218B Introduction to Manifolds and Geometry

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Contents

Introduction

This course will be about de Rham cohomology of differentiable manifolds. The prerequisite is 218A, so the student should already be familiar with the basics of differentiable manifolds. Topics will include differential forms, de Rham cohomology, Mayer-Vietoris sequence, Poincare duality, and the Künneth formula. Main references are [BT82, Lee13, Spi79, War83].

1 Lecture 1

1.1 Vectors and one-forms

Let M be a smooth manifold. A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M), \tag{1.1}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \tag{1.2}$$

is the coordinate partial. We will use the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

A 1-form is a section of the cotangent bundle, $X \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad \omega_i \in C^\infty(M). \tag{1.3}$$

Remark 1.1. Note that components of vector fields have upper indices, while components of 1-forms have lower indices. However, a collection of vector fields will be indexed by lower indices, $\{Y_1, \ldots, Y_p\}$, and a collection of 1-forms will be indexed by upper indices $\{dx^1, \ldots, dx^n\}$. This is one reason why we write the coordinates with upper indices.

Note that a smooth mapping $f : M \to N$ induces a mapping

$$f_* : TM \to TN, \tag{1.4}$$

called a "push-forward", which is linear on fibers and which makes the following diagram commute

$$\begin{array}{ccc} TM & \xrightarrow{f_*} & TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{1.5}$$

This mapping is defined as follows. If $X \in T_pM$, let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve satisfying $\gamma(0) = p$, $\gamma'(0) = X$. Then

$$f_*(X) = \frac{d}{dt}(f \circ \gamma)|_{t=0}. \tag{1.6}$$

Alternatively, since a tangent vector is equivalent to a linear derivation on germs of smooth functions around a point, we can define

$$(f_*X)_{f(p)}\phi = X(\phi \circ f), \tag{1.7}$$

where ϕ is a germ of a smooth function at $f(p)$.

Next, smooth mapping $f : M \to N$, induces a mapping

$$f^* : T^*_{f(p)}N \to T^*_pM \tag{1.8}$$

for each $p \in M$, called a "pull-back", defined by the following. If $\omega \in T^*_{f(p)}N$, and $v \in T_pM$, then

$$(f^*\omega)(v) \equiv \omega(f_*v). \tag{1.9}$$

Exercise 1.2. Prove that $f^*\omega$ is smooth if ω is.

In general, there is not a mapping $f^* : T^*N \to T^*M$, however, the pull-back operation does induce a mapping on sections

$$f^* : \Gamma(T^*N) \to \Gamma(T^*M), \tag{1.10}$$

defined by the following. If $\omega \in \Gamma(T^*N)$, $p \in M$, and $X \in T_pM$, then

$$(f^*\omega)_p(X) = \omega_{f(p)}(f_*X). \tag{1.11}$$

Note that in general there is *not* a mapping

$$f_* : \Gamma(TM) \to \Gamma(TN), \tag{1.12}$$

but later we will be able to make sense of the following: if $X \in \Gamma(TM)$, then

$$f_*X \in \Gamma(f^*TN), \tag{1.13}$$

where f^*TN is called a *pull-back bundle*.

Note also the following important proposition.

Proposition 1.3 (The chain rule)**.** *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)_* = h_* \circ f_* : TM \to TM' \tag{1.14}$$

and

$$(h \circ f)^* = f^* \circ h^* : \Gamma(T^*M') \to \Gamma(T^*M). \tag{1.15}$$

1.2 Pull-back bundles

Let us consider the above in a slightly more sophisticated way. Given a smooth mapping $f : M \to N$, define

$$f^*TN = \{(p, v) \in M \times TN \mid f(p) = \pi_N(v)\}. \tag{1.16}$$

Exercise 1.4. Prove that f^*TN is a vector bundle over M, with projection given by $\pi_1(p, v) = p$, and the fiber f^*TN over $p \in M$ is identified with the fiber $T_{f(p)}N$, i.e., the following diagram commutes

$$\begin{array}{ccc} f^*(TN) & \xrightarrow{\pi_2} & TN \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{1.17}$$

Next, define $(f_*)_B : TM \to f^*TN$ by

$$(f_*)_B(v_p) = (p, f_*v). \tag{1.18}$$

(the subscript B is short for "bundle mapping"). We have the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{(f_*)_B} & f^*TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{id} & M. \end{array} \tag{1.19}$$

Then if $X \in \Gamma(TM)$, then we can define $f_*X \in \Gamma(f^*TN)$, by

$$f_*X \equiv (f_*)_B \circ X. \tag{1.20}$$

In words: under smooth mappings, vector fields push-forward to sections of the pull-back bundle.

Noting that $(f^*(TN))^*$ is isomorphic to $f^*(T^*N)$, let us dualize the diagram (16.2) to obtain

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M. \end{array} \tag{1.21}$$

Note that if $\omega \in \Gamma(T^*N)$, then we can define $\omega \circ f \in \Gamma(f^*(T^*N))$ by

$$(\omega \circ f)(p) = (p, \omega_{f(p)}). \tag{1.22}$$

In words: sections of bundles can be pulled-back to a section of the pull-back bundle. Then, if $\omega \in \Gamma(T^*N)$, we can compose with the bundle mapping in (8.14) to define $f^*\omega \equiv f_B^* \circ (\omega \circ f) \in \Gamma(T^*M)$.

Exercise 1.5. Check that this definition agrees with the previous definition.

2 Lecture 2

Today we will just discuss linear algebra.

2.1 Direct sums

If $V_1, \ldots, V_k$ are vector spaces over $\mathbb{R}$, then the direct sum $V_1 \oplus \cdots \oplus V_k$ is the Cartesian product $V_1 \times \cdots \times V_k$ with the following vector space structure:

$$c(v_1, \ldots, v_k) = (cv_1, \ldots, cv_k) \tag{2.1}$$

$$(v_1, \ldots, v_k) + (v_1', \ldots, v_k') = (v_1 + v_1', \ldots, v_k + v_k'), \tag{2.2}$$

for $c \in \mathbb{R}$. The space $V_1 \oplus \cdots \oplus V_k$ satisfies the following "universal" mapping property. For $1 \leq i \leq k$, let $\iota_i : V_i \to V_1 \oplus \cdots \oplus V_k$ be the inclusion mapping

$$\iota_i : v \mapsto (0, \ldots, \overbrace{v}^{i}, \ldots, 0). \tag{2.3}$$

Let W be any vector space, and $f_i : V_i \to W$ be linear mappings for $1 \leq i \leq k$. Then there is a *unique* linear map $f : V_1 \oplus \cdots \oplus V_k \to W$ which makes the following diagram

$$\begin{array}{ccc} V_i & \xrightarrow{\iota_i} & V_1 \oplus \cdots \oplus V_k \\ & \underset{f_i}{\searrow} & \downarrow f \\ & & W \end{array} \tag{2.4}$$

commute for $1 \leq i \leq k$.

This property uniquely characterizes the direct sum. That is, a vector space with the above universal mapping property is isomorphic to the direct sum, where the ι_i are just required to be linear mappings. Note that obviously

$$\dim_{\mathbb{R}}(V_1 \oplus \cdots \oplus V_k) = \sum_{i=1}^{k} \dim_{\mathbb{R}}(V_i). \tag{2.5}$$

Exercise 2.1. Prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3). \tag{2.6}$$

Definition 2.2. Let $V_i, i \in \mathcal{I}$ be any collection of vector spaces. The Cartesian product $\Pi_{i \in \mathcal{I}} V_i$ is the collection of all functions

$$f : \mathcal{I} \to \cup_{i \in \mathcal{I}} V_i, \tag{2.7}$$

such that $f(i) \in V_i$ for all $i \in \mathcal{I}$. The direct product $\Pi_{i \in \mathcal{I}} V_i$ is the Cartesian product with the vector space structure

$$cf(i) = cf(i) \tag{2.8}$$

$$(f + g)(i) = f(i) + g(i). \tag{2.9}$$

The projection $\pi_i : \Pi_{i\in\mathcal{I}}V_i \to V_i$ is the mapping $\pi_i(f) = f(i)$. The above definition satisfies the following universal property. If V is any vector space and $\phi_i : V \to V_i$ are linear mappings for $i \in \mathcal{I}$, then there is a unique linear mapping $\phi : V \to \Pi_{i\in\mathcal{I}}V_i$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_i} & V_i \\ & \underset{\phi}{\searrow} & \uparrow{\scriptstyle \pi_i} \\ & & \Pi_{i\in\mathcal{I}}V_i \end{array} \tag{2.10}$$

commutes for each $i \in \mathcal{I}$. This property uniquely characterizes the direct product, where the π_i are just required to be linear mappings. That is, any vector space with the above universal mapping property with linear mappings π_i is isomorphic to the direct product.

