightarrow M$ be a vector bundle. A section of a bundle is a smooth mapping $s : M \rightarrow E$ such that $\pi \circ s = id_M$. The space of sections is denoted by $\Gamma(E)$.

In other words, $s(x) \in E_x$, $s$ maps $x$ to a vector in the fiber over $x$. In terms of local trivializations we have the following. Let

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \rightarrow \pi^{-1}(U_\alpha)$$

(1.18)

be a local trivialization. Then

$$s_\alpha = \pi_2 \circ \Phi_\alpha^{-1} \circ s : U_\alpha \rightarrow \mathbb{R}^k$$

(1.19)
is called a local representative of $s$ with respect to $\Phi_\alpha$. On $U_\beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$
\Phi_\beta : U_\beta \times \mathbb{R}^k \to \pi^{-1}(U_\beta).
$$

(1.20)

Recall that the transition functions of a bundle are

$$
\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})
$$

(1.21)

defined by

$$
\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)),
$$

(1.22)

for $v \in \mathbb{R}^k$. Then for any $e_x \in \pi^{-1}(x)$, we have

$$
\varphi_{\alpha\beta}(x)(\pi_2 \circ \Phi_\beta^{-1}(e_x)) = \pi_2 \circ \Phi_\alpha^{-1}(e_x).
$$

(1.23)

Choosing $e_x = s(x)$ we have

$$
\varphi_{\alpha\beta}(s)(\pi_2 \circ \Phi_\beta^{-1}(s(x))) = \pi_2 \circ \Phi_\alpha^{-1}(s(x)),
$$

(1.24)

or simply

$$
\varphi_{\alpha\beta}s = s, \text{ on } U_\alpha \cap U_\beta,
$$

(1.25)

which is the local transformation law for a section.

Conversely, if a bundle $\pi : E \to M$ is given to us in terms of transition functions, then any collection of functions

$$
s_\alpha : U_\alpha \to \mathbb{R}^k
$$

(1.26)

satisfying (1.25) gives a well-defined smooth section $s : M \to E$.

## 2 Lecture 2

### 2.1 Pull-back bundles

If $M$ and $N$ are smooth manifolds, and $\pi_N : E \to N$ is a vector bundle over $N$, then given a smooth mapping $f : M \to N$, define

$$
f^*E = \{(p, v) \in M \times E \mid f(p) = \pi_N(v)\}.
$$

(2.1)

**Proposition 2.1.** The pullback $f^*E$ is a vector bundle over $M$, with projection given by $\pi_1(p, v) = p$, and the fiber $f^*E$ over $p \in M$ is identified with the fiber $E_{f(p)}$, i.e., the following diagram commutes

$$
\begin{array}{ccc}
  f^*E & \xrightarrow{\pi_2} & E \\
  \downarrow{\pi_1} & & \downarrow{\pi_N} \\
  M & \xrightarrow{f} & N.
\end{array}
$$

(2.2)
Proof. Let $\Phi : U \times \mathbb{R}^k \to \pi_N^{-1}(U)$ be a local trivialization for $E$. The set $f^{-1}(U)$ is open since $f$ is continuous, and define

$$f^*\Phi : f^{-1}(U) \times \mathbb{R}^k \to \pi_1^{-1}(f^{-1}(U))$$

by

$$f^*\Phi(x, v) = (x, \Phi(f(x), v)).$$

The reader can verify that this is a local trivialization for $f^*E$. □

Next we note that sections can be pulled back to sections of the pullback bundle.

**Definition 2.2.** Let $f : M \to N$ be a smooth mapping between smooth manifolds, and $\pi : E \to N$ be a vector bundle over $N$. If $\sigma : N \to E$ is a section of $E$, then $(\sigma \circ f)(x) = (x, \sigma(f(x)))$ is a section of $\pi_1 : f^*E \to M$ and is called the pullback of $\sigma$ under $f$.

The fact that this is a section of the pullback bundle is almost obvious, we just need to check that

$$\pi_1(\sigma \circ f)(x) = \pi_1(x, \sigma(f(x))) = x.$$  

2.2 Push-forward of vector fields

Next, we restrict to tangent bundles. Let $f : M \to N$ be a smooth mapping between smooth manifolds. Then $f^*TN$ is a vector bundle over $M$. Define

$$(f_*)_B : TM \to f^*TN$$

by

$$(f_*)_B(v_p) = (p, f_*v).$$

(the subscript $B$ is short for “bundle mapping”). We have the commutative diagram

$$\begin{align*}
TM & \xrightarrow{(f_*)_B} f^*TN \\
\downarrow{\pi_M} & \quad \downarrow{\pi_1} \\
M & \xrightarrow{id} M.
\end{align*}$$

**Definition 2.3.** If $X \in \Gamma(TM)$, then we can define $f_*X \in \Gamma(f^*TN)$, by

$$f_*X \equiv (f_*)_B \circ X.$$  

In words: under smooth mappings, vector fields push-forward to sections of the pull-back bundle.
Remark 2.4. Note that for $f : M \to N$, although we can push-forward individual tangent vectors, in general there is not a mapping

$$f_* : \Gamma(TM) \to \Gamma(TN).$$

(2.10)

For example, $f$ might not even be surjective. This is one reason we had to consider pull-back bundles in the above discussion.

Example 2.5. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve. Then the tangent vector $X = \frac{\partial}{\partial t} \in \Gamma(T\mathbb{R})$, and $\gamma_* X \in \Gamma(\gamma^* T\mathbb{R}^n)$ is a vector field along the curve. Note the curve might have self-intersections, and it could even be a constant path, in which case the pull-back bundle is the trivial bundle.

2.3 Example: The Mobius bundle

Let $B = S^1$ be the base space. Of course, we have the trivial bundle $\pi : S^1 \times \mathbb{R} \to S^1$. Let us define another bundle over $S^1$. Consider $S^1 = \{ v \in \mathbb{R}^2 \mid |v| = 1 \}$. Consider $\mathbb{RP}^1 = S^1/\sim$ where $e^{i\theta} \sim e^{i(\theta + \pi)}$. The quotient space is clearly $S^1$. Note that a line through the origin is hits the unit circle in exactly 2 opposite points. Therefore we can identify $\mathbb{RP}^1$ with the space of lines through the origin in $\mathbb{R}^2$. Denote the line determined by $p \in \mathbb{RP}^1$ as $[p]$. We then define the following:

$$M = \{(p, v) \in \mathbb{RP}^1 \times \mathbb{R}^2 \mid v \in [p]\}.$$  

(2.11)

Define a local trivialization as follows. Cover $\mathbb{RP}^1$ by 2 open sets

$$U_1 = \{ \ell = re^{i\theta_1} \mid r \in \mathbb{R}, 0 < \theta_1 < \pi \}, \quad U_2 = \{ \ell = re^{i\theta_2} \mid r \in \mathbb{R}, \pi/2 < \theta_2 < 3\pi/2 \}$$

(2.12)

Then define $\Phi_1 : U_1 \times \mathbb{R} \to M$ by

$$\Phi_1(\theta_1, r) = \{[\ell_{\theta_1}], r e^{i\theta_1}\},$$

(2.13)

and $\Phi_2 : U_2 \times \mathbb{R} \to M$ by

$$\Phi_2(\theta_2, r) = \{[\ell_{\theta_2}], r e^{i\theta_2}\},$$

(2.14)

Next, we determine the overlap mapping

$$\varphi_{12} = \pi_2 \circ \Phi_1^{-1} \circ \Phi_2.$$  

(2.15)

Note that $U_1 \cap U_2 = V_1 \bigcup V_2$, where

$$V_1 = \{0 < \theta_1 < \pi/2\} = \{\pi < \theta_2 < 3\pi/2\},$$

(2.16)

$$V_2 = \{\pi/2 < \theta_1 < \pi\} = \{\pi/2 < \theta_2 < \pi\}.$$  

(2.17)

We have on $V_2$, $\varphi_{12} = Id_{\mathbb{R}}$, since $\theta_1 = \theta_2$ there. But on $V_1$, $\varphi_{12} = -Id_{\mathbb{R}}$, since $\theta_1 = \theta_2 - \pi$

there.

Proposition 2.6. The bundle $\pi : M \to S^1$ is not trivial.
Proof. If this bundle were trivial, there would exist a non-zero section. With respect to the above local trivializations, we have

\[ s_1 : U_1 \to \mathbb{R}, \quad s_2 : U_2 \to \mathbb{R}, \]  

(2.18)

with \( s_1 = \varphi_{12} s_2 \) on \( U_1 \cap U_2 \). So we have

\[ s_1 = \begin{cases} 
  s_2 & \text{on } V_2 \\
  -s_2 & \text{on } V_1 
\end{cases} \]  

(2.19)

Since \( s_2 \) is not zero on a connected set, it is either positive or negative. But this implies that \( s_1 \) is both positive and negative somewhere on \( U_1 \). But since \( U_1 \) is connected and \( s_1 \) is continuous, it would have to be zero somewhere.

\[ \square \]

Proposition 2.7. Let \( f : S^1 \to \mathbb{RP}^1 \) be the 2-fold covering mapping. Then \( \pi_1 : f^* M \to S^1 \) is a trivial bundle.

Proof. We have

\[ f^* M = \{(p, v) \in S^1 \times M \mid \pi(v) = f(p)\}. \]  

(2.20)

Consider the mapping \( \sigma : S^1 \to f^* M \) given by \( \sigma : p \mapsto (p, ([p], p)) \). Then

\[ \pi_1 \sigma(p) = \pi_1(p, [p]) = p, \]  

(2.21)

so \( \sigma \) is a section of \( f^* M \) which is nowhere-zero. We then have a global trivialization of \( f^* M \) by \( \Phi : S^1 \times \mathbb{R} \to f^* M \) by

\[ \Phi(p, r) = (p, r \cdot \sigma). \]  

(2.22)

\[ \square \]

3 Lecture 3

3.1 Direct sums

If \( V_1, \ldots, V_k \) are vector spaces over \( \mathbb{R} \), then the direct sum \( V_1 \oplus \cdots \oplus V_k \) is the Cartesian product \( V_1 \times \cdots \times V_k \) with the following vector space structure:

\[ c(v_1, \ldots, v_k) = (cv_1, \ldots, cv_k) \]  

(3.1)

\[ (v_1, \ldots, v_k) + (v'_1, \ldots, v'_k) = (v_1 + v'_1, \ldots, v_k + v'_k), \]  

(3.2)

for \( c \in \mathbb{R} \). The space \( V_1 \oplus \cdots \oplus V_k \) satisfies the following “universal” mapping property. For \( 1 \leq i \leq k \), let \( \iota_i : V_i \to V_1 \oplus \cdots \oplus V_k \) be the inclusion mapping

\[ \iota_i : v \mapsto (0, \ldots, 0, v, \ldots, 0). \]  

(3.3)
Let $W$ be any vector space, and $f_i : V_i \to W$ be linear mappings for $1 \leq i \leq k$. Then there is a unique linear map $f : V_1 \oplus \cdots \oplus V_k \to W$ which makes the following diagram commute for $1 \leq i \leq k$.

This property uniquely characterizes the direct sum. That is, a vector space with the above universal mapping property is isomorphic to the direct sum. Note that obviously

$$\dim_{\mathbb{R}}(V_1 \oplus \cdots \oplus V_k) = \sum_{i=1}^{k} \dim_{\mathbb{R}}(V_i).$$

**Exercise 3.1.** Prove that for 3 vector spaces $V_1, V_2, V_3$ we have

$$(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3).$$

**Definition 3.2.** Let $V_i, i \in I$ be any collection of vector spaces. The Cartesian product $\Pi_{i \in I} V_i$ is the collection of all functions

$$f : I \to \bigcup_{i \in I} V_i,$$

such that $f(i) \in V_i$ for all $i \in I$. The direct product $\Pi_{i \in I} V_i$ is the Cartesian product with the vector space structure

$$c f(i) = c f(i)$$

$$f + g)(i) = f(i) + g(i).$$

The projection $\pi_i : \Pi_{i \in I} V_i \to V_i$ is the mapping $\pi_i(f) = f(i)$. The above definition satisfies the following universal property. If $V$ is any vector space and $\phi_i : V \to V_i$ are linear mappings for $i \in I$, then there is a unique linear mapping $\phi : V \to \Pi_{i \in I} V_i$ such that the diagram

$$V \xrightarrow{\phi_i} V_i$$

$$\downarrow \phi$$

$$\Pi_{i \in I} V_i$$

commutes for each $i \in I$. This property uniquely characterizes the direct product. That is, any vector space with the above universal mapping property is isomorphic to the direct product.

**Definition 3.3.** Let $V_i, i \in I$ be any collection of vector spaces. The direct sum $\oplus_{i \in I} V_i$ is the subspace of the direct product consisting of the functions $f$ such that $f(i) \neq 0$ for only finitely many $i \in I$.

**Remark 3.4.** The direct sum satisfies the first universal property (3.4), but not the second (3.10), unless $I$ is finite. (We leave the proof to the interested reader.)
Definition 3.5. Given vector bundles $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$, the Cartesian product is the bundle $\pi_1 \times \pi_2 : E_1 \times E_2 \to M_1 \times M_2$, defined by $\pi_1 \times \pi_2(e_1, e_2) = (\pi_1(e_1), \pi_2(e_2))$.

Exercise 3.6. (i) Show that this is a vector bundle of rank equal to $\text{rank}(E_1) + \text{rank}(E_2)$ over $M_1 \times M_2$, with fiber over $(p_1, p_2)$ isomorphic to $\pi_1^{-1}(p_1) \oplus \pi_2^{-1}(p_2)$.
(ii) If $M_1$ and $M_2$ are smooth manifolds, then $T(M_1 \times M_2)$ is isomorphic to $T M_1 \times T M_2$.

Definition 3.7. Let $\Delta : M \to M \times M$ be the diagonal embedding, that is, $\Delta(p) = (p, p)$. Given vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$, define the direct sum $E_1 \oplus E_2 \equiv \Delta^*(E_1 \times E_2)$.

We can also give a description of the direct sum in terms of trivializations. If $\Phi_1 : U \times \mathbb{R}^{k_1} \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^{k_2} \to \pi_2^{-1}(U)$ are local trivializations then

$$\Phi : U \times (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2}) \to \pi^{-1}(U)$$

defined by

$$\Phi(x, (v_1, v_2)) = (\Phi_1(x, v_1), \Phi_2(x, v_2))$$

is a local trivialization for $E_1 \oplus E_2$. Note that the transition functions satisfy

$$\varphi^{E_1 \oplus E_2}_{\alpha \beta} = \varphi^{E_1}_{\alpha \beta} \oplus \varphi^{E_2}_{\alpha \beta} \in GL(k_1 + k_2, \mathbb{R}),$$

where this is the “block” matrix

$$\varphi^{E_1 \oplus E_2}_{\alpha \beta}(x, w) = \begin{pmatrix} \varphi^{E_1}_{\alpha \beta}(x) & 0 \\ 0 & \varphi^{E_2}_{\alpha \beta}(x) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

3.2 Tensor products

Definition 3.8. If $A$ is any set, then the free vector space over $A$ is

$$\mathcal{F}(A) = \oplus_{a \in A} \mathbb{R}.$$  

This can be thought of as the vector space with basis elements $a \in A$. That is, $\mathcal{F}(A)$ is the set of formal sums

$$\mathcal{F}(A) = \left\{ \sum_{a \in A} f_a a \mid f_a \neq 0 \text{ for only finitely many } a \in A \right\}$$

with vector space structure

$$c \sum_{a \in A} f_a a = \sum_{a \in A} (cf_a) a$$

$$\sum_{a \in A} f_a a + \sum_{a \in A} f'_a a = \sum_{a \in A} (f_a + f'_a) a.$$
**Definition 3.9.** If $V_1, \ldots, V_k$ are vector spaces over $\mathbb{R}$, then the tensor product $V_1 \otimes \cdots \otimes V_k$ is the free real vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$ modulo the subspace spanned by all elements of the form

$$ (v_1, \ldots, cv_i, \ldots, v_k) - c(v_1, \ldots, v_i, \ldots, v_k) \quad (3.19) $$

and

$$ (v_1, \ldots, v_i + v'_i, \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v'_i, \ldots, v_k), \quad (3.20) $$

for $c \in \mathbb{R}$.