Definition 2.3. Let $V_i, i \in \mathcal{I}$ be any collection of vector spaces. The direct sum $\oplus_{i\in\mathcal{I}}V_i$ is the subspace of the direct product consisting of the functions f such that $f(i) \neq 0$ for only finitely many $i \in \mathcal{I}$.

Exercise 2.4. The direct sum satisfies the first universal property (2.4), but not the second (2.10), unless $\mathcal{I}$ is finite.

2.2 Free vector spaces

The following notion will be used below to define tensor product spaces and exterior product spaces.

Definition 2.5. Let S be any set. The free (real) vector space over S, denoted by $\mathcal{F}(S)$, is the vector space of all formal finite linear combinations

$$\sum_S c_s s,\ \ c_s \in \mathbb{R}, \tag{2.11}$$

where only fintely many coefficients c_s are non-zero.

Note that there is a canonical mapping $\iota : S \to \mathcal{F}(S)$ by $s \mapsto e_s$ where e_s is just the linear combination of the single element s with coefficient 1. The free vector space satisfies the following universal property. Let V be any real vector space and $f : S \to V$ be an functions of sets. Then there exists a unique *linear* mapping $\tilde{f} : \mathcal{F}(S) \to V$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & V \\ {\scriptstyle \iota}\downarrow & \underset{\tilde{f}}{\nearrow} & \\ \mathcal{F}(S) & & \end{array}$$

commutes.

Exercise 2.6. Verify this universal property, and show that this uniquely characterizes $\mathcal{F}(S)$, that is, any vector space satisfying this universal property must be isomorphic to $\mathcal{F}(S)$, with the requirement that the mapping ι satisfies that $i(S)$ is a basis.

2.3 Tensor products

If $V_1, \dots, V_k$ are vector spaces over $\mathbb{R}$, then the tensor product $V_1 \otimes \cdots \otimes V_k$ is the free real vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$ modulo the subspace spanned by all elements of the form

$$(v_1, \dots, cv_i, \dots, v_k) - c(v_1, \dots, v_i, \dots, v_k) \tag{2.12}$$

$$(v_i, \dots, v_i + v_i', \dots, v_k) - (v_i, \dots, v_i, \dots, v_k) - (v_i, v_i', \dots, v_k), \tag{2.13}$$

for $c \in \mathbb{R}$. The space $V_1 \otimes \cdots \otimes V_k$ satisfies the universal mapping property as follows. Let W be any vector space, and $F : V_1 \times \cdots V_k \to W$ be a multilinear mapping, i.e., F is linear when restricted to each factor, with the other variables held fixed. Then there is a unique *linear* map $\tilde{F} : V_1 \otimes \cdots \otimes V_k$ which makes the following diagram

$$\begin{array}{ccc} V_1 \times \cdots \times V_k & \xrightarrow{\pi} & V_1 \otimes \cdots \otimes V_k \\ & {\scriptstyle F} \searrow & \downarrow {\scriptstyle \tilde{F}} \\ & & W \end{array}$$

commutative, where π is the projection to the quotient space, which we write as

$$\pi(v_1, \dots, v_k) = v_1 \otimes \cdots \otimes v_k. \tag{2.14}$$

This universal property characterizes the tensor product, where π is just required to be a multilinear mapping.

Exercise 2.7. Prove that

$$\dim_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k) = \dim_{\mathbb{R}}(V_1) \cdots \dim_{\mathbb{R}}(V_k). \tag{2.15}$$

Also, prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3). \tag{2.16}$$

2.4 Exterior algebra and wedge product

Let V be a real vector space, and $V^* = Hom(V, \mathbb{R})$ denote the dual vector space. The exterior algebra $\Lambda(V^*)$ is defined as

$$\Lambda(V^*) = \Big\{ \bigoplus_{k \geq 0} (V^*)^{\otimes^k} \Big\} / \mathcal{I} = \bigoplus_{k \geq 0} \Big\{ (V^*)^{\otimes^k} / \mathcal{I}_k \Big\} = \bigoplus_{k \geq 0} \Lambda^k(V^*) \tag{2.17}$$

where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $\alpha \otimes \alpha \in V^* \otimes V^*$, and $\mathcal{I}_k = (V^*)^{\otimes^k} \cap \mathcal{I}$. The wedge product of $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ is just the multiplication induced by the tensor product in this algebra, that is, lift α and β to $\tilde{\alpha} \in (V^*)^{\otimes^p}$, and $\tilde{\beta} \in (V^*)^{\otimes^q}$, and define $\alpha \wedge \beta = \pi(\tilde{\alpha} \otimes \tilde{\beta})$, where $\pi : (V^*)^{\otimes^{p+q}} \to \Lambda^{p+q}(V^*)$ is the projection. This is easily seen to be well-defined. We say that an element in $\Lambda^k(V^*)$ of the form $\alpha^1 \wedge \cdots \wedge \alpha^k$ is *decomposable*. A general element of $\Lambda^k(V^*)$ is not decomposable, but can always be written as a sum of decomposable elements.

The space $\Lambda^k(V^*)$ satisfies the universal mapping property as follows. Let W be any vector space, and let

$$F : \overbrace{V^* \times \cdots \times V^*}^{k} \to W \tag{2.18}$$

be an alternating multilinear mapping. That is, F is multilinear and $F(\alpha^1, \ldots, \alpha^k) = 0$ if $\alpha^i = \alpha^j$ for some $i \neq j$. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

$$\begin{array}{ccc} \overbrace{V^* \times \cdots \times V^*}^{k} & \xrightarrow{\pi} & \Lambda^k(V^*) \\ & \searrow^{F} & \downarrow \tilde{F} \\ & & W \end{array}$$

commutative, where π is the projection, which we denote as

$$\pi(\alpha^1, \ldots, \alpha^k) = \alpha^1 \wedge \cdots \wedge \alpha^k. \tag{2.19}$$

This universal property characterizes the kth exterior product, where π is required to be an alternating multilinear mapping.

Some important properties of the wedge product:

- Bilinearity: $(\alpha^1 + \alpha^2) \wedge \beta = \alpha^1 \wedge \beta + \alpha^2 \wedge \beta$, and $(c\alpha) \wedge \beta = c(\alpha \wedge \beta)$ for $c \in \mathbb{R}$.
- If $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.
- Associativity $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

Exercise 2.8. If $\dim_{\mathbb{R}}(V) = n$, prove that $\Lambda^k(V^*) = \{0\}$ if $k > n$,

$$\dim(\Lambda^k(V^*)) = \binom{n}{k} \quad \text{if } 0 \leq k \leq n, \tag{2.20}$$

and

$$\dim(\Lambda(V^*)) = 2^n, \tag{2.21}$$

We could just stick with the above definition and try and prove all results using only this definition. However, it is very useful to think of elements of $\Lambda^k(V^*)$ as alternating multilinear maps as follows. One first has to choose a pairing

$$\Lambda^k(V^*) \cong (\Lambda^k(V))^*. \tag{2.22}$$

The pairing we will choose is as follows. If $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $v = v_1 \wedge \cdots \wedge v_k$, then

$$\alpha(v) = \det(\alpha^i(v_j)), \tag{2.23}$$

(note this is not canonical). For example,

$$\alpha^1 \wedge \alpha^2(v_1 \wedge v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{2.24}$$

We would then like to view an element of $(\Lambda^k(V))^*$ as an alternating multilinear mapping from

$$\overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}. \tag{2.25}$$

For this, we specify that if $\alpha \in (\Lambda^k(V))^*$, then

$$\alpha(v_1, \ldots, v_k) \equiv \alpha(v_1 \wedge \cdots \wedge v_k). \tag{2.26}$$

For example

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{2.27}$$

With this convention, if $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ then

$$\alpha \wedge \beta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}). \tag{2.28}$$

This then agrees with the definition of the wedge product given in [Spi79, Chapter 7].

It is convenient to have our 2 definitions of the wedge product because some proofs can be easier using one of the definitions, but harder using the other (for example, associativity of the wedge product).

3 Lecture 3

3.1 Induced mappings

Recall that if $L : V \to W$ is a linear mapping between vector spaces, then there is a mapping, $L^* : W^* \to V^*$ called the *transpose*, defined by the following. If $\omega \in W^*$, and $v \in V$, then

$$(L^*\omega)(v) = \omega(Lv). \tag{3.1}$$

This is called the transpose for the following reason. Let $dim(V) = n$, and $\dim(W) = m$. Let $e_1, \ldots, e_n$ be a basis of V and $f_1, \ldots, f_n$ be a basis of W. Let $e^1, \ldots, e^n$, and $f^1, \ldots, f^n$ denote the dual bases, that is

$$e^i(e_j) = \delta^i_j, \ 1 \leq i, j \leq n \tag{3.2}$$

$$f^i(f_j) = \delta^i_j, \ 1 \leq i, j \leq m. \tag{3.3}$$

We define the quantities L^j_i, $1 \leq i \leq n$, $1 \leq j \leq m$, by

$$Le_i = L^j_i f_j. \tag{3.4}$$

Note that if we write $v \in V$ as $v = v^i e_i$, and $w \in W$ as $w = w^i f_i$, then

$$Lv = L(v^i e_i) = v^i L(e_i) = (v^i L^j_i) f_j = \tag{3.5}$$

So the components of a vector transform like

$$\{v^i\} \mapsto \{L^j_i v^i\}, \tag{3.6}$$

which is the matrix corresponding to the transformation L.

We define the quantities $(L^*)^i_j$, $1 \leq i \leq m,\ 1 \leq j \leq n$, by

$$L^* f^i = (L^*)^i_j e^j \tag{3.7}$$

Plugging in the dual bases, we compute

$$(L^* f^i)(e_k) = (L^*)^i_j e^j(e_k) = (L^*)^i_j \delta^j_k = (L^*)^i_k. \tag{3.8}$$

However, by the definition of the transpose mapping, we have

$$(L^* f^i)(e_k) = f^i(Le_k) = f^i L^j_k f_j = L^j_k f^i(f_j) = L^j_k \delta^i_j = L^i_k \tag{3.9}$$

So if we write $\omega \in V^*$ as $\omega_i e^i$ and $\eta \in W^*$ as $\eta_j f^j$, the components of a dual vector transform like

$$\{\eta_j\} \mapsto \{L^i_j \eta_i\} \tag{3.10}$$

So the matrix corresponding to L^* in the dual basis is indeed the transpose matrix.

The mapping $L^* : W^* \to V^*$ induces a mapping

$$L^* : \overbrace{W^* \times \cdots \times W^*}^{p} \to (V^*)^{\otimes p} \tag{3.11}$$

by

$$L^*(\alpha^1, \ldots, \alpha^p) \equiv (L^*\alpha^1) \otimes \cdots \otimes (L^*\alpha^p). \tag{3.12}$$

This mapping is a multilinear mapping, so by the universal property of tensor products, this induces a unique mapping

$$L^* : (W^*)^{\otimes p} \to (V^*)^{\otimes p}. \tag{3.13}$$

By composing with the projection $\pi : (V^*)^{\otimes p} \to \Lambda^p(V^*)$, we obtain an alternating multilinear mapping

$$L^* : (W^*)^{\otimes p} \to \Lambda^p(V^*). \tag{3.14}$$

Now by the universal property of exterior products, this induces a mapping

$$L^* : \Lambda^p(W^*) \to \Lambda^p(V^*). \tag{3.15}$$

Note that by taking the direct sum on all p-s, we obtain a mapping between the full exterior algebras

$$L^* : \Lambda(W^*) \to \Lambda(V^*) \tag{3.16}$$

which is an algebra homomorphism, that is

$$L^*(\alpha \wedge \beta) = (L^*\alpha) \wedge (L^*\beta). \tag{3.17}$$

3.2 Pull-back of differential forms

A differential form is a section of $\Lambda^p(T^*M)$. I.e., a differential form is a smooth mapping $\omega : M \to \Lambda^p(T^*M)$ such that $\pi \circ \omega = Id_M$, where $\pi : \Lambda^p(T^*M) \to M$ is the bundle projection map. We will write $\omega \in \Gamma(\Lambda^p(T^*M))$, or $\omega \in \Omega^p(M)$.

If $f : M \to N$ is a smooth mapping, recall the bundle mapping (8.14) from above

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M. \end{array} \tag{3.18}$$

By the discussion in the previous section, we obtain induced mappings

$$\begin{array}{ccc} f^*(\Lambda^p(T^*N)) & \xrightarrow{f_B^*} & \Lambda^p(T^*M) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M, \end{array} \tag{3.19}$$

which is linear on fibers, and is therefore f_B^* is a smooth mapping.

Definition 3.1 (Pull-back of a differential form)**.** If $f : M \to N$ is a smooth mapping, and $\omega \in \Lambda^p(T^*N)$, then define $\omega \circ f \in \Gamma(f^*(\Lambda^p(T^*N))$ by $\omega \circ f(p) = (p, \omega_{f(p)})$. Then define

$$f^*\omega \equiv f_B^*(\omega \circ f) \in \Gamma(\Lambda^p(T^*M)). \tag{3.20}$$

Remark 3.2. If we view differential forms as multilinear mappings, for $f : M \to N$, and $\omega \in \Omega^k(N)$, then we have the following "formula". If $p \in M$ and $X_1, \cdots X_k \in T_pM$, then

$$(f^*\omega)(X_1, \cdots, X_k) = \omega_{f(p)}(f_*X_1, \cdots, f_*X_k). \tag{3.21}$$

We could have *defined* pulback of forms this way, but we would need an extra step to show the pullback of a smooth form is smooth. It is easy to see that this is in fact the same definition as above.

Example 3.3. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x^2 + y^2, x^2 - y^2, x^3). \tag{3.22}$$

Denote the coordinates on $\mathbb{R}^3$ as (u, v, w), and let

$$\alpha = wdu \wedge dv - vdu \wedge dw + udv \wedge dw. \tag{3.23}$$

Then

$$f^*\alpha = 4x^4 y dx \wedge dy. \tag{3.24}$$

We compute this using (3.21). (Details done in lecture.)

For any manifold M, define

$$\Omega(M) = \Gamma(\Lambda(T^*M)) = \Gamma\Big(\bigoplus_{p\geq 0} \Lambda^p(T^*M)\Big) = \bigoplus_{p\geq 0} \Gamma(\Lambda^p(T^*M)) = \bigoplus_{p\geq 0} \Omega^p(M). \tag{3.25}$$

By taking the direct sum of the exterior powers, we obtain a mapping

$$f^* : \Omega(N) \to \Omega(M), \tag{3.26}$$

which by (3.17) satisfies

$$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta). \tag{3.27}$$

We define the following differential of a function.

Definition 3.4. For any function $\phi : N \to \mathbb{R}$, we define a 1-form $d\phi \in \Omega^1(N)$ by $d\phi(X) = \phi_* X$ for any vector field X on N, where we identify the tangent space of $\mathbb{R}$ at any point with $\mathbb{R}$ itself.

If $f : M \to N$, then we have the formula $f^* d\phi = d(\phi \circ f)$. This is because

$$f^* d\phi(X) = (d\phi)(f_* X) = \phi_* f_*(X) = (\phi \circ f)_*(X) = d(\phi \circ f)(X), \tag{3.28}$$

where we used the chain rule (1.14). Now, we re-do the above example in an easier way using this formula. (Details done in lecture).

We also have the chain rule for pullbacks of differential forms.

Proposition 3.5 (The chain rule)**.** *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)^* = f^* \circ h^* : \Omega(M') \to \Omega(M). \tag{3.29}$$

Proof. For $\alpha \in \Omega^k(M')$, and vector fields $X_1, \dots, X_k \in \Gamma(TM)$, we have

$$\begin{aligned} (h \circ f)^*\alpha(X_1, \cdots, X_k) &= \alpha((h \circ f)_* X_1, \cdots, (h \circ f)_* X_k) = \alpha(h_* f_* X_1, \cdots, h_* f_* X_k) \\ &= h^*\alpha(f_* X_1, \cdots, f_* X_k) = f^* h^* \alpha(X_1, \cdots, X_k), \end{aligned} \tag{3.30}$$

where we used the chain rule (1.14). □

4 Lecture 4

4.1 The exterior derivative

Choose a coordinate system (U, x), and let $\frac{\partial}{\partial x^i}$ denote the coordinate vector field. Recall that viewing vector fields as derivations on germs of functions, this is characterized by

$$\frac{\partial}{\partial x^i}(x^j) = \delta_i^j. \tag{4.1}$$

We then define a local basis of 1-forms dx^i by

$$dx^i\Big(\frac{\partial}{\partial x^j}\Big) = \delta^i_j. \tag{4.2}$$

which is just the dual basis.