The space $V_1 \otimes \cdots \otimes V_k$ satisfies the universal mapping property as follows. Let $W$ be any vector space, and $F : V_1 \times \cdots \times V_k \to W$ be a multilinear mapping, i.e., $F$ is linear when restricted to each factor, with the other variables held fixed. Then there is a unique linear map $\tilde{F} : V_1 \otimes \cdots \otimes V_k$ which makes the following diagram commutative, where $\pi$ is the projection to the quotient space, which we write as

$$ \pi(v_1, \ldots, v_k) = v_1 \otimes \cdots \otimes v_k. \quad (3.22) $$

We say that an element in $V_1 \otimes \cdots \otimes V_k$ of the form $v_1 \otimes \cdots \otimes v_k$ is *decomposable*. A general element of $V_1 \otimes \cdots \otimes V_k$ is not decomposable, but can always be written as a finite sum of decomposable elements.

**Exercise 3.10.** (i) Prove that

$$ \dim_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k) = \dim_{\mathbb{R}}(V_1) \cdots \dim_{\mathbb{R}}(V_k). \quad (3.23) $$

(ii) Prove that for 3 vector spaces $V_1, V_2, V_3$ we have

$$ (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3). \quad (3.24) $$

**Definition 3.11.** The tensor product of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is the vector bundle $\pi : E_1 \otimes E_2 \to M$ defined by $\pi^{-1}(p) = \pi_1^{-1}(p) \otimes \pi_2^{-1}(p)$. If $\Phi_1 : U \times \mathbb{R}^{k_1} \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^{k_2} \to \pi_1^{-1}(U)$ are local trivializations then consider

$$ F : U \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}) \to \pi^{-1}(U) \quad (3.25) $$

defined by

$$ F(x, (v_1, v_2)) = \Phi_1(x, v_1) \otimes \Phi_2(x, v_2). \quad (3.26) $$

This is clearly a multilinear mapping on each fiber, so by the universal property of tensor products, there is a unique induced mapping

$$ \tilde{F} : U \times (\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}) \to \pi^{-1}(U) \quad (3.27) $$

which, using an isomorphism $\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \cong \mathbb{R}^{k_1k_2}$, defines a local trivialization for $E_1 \otimes E_2$. 

\[12\]
We could have equivalently defined the tensor product in terms of transition functions. To do this, note the following. If \( \phi_1 \in GL(k_1, \mathbb{R}) \) and \( \phi_2 \in GL(k_2, \mathbb{R}) \) then define

\[
\phi_1 \times \phi_2 : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}
\] (3.28)

by

\[
(\phi_1 \times \phi_2)(v_1, v_2) = \phi_1(v_1) \otimes \phi_2(v_2).
\] (3.29)

This is clearly a multilinear mapping, so by the universal property for tensor products, there is a unique induced mapping

\[
\phi_1 \otimes \phi_2 : \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \to \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}.
\] (3.30)

Given transition functions for \( E_1 \)

\[
\phi^E_{\alpha \beta} : U_\alpha \cap U_\beta \to GL(k_1, \mathbb{R}),
\] (3.31)

and transition functions for \( E_2 \)

\[
\phi^E_{\alpha \beta} : U_\alpha \cap U_\beta \to GL(k_2, \mathbb{R}),
\] (3.32)

we define

\[
\varphi^{E_1 \otimes E_2}_{\alpha \beta} = \varphi^E_{\alpha \beta} \otimes \varphi^E_{\alpha \beta} \in GL(k_1 k_2, \mathbb{R}),
\] (3.33)

where we choose some isomorphism \( \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \cong \mathbb{R}^{k_1 k_2} \).

### 3.3 Dual bundles

**Definition 3.12.** The dual of a vector space \( V \) is \( V^* = \text{Hom}(V, \mathbb{R}) \), which is the space of all linear mappings from \( V \) to \( \mathbb{R} \).

**Remark 3.13.** If \( V \) is finite-dimensional, we have that \( V^* \cong V \), equivalently, \( \text{dim}_\mathbb{R}(V^*) = \text{dim}_\mathbb{R}(V) \). Warning: this is not true over \( \mathbb{C} \). In this case, we have \( V^* \cong \overline{V} \), more on this later.

**Definition 3.14.** The dual of a vector bundle \( \pi : E \to M \) is the vector bundle \( \Pi : E^* \to M \) defined by \( \Pi^{-1}(p) = (\pi^{-1}(p))^* \). If \( \Phi : U \times \mathbb{R}^k \to \pi^{-1}(U) \) is a local trivialization then

\[
\Phi^* : U \times (\mathbb{R}^k)^* \to \Pi^{-1}(U)
\] (3.34)

defined by

\[
\Phi^*(x, f)(v_p) = f(\pi_2 \circ \Phi^{-1}(v_p))
\] (3.35)

is a local trivialization for \( E^* \).

**Exercise 3.15.** Show that the transition functions of \( E^* \) are

\[
\varphi^{E^*}_{\alpha \beta} = ((\varphi^E_{\alpha \beta})^{-1})^T = (\varphi^E_{\beta \alpha})^T.
\] (3.36)
4 Lecture 4

4.1 Riemannian metrics on real vector bundles

If \( \pi : E \to M \) is a real vector bundle, a Riemannian metric on \( E \) is a choice of smoothly varying positive definite symmetric inner product on each fiber. That is \( g \in \Gamma(E^* \otimes E^*) \) satisfying

\[
g(e_1, e_2) = g(e_2, e_1), \tag{4.1}
\]

and

\[
g(e, e) > 0 \text{ for } e \neq 0. \tag{4.2}
\]

**Proposition 4.1.** If \( E \) is any real vector bundle, then \( E \) admits a Riemannian metric.

**Proof.** Let

\[
\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha) \tag{4.3}
\]

be a local trivialization for \( U_\alpha \) an open covering of \( M \) which is locally finite. For \( x \in U_\alpha \) and \( e_1, e_2 \in E_x \), define

\[
g_\alpha(e_1, e_2) = \langle \pi_2 \circ \Phi_\alpha^{-1}(e_1), \pi_2 \circ \Phi_\alpha^{-1}(e_2) \rangle, \tag{4.4}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product on \( \mathbb{R}^k \). Next, let \( \chi_\alpha \) be a partition of unity subordinate to the cover \( U_\alpha \), that is

\[
\text{supp}(\chi_\alpha) \subset U_\alpha, \quad 0 \leq \chi_\alpha \leq 1, \quad \text{and} \quad \sum_\alpha \chi_\alpha = 1. \tag{4.5}
\]

Define

\[
g(e_1, e_2) = \sum_\alpha \chi_\alpha g_\alpha(e_1, e_2). \tag{4.6}
\]

This is clearly symmetric since each \( g_\alpha \) is symmetric. It is positive definite since for \( e_1 = e_2 = v \in \pi^{-1}(x) \setminus \{0\} \), the right hand side is a finite sum of nonnegative terms, with at least one strictly positive term. \( \square \)

**Corollary 4.2.** For any real vector bundle \( E \), \( E^* \cong E \).

**Proof.** Choose a Riemannian metric \( g \) on \( E \). Then the mapping \( b : E \to E^* \) defined by

\[
b(e_1)(e_2) = g(e_1, e_2) \tag{4.7}
\]

is an isomorphism on fibers, and covers the identity map. \( \square \)

**Definition 4.3.** Given vector bundles \( \pi_1 : E_1 \to M \) and \( \pi_2 : E_2 \to M \) over the same base space \( M \), and assume that \( E_1 \subset E_2 \). We say that \( E_1 \) is a subbundle of \( E_2 \), if each fiber \( \pi_1^{-1}(x) \subset \pi_2^{-1}(x) \) is a vector subspace.
In bundle terms, existence of a Riemannian metric $g$ implies that there is always a non-zero section of $E^* \otimes E^*$, which says that $E^* \otimes E^*$ always admits a trivial 1-dimensional subbundle $A = c \cdot g$ for $c \in \mathbb{R}$. That is, span$(g(x))$ defines a 1-dimensional subspace of every fiber, and noting that any 1-dimensional bundle with a non-vanishing section must be a trivial bundle. Of course, the metric gives an isomorphism

$$E^* \otimes E^* \cong E^* \otimes E \cong \text{Hom}(E, E).$$

(4.8)

The latter bundle always admits the identity section $I : E \to E$, so $c \cdot I$ for $c \in \mathbb{R}$ defines a 1-dimensional trivial subbundle of $\text{Hom}(E, E)$. The latter choice is canonical, but the sub-bundle $A$ is not.

**Remark 4.4.** In the special case of a real line bundle $\pi : L \to M$, the bundle $\text{Hom}(L, L)$ must be a trivial line bundle. So $L^* \otimes L^*$ is always a trivial bundle, and a Riemannian metric can simply be viewed as a positive function on $M$.

**Definition 4.5.** If $E_1 \subset E_2$ is a subbundle, then the quotient bundle $E_2/E_1$ is the vector bundle with fiber $\pi_2^{-1}(x)/\pi_1^{-1}(x)$ over $x$.

**Exercise 4.6.** Prove that the quotient bundle is a vector bundle. That is, find local trivializations for $E_2/E_1$.

Note the following corollary.

**Corollary 4.7.** If $E_1 \subset E$ is a sub-bundle, then there exists a subbundle $E_2 \subset E$ such that

$$E \cong E_1 \oplus E_2.$$  

(4.9)

Furthermore, the quotient bundle $(E/E_1) \cong E_2$.

**Proof.** Choose a Riemannian metric $g$ on $E$, and let $E_2 = (E_1)\perp$. Use Gram-Schmidt to construct local trivializations for $(E_1)\perp$ to show this is indeed a subbundle. The rest is just linear algebra. \qed

**Example 4.8.** If $f : M^k \to \mathbb{R}^n$ is an embedded (or immersed) submanifold, then define the normal bundle

$$\nu_M = \{(p, v) \in M \times T\mathbb{R}^n \mid v \in T_{f(p)}\mathbb{R}^n, v \perp f_*(T_pM)\},$$

(4.10)

where where use the Euclidean metric on $T\mathbb{R}^n$. This is a bundle of rank $n - k$ over $M$, since $f_*$ is injective at any point. We then have the decomposition

$$f^*T\mathbb{R}^n = TM \oplus \nu_M.$$  

(4.11)

For example, take $\iota : S^n \to \mathbb{R}^{n+1}$ to be the standard inclusion. The radial vector field is a nontrivial normal vector field, so we have

$$T\mathbb{R}^{n+1} = TS^n \oplus (S^n \times \mathbb{R}),$$

(4.12)

where the latter factor is just the trivial line bundle over $S^n$. Note this shows that a trivial vector bundle can have a non-trivial sub-bundle (for non-parallelizable spheres).

There is nothing special about $\mathbb{R}^n$ in the above: if $f : M^k \to N^n$ is an immersed submanifold, and $g$ is a Riemannian metric on $TN$, then we similarly have

$$f^*TN = TM \oplus \nu_M,$$

(4.13)

where we use Riemannian metric on $N$ to define the orthogonal complement.
4.2 Reduction of structure group

Definition 4.9. If a bundle \( \pi : E \to M \) is equivalent to a bundle which has transition functions \( \varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to K \), where \( K \) is a subgroup of \( GL(k, \mathbb{R}) \), then we say that the structure group of \( E \) can be reduced to \( K \).

Another way to state the results from the previous section is as follows.

Proposition 4.10. We have the following.

- A bundle is trivial if and only if its structure group can be reduced to \( \{ Id \} \).
- The structure group of any real vector bundle \( \pi : E \to M \) of rank \( k \) can be reduced to \( O(k) \).

Proof. If \( E \) is trivial, there is a global trivialization \( \Phi : M \times \mathbb{R}^k \to E \). Given an open covering \( \{ U_\alpha \}, \alpha \in \mathcal{I} \) of \( M \), then \( \Phi_\alpha = \Phi \big|_{U_\alpha \times \mathbb{R}^k} \) is a system of local trivializations which has overlap mappings \( \phi_{\alpha\beta} = Id \). Conversely, a system of local trivializations which have identity overlap mappings patch together to give a global trivialization.

For the second case, from above \( E \) admits a Riemannian metric. By Gram-Schmidt, for any point \( x \in M \), there exists a neighborhood \( U_x \) and a local basis of sections \( \{ e_1, \ldots, e_k \} \) which are orthonormal at every point in \( U_x \). Define local trivializations by

\[
\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^{k} v^i e_i. \tag{4.14}
\]

Then overlaps maps then necessarily satisfy

\[
\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k), \tag{4.15}
\]

where \( O(k) \) is the orthogonal group of \( k \times k \) real matrices satisfying \( AA^T = I_k \).

4.3 Real line bundles

Note for a real 1-dimensional line bundle \( \pi : L \to M \), we have that the structure group can be reduced to \( O(1) = \{ \pm 1 \} \). Consider the set

\[
\tilde{M} = \{ v \in L \mid g(v, v) = 1 \}. \tag{4.16}
\]

Since there are exactly two unit norm vectors in any fiber, we have that \( \pi : \tilde{M} \to M \) is a 2-fold covering space. So any real line bundle give an associated 2-fold covering space. Conversely, any 2-fold covering space gives a real line bundle, which is uniquely determined up to equivalence. To see this, note that a 2-fold covering space can be viewed as a fiber bundle with group \( \mathbb{Z}_2 \), and viewing \( \mathbb{Z}_2 = \{ \pm 1 \} \subset GL(1, \mathbb{R}) \), we naturally obtain an associated real line bundle.
Therefore real line bundles over $M$ are in one-to-one correspondence with 2-fold covering spaces of $M$, up to equivalence. Using some covering space theory, the 2-fold coverings correspond to index 2 subgroups of $\pi_1(M)$, which is

$$\text{Hom}(\pi_1(M), \mathbb{Z}_2).$$

(4.17)

Since the abelianization of $\pi_1(M)$ is $H_1(M, \mathbb{Z}_2)$, and from the universal coefficient theorem, we have the isomorphisms

$$\text{Hom}(\pi_1(M), \mathbb{Z}_2) \cong \text{Hom}(H_1(M), \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2),$$

(4.18)

the first cohomology group with $\mathbb{Z}_2$ coefficients. Consequently, we have proved the following.

**Proposition 4.11.** The real line bundles on $M$ up to bundle equivalence, are in one-one correspondence with $H^1(M, \mathbb{Z}_2)$.

## 5 Lecture 5

### 5.1 Čech cohomology with $\mathbb{Z}_2$ coefficients

The above used a lot of topology, so we give another explanation for this isomorphism. Given an open covering $\{U_\alpha\}, \alpha \in \mathcal{I}$, define $C^0(\mathcal{U}, \mathbb{Z}_2)$ to be the free vector space over $\mathcal{I}$ with $\mathbb{Z}_2$ coefficients. We can think of this as a choice $f_\alpha \in \mathbb{Z}_2$, for each $\alpha \in \mathcal{I}$. Define $C^1(\mathcal{U}, \mathbb{Z}_2)$ to be the free vector space over $U_\alpha \cap U_\beta$ for nontrivial intersections with $\mathbb{Z}_2$ coefficients, which we can think of as a choice $f_{\alpha \beta} \in \mathbb{Z}_2$, $\alpha, \beta \in \mathcal{I}$, and $U_\alpha \cap U_\beta \neq \emptyset$. Define $C^2(\mathcal{U}, \mathbb{Z}_2)$ to be the free vector space over $U_\alpha \cap U_\beta \cap U_\gamma$ with $\mathbb{Z}_2$ coefficients, which we can think of as a choice $f_{\alpha \beta \gamma} \in \mathbb{Z}_2$, $\alpha, \beta, \gamma \in \mathcal{I}$, and $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Finally, define $C^3(\mathcal{U}, \mathbb{Z}_2)$ to be the free vector space over $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ with $\mathbb{Z}_2$ coefficients, which we can think of as a choice $f_{\alpha \beta \gamma \delta} \in \mathbb{Z}_2$, $\alpha, \beta, \gamma, \delta \in \mathcal{I}$, and $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \neq \emptyset$.

Next, we define linear operators $\delta^0 : C^0(\mathcal{U}, \mathbb{Z}_2) \to C^1(\mathcal{U}, \mathbb{Z}_2)$ by $(\delta^0 f)_{\alpha \beta} = f_\beta - f_\alpha$, and $\delta^1 : C^1(\mathcal{U}, \mathbb{Z}_2) \to C^2(\mathcal{U}, \mathbb{Z}_2)$ by $(\delta^1 f)_{\alpha \beta \gamma} = f_{\beta \gamma} - f_{\alpha \gamma} + f_{\alpha \beta}$, and $\delta^2 : C^2(\mathcal{U}, \mathbb{Z}_2) \to C^3(\mathcal{U}, \mathbb{Z}_2)$ by $(\delta^2 f)_{\alpha \beta \gamma \delta} = f_{\beta \gamma \delta} - f_{\alpha \gamma \delta} + f_{\alpha \beta \delta} - f_{\alpha \beta \gamma}$. We check that

$$((\delta^1 \circ \delta^0) f)_{\alpha \beta \gamma} = (\delta^1 f_{\beta} - f_\alpha)_{\alpha \beta \gamma} = f_\gamma - f_\beta - f_\alpha + f_\beta - f_\alpha = 0,$$

(5.1)

and

$$(\delta^2 \circ \delta^1 f)_{\alpha \beta \gamma \delta} = (\delta^1 f)_{\beta \gamma \delta} - (\delta^1 f)_{\alpha \gamma \delta} + (\delta^1 f)_{\alpha \beta \delta} - (\delta^1 f)_{\alpha \beta \gamma} = f_\gamma \delta - f_\beta \delta + f_\beta \gamma - (f_\gamma \delta - f_\alpha \delta + f_\alpha \gamma)$$

$$+ f_\beta \delta - f_\alpha \delta + f_{\alpha \beta} - (f_\beta \gamma - f_\alpha \gamma + f_{\alpha \beta}) = 0.$$

(5.2)

This allows us to define

$$H^0(\mathcal{U}, \mathbb{Z}_2) = \ker(\delta^0), \quad H^1(\mathcal{U}, \mathbb{Z}_2) = \ker(\delta^1)/\text{image}(\delta^0), \quad H^2(\mathcal{U}, \mathbb{Z}_2) = \ker(\delta^2)/\text{image}(\delta^1).$$

(5.3)
It is easy to see that if $M$ is connected, then
\begin{equation}
H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.4}
\end{equation}

Back to real line bundles: after reduction the structure group to $\mathbb{Z}_2$, the transition functions of the bundle are given by
\begin{equation}
\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2. \tag{5.5}
\end{equation}
Since $\mathbb{Z}_2$ with multiplication is isomorphic to $\mathbb{Z}_2$ with addition, the condition on transition functions
\begin{equation}
\phi_{\alpha\gamma} = \phi_{\alpha\beta}\phi_{\beta\gamma} \tag{5.6}
\end{equation}
says that $\phi_{\alpha\beta}$ form a Čech 1-cocycle, so
\begin{equation}
\phi_{\alpha\beta} \in \check{H}^1_\mathfrak{U}(M, \mathbb{Z}_2). \tag{5.7}
\end{equation}
The condition that $\phi_{\alpha\beta}$ be a coboundary is that there exists a 0-cochain $f_\alpha : U_\alpha \to \mathbb{Z}_2$ so that
\begin{equation}
\phi_{\alpha\beta} = f_\beta f_\alpha^{-1} \tag{5.8}
\end{equation}
on $U_\alpha \cap U_\beta$ which is exactly the condition for the bundle to the equivalent to a trivial bundle.

However, note there is a slight difference in the definitions because the transition functions in the bundle definition are smooth, so the $\phi_{\alpha\beta}$ in (5.5) are constant on each components of $U_\alpha \cap U_\beta$, with possibly different values on different components. But a Čech 1-cocycle is the same constant on all components of $U_\alpha \cap U_\beta$.

**Remark 5.1.** The Čech cohomology as defined above obviously depends on the open cover. It turns out that if the cover is sufficiently nice, it is independent of the cover. Such a cover is called a “good” cover, which is a covering so that all open sets in the cover and all nontrivial intersections are contractible. We will not prove this right now (maybe later).

**Example 5.2** (Case of $S^1$). If $\mathfrak{U}$ is a covering with 2 intervals so that the intersection is 2 intervals. So
\begin{equation}
C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.9}
\end{equation}
We conclude that
\begin{equation}
H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(\mathfrak{U}, \mathbb{Z}_2) = 0. \tag{5.10}
\end{equation}
However, if cover by 3 intervals, so that the intersections are connected intervals, we then have
\begin{equation}
C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^\oplus 3, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^\oplus 3. \tag{5.11}
\end{equation}
We conclude that
\begin{equation}
H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{5.12}
\end{equation}
Therefore we know all real line bundles on $S^1$: the trivial bundle and the Mobius bundle.
Example 5.3 (Case of $S^2$). We can find a good cover of $S^2$ by slightly enlarging the faces of tetrahedron, call these $U_0, U_1, U_2, U_3, U_{a\beta} = U_\alpha \cap U_\beta$, and $U_{a\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. We see that

$$C^0(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^4, \quad C^1(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{10}, \quad C^2(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{10}.$$

We compute that

$$\text{Ker} \delta^1 = \{f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23} | \quad f_{12} - f_{02} + f_{01} = 0, f_{13} - f_{03} + f_{01} = 0, f_{23} - f_{03} + f_{02} = 0, f_{23} - f_{13} + f_{12} = 0\}.$$

This shows that a kernel element is determined by $f_{01}, f_{02}, f_{03}$. We conclude that

$$H^0(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(\mathcal{U}, \mathbb{Z}_2) = \{0\}, \quad H^2(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2.$$  \hspace{1cm} (5.13)

Consequently, every real line bundle on $S^2$ is trivial.

Remark 5.4. Instead of a tetrahedron, one could also use a cube to construct a good cover of $S^2$. We leave it to the interested student to compute the Čech cohomology of this cover, and verify you get the same answer for the cohomology groups.

Example 5.5 (Case of $T^2$). We can find a good cover of $T^2$ by viewing $T^2$ as a square with opposite sides identified, dividing into 9 squares, and slightly enlarging each square. We see that

$$C^0(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^9, \quad C^1(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{36}, \quad C^2(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{36}, \quad C^3(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2^9.$$  \hspace{1cm} (5.15)

Since $T^2$ is a 2-manifold, it is possible to show that $\delta^2$ is surjective. This implies that $\text{Ker} (\delta^2) = \mathbb{Z}_2^7$. Next, by consider a “dual” good cover (where faces become vertices and vice-versa), we can show that $H^2(\mathcal{U}, \mathbb{Z}_2) = H^0(\mathcal{U}, \mathbb{Z}_2) = \mathbb{Z}_2$. Consequently, $\text{Image}(\delta^1) = \mathbb{Z}_2^{26}$, so $\text{Ker}(\delta^1) = \mathbb{Z}_2^{10}$. We already know that $\text{Image}(\delta^0) = \mathbb{Z}_2^8$, so we conclude the following:

$$H^k(T^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{k = 0, 2} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{k = 1} \end{cases}.$$  \hspace{1cm} (5.16)

We can write down generators using the following. Let $\pi_i : S^1 \times S^1 \rightarrow S^1$ be the projection mappings, and let $\gamma$ denote the Mobius bundle over $S^1$. In addition to the trivial bundle, we have $\pi_1^* \gamma, \pi_2^* \gamma, \text{ and } \pi_1^* \gamma \oplus \pi_2^* \gamma$. To get the Čech generator(s), we take a 1-cocycle which is +1 along the intersections on a vertical (horizontal) strip of 3 squares, and zero elsewhere.

Remark 5.6. In the previous example, we found there are exactly 4 real line bundles on $T^2$, up to bundle equivalence. Recall that this equivalence only considers bundle isomorphisms covering the identity mapping. If we had allowed arbitrary diffeomorphisms of the base then pullback under the mapping $f(\theta_1, \theta_2) = (\theta_2, \theta_1)$ would identify $\pi_1^* \gamma$ and $\pi_2^* \gamma$. So with this notion of equivalence, there would only be 3 line bundles over $T^2$.

Example 5.7 (Klein bottle). The Klein bottle can be constructed similar to a torus by identifying opposite sides of a square, but with a twist on one pair of opposite sides. Similar to the above computation for a torus, we can compute that

$$H^k(K^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{k = 0, 2} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{k = 1} \end{cases}.$$  \hspace{1cm} (5.17)
Example 5.8. (Tautological bundle on $\mathbb{RP}^n$) Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$ (v_1, \ldots, v_{n+1}) \sim (cv_1, \ldots, cv_{n+1}), \quad c \neq 0. $$

(5.18)

Define

$$ \gamma_1^n = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in [x]\} $$

(5.19)

We claim that $\gamma_1^n$ is a nontrivial 1-dimensional bundle over $\mathbb{RP}^n$. Assume by contradiction that it were the trivial bundle. Then there would exist a nowhere vanishing section $\sigma : \mathbb{RP}^n \to \gamma_1^n$. This is a mapping

$$ \sigma : \mathbb{RP}^n \to \mathbb{RP}^n \times \mathbb{R}^{n+1} $$

(5.20)

of the form for $x \in S^n$,

$$ \sigma([x]) = ([x], c(x) \cdot x) $$

(5.21)

For this to be well-defined, we require that $c(x) : S^n \to \mathbb{R}$ is a function satisfying $c(-x) = -c(x)$. Since $c$ must take negative and positive values, by the intermediate value theorem, $c(x_0) = 0$ for some $x_0$, which is a contradiction.

Example 5.9 (Case of $\mathbb{RP}^2$). We can construct $\mathbb{RP}^2$ by identifying opposite sides of a square, but twisting on both pairs of sides. To find a good cover: divide the square into 9 squares, but shave off the corner squares to become triangles, so that we have an octagon on the boundary. We slightly enlarge the each square and triangle to obtain a good cover. We see that

$$ C^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 9}, \quad C^1(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 32}, \quad C^2(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 32}, \quad C^3(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2^{\oplus 8} $$

(5.22)

Since $\mathbb{RP}^2$ is a 2-manifold, it is possible to show that $\delta_2$ is surjective. This implies that $\text{Ker}(\delta_2) = \mathbb{Z}_2^{24}$. Next, by consider the “dual” open cover, we can show that $H^2(\mathfrak{U}, \mathbb{Z}_2) = H^0(\mathfrak{U}, \mathbb{Z}_2) = \mathbb{Z}_2$. Consequently, $\text{Image}(\delta_1) = \mathbb{Z}_2^{24}$, so $\text{Ker}(\delta_1) = \mathbb{Z}_2^8$. We already know that $\text{Image}(\delta_0) = \mathbb{Z}_2^8$, so we conclude the following:

$$ H^k(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2, \quad k = 0, 1, 2. $$

(5.23)

So there are exactly 2 real line bundles over $\mathbb{RP}^2$: the trivial bundle and the tautological bundle $\gamma_1^2$.

6 Lecture 6

6.1 Exterior powers

Let $V$ be a real vector space. The exterior algebra $\Lambda(V)$ is defined as

$$ \Lambda(V) = \left\{ \bigoplus_{k \geq 0} V^{\otimes k} \right\}/\mathcal{I} = \bigoplus_{k \geq 0} \left(V^{\otimes k}/\mathcal{I}_k\right) = \bigoplus_{k \geq 0} \Lambda^k V, $$

(6.1)
where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $v \otimes v \in V \otimes V$, and $\mathcal{I}_k = V^{\otimes k} \cap \mathcal{I}$. The wedge product of $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$ is just the multiplication induced by the tensor product in this algebra, that is, lift $v$ and $w$ to $\tilde{v} \in V^{\otimes p}$, and $\tilde{w} \in V^{\otimes q}$, and define $v \wedge w = \pi(\tilde{v} \otimes \tilde{w})$, where $\pi : V^{\otimes p+q} \to \Lambda^{p+q}V$ is the projection. This is easily seen to be well-defined. We say that an element in $\Lambda^k(V)$ of the form $v_1 \wedge \cdots \wedge v_k$ is decomposable. A general element of $\Lambda^k(V)$ is not decomposable, but can always be written as a sum of decomposable elements.

The space $\Lambda^k(V)$ satisfies the universal mapping property as follows. Let $W$ be any vector space, and let

$$F : V \times \cdots \times V \to W$$

be an alternating multilinear mapping. That is, $F$ is multilinear and $F(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

$$
\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{\pi} & \Lambda^k(V) \\
\text{x} & \downarrow & \text{x} \\
F & \downarrow & \tilde{F} \\
W & \text{y}
\end{array}
$$

commutative, where $\pi$ is the projection, which we denote as

$$\pi(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k. \quad (6.3)$$

**Exercise 6.1.** Prove the following properties of the wedge product.

- Bilinearity: $(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w$, and $(cv) \wedge w = c(v \wedge w)$ for $c \in \mathbb{R}$.
- If $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$, then $v \wedge w = (-1)^{pq}w \wedge v$.
- Associativity $(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3)$.

**Exercise 6.2.** If $\dim_{\mathbb{R}}(V) = n$, prove that $\Lambda^k(V) = \{0\}$ if $k > n$,

$$\dim(\Lambda^k(V)) = \binom{n}{k} \text{ if } 0 \leq k \leq n, \quad (6.4)$$

and

$$\dim(\Lambda(V)) = 2^n, \quad (6.5)$$

**Definition 6.3.** For a real vector bundle $\pi : E \to M$, we define $\Pi : \Lambda^p(E) \to M$ by $\Pi^{-1}(x) = \Lambda^p(\pi^{-1}(x))$. If $\Phi : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ is a local trivialization for $E$, then consider the mapping

$$F : U \times \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \Pi^{-1}(U) \quad (6.6)$$
defined by
\[ F(x,(v_1,\ldots,v_p)) = \Phi(x,v_1) \wedge \cdots \wedge \Phi(x,v_k) \] (6.7)
This is clearly an alternating multilinear mapping on fibers, so by the universal property, there is a unique induced mapping
\[ \tilde{F} : U \times \Lambda^p(\mathbb{R}^k) \to \Pi^{-1}(U) \] (6.8)
which is a local trivialization for \( \Lambda^p(E) \).