Given a function $f \in C^\infty(M, \mathbb{R})$ we define $df \in \Omega^1(M)$ in two ways. First, viewing vector fields as derivations on smooth functions, we can define

$$df(X) \equiv X(f). \tag{4.3}$$

Alternatively, since $f : M \to \mathbb{R}$, we have $f_* : TM \to T\mathbb{R}$. But there is a natural identification $T_p\mathbb{R} \cong \mathbb{R}$ for any $p \in \mathbb{R}$, so we can view

$$f_* : TU \to \mathbb{R}, \tag{4.4}$$

which is naturally an element in $f_* \in \Omega^1(U)$.

Proposition 4.1. *These two definitions agree, that is*

$$f_* = df. \tag{4.5}$$

Proof. First, note that

$$f_*(\frac{\partial}{\partial x^j}) = \frac{\partial f}{\partial x^j}, \tag{4.6}$$

so we have

$$f_* = \sum_j \frac{\partial f}{\partial x^j} dx^j. \tag{4.7}$$

So the left hand side of (4.5) is

$$f_*(X) = \frac{\partial f}{\partial x^i} dx^i\left(X^j \frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i} X^i. \tag{4.8}$$

For the right hand side of (4.5), given $p \in M$, let $\gamma : (-\epsilon, \epsilon) \to M$ satisfy $\gamma(0) = p$, $\gamma'(0) = X_p$, then

$$X(f) = \frac{d}{dt}(f \circ \gamma)|_{t=0} = \frac{\partial f}{\partial x^i}\frac{d\gamma^i}{dt}|_{t=0} = \frac{\partial f}{\partial x^i} X^i. \tag{4.9}$$

□

Remark 4.2. Note that from (4.5),

$$d(x^i)(\frac{\partial}{\partial x^j}) = \frac{\partial x^i}{\partial x^j} = \delta^i_j, \tag{4.10}$$

so by (4.2), we have that $dx^i = d(x^i)$ which is great!

An element $\alpha \in \Omega^p(U)$ can be written as

$$\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{4.11}$$

where the coefficients $\alpha_{i_1 \dots i_p} : U \to \mathbb{R}$ are well-defined functions. Note these coefficients are only defined for strictly increasing sequences $i_1 < \cdots < i_p$. Using our identification of $\Lambda^p(T^*M)$ with $Alt^p(TM)$, the alernating multilinear maps from $TM^{\times p} \to \mathbb{R}$, we can define the coefficient functions for *any* multi-index $(i_1, \dots, i_p)$ by

$$\alpha_{i_1 \dots i_p} = \alpha\Big(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_p}}\Big). \tag{4.12}$$

Then we also have

$$\alpha = \frac{1}{p!} \sum_{1 \leq i_1, i_2, \dots, i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \tag{4.13}$$

where the sum is over ALL indices.

However, if we want to think of α as a multilinear mapping from $TM^{\otimes p} \to \mathbb{R}$, then we extend the coefficients $\alpha_{i_1 \dots i_p}$, which are only defined for strictly increasing multi-indices $i_1 < \cdots < i_p$, to ALL indices by skew-symmetry. Then we have

$$\alpha = \sum_{1 \leq i_1, i_2, \dots, i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p}. \tag{4.14}$$

This convention is slightly annoying because then the projection to the exterior algebra of this is $p!$ times the original α, but has the positive feature that coefficients depending upon p do not enter into various formulas.

We next define the exterior derivative operator [War83, Theorem 2.20].

Proposition 4.3. *There exists an exterior derivative operator*

$$d : \Omega^p(M) \to \Omega^{p+1}(M) \tag{4.15}$$

which is the unique linear mapping satisfying

- *For* $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.
- $d^2 = 0$.
- *If* $f \in C^\infty(M, \mathbb{R})$ *then* df *is the differential of* f *defined above.*

Proof. Note that the differential of a function is given locally by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i. \tag{4.16}$$

To see this, we have $df = \sum c_i dx^i$, and plugging in the coordinate vector field identifies the coefficient c_i. Since we gave a global definition of df, this is obviously well-defined and independent of the coordinate system. Given a p-form α, write α locally as in (4.11), and then define

$$\begin{aligned} d\alpha &= \sum_{1\leq i_1<i_2<\cdots<i_p\leq n} d\alpha_{i_1\ldots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{1\leq i_1<i_2<\cdots<i_p\leq n} \sum_{i=1}^{n} \frac{\partial \alpha_{i_1\ldots i_p}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{4.17}$$

The first "anti-derivation" property is easily verified by computation. The second property holds on functions, because

$$d(df) = d\Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i\Big) = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0, \tag{4.18}$$

since the Hessian of a smooth function is symmetric.

For existence, we need to check that this definition is independent of the coordinate system. Let d' be the operator defined with respect to another coordinate system $x' : U \to \mathbb{R}^n$. Then

$$\begin{aligned} d'(\alpha) &= d'\Big(\sum_{1\leq i_1<i_2<\cdots<i_p\leq n} \alpha_{i_1\ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\Big) \\ &= \sum_{|I|=p} (d'\alpha_{i_1\ldots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &\quad + \sum_{|I|=p} \alpha_{i_1\ldots i_p} \sum_{k} (-1)^{k-1} dx^{i_1} \wedge \cdots \wedge d'(dx^{i_k}) \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{|I|=p} (d\alpha_{i_1\ldots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = d(\alpha), \end{aligned} \tag{4.19}$$

since d and d' agree on functions, and since $d'dx^i = d'd'x^i = 0$.

Then for any p-form α,

$$d(d\alpha) = d\Big(\sum_{|I|=p} (d\alpha_I) \wedge dx^I\Big) = \sum_{|I|=p} (d^2\alpha_I) \wedge dx^I - d\alpha_I \wedge d(dx^I) = 0. \tag{4.20}$$

□

Exercise 4.4. Prove the uniqueness statement.

An important fact is that d commutes with pull-back.

Proposition 4.5. *If $f : M \to N$ is a smooth mapping, and $\omega \in \Omega^p(N)$, then*

$$f^*(d_N\omega) = d_M(f^*\omega). \tag{4.21}$$

Proof. If ω is a 0-form, which is a function, then $f^*\omega = \omega \circ f$. So by above, we have

$$d(f^*\omega) = d(\omega \circ f) = (\omega \circ f)_*. \tag{4.22}$$

By the chain rule, we then have

$$d(f^*\omega) = \omega_* \circ f_*. \tag{4.23}$$

On the other hand, we have that

$$f^*(d\omega)(X) = d\omega(f_*(X)) = \omega_* \circ f_*(X). \tag{4.24}$$

So the claim is true on functions. Then if ω is a p-form, write

$$\omega = \sum_{|I|=p} \omega_I dx^I. \tag{4.25}$$

Since the pull-back operation is an algebra homomorphism, we have

$$f^*\omega = \sum_{|I|=p} (f^*\omega_I) f^* dx^I = \sum_{|I|=p} (\omega_I \circ f) d(x^I \circ f). \tag{4.26}$$

Then

$$d(f^*\omega) = \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f). \tag{4.27}$$

On the other hand, we have

$$d\omega = \sum_{|I|=p} (d\omega_I) \wedge dx^I, \tag{4.28}$$

so

$$\begin{aligned} f^*(d\omega) = \sum_{|I|=p} f^*(d\omega_I) \wedge f^* dx^I &= \sum_{|I|=p} d(f^*\omega_I) \wedge d(f^* x^I) \\ &= \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f) = d(f^*\omega). \end{aligned} \tag{4.29}$$

□

4.2 Classical tensor calculus

A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$, and the components of X with respect to a coordinate system $x : U \to \mathbb{R}^n$ are functions $X^i : U \to \mathbb{R}$, $i = 1 \dots n$, defined by

$$X = X^i \frac{\partial}{\partial x^i} \tag{4.30}$$

on U, where $\frac{\partial}{\partial x^i}$ is the ith coordinate partial, which is a vector field on TU. Given another overlapping coordinate system $\tilde{x} : U \to \mathbb{R}^n$, we can write

$$X = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}. \tag{4.31}$$

Proposition 4.6. *The components of a vector field are related by*

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^i} X^i. \tag{4.32}$$

Conversely, any collection of locally defined functions satisfying this relation gives a well-defined vector field $X \in \Gamma(TM)$.