We can equivalently define the \( p \)th exterior power in terms of transition functions. To do this, note that for any linear map \( f : \mathbb{R}^k \to \mathbb{R}^k \), there is a naturally induced mapping
\[ \Lambda^p f : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \] (6.9)
define as follows. Define
\[ F : \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \Lambda^p(\mathbb{R}^k) \] (6.10)
by
\[ F(v_1,\ldots,v_p) = f(v_1) \wedge \cdots \wedge f(v_p) \] (6.11)
This is clearly an alternating multilinear mapping, so by the universal property, there exists a unique mapping
\[ \Lambda^p f = \tilde{F} : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k). \] (6.12)
therefore for any vector bundle \( E \), the \( p \)th exterior power \( \Lambda^p(E) \) is defined to be the bundle with transition functions
\[ \varphi^\Lambda_{\alpha\beta}(E) = \Lambda^p(\varphi_{\alpha\beta}^E). \] (6.13)
Putting all of these together, we can define the following.

**Definition 6.4.** For a real vector bundle \( \pi : E \to M \), define the exterior algebra bundle \( \Lambda(E) = \bigoplus_{p=0}^k \Lambda^p(E) \).

Note in the above discussion, if we sum together all of the \( \Lambda^p f \) mappings, we get an induced mapping between the exterior algebras
\[ \Lambda(f) : \Lambda(\mathbb{R}^k) \to \Lambda(\mathbb{R}^k) \] (6.14)
which satisfies
\[ \Lambda(f)(\alpha \wedge \beta) = \Lambda(f)(\alpha) \wedge \Lambda(f)(\beta) \] (6.15)
Therefore, the wedge product gives an algebra structure on each fiber of \( \Lambda(E) \).
6.2 Differential forms

Definition 6.5. Given a smooth manifold \( M \), a differential \( k \)-form on \( M \) is smooth section of the \( k \)th exterior power of the cotangent bundle, that is, \( \omega \in \Gamma(\Lambda^k(T^*M)) \equiv \Omega^k(M) \).

Given a function \( f \in C^\infty(M, \mathbb{R}) \) we define \( df \in \Omega^1(M) \) in two ways. First, viewing vector fields as derivations on smooth functions, we can define
\[
df(X) \equiv X(f).
\] (6.16)

Alternatively, since \( f : M \to \mathbb{R} \), we have \( f_* : TM \to T\mathbb{R} \). But there is a natural identification \( T_p\mathbb{R} \cong \mathbb{R} \) for any \( p \in \mathbb{R} \), so we can view
\[
f_* : TU \to \mathbb{R},
\] (6.17)
which is naturally an element in \( df \in \Omega^1(U) \).

Exercise 6.6. Verify that these two definitions agree.

Given a coordinate system \( x : U \to \mathbb{R}^n \), write the component functions as \( x^i : U \to \mathbb{R} \), for \( i = 1, \ldots, n \). We then have that
\[
dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n
\] (6.18)
are a basis of local sections of \( \Lambda^k(T^*U) \). That is, any \( \omega \in \Omega^k(U) \) can be written as a linear combination
\[
\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\] (6.19)
for some functions \( f_{i_1 \ldots i_k} \in \Omega^0(U) = C^\infty(U, \mathbb{R}) \).

Example 6.7. Let \( M = \mathbb{R}^4 \), and \( f, g \in \Omega^0(\mathbb{R}^4) \), and define
\[
\omega_1 = f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)
\] (6.20)
\[
\omega_2 = g(dx^1 \wedge dx^3 - dx^2 \wedge dx^4).
\] (6.21)

Then
\[
\omega_1 \wedge \omega_1 = f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)
\]
\[
= f^2(dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^1 \wedge dx^2 + dx^3 \wedge dx^4)
\]
\[
+ dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^4)
\] (6.22)
\[
= f^2(0 + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 + 0)
\]
\[
= 2f^2 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.
\]

Also,
\[
\omega_1 \wedge \omega_2 = f(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge g(dx^1 \wedge dx^3 - dx^2 \wedge dx^4)
\]
\[
= fg(dx^1 \wedge dx^2 \wedge dx^3 - dx^1 \wedge dx^2 \wedge dx^4)
\]
\[
+ dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 - dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^4)
\] (6.23)
\[
= 0.
\]
6.3 Differential forms as multilinear mappings

We could just stick with the above definition of the exterior algebra and prove all results using only this definition. However, it is very useful to think of elements of $\Lambda^k(V^*)$ as alternating multilinear maps on $V$ as follows. One first has to choose a pairing

$$\Lambda^k(V^*) \cong (\Lambda^k(V))^*.$$  \hfill (6.24)

The pairing we will choose is as follows. If $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $v = v_1 \wedge \cdots \wedge v_k$, then

$$\alpha(v) = \det(\alpha^i(v_j)),$$  \hfill (6.25)

(note this is not canonical). For example,

$$\alpha^1 \wedge \alpha^2(v_1 \wedge v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1).$$  \hfill (6.26)

We would then like to view an element of $(\Lambda^k(V))^*$ as an alternating multilinear mapping from

$$\underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}.$$  \hfill (6.27)

For this, we specify that if $\alpha \in (\Lambda^k(V))^*$, then

$$\alpha(v_1, \ldots, v_k) \equiv \alpha(v_1 \wedge \cdots \wedge v_k).$$  \hfill (6.28)

For example

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1).$$  \hfill (6.29)

With this convention, if $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ then

$$\alpha \wedge \beta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \text{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}).$$  \hfill (6.30)

This then agrees with the definition of the wedge product given in [Spi79, Chapter 7].

It is convenient to have our 2 definitions of the wedge product because some proofs can be easier using one of the definitions, but harder using the other (for example, associativity of the wedge product).

6.4 Orientability of real bundles

Note that if $V$ is an $n$-dimensional vector space, then $\Lambda^n V$ is 1-dimensional. So if $L : V \to V$ is a linear transformation then $\Lambda^n L : \Lambda^n V \to \Lambda^n V$ is an endomorphism of a 1-dimensional vector space. Therefore $\Lambda^n L(\omega) = c \cdot \omega$ for some scalar $c$. So we can make the following definition:

**Definition 6.8.** For a linear transformation $L : V \to V$, define $\det(L)$ to be the real number so that

$$\Lambda^n L(\omega) = \det(L) \cdot \omega.$$  \hfill (6.31)
Exercise 6.9. Show that this definition of determinant agrees with the usual linear algebra definition of determinant.

Proposition 6.10. Let $\pi : E \to M$ be a real vector bundle of rank $k$. The following are equivalent.

- The line bundle $\Lambda^k(E)$ is trivial.
- $\Lambda^k(E)$ admits a nowhere zero section.
- The double cover $\tilde{M}$ corresponding to $\Lambda^k(E)$ is a trivial 2-fold covering space.
- The structure group of $E$ can be reduced to $GL_+(k, \mathbb{R}) \equiv \{ A \in GL(k, \mathbb{R}) \mid \det(A) > 0 \}$ \hspace{1cm} (6.32)

- The structure group of $E$ can be reduced to $SO(k)$

Proof. The proof follows from the above discussion, with the following remarks. If $e_1, \ldots, e_k$ is a local basis of sections, we say that $\{e_1, \ldots, e_k\}$ is oriented if

$$e_1 \wedge \cdots \wedge e_k = f \omega,$$

with $f > 0$ and $\omega \in \Lambda^k(E)$ is the nowhere zero section. Restricting to local trivializations arising from oriented local bases of sections will give a reduction of structure group to $GL_+(k, \mathbb{R})$.

Definition 6.11. We say that a real vector bundle $\pi : E \to M$ is orientable if any of the equivalent conditions in Proposition 6.10 are satisfied.

Remark 6.12. If we use the 2-fold covering notion, then we see that if $\pi_1(M) = \{e\}$ then every vector bundle over $M$ is orientable. This is because any covering of a simply connected space is trivial. (Actually, we just need to assume that $H^1(M, \mathbb{Z}_2) = 0$.) Thus, every vector bundle over $S^n$ is orientable for $n \geq 2$.

Example 6.13. Returning to $\mathbb{R}P^n$, since $\mathbb{R}P^n$ is double covered by $S^n$, we have $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$. Therefore there are exactly 2 real line bundles over $\mathbb{R}P^n$, the trivial bundle and the tautological line bundle. Note that if we put a Riemannian metric on the tautological bundle $\pi : \gamma_2^1 \to \mathbb{R}P^n$, then the total space of the unit sphere bundle is just $S^n$. But for the trivial bundle over $\mathbb{R}P^n$, the unit sphere bundle is just 2 copies of $\mathbb{R}P^n$.

7 Lecture 7

7.1 Induced mappings

Recall that if $L : V \to W$ is a linear mapping between vector spaces, then there is a mapping, $L^* : W^* \to V^*$ called the transpose, defined by the following. If $\omega \in W^*$, and $v \in V$, then

$$(L^* \omega)(v) = \omega(Lv). \hspace{1cm} (7.1)$$
This is called the transpose for the following reason. Let \text{dim}(V) = n, and \text{dim}(W) = m. Let \(e_1, \ldots, e_n\) be a basis of \(V\) and \(f_1, \ldots, f_n\) be a basis of \(W\). Let \(e^1, \ldots, e^n\), and \(f^1, \ldots, f^n\) denote the dual bases, that is
\[
e^i(e_j) = \delta^i_j, \quad 1 \leq i, j \leq n
\] (7.2)
\[
f^i(f_j) = \delta^i_j, \quad 1 \leq i, j \leq m.
\] (7.3)

We define the quantities \(L^j_i, 1 \leq i \leq n, 1 \leq j \leq m\), by
\[
Le_i = L^j_i f_j.
\] (7.4)

Note that if we write \(v \in V\) as \(v = v^i e_i\), and \(w \in W\) as \(w = w^j f_j\), then
\[
Lv = L(v^i e_i) = v^i L(e_i) = (v^i L^j_i) f_j.
\] (7.5)

So the components of a vector transform like
\[
\{v^i\} \mapsto \{L^j_i v^i\},
\] (7.6)
which is the matrix corresponding to the transformation \(L\).

We define the quantities \((L^*)^j_i, 1 \leq i \leq m, 1 \leq j \leq n\), by
\[
L^* f^i = (L^*)^j_i e^j
\] (7.7)

Plugging in the dual bases, we compute
\[
(L^* f^i)(e_k) = (L^*)^j_i e^j(e_k) = (L^*)^j_i \delta^j_k = (L^*)^i_k.
\] (7.8)

However, by the definition of the transpose mapping, we have
\[
(L^* f^i)(e_k) = f^i(Le_k) = f^i L^j_i f_j = L^j_i f^i(f_j) = L^j_i \delta^i_j = L^i_k
\] (7.9)

So if we write \(\omega \in V^*\) as \(\omega^i e^i\) and \(\eta \in W^*\) as \(\eta^j f^j\), the components of a dual vector transform like
\[
\{\eta^j\} \mapsto \{L^j_i \eta^i\}
\] (7.10)

So the matrix corresponding to \(L^*\) in the dual basis is indeed the transpose matrix.

The mapping \(L^*: W^* \to V^*\) induces a mapping
\[
(L^*)^p: W^* \times \cdots \times W^* \to (V^*)^\otimes p
\] (7.11)
by
\[
(L^*)^p(\alpha^1, \ldots, \alpha^p) \equiv (L^* \alpha^1) \otimes \cdots \otimes (L^* \alpha^p).
\] (7.12)

This mapping is a multilinear mapping, so by the universal property of tensor products, this induces a unique mapping
\[
(L^*)^p: (W^*)^\otimes p \to (V^*)^\otimes p.
\] (7.13)
By composing with the projection \( \pi : (V^*)^\otimes p \to \Lambda^p(V^*) \), we obtain an alternating multilinear mapping

\[
(L^*)^\times p : (W^*)^\otimes p \to \Lambda^p(V^*).
\] (7.14)

Now by the universal property of exterior products, this induces a mapping

\[
\Lambda^p(L^*) : \Lambda^p(W^*) \to \Lambda^p(V^*).
\] (7.15)

Note that by taking the direct sum on all \( p \)-s, we obtain a mapping between the full exterior algebras

\[
\Lambda(L^*) : \Lambda(W^*) \to \Lambda(V^*)
\] (7.16)

which is an algebra homomorphism, that is,

\[
\Lambda(L^*)(\alpha \wedge \beta) = (\Lambda(L^*)\alpha) \wedge (\Lambda(L^*)\beta).
\] (7.17)

### 7.2 Pull-back of differential forms

Recall that if \( f : M \to N \) is a smooth mapping between smooth manifolds, then the mapping \((f_*)_B : TM \to f^*TN\) defined by \((f_*)_B(v_p) = (p, f_*v)\) makes the following diagram commute

\[
\begin{array}{ccc}
TM & \xrightarrow{(f_*)_B} & f^*TN \\
\downarrow \pi_M & & \downarrow \pi_1 \\
M & \xrightarrow{id} & M
\end{array}
\] (7.18)

Noting that \((f^*(TN))^*\) is naturally isomorphic to \(f^*(T^*N)\), let us dualize (7.18) to obtain the commutative diagram

\[
\begin{array}{ccc}
f^*(T^*N) & \xrightarrow{f^*B} & T^*M \\
\downarrow \pi_1 & & \downarrow \pi_M \\
M & \xrightarrow{id} & M
\end{array}
\] (7.19)

Next, by the diagram (7.19) and the above discussion, we obtain bundle mappings

\[
\begin{array}{ccc}
f^*(\Lambda^p(T^*N)) & \xrightarrow{\Lambda^p(f^*B)} & \Lambda^p(T^*M) \\
\downarrow \pi_1 & & \downarrow \pi_M \\
M & \xrightarrow{id} & M
\end{array}
\] (7.20)

**Definition 7.1** (Pull-back of a differential form). If \( f : M \to N \) is a smooth mapping, and \( \omega \in \Lambda^p(T^*N) \), then define \( \omega \circ f \in \Gamma(f^*(\Lambda^p(T^*N))) \) by \( \omega \circ f(p) = (p, \omega_{f(p)}) \). Then define

\[
f^*\omega \equiv \Lambda^p(f^*_B)(\omega \circ f) \in \Gamma(\Lambda^p(T^*M)).
\] (7.21)
Remark 7.2. If we view differential forms as multilinear mappings, for \( f: M \rightarrow N \), and \( \omega \in \Omega^k(N) \), then we have the following “formula”. If \( p \in M \) and \( X_1, \cdots, X_k \in T_p M \), then
\[
(f^\ast \omega)(X_1, \cdots, X_k) = \omega_{f(p)}(f_\ast X_1, \cdots, f_\ast X_k).
\] (7.22)

We could have defined pullback of forms this way, but we would need an extra step to show the pullback of a smooth form is smooth.

For any manifold \( M \), define
\[
\Omega(M) = \Gamma(\Lambda(T^\ast M)) = \bigoplus_{p \geq 0} \Gamma(\Lambda^p(T^\ast M)) = \bigoplus_{p \geq 0} \Omega^p(M).
\] (7.23)

By taking the direct sum of the above mappings on each exterior power, we obtain a mapping
\[
f^\ast: \Omega(N) \rightarrow \Omega(M),
\] (7.24)
which by (7.17) satisfies
\[
f^\ast(\alpha \wedge \beta) = (f^\ast \alpha) \wedge (f^\ast \beta).
\] (7.25)

Proposition 7.3 (The chain rule). If \( f: M \rightarrow N \), and \( h: N \rightarrow M' \) are smooth maps, then
\[
(h \circ f)^\ast = f^\ast \circ h^\ast: \Omega(M') \rightarrow \Omega(M).
\] (7.26)

Proof. We have that \( f_\ast: T M \rightarrow T N \) is a bundle mapping covering \( f \), and \( h_\ast: T N \rightarrow T M' \) is a bundle mapping covering \( h \). The above chain rule for the differential says that the bundle mapping \( (h \circ f)_\ast: T M \rightarrow T M' \) is given by \((h \circ f)_\ast = h_\ast \circ f_\ast\). Next, we have the mappings \((f_\ast)_B: T M \rightarrow f^\ast T N\), \((h_\ast)_B: T N \rightarrow h^\ast T M'\), and \(((h \circ f)_\ast)_B: T M \rightarrow (h \circ f)^\ast T M'\). The mapping \((h_\ast)_B\) induces a mapping \((h_\ast)_B \circ f: f^\ast T N \rightarrow f^\ast h^\ast T M'\). Since \((h \circ f)^\ast T M' = f^\ast(h^\ast T M')\), the chain rule implies that \(((h \circ f)_\ast)_B = ((h_\ast)_B \circ f) \circ (f_\ast)_B\). Dualizing and taking the induced mapping on exterior powers then implies the result. 

Example 7.4. Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be defined by
\[
f(x, y) = (x^2 + y^2, x^2 - y^2, x^3).
\] (7.27)
Denote the coordinates on \( \mathbb{R}^3 \) as \((u, v, w)\), and let
\[
\alpha = wdu \wedge dv - vdu \wedge dw + udv \wedge dw.
\] (7.28)
Then
\[
f^\ast \alpha = 4x^4 y dx \wedge dy.
\] (7.29)
(Details were done in lecture.)

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8 Lecture 8

8.1 The exterior derivative

Choose a coordinate system \((U, x)\), and let \(\frac{\partial}{\partial x^i}\) denote the coordinate vector field. Recall that viewing vector fields as derivations on germs of functions, this is characterized by

\[
\frac{\partial}{\partial x^i}(x^j) = \delta^j_i. \tag{8.1}
\]

We then define a local basis of 1-forms \(dx^i\) by

\[
dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j. \tag{8.2}
\]

Note this is just the dual basis, but these are also \(d(x^i)\) as defined above in (6.16).

An element \(\alpha \in \Omega^p(U)\) can be written as

\[
\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{8.3}
\]

where the coefficients \(\alpha_{i_1 \cdots i_p} : U \to \mathbb{R}\) are well-defined functions. Note these coefficients are only defined for strictly increasing sequences \(i_1 < \cdots < i_p\). Using our identification of \(\Lambda^p(T^*M)\) with \(\text{Alt}^p(TM)\), the alternating multilinear maps from \(TM^\times p \to \mathbb{R}\), we have that the coefficient functions are given by

\[
\alpha_{i_1 \cdots i_p} = \alpha\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_p}}\right). \tag{8.4}
\]

We next define the exterior derivative operator [War83, Theorem 2.20].

**Proposition 8.1.** There exists an exterior derivative operator

\[
d : \Omega^p(M) \to \Omega^{p+1}(M) \tag{8.5}
\]

which is the unique linear mapping satisfying

- For \(\alpha \in \Omega^p(M)\), \(d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta\).

- \(d^2 = 0\).

- If \(f \in C^\infty(M, \mathbb{R})\) then \(df\) is the differential of \(f\) defined above.

**Proof.** Note that the differential of a function is given locally by

\[
df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \tag{8.6}
\]

To see this, we have \(df = \sum c_i dx^i\), and plugging in the coordinate vector field identifies the coefficient \(c_i\). Since we gave a global definition of \(df\), this is obviously well-defined and
independent of the coordinate system. Given a $p$-form $\alpha$, write $\alpha$ locally as in (8.3), and then define

$$d\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} d\alpha_{i_1 \ldots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

(8.7)

The first “anti-derivation” property is easily verified by computation. The second property holds on functions, because

$$d(d(f)) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i\right) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0,$$

(8.8)

since the Hessian of a smooth function is symmetric.

For existence, we need to check that this definition is independent of the coordinate system. Let $d'$ be the operator defined with respect to another coordinate system $x': U \to \mathbb{R}^n$. Then

$$d'(\alpha) = d'\left(\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right)$$

$$= \sum_{|I|=p} (d'\alpha_{i_1 \ldots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$+ \sum_{|I|=p} \alpha_{i_1 \ldots i_p} \sum_k (-1)^{k-1} dx^{i_1} \wedge \cdots \wedge d'(dx^{i_k}) \wedge \cdots \wedge dx^{i_p}$$

(8.9)

$$= \sum_{|I|=p} (d\alpha_{i_1 \ldots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = d(\alpha),$$

since $d$ and $d'$ agree on functions, and since $d' dx^i = d'dx^i = 0$.

Then for any $p$-form $\alpha$,

$$d(d\alpha) = d\left(\sum_{|I|=p} (d\alpha_I) \wedge dx^I\right) = \sum_{|I|=p} (d^2 \alpha_I) \wedge dx^I - d\alpha_I \wedge d(dx^I) = 0.$$

(8.10)

Uniqueness will be left as an (optional) exercise. □

An important fact is that $d$ commutes with pull-back.

**Proposition 8.2.** If $f : M \to N$ is a smooth mapping, and $\omega \in \Omega^p(N)$, then

$$f^*(d_N \omega) = d_M (f^* \omega).$$

(8.11)

**Proof.** If $\omega$ is a 0-form, which is a function, then $f^* \omega = \omega \circ f$. So by above, we have

$$d(f^* \omega) = d(\omega \circ f) = (\omega \circ f)_*.$$

(8.12)
By the chain rule, we then have
\[ d(f^*\omega) = \omega \circ f_. \quad (8.13) \]
On the other hand, we have that
\[ f^*(d\omega)(X) = d\omega(f_*(X)) = \omega \circ f_. \quad (8.14) \]
So the claim is true on functions. Then if \( \omega \) is a \( p \)-form, write
\[ \omega = \sum_{|I|=p} \omega_I dx^I. \quad (8.15) \]
Since the pull-back operation is an algebra homomorphism, we have
\[ f^*\omega = \sum_{|I|=p} (f^*\omega_I) f^* dx^I = \sum_{|I|=p} (\omega_I \circ f) d(x^I \circ f). \quad (8.16) \]
Then
\[ d(f^*\omega) = \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f). \quad (8.17) \]
On the other hand, we have
\[ d\omega = \sum_{|I|=p} (d\omega_I) \wedge dx^I, \quad (8.18) \]
so
\[ f^*(d\omega) = \sum_{|I|=p} f^*(d\omega_I) \wedge f^* dx^I = \sum_{|I|=p} d(f^*\omega_I) \wedge d(f^*x^I) \]
\[ = \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f) = d(f^*\omega). \quad (8.19) \]

### 8.2 Lie derivatives

Given a vector field \( X \in \Gamma(TM) \), the Lie derivative of \( Y \) with respect to \( X \) is
\[ \mathcal{L}_X Y = [X, Y], \quad (8.20) \]
where \([X, Y]f = X(Yf) - Y(Xf)\)

**Proposition 8.3.** For \( X, Y \in \Gamma(TM) \), we have \([X, Y] \in \Gamma(TM)\).
Proof. In a local coordinate system, write
\[ X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}, \tag{8.21} \]
then
\[ [X,Y]f = X^i \frac{\partial}{\partial x^i} \left( Y^j \frac{\partial f}{\partial x^j} \right) - Y^j \frac{\partial}{\partial x^j} \left( X^i \frac{\partial f}{\partial x^i} \right) \]
\[ = X^i \left( \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + Y^j \frac{\partial^2 f}{\partial x^j \partial x^i} \right) - Y^j \left( \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} + X^i \frac{\partial^2 f}{\partial x^i \partial x^j} \right). \tag{8.22} \]
Since \( f \) is smooth, we have equality of the mixed partials, so
\[ [X,Y]f = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i}. \tag{8.23} \]
This shows that \([X,Y] \) is a derivation on germs of functions, so is a well-defined vector field.

Next, for \( X, Y \in \Gamma(TM) \), and \( \omega \in \Gamma(T^*M) \), define
\[ L_X \omega(Y) = X(\omega(Y)) - \omega(L_X Y). \tag{8.24} \]

**Proposition 8.4.** If \( X \in \Gamma(TM) \) and \( \omega \in \Gamma(T^*M) \), then \( L_X \omega \in \Gamma(T^*M) \).

**Proof.** Let \( f : M \to \mathbb{R} \). Then
\[ L_X \omega(fY) = X(\omega(fY)) - \omega(L_X (fY)) \]
\[ = X(f \omega(Y) - \omega([X, fY])) \]
\[ = (Xf \omega(Y) + fX(\omega(Y))) - \omega(f[X,Y] - (Xf)Y) \]
\[ = fX(\omega(Y)) - \omega(f[X,Y]) = fL_X \omega(Y). \tag{8.25} \]
Since this expression is linear over \( C^\infty \) functions, it is a well-defined tensor. To see this, let \( \alpha : \Gamma(TM) \to C^\infty(M) \) be a mapping which is linear over \( C^\infty \)-functions. It suffices to show that \( \alpha(X)(p) = 0 \) if \( X_p = 0 \). This is because if we let \( X \) and \( \tilde{X} \) be any smooth extensions of \( X_p \), then since \( X - \tilde{X} \) vanishes at \( p \)
\[ \omega(X - \tilde{X})(p) = 0, \tag{8.26} \]
so \( \omega(X)(p) = \omega(\tilde{X})(p) \) has a well-defined value, independent of the extension of \( X_p \). To proceed, given a coordinate system around \( p \), choose a cutoff function which is 1 in a coordinate neighborhood of \( p \), and 0 outside. Then
\[ X = (\phi X^i) \left( \phi \frac{\partial}{\partial x^i} \right) + (1 - \phi^2)X. \tag{8.27} \]
Both terms in the above are smooth vector fields on \( M \), so using linearity,
\[ \alpha(X)(p) = (\phi(p) X^i(p)) \alpha \left( \phi \frac{\partial}{\partial x^i} \right)(p) + (1 - \phi^2)(p) \alpha(X)(p) = 0. \tag{8.28} \]
9 Lecture 9

9.1 Lie derivative of differential forms

For a function, we define \( L_X f = X f \). In the previous lecture, we defined \( L_X \omega \) for \( \omega \in \Omega^1(M) \). We now extend the Lie derivative operator to differential forms in \( \Omega^p(M) \) for \( p > 1 \) by

\[
L_X (\omega^1 \wedge \cdots \wedge \omega^p) = \sum_{i=1}^p \omega^i \wedge \cdots \wedge (L_X \omega^i) \wedge \cdots \wedge \omega^p,
\]

for \( \omega^i \in \Gamma(T^*M) \). Note this is for decomposable forms, but extends to arbitrary forms by linearity. Tensorality is proved similar to the 1-form case. The analog to (8.24) is

\[
(L_X \omega)(X_1, \ldots, X_p) = X(\omega(X_1, \ldots, X_p)) + \sum_{i=1}^p (-1)^i \omega([X, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_p).
\]

Proposition 9.1. For \( \omega \in \Omega^p(M) \) and \( X \in \Gamma(TM) \), we have \( L_X d\omega = dL_X \omega \).

Proof. First, for a function, this is

\[
L_X df = df.
\]

To see this, take \( Y \in \Gamma(TM) \), and compute

\[
(L_X df)(Y) = X(df(Y)) - df(L_X Y)
= X(Yf) - (L_X Y)f
= X(Yf) - X(Y f) + Y(X f) = Y(X f)
\]

Plugging \( Y \) into the right hand side of (9.3) yields

\[
d(X f)(Y) = Y(X f),
\]

which proves this for functions.

Next, we consider the case that \( p \geq 1 \). Since both sides of this equation are tensors, it suffices to prove this in a local coordinate system, and we can assume that \( \omega \) is of the form \( \omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_p} \). We have \( d\omega = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \), so

\[
L_X d\omega = (L_X df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^p df \wedge dx^{i_1} \wedge \cdots \wedge L_X dx^{i_k} \wedge \cdots \wedge dx^{i_p}
\]

Next, we claim that for \( \omega \in \Omega^1(M) \), we have

\[
L_X (f \omega) = (X f)\omega + f L_X \omega
\]
To see this, take $Y \in \Gamma(TM)$ and plug into (9.7)
\[
\mathcal{L}_X(f\omega)(Y) = X(f\omega(Y)) - f\omega(\mathcal{L}_X Y)
= (Xf)\omega(Y) + f X(\omega(Y)) - f\omega(\mathcal{L}_X Y)
= ((Xf)\omega + f\mathcal{L}_X \omega)(Y).
\]
(9.8)

So we then have
\[
d\mathcal{L}_X \omega = d(\mathcal{L}_X f dx^{i_1} \wedge \cdots \wedge dx^{i_p})
= d\left(\mathcal{L}_X (f dx^{i_1}) \wedge \cdots \wedge dx^{i_p} + \sum_{k=2}^{p} f dx^{i_1} \wedge \cdots \wedge \mathcal{L}_X dx^{i_k} \wedge \cdots \wedge dx^{i_p}\right)
= d\left(\mathcal{L}_X (f dx^{i_1}) \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} f dx^{i_1} \wedge \cdots \wedge d(Xx^{i_k}) \wedge \cdots \wedge dx^{i_p}\right)
= d(Xf) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} df \wedge dx^{i_1} \wedge \cdots \wedge d(Xx^{i_k}) \wedge \cdots \wedge dx^{i_p}
= (\mathcal{L}_X df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^{p} df \wedge dx^{i_1} \wedge \cdots \wedge \mathcal{L}_X dx^{i_k} \wedge \cdots \wedge dx^{i_p}.
\]
(9.9)

**Definition 9.2.** Given $\omega \in \Lambda^p(T_x^*M)$, and $X \in T_xM$, define the interior product
\[
X \lrcorner : \Lambda^p(T_x^*M) \to \Lambda^{p-1}(T_x^*M)
\]
by
\[
X \lrcorner \alpha(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}).
\]
(9.11)

**Exercise 9.3.** Prove that this is equivalent to the following definition. Given $v \in T_xM$, we get a mapping $\iota_v : \Lambda^{p-1}(T_xM) \to \Lambda^p(T_xM)$ by
\[
\iota_v(\alpha) = v \wedge \alpha.
\]
(9.12)

The transpose mapping is
\[
\iota_v^* : (\Lambda^p(T_xM))^* \to (\Lambda^{p-1}(T_xM))^*
\]
(9.13)

Above, we chose an identification of $(\Lambda^k(T_xM))^* \cong \Lambda^k(T_x^*M)$, so using this we get a mapping
\[
\iota_v^* : \Lambda^p(T_x^*M) \to \Lambda^{p-1}(T_x^*M).
\]
(9.14)

Show that $v \lrcorner = (\iota_v)^*$. Also, show that $v \lrcorner$ is an anti-derivation, that is
\[
v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner \beta)
\]
(9.15)

if $\alpha \in \Lambda^p(T_x^*M)$. 

\[34\]
An important formula is Cartan’s formula relating the Lie derivative and the exterior derivative.