Proof. Since vector fields are derivations on germs of functions, plug in the function $\tilde{x}^j$ to the equality

$$X^i \frac{\partial}{\partial x^i} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}, \tag{4.33}$$

to obtain

$$X^i \frac{\partial}{\partial x^i}(\tilde{x}^j) = \tilde{X}^j. \tag{4.34}$$

□

Similarly, a 1-form is a section of the cotangent bundle, $\omega \in \Gamma(T^*M)$, and the components of ω with respect to a coordinate system $x : U \to \mathbb{R}^n$ are functions $\omega_i : U \to \mathbb{R}$, $i = 1 \dots n$, defined by

$$\omega = \omega_i dx^i \tag{4.35}$$

on U. Given another overlapping coordinate system $\tilde{x} : U \to \mathbb{R}^n$, we can write

$$\omega = \tilde{\omega}_i d\tilde{x}^i. \tag{4.36}$$

Proposition 4.7. *The components of a 1-form are related by*

$$\tilde{\omega}_j = \frac{\partial x^i}{\partial \tilde{x}^j} \omega_i. \tag{4.37}$$

Conversely, any collection of locally defined functions satisfying this relation gives a well-defined 1-form $\omega \in \Gamma(T^*M)$.

Proof. Plug in the vector field $\frac{\partial}{\partial \tilde{x}^j}$ to the equality

$$\omega_i dx^i = \tilde{\omega}_i d\tilde{x}^i, \tag{4.38}$$

to obtain

$$\omega_i dx^i \left(\frac{\partial}{\partial \tilde{x}^j} \right) = \tilde{\omega}_j. \tag{4.39}$$

But recall the definition of df, where $f : U \to \mathbb{R}$ is a function. We claim that

$$df(X) = X(f). \tag{4.40}$$

To see this, the left hand side is

$$df(X) = \frac{\partial f}{\partial x^i} dx^i \left(X^j \frac{\partial}{\partial x^j} \right) = \frac{\partial f}{\partial x^i} X^i. \tag{4.41}$$

For the right hand side, let $\gamma : (-\epsilon, \epsilon) \to M$ satisfy $\gamma(0) = p$, $\gamma'(0) = X_p$, then

$$X(f) = \frac{d}{dt}(f \circ \gamma)|_{t=0} = \frac{\partial f}{\partial x^i} \frac{d\gamma^i}{dt}_{t=0} = \frac{\partial f}{\partial x^i} X_p^i. \tag{4.42}$$

Then plugging (4.40) into (4.39), we have

$$\tilde{\omega}_j = \omega_i \frac{\partial x^i}{\partial \tilde{x}^j}. \tag{4.43}$$

□

Next, consider a (p, q)-*tensor field*

$$T \in \Gamma\Big((TM)^{\otimes p} \otimes (T^*M)^{\otimes q} \Big). \tag{4.44}$$

We can locally write

$$T = T^{j_1 \dots j_p}_{i_1 \dots i_q} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_q}, \tag{4.45}$$

and in another coordinate system

$$T = \tilde{T}^{j_1 \dots j_p}_{i_1 \dots i_q} \frac{\partial}{\partial \tilde{x}^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial \tilde{x}^{j_p}} \otimes d\tilde{x}^{i_1} \otimes \cdots \otimes d\tilde{x}^{i_q}. \tag{4.46}$$

The above transformation formulas combine to give the following.

Proposition 4.8. *The components of T satisfy the transformation formulas*

$$\tilde{T}^{j_1 \dots j_p}_{i_1 \dots i_q} = \frac{\partial \tilde{x}^{j_1}}{\partial x^{l_1}} \cdots \frac{\partial \tilde{x}^{j_p}}{\partial x^{l_p}} \frac{\partial x^{k_1}}{\partial \tilde{x}^{i_1}} \cdots \frac{\partial x^{k_q}}{\partial \tilde{x}^{i_q}} T^{l_1 \dots l_p}_{k_1 \dots k_q} \tag{4.47}$$

Conversely, any collection of locally defined functions satisfying this relation gives a well-defined tensor $T \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$.

Exercise 4.9. Show that the Kronecker δ symbol, defined by

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{4.48}$$

defines a tensor. That is,

$$T = \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j \tag{4.49}$$

is a well-defined $(1,1)$-tensor $\delta \in \Gamma(TM \otimes T^*M)$. Under the canonical isomorphisms

$$TM \otimes T^*M \cong T^*M \otimes TM \cong Hom(TM, TM), \tag{4.50}$$

identify what is the image of δ.

5 Lecture 5

We note that for an n-form, we can write

$$\omega = \omega_{1\dots n} dx^1 \wedge \cdots \wedge dx^n, \tag{5.1}$$

In another coordinate system, we can write

$$\omega = \tilde{\omega}_{1\dots n} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n. \tag{5.2}$$

Exercise 5.1. Show that these components are related by

$$\tilde{\omega}_{1\dots n} = \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right)\omega_{1\dots n}, \tag{5.3}$$

This is the key to defining the integral of an n-form on an orientable manifold, but we first need to discuss orientations.

5.1 Orientability

First, an orientation on a n-dimensional vector space V is a choice of ordered basis $(v_1, \dots, v_n)$ with equivalence relation if 2 ordered bases are related by a change of basis matrix with positive determinant. There are exactly 2 such equivalence classes, and if M is a manifold, the oriented double cover of M denoted by $\pi : \tilde{M} \to M$ is the double covering obtained by replacing a point p with the 2 orientations on T_pM (we choose the topology on $\tilde{M}$ which makes π continuous and open).

Definition 5.2. A manifold M is orientable if any of the following equivalent conditions are satisfied.

- M admits an coordinate atlas (U_α, ϕ_α) such that the overlap maps are orientation-preserving $\phi_\alpha \circ \phi_\beta^{-1}$, that is, the Jacobian $(\phi_\alpha \circ \phi_\beta^{-1})_*$ has positive determinant.
- M admits a nowhere-zero n-form.
- The oriented double cover $\tilde{M} \to M$ is trivial, i.e., it has 2 components.

If M is orientable, the choice of one of the components of $\tilde{M}$ is called an *orientation* on M. An orientation on M induces well-defined orientation on each tangent space T_xM.

Exercise 5.3. Prove that the bullet points are equivalent, and prove that for any smooth manifold M, the orientable double cover $\tilde{M}$ is always orientable.

Let's illustrate Definition 5.2 with an example:

Example 5.4. We observe that the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is orientable. We have several ways to see this:

- We can cover S^{n-1} by 2 coordinate charts (generalized stereographic projection), with intersection $\mathbb{R}^{n-1} \setminus \{0\}$. By changing the orientation of one of these charts, we can arrange that the overlap mapping is orientation-preserving.

- Let ν be the outer unit normal to S^{n-1}, and define $\omega_{S^{n-1}} = \nu \lrcorner (dx^1 \wedge \cdots \wedge dx^n)$. We claim that ω is non-zero at every point. This form is invariant under rotations, so we only need to check this claim at a point. So let $p = (0, \dots, 0, 1)$ be the north pole, and $\nu_p = \frac{\partial}{\partial x^n}$. Then

$$\omega_p = dx^1 \wedge \cdots \wedge dx^n \Big(\frac{\partial}{\partial x^n}, \cdot, \cdots, \cdot \Big) = (-1)^{n-1} dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{5.4}$$

But $T_p S^{n-1} = \mathrm{span}\Big\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}} \Big\}$, so ω_p is clearly non-zero at p, therefore it is everywhere non-zero.

Exercise 5.5. Let $\iota : M^{n-1} \to \mathbb{R}^n$ be an embedded submanifold. Assume that $\mathbb{R}^n \setminus \iota(M^{n-1})$ has exactly 2 components. Prove that M^{n-1} is orientable. (Hint: define a non-zero $(n-1)$-form on M^{n-1} similar to the second bullet point in the previous example).

Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}), \ c \neq 0. \tag{5.5}$$

Exercise 5.6. Prove that the real projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n which is orientable if n is odd, but non-orientable if n is even.