**Proposition 9.4** (Cartan’s magic formula). If \( \omega \in \Omega^p(M) \), then
\[
\mathcal{L}_X \omega = d(X \lrcorner \omega) + X \lrcorner d\omega, \tag{9.16}
\]

**Proof.** Define the operator \( H : \Omega^p(M) \to \Omega^p(M) \) by
\[
H \omega = d(X \lrcorner \omega) + X \lrcorner d\omega. \tag{9.17}
\]

We claim that \( H \) is a derivation, that is
\[
H(\alpha \wedge \beta) = H(\alpha) \wedge \beta + \alpha \wedge H(\beta). \tag{9.18}
\]

To prove this, assume that \( \alpha \in \Omega^p(M) \), then
\[
H(\alpha \wedge \beta) = d(X \lrcorner (\alpha \wedge \beta)) + X \lrcorner d(\alpha \wedge \beta)
= d((X \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (X \lrcorner \beta)) + X \lrcorner (d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta)
= d((X \lrcorner \alpha)) \wedge \beta + (-1)^{p-1} (X \lrcorner \alpha) \wedge d\beta + (-1)^p \alpha \wedge (X \lrcorner \beta) + (-1)^p \alpha \wedge (X \lrcorner d\beta)
+ (X \lrcorner d\alpha) \wedge \beta + (-1)^{p+1} d\alpha \wedge (X \lrcorner \beta) + (-1)^p (X \lrcorner \alpha) \wedge d\beta + (-1)^p \alpha \wedge (X \lrcorner d\beta)
= H(\alpha) \wedge \beta + \alpha \wedge H(\beta). \tag{9.19}
\]

Then the operators \( \mathcal{L}_X \) and \( H \) are derivations which commute with \( d \), and agree on functions. So they must be the same operator on forms, since they agree in local coordinates. \( \square \)

We also have the following formula for the exterior derivative which agrees with the formula for \( d \) given in \cite{Spi79}, Chapter 7.

**Proposition 9.5.**
\[
d\omega(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j X_j \left( \omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \right)
+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \tag{9.20}
\]

**Proof.** This formula follows from (9.2) and (9.16), with an induction argument. To see this, if \( \omega \in \Omega^1(M) \), then
\[
d\omega(X_0, X_1) = (X_0 \lrcorner d\omega)(X_1) = (\mathcal{L}_{X_0} \omega)(X_1) - d(\omega(X_0))(X_1)
= X_0(\omega(X_1)) - \omega([X_0, X_1]) - X_1(\omega(X_0)) \tag{9.21}
= X_0(\omega(X_1)) - X_1(\omega(X_0)) - \omega([X_0, X_1]).
\]

Then we use induction to get the formula for higher degree forms. \( \square \)

**Remark 9.6.** The Lie derivative operator can be defined by using the 1-parameter group of diffeomorphisms generated by \( X \) via
\[
\mathcal{L}_X Y = \left. \frac{d}{dt} (\Phi_t)_* Y \right|_{t=0} \tag{9.22}
\]
\[
\mathcal{L}_X \omega = \left. \frac{d}{dt} (\Phi_t)^* \omega \right|_{t=0}, \tag{9.23}
\]
but for this, we next first need to discuss the flow of a vector field.
9.2 The flow of a vector field

Given a vector field \( X \in \Gamma(TM) \), an integral curve of \( X \) is a mapping \( \gamma: (-\epsilon, \epsilon) \to M \) such that \( \gamma'(t) = X(\gamma(t)) \). Let’s look at the case of the real line first, and more general cases next time.

9.3 The real line

Let \( M = \mathbb{R} \). Then a vector field \( X = f(t) \frac{\partial}{\partial t} \). The differential equation for an integral curve is \( \frac{d\gamma}{dt} = f(\gamma) \). We can rewrite this as

\[
\frac{d\gamma}{f(\gamma)} = dt, \tag{9.24}
\]

which is a separable equation, and we get

\[
F(\gamma) = t + C, \tag{9.25}
\]

where \( F \) is an anti-derivative for \( 1/f \), and \( C \) is a constant of integration. If we plug in \( \gamma(0) = t_0 \), then \( C = F(t_0) \), so we then have

\[
\gamma(t) = F^{-1}(t + F(t_0)), \tag{9.26}
\]

where \( F^{-1} \) is an inverse function to \( F \). Thinking of \( t_0 \) as a parameter, we get

\[
\Phi(t, t_0) = F^{-1}(t + F(t_0)). \tag{9.27}
\]

The standard existence and uniqueness theorem for ODEs says that the domain of \( \Phi \) is an open set in \( \mathbb{R}^2 \) containing the \( t_0 \)-axis.

Example 9.7. We write down a few explicit examples.

- If \( f = 1 \), then \( \Phi(t, t_0) = t + t_0 \). For any \( t_0 \), the flow is defined for all \( t \).
- If \( f = t \), then \( \Phi(t, t_0) = t_0e^t \). For any \( t_0 \), the flow is defined for att \( t \)
- If \( f = t^2 \), then \( \Phi(t, t_0) = \frac{t_0}{1-t_0 t} \). The flow is only defined from \((-\infty, t_0^{-1})\) if \( t_0 > 0 \) and from \((t_0^{-1}, -\infty)\) if \( t_0 < 0 \). We say this is incomplete.
- If \( f = t^2 + 1 \), then \( \Phi(t, t_0) = \tan(t + \tan^{-1}(t_0)) \), also incomplete.

Proposition 9.8. We have \( \Phi(t+s, t_0) = \Phi(t, \Phi(s, t_0)) \)

Proof. From the formula (9.27), we have

\[
\Phi(t+s, t_0) = F^{-1}(t+s + F(t_0)) = F^{-1}(t + (s + F(t_0)). \tag{9.28}
\]

Let us write \( s + F(t_0) = F(t'_0) \). Then \( t'_0 = F^{-1}(s + F(t_0)) \), so

\[
\Phi(t+s, t_0) = F^{-1}(t + F(t'_0)) = \Phi(t, t'_0) = \Phi(t, \Phi(s, t_0)). \tag{9.29}
\]
Corollary 9.9. If the flows at time $t$ and $-t$ are defined for all $x \in \mathbb{R}$, then the mapping $x \mapsto \Phi(t,x)$ is a diffeomorphism.

Proof. From the previous proposition, we have

$$x = \Phi(t, \Phi(-t, x)) = \Phi(-t, \Phi(t, x)),$$

(9.30)

so the mappings $x \mapsto \Phi(t, x)$ and $x \mapsto \Phi(-t, x)$ are inverses of each other. Smoothness follows from the standard ODE existence and uniqueness theorem. \qed

10 Lecture 10

10.1 The flow of a vector field

Given a vector field $X \in \Gamma(TM)$, an integral curve of $X$ is a mapping $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma'(t) = X(\gamma(t))$. Define $\Phi(t, x) = \gamma(t)$, where $\gamma(t)$ is the integral curve satisfying $\gamma(0) = x$. The fundamental theorem is the following.

Proposition 10.1. Assume $X$ is smooth. Then through each $x \in M$ there passes a unique integral curve of $X$. The domain of $\Phi(t, x)$ is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$, and $\Phi : U \to M$ is smooth. We have

$$\Phi(t + s, x) = \Phi(t, \Phi(s, x)),$$

(10.1)

for $t, s, x$ for which the above is defined.

Proof. We discuss the uniqueness, since this is a local property, we assume that $M$ is an open subset of $\mathbb{R}^n$, and the differential equation is

$$y'(t) = X(y(t)), \quad y(0) = x.$$  

(10.2)

Assume we have 2 solutions $y_1$ and $y_2$. Integrating, we obtain

$$y_i(t) = x + \int_0^t X(y_i(s))ds,$$

(10.3)

for $i = 1, 2$. We assume that $t > 0$, the argument for $t < 0$ is similar. First, we have

$$|y_1(t) - y_2(t)| = \left| \int_0^t (X(y_1(s)) - X(y_2(s)))ds \right|$$

$$\leq \int_0^t |X(y_1(s)) - X(y_2(s))|ds$$

(10.4)

$$\leq K \int_0^t |y_1(s) - y_2(s)|ds,$$

using that $X$, being smooth, is necessarily Lipschitz.
Next, define

\[ U(t) = \int_0^t |y_1(s) - y_2(s)| ds. \] (10.5)

Then \( U(t) \geq 0 \) for \( t \geq 0 \), and (10.4) says that

\[ U''(t) \leq KU(t). \] (10.6)

or \( U''(t) - KU(t) \leq 0 \). Multiplying by \( e^{-Kt} \) yields

\[ (e^{-Kt}U(t))' \leq 0. \] (10.7)

Integrating from 0 to \( t \), we obtain

\[ e^{-Kt}U(t) \leq 0, \] (10.8)

which says that \( U(t) \leq 0 \), so \( U(t) = 0 \) for \( t \geq 0 \) which implies that \( y_1(t) = y_2(t) \) for \( t \geq 0 \).

Next, we prove (10.1). Note that for \( x \) and \( s \) fixed, we define \( L(t) = \Phi(t + s, x) \). Then

\[ \frac{d}{dt}L(t)|_{t=0} = \frac{d}{dt}\Phi(t + s, x)|_{t=0} = \frac{d}{dt}\Phi(t, x)|_{t=s} = X(\Phi(s, x)), \] (10.9)

and \( L(0) = \Phi(s, x) \). Define \( R(t) = \Phi(t, \Phi(s, x)) \). Then

\[ \frac{d}{dt}R(t)|_{t=0} = X(\Phi(s, x)), \] (10.10)

and \( R(0) = \Phi(s, x) \). So \( L(t) \) and \( R(t) \) are integral curves of \( X \) passing through the same point at \( t = 0 \), so they are equal by the uniqueness theorem.

Existence of solutions is proved by writing as an integral equation

\[ y(t) = x + \int_0^t X(y(s)) ds. \] (10.11)

Let

\[ y_0(t) = x, \] (10.12)
\[ y_1(t) = x + \int_0^t X(x) ds \] (10.13)
\[ y_2(t) = y_0 + \int_0^t X(y_1(s)) ds \] (10.14)
\[ \vdots \] (10.15)
\[ y_n(t) = y_0 + \int_0^t X(y_{n-1}(s)) ds. \] (10.16)

One then proves this converges to a solution defined on some small interval (details omitted, this is called Picard’s iteration method).
Next, we discuss continuous dependence on initial conditions. We have

\[
\Phi(t, x) = x + \int_0^t X(\Phi(s, x))ds
\]
(10.17)

\[
\Phi(t, x') = x' + \int_0^t X(\Phi(s, x'))ds.
\]
(10.18)

Then

\[
|\Phi(t, x) - \Phi(t, x')| \leq |x - x'| + K \int_0^t |\Phi(s, x) - \Phi(s, x')|ds
\]
(10.19)

Letting

\[
U(t) = |x - x'| + K \int_0^t |\Phi(s, x) - \Phi(s, x')|ds,
\]
(10.20)

we have \(U(t) \geq 0\) and \(U(0) = |x - x'|\). Then

\[
U'(t) = K|\Phi(t, x) - \Phi(t, x')| \leq KU(t).
\]
(10.21)

This implies that \(U'(t) - KU(t) \leq 0\). Similar to above, it follows that

\[
(e^{-Kt}U(t))' \leq 0.
\]
(10.22)

Integrating from 0 to \(t\), we obtain

\[
e^{-Kt}U(t) \leq |x - x'|.
\]
(10.23)

This implies that

\[
|\Phi(t, x) - \Phi(t, x')| \leq e^{Kt}|x - x'|
\]
(10.24)

Assuming \(t < t'\), we also estimate

\[
|\Phi(t, x') - \Phi(t', x')| \leq \int_t^{t'} |X(\Phi(s, x')) - X(\Phi(s, x'))|ds \leq M|t - t'|.
\]
(10.25)

Finally, we have

\[
|\Phi(t, x) - \Phi(t', x')| \leq |\Phi(t, x) - \Phi(t, x')| + |\Phi(t, x') - \Phi(t', x')| \leq e^{Kt}|x - x'| + M|t - t'|
\]
(10.26)

Given \(\epsilon > 0\), choosing

\[
\delta = \frac{\epsilon}{2(e^K + M)},
\]
(10.28)

proves the continuity. Higher derivatives are estimated in a similar way by differentiating the equation, details are omitted.
Proposition 10.2. If $M$ is compact, then the domain of $\Phi$ is $\mathbb{R} \times M$. In other words, every vector field on a compact manifold is complete.

Proof. The previous result says the domain of definition of $\Phi$ is an open subset $U$ of $\mathbb{R} \times M$ containing $\{0\} \times M$. Since $M$ is compact, the “Tube Lemma” from basic topology says that $U$ contains $(-\epsilon, \epsilon) \times M$ for some $\epsilon > 0$. Given any $t \in \mathbb{R}$, write $t = k\epsilon + r$, where $0 \leq r < \epsilon$.

Writing $\phi_t(x) = \Phi(t, x)$, we define

$$\phi_t(x) = \phi_{\epsilon} \circ \cdots \circ \phi_{\epsilon} \circ \phi_r(x)$$

if $k \geq 0$, and

$$\phi_t(x) = \phi_{-\epsilon} \circ \cdots \circ \phi_{-\epsilon} \circ \phi_r(x)$$

if $k < 0$.

Theorem 10.3. Let $X$ be a vector field on $M$ such that $X(p) \neq 0$. Then there exists a local coordinate system $(x^1, \ldots, x^n)$ around $p$ such that $x_*X = \frac{\partial}{\partial x^1}$.

Proof. This is local, so we can assume we are in $\mathbb{R}^n$, with coordinates $y^i$. We can also assume that $X(0) = \frac{\partial}{\partial y^1}|_0$. We then define coordinates $z^i(y^1, \ldots, y^n)$ by

$$(z^1, \ldots, z^n) = \phi_{y^1}(0, y^2, \ldots, y^n)$$

We compute that

$$z_*\left(\frac{\partial}{\partial y^1}\right)f = \frac{\partial}{\partial y^1}(f \circ z)$$

$$= \lim_{h \to 0} \frac{1}{h}(f \circ \phi_{y^1+h}(0, y^2, \ldots, y^n) - f \circ \phi_{y^1}(0, y^2, \ldots, y^n))$$

$$= \lim_{h \to 0} \frac{1}{h}(f \circ \phi_h \circ z - f \circ z)$$

$$= Xf \circ z.$$  

Clearly, we have

$$z_*\left(\frac{\partial}{\partial y^i}|_0\right) = \frac{\partial}{\partial y^i}|_0, \quad i > 1.$$  

Thus we have that $z_*|0 = Id$. By the inverse function theorem, $x = z^{-1}$ exists in a neighborhood of the origin. Above, we showed that

$$z_*\left(\frac{\partial}{\partial y^1}\right) = X \circ z$$

so applying $x_*$ to this equation yields

$$\frac{\partial}{\partial y^1} = x_* \circ z_*\left(\frac{\partial}{\partial y^1}\right) = x_*(X \circ z) = x_*X \circ z \circ x = x_*X.$$
11 Lecture 11

11.1 Method of characteristics

The above proof seems magical, so let’s give another explanation of the proof of Theorem 10.3. For simplicity, let’s just consider the case of $n = 2$, and our vector field is written near the origin as

$$X = f(x, y) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y}$$ (11.1)

with $f(0, 0) = 1$ and $g(0, 0) = 0$. We want to find coordinates $u(x, y)$ and $v(x, y)$ so that

$$X = \frac{\partial}{\partial u} = f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) + g \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right)$$

$$= \left( f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} \right) \frac{\partial}{\partial u} + \left( f \frac{\partial v}{\partial x} + g \frac{\partial v}{\partial y} \right) \frac{\partial}{\partial v}$$ (11.2)

So we want to solve the equations

$$f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} = 1, \quad f \frac{\partial v}{\partial x} + g \frac{\partial v}{\partial y} = 0$$ (11.4)

which are more simply

$$Xu = 1, \quad Xv = 0.$$ (11.5)

together with a condition that $u(x, y), v(x, y)$ form a coordinate system near $(0, 0)$. This will be true if the Jacobian determinant

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} (0, 0) \neq 0.$$ (11.6)

Remark 11.1. The pair of equations (11.5) is obviously a completely uncoupled system of 2 first order linear PDEs for 2 functions of 2 variables. This is a “determined” system. There is some slight coupling of the “initial conditions” in (11.6).