5.2 Integration of differential forms

On an oriented n-dimensional manifold, the integral of $\omega \in \Omega^n(M)$ is defined as follows. Choose an oriented coordinate atlas (U_α, ϕ_α). First, assume that $\omega \in \Omega^n(M)$ has compact support in a single coordinate system U_α. Then

$$(\phi_\alpha)_*(\omega) = f dx^1 \wedge \cdots \wedge dx^n, \tag{5.6}$$

where $f : \phi_\alpha(U_\alpha) \to \mathbb{R}$ has compact support. Define

$$\int_M \omega \equiv \int_{\phi_\alpha(U_\alpha)} f dx^1 \dots dx^n. \tag{5.7}$$

By the change-of-variables formula for integrals and the formula

$$\tilde{\omega}_{1\dots n} = \det \left(\frac{\partial x^i}{\partial \tilde{x}^j} \right) \omega_{1 \dots n}, \tag{5.8}$$

from above (5.3), this definition is independent of coordinate system containing the support of ω.

Next, if M is compact, or if ω has compact support, let χ_α be a partition of unity subordinate to $\{U_\alpha\}$, and define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega. \tag{5.9}$$

Since the sum is finite, this definition is independent of the choice of coordinate atlas and choice of partition of unity. To see this, let U_α and V_β be open covers with subordinate partitions of unity ρ_α, χ_β, respectively. Then

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \sum_\beta \chi_\beta \rho_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \omega = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta \omega, \tag{5.10}$$

since $\rho_\alpha \chi_\beta$ is suported in V_β. Therefore

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \sum_\alpha \rho_\alpha \chi_\beta \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega. \tag{5.11}$$

5.3 Stokes' Theorem

Definition 5.7. A manifold with boundary $M = (M \setminus \partial M) \coprod \partial M$, can be covered by usual manifold coordinate charts in the interior $M \setminus \partial M$, together with coordinate charts near points in ∂M of the the form (U_i, ϕ_i), where $\phi_i : U_i \to H^n$, where

$$H^n = \{(x^1, \dots x^n) \in \mathbb{R}^n \mid x^n \geq 0\}. \tag{5.12}$$

is the upper half space in $\mathbb{R}^n$, such that

$$\phi_i : U_i \cap \partial M \to \mathbb{R}^{n-1} \tag{5.13}$$

is a coordinate chart on ∂M viewed as an $(n-1)$-dimensional smooth manifold.

Integration by parts on manifolds is the following.

Theorem 5.8 (Stokes' Theorem for manifolds with boundary)**.** *Let $(M, \partial M)$ be an oriented manifold with boundary of dimension n. If $\omega \in \Omega^{n-1}(M)$ has compact support, then*

$$\int_{\partial M} \omega = \int_M d\omega, \tag{5.14}$$

where the boundary has the orientation induced from the outer normal, i.e., if $v_i \in T_p(\partial M)$, then the ordered basis $(v_1, \dots v_{n-1})$ is oriented if $(v, v_1, \dots, v_{n-1})$ is positively oriented, for any outward pointing normal vector v.

Proof. We first consider forms compactly supported in a coordinate chart (either an interior chart or a boundary chart). Then just consider an $(n-1)$-form of the form

$$\omega = f dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n. \tag{5.15}$$

Note that

$$d\omega = (-1)^{i-1} \partial_i f dx^1 \wedge \dots \wedge dx^n \tag{5.16}$$

If $i < n$, then ω restricted to the boundary is zero, and

$$\int_{H^n} d\omega = (-1)^{i-1} \int_{H^n} \partial_i f dx^1 \cdots dx^n = 0, \tag{5.17}$$

by Fubini's Theorem and the fundamental theorem of calculus, since f has compact support. If $i = n$, then

$$\begin{aligned} \int_{H^n} d\omega &= (-1)^{n-1} \int_{H^n} \partial_n f dx^1 \cdots dx^n \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_n f dx^1 \cdots dx^n \\ &= (-1)^n \int_{\mathbb{R}^{n-1}} \omega(x^1, \ldots, x^n, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{\partial H^n} \omega, \end{aligned} \tag{5.18}$$

since the outward normal is $-e_n$, so $\{-e_n, e_1, \ldots, e_{n-1}\}$ is oriented, which is equivalent to $(-1)^n$ times $\{e_1, \ldots, e_n\}$. In general ω is a sum of n-terms of the above type, so this proves Stokes' Theorem for $\omega \in \Omega^{n-1}(H^n)$ with compact support.

Next, we choose a partition of unity χ_i subordinate to the cover (U_i, ϕ_i), $\phi_i : U_i \to \mathbb{R}^n$, and write $\omega = \sum_i \chi_i \omega$. Let $\omega_i = \chi_i \omega$. Then for each i in the index set, we have

$$\begin{aligned} \int_M d\omega_i = \int_{U_i} d\omega_i &= \int_{\phi_i^{-1}(U_i)} (\phi_i^{-1})^*(d\omega_i) = \int_{\phi_i^{-1}(U_i)} d(\phi_i^{-1})^*(\omega_i) \\ &= \int_{H^n} d(\phi_i^{-1})^*(\omega_i) = \int_{\partial H^n} (\phi_i^{-1})^*(\omega_i) = \int_{\partial M} \omega_i, \end{aligned} \tag{5.19}$$

where the last equality holds since $\phi_i|_{\partial M}$ is a coordinate chart on ∂M as a $(n-1)$-dimensional manifold. Finally, we have

$$\int_M d\omega = \int_M d\Big(\sum_i \omega_i\Big) = \sum_i \int_M d\omega_i = \sum_i \int_{\partial M} \omega_i = \int_{\partial M} \sum_i \omega_i = \int_{\partial M} \omega. \tag{5.20}$$

□

6 Lecture 6

6.1 de Rham cohomology

Let M be a smooth manifold of dimension n. Since $d^2 = 0$, we have a "cochain" complex

$$\cdots \xrightarrow{d^{p-2}} \Omega^{p-1}(M) \xrightarrow{d^{p-1}} \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \xrightarrow{d^{p+1}} \cdots . \tag{6.1}$$

which terminates at $\Omega^n(M)$, where $n = \dim(M)$. Clearly we have that Image$(d^{p-1}) \subset$ Ker(d^p), so we can define the following vector spaces.

Definition 6.1. For $0 \leq p \leq n$, the pth de Rham cohomology group is

$$H^p_{dR}(M) = \frac{\mathrm{Ker}\{d^p : \Omega^p(M) \to \Omega^{p+1}(M)\}}{\mathrm{Image}\{d^{p-1} : \Omega^{p-1}(M) \to \Omega^p(M)\}}. \tag{6.2}$$

Example 6.2. If $p = 0$, then $H^0_{dR}(M) = \{f : M \to \mathbb{R} \mid df = 0\}$. Since in local coordinates,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j} dx^j, \tag{6.3}$$

it follows that f is constant on connected components of M. Consequently, $\dim(H^0_{dR}(M))$ is equal to the number of components of M.

Example 6.3. Let $M = \mathbb{R}^n \setminus \{0\}$, and consider

$$\omega_{\mathbb{R}^{n-1}} = \frac{1}{r^n} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \in \Omega^{n-1}(M). \tag{6.4}$$

By direct computation, we showed in class that $d\omega_{\mathbb{R}^{n-1}} = 0$. We claim that $\omega_{\mathbb{R}^{n-1}}$ cannot be written in the form $\omega_{\mathbb{R}^{n-1}} = d\alpha_{n-2}$ for any $\alpha_{n-2} \in \Omega^{n-2}(M)$. To see this, assume by contradiction that this is true. Let $\iota : S^{n-1} \to \mathbb{R}^n$ be the inclusion of the unit sphere. We showed above that S^{n-1} is orientable, so we can integrate $(n-1)$-forms on S^{n-1}, once we choose an orientation. Then

$$\int_{S^{n-1}} \iota^* \omega_{\mathbb{R}^{n-1}} = \int_{S^{n-1}} \iota^* d\alpha_{n-2} = \int_{S^{n-1}} d\iota^* \alpha_{n-2} = 0, \tag{6.5}$$

by Stokes' Theorem 5.8. However,

$$\iota^* \omega_{\mathbb{R}^{n-1}} = \omega_{S^{n-1}}, \tag{6.6}$$

which we defined above, is non-zero at every point, so the integral must be non-zero. This contradiction proves 2 things:

$$H^{n-1}_{dR}(\mathbb{R}^n \setminus \{0\}) \neq \{0\} \tag{6.7}$$

$$H^{n-1}_{dR}(S^{n-1}) \neq \{0\}. \tag{6.8}$$

Note the latter part of this example proves the following.