The method used in Theorem 10.3 is a special case of the “method of characteristics”, which we explain next. This method reduces solving a first order linear PDE to solving an ODE. The only “drawback” of this method is that the ODE is nonlinear.

First, consider the equation $Xu = 1$, which is just

$$f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} = 1.$$ (11.7)

Consider the graph of the solution as a hypersurface in $\mathbb{R}^3$:

$$G = \{(x, y, u(x, y)) \mid (x, y) \in U \}.$$ (11.8)
A normal vector field to the graph is given by
\[ \vec{N} = (u_x, u_y, -1). \]  
(11.9)

Define the vector field along \( G \) by
\[ \vec{F} = (f, g, 1). \]  
(11.10)

Then
\[ \vec{F} \cdot \vec{N} = f u_x + g u_y - 1 = 0. \]  
(11.11)

So the vector field \( \vec{F} \) is everywhere tangent to the graph \( G \). Consequently, \( G \) is stratified by the integral curves of \( \vec{F} \). We then solve the ODE system:
\[ \frac{dx}{ds} = f(x, y), \quad \frac{dy}{ds} = g(x, y), \quad \frac{du}{ds} = 1, \]  
(11.12)

with initial conditions
\[ x(0) = 0, \quad y(0) = v, \quad u(0) = 0. \]  
(11.13)

The last equation gives \( u = s + C \), which yields \( u = s \). The first 2 equations are just the flow of the vector field \( X \), with initial conditions along the \( y \)-axis. The solution is of the form
\[ (x(s, v), y(s, v), u(s)) = (\Phi(s, (0, v)), s) \]  
(11.14)

so
\[ (x(u, v), y(u, v)) = \Phi(u, (0, v)). \]  
(11.15)

This determines the variables \((u, v)\) implicitly as functions of the variables \(x\) and \(y\). This is solvable by the inverse function theorem, provided that
\[ \det \left( \begin{align*} \frac{\partial \Phi_1(u, (0, v))}{\partial u} & \frac{\partial \Phi_1(u, (0, v))}{\partial v} \\ \frac{\partial \Phi_2(u, (0, v))}{\partial u} & \frac{\partial \Phi_2(u, (0, v))}{\partial v} \end{align*} \right) \bigg|_{(0, 0)} \neq 0. \]  
(11.16)

But we have
\[ \frac{\partial \Phi_1(u, (0, v))}{\partial u} \bigg|_{(0, 0)} = \frac{\partial x}{\partial s} \bigg|_{(0, 0)} = f(0, 0) = 1, \]  
(11.17)
\[ \frac{\partial \Phi_2(u, (0, v))}{\partial u} \bigg|_{(0, 0)} = \frac{\partial y}{\partial s} \bigg|_{(0, 0)} = g(0, 0) = 0, \]  
(11.18)
\[ \frac{\partial \Phi_2(u, (0, v))}{\partial v} \bigg|_{(0, 0)} = \frac{\partial y}{\partial v} \bigg|_{(0, 0)} = \lim_{h \to 0} \frac{y(0, h) - y(0, 0)}{h} = 1, \]  
(11.19)

so the determinant is equal to 1 at \((0, 0)\), and we can indeed solve for \(u\) and \(v\) as functions of \(x\) and \(y\).
Finally, since
\[
\begin{pmatrix} u(x(u, v), y(u, v)), v(x(u, v), y(u, v)) \end{pmatrix} = (u, v),
\] (11.20)
by the inverse function theorem,
\[
\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (11.21)
In particular, we have
\[0 = v_x u_x + v_y u_y = f v_x + g v_y.\] (11.22)
Consequently, this \(v\) solves the equation \(X v = 0\), and we are done since we already know that \((u, v)\) forms a coordinate system in a neighborhood of the origin.

**Remark 11.2.** Note that the initial conditions (11.13) specify the solution to vanish along the \(y\)-axis, and the parameter \(u\) was just a parametrization of the \(y\)-axis. We could also get a solution of \(X u = 1\), with initial data specified along a non-characteristic curve. That is, a curve passing through \((0, 0)\) which is not tangent to \(X\) at the origin. Then our initial condition would be
\[x(0) = a(v), \quad y(0) = b(v), \quad u(0) = c(v).\] (11.23)
Here we have parametrized the curve by the parameter \(v\), and we require that \(b'(0) \neq 0\). We then have \(u = s + c(v)\), so the solution is of the form
\[(x(s, v), y(s, v), u(s)) = \left(\Phi(s, (a(v), b(v))), s + c(v)\right)\] (11.24)
so
\[(x(u, v), y(u, v)) = \Phi(u - c(v), (a(v), b(v))).\] (11.25)
The \(c(v)\) term can be eliminated by the transform
\[
\tilde{u} = u - c(v), \quad \tilde{v} = v,
\] (11.26)
which has Jacobian determinant
\[
\det \begin{pmatrix} 1 & -c'(v) \\ 0 & 1 \end{pmatrix} = 1 \neq 0.
\] (11.27)
Then the main modification to the above is
\[
\left. \frac{\partial \Phi_2(u, (a(v), b(v)))}{\partial v} \right|_{(0,0)} = \left. \frac{\partial y}{\partial v} \right|_{(0,0)} = \lim_{h \to 0} \frac{y(a(h), b(h)) - y(0, 0)}{h}
= \lim_{h \to 0} \frac{b(h) - 0}{h} = b'(0) \neq 0.
\] (11.28)
Example 11.3. Let’s do the above “straightening” procedure for the vector field

\[ X = (1 + y) \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]  \hspace{1cm} (11.29)

If we write an integral curve as \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \), the flow ODE is

\[ \gamma'_1 = 1 + \gamma_2, \quad \gamma'_2 = \gamma_1 \]  \hspace{1cm} (11.30)

Differentiating the first equation yields

\[ \gamma''_1 = \gamma'_2 = \gamma_1, \]  \hspace{1cm} (11.31)

so \( \gamma_1 = c_1 \cosh(t) + c_2 \sinh(t) \). The second equation then gives \( \gamma_2 = c_1 \sinh(t) + c_2 \cosh(t) + c_3 \). The first equation shows that \( c_3 = -1 \). The initial conditions are \( \gamma(0) = (0, y) \), which yields

\[ \gamma_1(t) = (y + 1) \sinh(t), \quad \gamma_2(t) = -1 + (y + 1) \cosh(t) \]  \hspace{1cm} (11.32)

So we have

\[ (x(u, v), y(u, v)) = \Phi(u, (0, v)) = ((v + 1) \sinh(u), -1 + (v + 1) \cosh(u)). \]  \hspace{1cm} (11.33)

This can be inverted explicitly

\[ u(x, y) = \tanh^{-1} \left( \frac{x}{y + 1} \right), \quad v(x, y) = -1 + \sqrt{(y + 1)^2 - x^2}. \]  \hspace{1cm} (11.34)

11.2 Lie derivatives

Now that we understand the flow of a vector field, we give an alternative characterization of the Lie derivative. First, for vector fields. Recall that for \( X, Y \in \Gamma(TM) \), we previously defined \( L^X f = \left[ X, f \right] \), where

\[ \left[ X, f \right] = X(Yf) - Y(Xf). \]  \hspace{1cm} (11.35)

Furthermore, if \( \omega \in \Omega^1(M) \), we defined

\[ L^X \omega(Y) = X\omega(Y) - \omega(L^X Y). \]  \hspace{1cm} (11.37)

Now we will give the new definitions

**Definition 11.4.** For \( X, Y \in \Gamma(TM) \), \( f \in \Omega^0(M) \), and \( \omega \in \Omega^1(M) \), define

\[ L^X' f(p) \equiv \lim_{h \to 0} h^{-1} \left( f \circ \phi_h(p) - f(p) \right) \]  \hspace{1cm} (11.38)

\[ L^X' \omega(p) \equiv \lim_{h \to 0} h^{-1} \left( (\phi_h^* \omega)_p - \omega(p) \right) \]  \hspace{1cm} (11.39)

\[ L^X' Y(p) \equiv \lim_{h \to 0} h^{-1} \left( Y_p - ((\phi_h)_* Y)_p \right). \]  \hspace{1cm} (11.40)
Note that obviously \( L'_X f = Xf \).

**Proposition 11.5.** The operator \( L'_X \) acts as a derivation, that is,

\[
L'_X(fY) = (Xf)Y + fL'_X Y \tag{11.41}
\]
\[
L'_X(f\omega) = (Xf)\omega + fL'_X \omega \tag{11.42}
\]
\[
L'_X \omega(Y) = X\omega(Y) - \omega(L'_X Y). \tag{11.43}
\]

**Proof.** We prove (11.41), the others are proved similarly. We have

\[
L'_X(fY) = \lim_{h \to 0} h^{-1} \left( (fY)_p - ((\phi_h)_* fY)_p \right)
= \lim_{h \to 0} h^{-1} \left( f(p)Y_p - (\phi_h)_*(fY)_{\phi_h(p)} \right)
= \lim_{h \to 0} h^{-1} \left( f(p)Y_p - f(\phi_h(p))(\phi_h)_* Y_{\phi_h(p)} \right)
= \lim_{h \to 0} h^{-1} \left( f(p)Y_p - f(p)(\phi_h)_* Y_{\phi_h(p)} + f(p)(\phi_h)_* Y_{\phi_h(p)} - f(\phi_h(p))(\phi_h)_* Y_{\phi_h(p)} \right)
= f(p) \lim_{h \to 0} h^{-1} \left( Y_p - (\phi_h)_* Y_{\phi_h(p)} \right) + \lim_{h \to 0} h^{-1} \left( f(p) - f(\phi_h(p)) \right)(\phi_h)_* Y_{\phi_h(p)}
= f(p)L'_X Y + \lim_{h \to 0} h^{-1} \left( f(p) - f(\phi_h(p)) \right) \cdot \lim_{h \to 0} (\phi_h)_* Y_{\phi_h(p)}
\]

\( (11.44) \)

Note that by letting \( k = -h \), we have

\[
\lim_{h \to 0} h^{-1} \left( f(p) - f(\phi_h(p)) \right) = \lim_{k \to 0} -k^{-1} \left( f(p) - f(\phi_k(p)) \right)
= \lim_{k \to 0} -k^{-1} \left( f(\phi_k(p)) - f(p) \right) = Xf(p).
\]

(11.45)

Finally, we have

\[
\lim_{h \to 0} (\phi_h)_* Y_{\phi_h(p)} = \lim_{h \to 0} \left( (\phi_h)_* Y_{\phi_h(p)} - Y_{\phi_h(p)} + Y_{\phi_h(p)} \right)
= \lim_{h \to 0} \left( (\phi_h)_* - Id \right) Y_{\phi_h(p)} + \lim_{h \to 0} Y_{\phi_h(p)} = Y(p),
\]

(11.46)

and we are done. \( \square \)

**Exercise 11.6.** Prove (11.42) and (11.43).

**Proposition 11.7.** For \( X, Y \in \Gamma(TM) \) and \( \omega \in \Omega^1(M) \), we have

\[
L_X Y = L'_X Y, \quad L_X \omega = L'_X \omega. \tag{11.47}
\]

**Proof.** From (11.43), we have

\[
0 = L'_X \delta^i_j = L'_X(dx^j(\partial_j)) = L'_X(dx^i)(\partial_j) + dx^i(L'_X \partial_j), \tag{11.48}
\]

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so
\[ L'_X \partial_j = - \left( L'_X (dx^i)(\partial_j) \right) \partial_i. \] (11.49)

We compute
\[
L'_X (dx^i)(\partial_j) = \lim_{h \to 0} h^{-1} \left( \phi^*_h dx^i - dx^i(\partial_j) \right)
= \lim_{h \to 0} h^{-1} \left( (\phi^*_h dx^i)(\partial_j) - dx^i(\partial_j) \right)
= \lim_{h \to 0} h^{-1} \left( d(x^i \circ \phi_h)(\partial_j) - dx^i(\partial_j) \right)
= \lim_{h \to 0} h^{-1} \left( d(x^i \circ \phi_h - x^i)(\partial_j) \right)
= d \left( \lim_{h \to 0} h^{-1} (x^i \circ \phi_h - x^i) \right)(\partial_j)
= d(X x^i)(\partial_j) = \frac{\partial}{\partial x^j} X^i.
\] (11.50)

So we have
\[
L'_X Y = L'_X (Y^j \partial_j) = (X(Y^j))\partial_j + Y^j L'_X(\partial_j)
= X(Y^j)\partial_j - Y^j \left( \frac{\partial}{\partial x^j} X^i \right) \partial_i
= X^i \left( \frac{\partial}{\partial x^i} Y^j \right) \partial_j - Y^j \left( \frac{\partial}{\partial x^j} X^i \right) \partial_i
= \left( X^i \left( \frac{\partial}{\partial x^i} Y^j \right) - Y^j \left( \frac{\partial}{\partial x^j} X^i \right) \right) \partial_i,
\] (11.51)

which agrees with (11.36). This proves \( L_X = L'_X \) on vector fields. Since they both satisfy the Leibniz rule on 1-forms,
\[
L'_X \omega(Y) = X \omega(Y) - \omega(L'_X Y)
L_X \omega(Y) = X \omega(Y) - \omega(L_X Y),
\] (11.52, 11.53)

they also agree on 1-forms.

\[ \square \]

Exercise 11.8. Show that for \( \omega \in \Omega^k \) and \( k > 1 \), we have \( L_X \omega = L'_X \omega \).

12 Lecture 12

12.1 Frobenius Theorem (local version)

Today we address the following question. Assume we are given vector fields \( X_1, \ldots, X_k \in \Gamma(TU) \) which are linearly independent at every point. Then does there exist a coordinate system \( (x^1, \ldots, x^n) \) such that \( X_i = \partial_i \) for \( 1 \leq i \leq k \). Before we give the answer, we need a few auxiliary results.
**Proposition 12.1.** Let \( X \in \Gamma(TM) \) with 1-parameter group \( \phi_t \). If \( \alpha : M \to M \) is a diffeomorphism, then the 1-parameter group of \( \alpha_* X \) is given by \( \alpha \circ \phi_t \circ \alpha^{-1} \).

**Proof.** Recall the formula that
\[
(\alpha_* X)_q = (\alpha_*)_{\alpha^{-1}(q)} X_{\alpha^{-1}(q)},
\]
for any \( q \in M \). Given a point \( p \in M \), consider the curve
\[
\gamma(t) = \alpha \circ \phi_t \circ \alpha^{-1}(p) = \alpha \circ \Phi(t, \alpha^{-1}(p)).
\]
This satisfies
\[
\gamma'(t) = \gamma_*(\partial_t) = (\alpha_*)_{\phi_{t} \circ \alpha^{-1}(p)} \circ X_{\phi_{t} \circ \alpha^{-1}(p)}
= (\alpha_* \alpha_{t} \circ \alpha^{-1}(p) \circ X_{\alpha_{t} \circ \alpha^{-1}(p)}
= (\alpha_* X)_{\alpha_{t} \circ \alpha^{-1}(p)},
\]
using \((12.1)\) with \( q = \alpha \circ \phi_t \circ \alpha^{-1}(p) \). Therefore \( \gamma(t) \) is an integral curve of \( \alpha_* X \), so by uniqueness of integral curves, we are done.

**Proposition 12.2.** Let \( X, Y \in \Gamma(TM) \) with 1-parameter groups \( \phi_t, \psi_t \), respectively. Then \([X, Y] = 0\) if and only if \( \phi_t \circ \psi_s = \psi_s \circ \phi_t \) for all \( s, t \).