Proposition 6.4. *Let M be a compact oriented n-dimensional manifold. Then*

$$H^n_{dR}(M) \neq 0. \tag{6.9}$$

Proof. If M is oriented, then we know there exists a nowhere-zero $\omega \in \Omega^n(M)$ which determines the orientation. In any oriented coordinate system (U, ϕ), we have $\phi_* \omega = f dx^1 \wedge \cdots \wedge dx^n$ where $f > 0$. Therefore we must have

$$\int_M \omega > 0. \tag{6.10}$$

If $\omega = d\alpha_{n-1}$ for $\alpha \in \Omega^{n-1}(M)$, then Stokes' Theorem would say that

$$\int_M \omega = \int_M d\alpha_{n-1} = \int_{\partial M} \alpha_{n-1} = 0, \tag{6.11}$$

since $\partial M = \emptyset$. □

6.2 Diffeomorphism invariance

Note that

$$H^*_{dR}(M) \equiv \bigoplus_{p=0}^{n} H^p_{dR}(M) \tag{6.12}$$

has an algebra structure induced by the wedge product. To see this, for $[\alpha] \in H^p_{dR}(M)$ and $[\beta] \in H^q_{dR}(M)$, represented by $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, we have that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0, \tag{6.13}$$

so we define

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]. \tag{6.14}$$

To see that this is well-defined, we have

$$(\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d\gamma \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta), \tag{6.15}$$

since β is closed, so

$$[(\alpha + d\gamma) \wedge \beta] = [\alpha \wedge \beta]. \tag{6.16}$$

Well-definedness in the other factor is similar, or just use the skew-symmetry property of the wedge product. Therefore we have

$$\wedge : H^p_{dR}(M) \otimes H^q_{dR}(M) \to H^{p+q}_{dR}(M). \tag{6.17}$$

Note that from Proposition 4.3, we have

$$[\alpha] \wedge [\beta] = (-1)^{pq} [\beta] \wedge [\alpha]. \tag{6.18}$$

Next, let $f : X \to Y$ be a smooth mapping between smooth manifolds. As discussed before, we have a pullback operation on differential forms, $f^* : \Omega^*(Y) \to \Omega^*(X)$, which makes the following diagram commute

$$\begin{array}{ccc} \Omega^p(Y) & \xrightarrow{d^p_Y} & \Omega^{p+1}(Y) \\ \downarrow{\scriptstyle (f^*)^p} & & \downarrow{\scriptstyle (f^*)^{p+1}} \\ \Omega^p(X) & \xrightarrow{d^p_X} & \Omega^{p+1}(X). \end{array} \tag{6.19}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The de Rham cohomology algebra is a diffeomorphism invariant.

Corollary 6.5. *If $f : X \to Y$ then there are induced mappings*

$$(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X). \tag{6.20}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{6.21}$$

Consequently, if X and Y are diffeomorphic, then $H^p_{dR}(X) \cong H^p_{dR}(Y)$ for every $p \geq 0$, and moreover, the cohomology algebras are isomorphic $H^*_{dR}(X) \cong H^*_{dR}(Y)$.

Proof. We first note that any smooth mapping $f : X \to Y$ induces a well-defined mapping on cohomology $(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X)$ by the following. If $[\alpha] \in H^p_{dR}(Y)$ is represented by a form α, such that $d^p_Y \alpha = 0$, then we have

$$d^p_X (f^*)^p \alpha = (f^*)^{p+1} d^p_Y \alpha = (f^*)^{p+1} 0 = 0, \tag{6.22}$$

so we can define $f^*[\alpha] = [f^*\alpha]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$(f^*)^p(\alpha + d^{p-1}_Y \beta) = (f^*)^p \alpha + (f^*)^p d^{p-1}_Y \beta = (f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1} \beta, \tag{6.23}$$

so $[(f^*)^p(\alpha + d^p_Y \beta)] = [(f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1}\beta] = [(f^*)^p \alpha]$.

If f is a diffeomorphism, then f^{-1} exists and is smooth, so we have

$$f \circ f^{-1} = id_Y, \quad f^{-1} \circ f = id_X, \tag{6.24}$$

and from Proposition 3.5, the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^*_{dR}(X)}, \quad (f^{-1})^* \circ f^* = id_{H^*_{dR}(Y)}, \tag{6.25}$$

Finally, since $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$, together these mappings form an algebra homomorphism on cohomology algebras, which will be an algebra isomorphism if X and Y are diffeomorphic. □

7 Lecture 7

7.1 The Poincaré Lemma

Let M be a smooth manifold, possibly noncompact, and let $N = M \times [0, 1]$, which is an $(n + 1)$-dimensional manifold with boundary. Let $\pi : N \to M$ be the projection $\pi(x, t) = x$. Also, let $\iota_t : M \to M \times [0, 1]$ be the inclusion $\iota_t(x) = (x, t)$. Define a mapping

$$I^k : \Omega^k(M \times [0, 1]) \to \Omega^{k-1}(M) \tag{7.1}$$

by the following. Since

$$T_{(p,t)}N = \mathrm{Ker}(\tilde{\pi}_*)_{(p,t)} \oplus \mathrm{Ker}(\pi_*)_{(p,t)} \cong T_pM \oplus T_t[0, 1] \tag{7.2}$$

where $\tilde{\pi}(x, t) = t$, any k-form on N can be uniquely written as

$$\omega = h(x, t)\pi^*\phi_k + f(x, t)dt \wedge (\pi^*\phi_{k-1}), \tag{7.3}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0(N)$. The mapping I^k is then defined by

$$I^k(\omega) = \Big(\int_0^1 f(x, t)dt\Big)\phi_{k-1}. \tag{7.4}$$

Proposition 7.1. *For $\omega \in \Omega^k(N)$, we have*

$$(\iota_1)^*\omega - (\iota_0)^*\omega = d_M I^k \omega + I^{k+1} d_N \omega. \tag{7.5}$$

Proof. Writing ω in the form (7.3), since $\iota_t^* dt = 0$, and $\pi \circ \iota_t = id_M$, the left hand side of (7.5) is

$$\begin{aligned}(\iota_1)^*\omega - (\iota_0)^*\omega &= (\iota_1)^* h(x,t)\pi^*\phi_k - (\iota_0)^* h(x,t)\pi^*\phi_k \\ &= (h(x,1) - h(x,0))\phi_k.\end{aligned} \tag{7.6}$$

Next, assume that ω is just of the form

$$\omega = h(x,t)\pi^*\phi_k. \tag{7.7}$$

Then, choosing a local coordinate system $\{x^i\}$ on M near any point, we have

$$d_N\omega = \Big(\sum_{i=1}^{m} \frac{\partial h}{\partial x^i} dx^i + \frac{\partial h}{\partial t} dt\Big) \wedge \pi^*\phi_k + h(x,t)\pi^* d_M \phi_k. \tag{7.8}$$

By the definition of I^k, we have $I^k\omega = 0$, so obviously

$$d_M I^k \omega = 0, \tag{7.9}$$

and

$$\begin{aligned}I^{k+1} d_N \omega &= I^{k+1}\Big\{\Big(\sum_{i=1}^{n} \frac{\partial h}{\partial x^i} dx^i + \frac{\partial h}{\partial t} dt\Big) \wedge \pi^*\phi_k + h(x,t)\pi^* d_M\phi_k\Big\} \\ &= I^{k+1}\Big\{\frac{\partial h}{\partial t} dt \wedge \pi^*\phi_k\Big\} = \Big(\int_0^1 \frac{\partial h}{\partial t} dt\Big)\phi_k = (h(x,1) - h(x,0))\phi_k.\end{aligned} \tag{7.10}$$

So the proposition holds for forms of this type.

Next, assume that ω is just of the form

$$\omega = f(x,t) dt \wedge (\pi^*\phi_{k-1}). \tag{7.11}$$

From (7.6) above, we have

$$(\iota_1)^*\omega - (\iota_0)^*\omega = 0. \tag{7.12}$$

Note that

$$\begin{aligned}d_N\omega &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dt \wedge (\pi^*\phi_{k-1}) - f(x,t) dt \wedge \pi^*(d_M\phi_{k-1}) \\ &= -\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dt \wedge \pi^*\Big(dx^i \wedge \phi_{k-1}\Big) - f dt \wedge \pi^*\Big(d_M\phi_{k-1}\Big).\end{aligned} \tag{7.13}$$

So by definition of I^{k+1} and (7.13), we have

$$I^{k+1} d_N \omega = -\Big(\sum_{i=1}^{n} \int_0^1 \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} - \Big(\int_0^1 f dt \Big) d_M \phi_{k-1}. \tag{7.14}$$

Next, by defintion of I^k,

$$\begin{aligned} d_M I^k \omega &= d_M \Big\{ \Big(\int_0^1 f(x,t) dt \Big) \phi_{k-1} \Big\} \\ &= \Big(\int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} + \Big(\int_0^1 f dt \Big) d_M \phi_{k-1}. \end{aligned} \tag{7.15}$$

Consequently, on forms of this type, we have

$$d_M I^k \omega + I^{k+1} d_N \omega = 0. \tag{7.16}$$

So the proposition is true for forms of the second type. By linearity, the proposition holds for all forms, and we are done. □

Remark 7.2. Note that we used a coordinate system in the above proof. This is OK since these are local expressions of global quantities, so the local identity therefore implies the global identity.