**Proof.** Assume that \([X, Y] = 0\). Then for \( q \in M \),
\[
0 = \lim_{h \to 0} h^{-1} (Y_q - (\phi_h)_* Y_q).
\]
Next, given any \( p \in M \), consider the curve \( \gamma : (-\epsilon, \epsilon) \to T_p M \) defined by
\[
c(t) = ((\phi_t)_* Y)_p.
\]
We then compute
\[
c'(t) = \lim_{h \to 0} h^{-1} (c(t + h) - c(t))
= \lim_{h \to 0} h^{-1} ((\phi_{t+h})_* Y)_p - ((\phi_t)_* Y)_p
= \lim_{h \to 0} h^{-1} ((\phi_t \circ \phi_h)_* Y)_p - ((\phi_t)_* Y)_{\phi_{t}^{-1}(p)}
= \lim_{h \to 0} h^{-1} ((\phi_t)_* ((\phi_h)_* Y)_{\phi_{t}^{-1}(p)} - ((\phi_t)_* Y)_{\phi_{t}^{-1}(p)})
= \lim_{h \to 0} h^{-1} ((\phi_h)_* Y)_{\phi_{t}^{-1}(p)} - Y_{\phi_{t}^{-1}(p)}) = 0,
\]
using \((12.4)\) with \( q = \phi_{-t}(p) \). Therefore \( c(t) \) is constant, and \( c(t) = c(0) = Y_p \). This implies that \((\phi_t)_* Y = Y \). By Proposition 12.1 the flow of \( Y \), \( \psi_t \), must be equal to \( \phi_t \circ \psi_s \circ \phi_t^{-1} \).

For the converse let \( \alpha = \psi_s \), and by assumption \( \phi_t = \alpha \circ \phi_t \circ \alpha^{-1} \). By Proposition 12.1 \( \phi_t \) must be the 1-parameter group generated by \( \alpha_* X = (\psi_s)_* X \). So we have \((\psi_s)_* X = X \), which obviously implies that \([X, Y] = L^X_{\epsilon} Y = 0 \).
The main result is the following.

**Theorem 12.3.** Assume we are given vector fields $X_1, \ldots, X_k \in \Gamma(TU)$ which are linearly independent in a neighborhood $U$ of $p \in M$ and which satisfy $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Then there exists a local coordinate system $(x^1, \ldots, x^n)$ such that $X_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq k$.

**Proof.** Without loss of generality, we can assume there is a coordinate system $(t^1, \ldots, t^n)$ such that $X_i(0) = \frac{\partial}{\partial t^i}(0)$ for $1 \leq i \leq k$. Call the 1-parameter group of $X_i$ by $\phi^i_t$ for $1 \leq i \leq k$.

Define

$$
(t^1(x^1, \ldots, x^n), \ldots, t^n(x^1, \ldots, x^n)) = \phi^1_{x^1}(\phi^2_{x^2}(\cdots (\phi^k_{x^k}(0, \ldots, 0, x^{k+1}, \ldots, x^n) \cdots )\cdots )). \tag{12.6}
$$

It is easy to see that the Jacobian as 0 is the identity, so by the inverse function theorem we can solve for the $x^i = x^i(t^1, \ldots, t^n)$ as functions of the $t^i$ for $1 \leq i \leq n$, in some possibly smaller neighborhood $U$ of $p$. By (12.2), given any $1 \leq i \leq k$, we can write

$$
(t^1(x^1, \ldots, x^n), \ldots, t^n(x^1, \ldots, x^n)) = \phi^i_{x^1}(\phi^2_{x^2}(\cdots (\phi^k_{x^k}(0, \ldots, 0, x^{k+1}, \ldots, x^n) \cdots )\cdots )). \tag{12.7}
$$

so by the same proof as straightening 1 vector field, we have that $X_i = \frac{\partial}{\partial x^i}$. \hfill \Box

**Remark 12.4.** We can also phrase the above as follows, which we state for 2 vector fields for simplicity. Given $X_1$ and $X_2$ linearly independent vector fields, we require functions $x^1, \ldots, x^n$ such that

$$
X_1 x^1 = 1, \quad X_1 x^i = 0, \quad i > 1 \tag{12.8}
$$
$$
X_2 x^2 = 1, \quad X_2 x^i = 0, \quad i \neq 2, \tag{12.9}
$$

together with the condition that these form a coordinate system. We know that separately, each line is a determined system. However, together these equations have twice as many equations as unknowns, so this is called an overdetermined system. In general, one does not expect solution to exist for overdetermined systems, unless some extra conditions are satisfied which in this case are exactly the condition that the vector fields commute.

**Exercise 12.5.** Define the vector fields on $\mathbb{R}^3$ by

$$
X_1 = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x \partial_x + y \partial_y + z \partial_z) \tag{12.10}
$$
$$
X_2 = -y \partial_x + x \partial_y \tag{12.11}
$$
$$
X_3 = \frac{xz}{\sqrt{x^2 + y^2}} \partial_x + \frac{yz}{\sqrt{x^2 + y^2}} \partial_y - \sqrt{x^2 + y^2} \partial_z. \tag{12.12}
$$

Show that these vector fields commute, and find a coordinate system which simultaneously straightens $X_1, X_2, X_3$.

**Exercise 12.6.** Define the vector fields on $\mathbb{R}^4$ by the following.

$$
X_1 = x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3 + x^4 \partial_4 \tag{12.13}
$$
$$
X_2 = -x^4 \partial_1 - x^3 \partial_2 + x^2 \partial_3 + x^1 \partial_4 \tag{12.14}
$$
$$
X_3 = x^3 \partial_1 - x^4 \partial_2 - x^1 \partial_3 + x^2 \partial_4 \tag{12.15}
$$
$$
X_4 = -x^2 \partial_1 + x^1 \partial_2 - x^4 \partial_3 + x^3 \partial_4. \tag{12.16}
$$
Show that $X_2, X_3,$ and $X_4$ commute with $X_1$, but do not commute with each other. Furthermore, find functions $f_1, f_2, f_3, f_4$ such that

$$X_1 f_1 = 1, X_1 f_2 = 0, X_2 f_1 = 0, X_2 f_2 = 1, \quad (12.17)$$
$$X_1 f_3 = 0, X_3 f_1 = 0, X_3 f_3 = 1 \quad (12.18)$$
$$X_1 f_4 = 0, X_4 f_1 = 0, X_4 f_4 = 1. \quad (12.19)$$

13 Lecture 13

13.1 Frobenius theorem (geometric version)

We begin with a lemma.

**Lemma 13.1.** Let $\phi : M \to N$ be a smooth mapping and $X, Y \in \Gamma(TM)$. Assume that $\phi_*X$ and $\phi_*Y$ are smooth vector fields on $N$. Then

$$\phi_*[X, Y] = [\phi_*X, \phi_*Y] \quad (13.1)$$

**Proof.** Let $f \in C^\infty(N, \mathbb{R})$. Then by definition of the push-forward, we have

$$(\phi_*X)f = X(f \circ \phi). \quad (13.2)$$

Applying (13.2) several times, we then compute

$$[\phi_*X, \phi_*Y]f = (\phi_*X)((\phi_*Y)(f)) - (\phi_*Y)((\phi_*X)(f))$$
$$= X\left(\left((\phi_*Y)(f)\right) \circ \phi\right) - Y\left(\left((\phi_*X)(f)\right) \circ \phi\right)$$
$$= X(Y(f \circ \phi)) - Y(X(f \circ \phi))$$
$$= [X, Y](f \circ \phi) = (\phi_*[X, Y])f. \quad (13.3)$$

\[\square\]

**Definition 13.2.** A distribution of rank $k$, $\Delta \subset TM$ is a sub-bundle of the tangent bundle of rank $k$. The distribution $\Delta$ is said to be **integrable** if for any two local sections $X, Y \in \Gamma(\Delta|U)$, we have $[X, Y] \in \Gamma(\Delta|U)$.

**Definition 13.3.** An immersed submanifold $N \subset M$ is called an integral manifold of $\Delta$ if $T_pN = \Delta_p$ for all $p \in N$.

**Theorem 13.4.** If $\Delta$ is an integrable rank $k$ distribution, then around any point $p \in M$, there exists a local coordinate system $(X, U)$ such that for $q \in U$,

$$x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n \quad (13.4)$$

is an integral manifold of $\Delta$, for each $(a^{k+1}, \ldots, a^n)$ with $|a^i| < \epsilon$ for $k + 1 \leq i \leq n$. 49
Proof. For the first part, clearly we can assume that $M$ is an open subset of $\mathbb{R}^n$, and $p = 0$. WLOG, assume that $\Delta_0$ is spanned by

$$\frac{\partial}{\partial t^i}(0), \quad 1 \leq i \leq k. \quad (13.5)$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first $k$ factors. Then $\pi_* : \Delta_0 \to \mathbb{R}^k$ is an isomorphism, so by continuity for $q$ sufficiently near $p$, $\pi_* : \Delta_q \to \mathbb{R}^k$ is an isomorphism (because the mapping $q \mapsto \det(\pi_*|_{\Delta_q})$ is continuous, and nonzero at 0). So we can choose $X_i(q) \in \Delta_q$ such that $\pi_* X_i = \frac{\partial}{\partial t^i}$ for $1 \leq i \leq k$. The $X_i$ are smooth vector fields in a neighborhood of the origin in $\mathbb{R}^n$, so by Lemma [13.1] we have

$$\pi_* [X_i, X_j]_q = \left[ \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \right]_{\pi(q)} = 0. \quad (13.6)$$

Since we assumed that $\Delta$ is integrable, $[X_i, X_j]_q \in \Delta_q$, so $[X_i, X_j] = 0$ because $\pi_*$ is injective. Then we can use Theorem [12.3] to find a coordinate system $(x, U)$ so that $X_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq k$, and we are done.

\[ \square \]

Remark 13.5. One can show that every point $p \in M$ lies on a unique connected maximal integral submanifold. A basic example is lines of some constant slope on a square torus. If the slope is rational, the maximal integral submanifolds are imbedded circles. However, if the slope is irrational, then the maximal integral submanifolds are the real line, since they never close up.

Finally, let’s discuss one of the homework problems from a previous lecture. Given $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^1(M)$, we have

$$L'_X \omega(Y) = X\omega(Y) - \omega(L'_X Y), \quad \text{(13.7)}$$

where for $p \in M$ we have

$$\begin{align*}
(L'_X \omega)_p &= \lim_{h \to 0} h^{-1} \left( (\phi_h^* \omega)_p - \omega_p \right) \\
(L'_X Y)_p &= \lim_{h \to 0} h^{-1} \left( Y_p - ((\phi_h)_* Y)_p \right). \quad \text{(13.8)}
\end{align*}$$

First, note that by definition $L'_X \omega$ is a tensor, i.e., $L'_X \omega \in \Omega^1(M)$. This means that for any vector field $Y \in \Gamma(TM)$, the expression $L'_X \omega(Y)_p$ depends only upon $Y_p$. For the left hand side, using [11.41], we compute

$$X\omega(fY) - \omega(L'_X(fY)) = X(f\omega(Y)) - \omega \left( (Xf)Y + f(L'_X Y) \right) = Xf \cdot \omega(Y) + fX(\omega(Y)) - Xf \cdot \omega(Y) + f\omega(L'_X Y) = f \left( X\omega(fY) - \omega(L'_X(fY)) \right). \quad \text{(13.10)}$$

So the left hand side is also a tensor since it is linear over $C^\infty$ functions. Since both sides of [13.7] are tensors, it suffices to prove in a coordinate system. Furthermore, without loss of
generality, we may assume that \( Y = \partial_i \), for \( 1 \leq i \leq n \). First, consider the case that \( \omega = dx^j \) for some \( 1 \leq j \leq n \), and write \( X = \sum_k X^k \partial_k \).

The left hand side of (13.7) is

\[
L'_X \omega (Y) = L'_X dx^j (\partial_i) = \frac{\partial X^j}{\partial x^i} (p),
\]

which we had already computed above in (11.50). Next, let us compute the right hand side of (13.7),

\[
X \omega (Y) - \omega (L'_X Y) = X^k \partial_k dx^j (\partial_i) - dx^j (L'_X \partial_i) = -dx^j (L'_X \partial_i).
\]

Lemma 13.6. We have

\[
\phi^* (\partial_i)_p = \sum_i \frac{\partial (x^j \circ \phi)}{\partial x^i} \big|_p (\partial_j)_{\phi (p)}.
\]

Proof. We write

\[
\phi^* (\partial_i)_p = \sum_j a^i (\partial_j)_{\phi (p)},
\]

so by plugging in the coordinate function \( x^j \), we have that

\[
a^j = (\phi^* (\partial_i)_p) (x^j) = \frac{\partial (x^j \circ \phi)}{\partial x^i} \big|_p,
\]

Next, let’s compute

\[
(L'_X \partial_i)_p = \lim_{h \to 0} h^{-1} ((\partial_i)_p - (\phi_h)^* (\partial_i)_{\phi_h (p)})
\]

\[
= \lim_{h \to 0} h^{-1} ((\partial_i)_p - \sum_j \frac{\partial (x^j \circ \phi_h)}{\partial x^i} \big|_{\phi_h (p)} (\partial_j)_{\phi_h (p)})
\]

\[
= \lim_{h \to 0} h^{-1} \left( \sum_j \frac{\partial (x^j \circ \phi_0)}{\partial x^i} \big|_p - \sum_j \frac{\partial (x^j \circ \phi_h)}{\partial x^i} \big|_{\phi_h (p)} (\partial_j)_{\phi_h (p)} \right)
\]

\[
= \lim_{h \to 0} h^{-1} \left( \sum_j \frac{\partial (x^j \circ \phi_0)}{\partial x^i} \big|_p - \sum_j \frac{\partial (x^j \circ \phi_h)}{\partial x^i} \big|_{\phi_h (p)} \right) (\partial_j)_{\phi_h (p)}
\]

\[
+ \lim_{h \to 0} h^{-1} \left( \sum_j \frac{\partial (x^j \circ \phi_h)}{\partial x^i} \big|_p - \sum_j \frac{\partial (x^j \circ \phi_h)}{\partial x^i} \big|_{\phi_h (p)} \right) \partial_j_{\phi_h (p)}
\]

\[
= \frac{\partial}{\partial x^i} \big|_p \lim_{h \to 0} h^{-1} \left( \sum_j (x^j \circ \phi_0) - \sum_j (x^j \circ \phi_h) \right) (\partial_j)_{\phi_h (p)}
\]

\[
= -\frac{\partial}{\partial x^i} X (x^j) (\partial_j)_{\phi_h (p)} = -\frac{\partial X^j}{\partial x^i} (\partial_j)_{\phi_h (p)}.
\]
The interchange of limits is valid since the flow is smooth in both variables. The second limit vanishes because. . . . This shows that the right hand side is

$$X \omega(Y) - \omega(L'_X Y) = \frac{\partial X^j}{\partial x^i}.$$  \hspace{1cm} (13.17)

So we have verified the formula for any vector fields $X, Y$, but we assumed that $\omega = dx^j$. To finish, from (11.42), we have

$$L'_X(fdx^j) = Xfdx^j + fL'_X dx^j,$$  \hspace{1cm} (13.18)

so for any $Y$,

$$L'_X(fdx^j)(Y) = Xfdx^j(Y) + f(L'_X dx^j)(Y).$$  \hspace{1cm} (13.19)

We also have

$$X(fdx^j)(Y) - (fdx^j)(L'_X Y) = (Xf)dx^j(Y) + f(Xdx^j(Y) - fdx^j(L'_X Y)).$$  \hspace{1cm} (13.20)

Since the formula is true for $\omega = dx^j$, it is therefore true for $\omega = fdx^j$, so true for any sum $\omega = f_j dx^j$, and we are done.

References


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