7.2 Homotopy invariance of de Rham cohomology

Definition 7.3. Let X and Y be smooth manifolds. Smooth mappings $f, g : X \to Y$ are said to be smoothly homotopic if there exists a smooth mapping $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

Proposition 7.4. *Let X and Y be smooth manifolds. If $f, g : X \to Y$ are smoothly homotopic then*

$$f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X) \tag{7.17}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x,t)$, and note that

$$(\iota_t)^* : \Omega^*(X \times [0,1]) \to \Omega^*(X). \tag{7.18}$$

In Proposition 7.1, we constructed

$$I^k : \Omega^k(X \times [0,1]) \to \Omega^{k-1}(X) \tag{7.19}$$

satisfying

$$(\iota_1)^* - (\iota_0)^* = I^{k+1} d_{X \times [0,1]} + d_X I^k. \tag{7.20}$$

This clearly implies that as mappings on de Rham cohomology

$$(\iota_0)^* = (\iota_1)^* : H^k_{dR}(X \times [0,1]) \to H^k_{dR}(X). \tag{7.21}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, we have

$$f^* = (\iota_0)^* \circ F^*, \quad g^* = (\iota_1)^* \circ F^*, \tag{7.22}$$

therefore $f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X)$. □

Definition 7.5. Smooth manifolds X and Y have the same smooth homotopy type if there exist smooth mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is smoothly homotopic to Id_X and $f \circ g$ is smoothly homotopic to id_Y.

Corollary 7.6. *If X and Y have the same smooth homotopy type, then $H^*_{dR}(X) \cong H^*_{dR}(Y)$.*

Proof. From Proposition 7.4, we have

$$f^* \circ g^* = Id_{H^*_{dR}(X)} \tag{7.23}$$
$$g^* \circ f^* = Id_{H^*_{dR}(Y)}, \tag{7.24}$$

so f^* and g^* are isomorphisms. □

Some special cases of this are the following.

Definition 7.7. A smooth manifold X is smoothly contractible if X has the same smooth homotopy type as a point.

Corollary 7.8. *If X is smoothly contractible, then*

$$H^k_{dR}(X) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{7.25}$$

Proof. By definition, there is a mapping $f : X \to \{p\}$ and a mapping $g : \{p\} \to X$ so that $g \circ f$ is homotopic to Id_X. This is equivalent to the existence of $H : X \times [0,1] \to X$ so that $H(x,1) = x$ for all $x \in X$ and $H(x,0) = x_0$ where $x_0 = g(p)$. We already know that $H^0_{dR}(X) = \mathbb{R}$, so let $k \geq 1$, and $\omega \in \Omega^k(X)$ such that $d_X\omega = 0$. Plugging in $H^*\omega$ into the Poincaré Lemma yields

$$\begin{aligned} (\iota_1)^* H^*\omega - (\iota_0)^* H^*\omega &= (H \circ \iota_1)^*\omega - (H \circ \iota_0)^*\omega \\ &= I^{k+1} d_{X\times[0,1]} H^*\omega + d_X I^k H^*\omega = d_X I^k H^*\omega, \end{aligned} \tag{7.26}$$

because ω is closed and d commutes with pullback. However, $H \circ \iota_1 = Id_X$ and $H \circ \iota_0$ is a constant map, therefore we have

$$\omega = d_X I^k H^*\omega, \tag{7.27}$$

so ω is exact. □

Exercise 7.9. A domain $A \subset \mathbb{R}^n$ is star-shaped if there exists a $p \in A$ such that for any $x \in A$, the line segment between p and x is contained in A. In this case, let $H : A \times [0,1] \to \mathbb{R}^n$ be the mapping $H(x,t) = tx + (1-t)p$. This shows that A is (smoothly) contractible to a point, so A has the same de Rham cohomology groups as a point. Show that the Poincaré Lemma gives the explicit formula as follows. Without loss of generality, we can assume that $p = \{0\}$. Writing $\omega \in \Omega^k(A)$ for $k \geq 1$ with $d\omega = 0$ as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{7.28}$$

then

$$\gamma \equiv \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \Big(\int_0^1 t^{k-1} \omega_{i_1 \cdots i_k}(tx) dt \Big) x^{i_\alpha} dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_k} \tag{7.29}$$

is an explicit $(k-1)$-form solving $d\gamma = \omega$. (Hint: $\gamma = I^k H^* \omega$.)

Definition 7.10. A submanifold $i : A \hookrightarrow X$ is a smooth deformation retraction of X if there exists a smooth mapping $r : X \to X$ such that

$$r \circ i = id_A, \tag{7.30}$$

and $i \circ r$ is smoothly homotopic to Id_X.

Corollary 7.11. *If A is a smooth deformation retraction of X then*

$$H^k_{dR}(A) \cong H^k_{dR}(X), \tag{7.31}$$

for all $k \geq 0$.

Example 7.12. Consider $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \subset \mathbb{R}^n$ given by $r(x) = x/|x|$. The mapping $F(x,t) = (1-t)x + t(x/|x|)$ is a smooth homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, so S^{n-1} is a smooth deformation retraction of $\mathbb{R}^n \setminus \{0\}$ and we therefore have

$$H^k_{dR}(S^{n-1}) = H^k_{dR}(\mathbb{R}^n \setminus \{0\}). \tag{7.32}$$

From earlier, we know that this vector space is non-zero when $k = n-1$, and our next task is to prove that this cohomology group is 1-dimensional (and determine all of the other groups as well).

8 Lecture 8

8.1 Cochain complexes

A collection A^p of vector spaces for $p \geq 0$ and operators $\delta^p_A : A^p \to A^{p+1}$ for $p \geq 0$ satisfying $\delta^{p+1}_A \delta^p_A = 0$ is called a *cochain complex*.

$$\cdots \xrightarrow{\delta^{p-2}_A} A^{p-1} \xrightarrow{\delta^{p-1}_A} A^p \xrightarrow{\delta^p_A} A^{p+1} \xrightarrow{\delta^{p+1}_A} \cdots. \tag{8.1}$$

Definition 8.1. The pth cohomology of a chain complex is the vector space

$$H^p(A) = \frac{Ker\{\delta_A^p : A^p \to A^{p+1}\}}{Im\{\delta_A^{p-1} : A^{p-1} \to A^p\}} \tag{8.2}$$

Definition 8.2. A morphism $\alpha : A \to B$ of cochain complexes is a collection of mappings $\alpha^p : A^p \to B^p$ such that $\delta_B^p \alpha^p = \alpha^{p+1}\delta_A^p$ for $p \geq 0$. In other words, $\alpha : A \to B$ is a morphism if the following diagram commutes

$$\begin{array}{ccc} A^p & \xrightarrow{\delta_A^p} & A_{p+1} \\ \big\downarrow{\scriptstyle \alpha^p} & & \big\downarrow{\scriptstyle \alpha^{p+1}} \\ B^p & \xrightarrow{\delta_B^p} & B^{p+1}. \end{array} \tag{8.3}$$

Proposition 8.3. *A morphism of cochain complexes $\alpha : A \to B$ induces mappings $H^p\alpha^p : H^p(A) \to H^p(B)$.*

Proof. Given $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$, we have

$$\delta_B^p \alpha^p a^p = \alpha^{p+1}\delta_A^p a^p = 0, \tag{8.4}$$

therefore we can define $(H^p\alpha^p)[a^p] = [\alpha^p a^p]$. To check that this is well-defined,

$$[\alpha^p(a^p + \delta_A^{p-1}a^{p-1})] = [\alpha^p a^p + \alpha^p \delta_A^{p-1} a^{p-1}] = [\alpha^p a^p + \delta_B^{p-1}\alpha^{p-1}a^{p-1}] = [\alpha^p a^p]. \tag{8.5}$$

□

8.2 Exact sequences of cochain complexes

Definition 8.4. A sequence of vector spaces A, B, C, with linear mappings $\alpha : A \to B$, $\beta : B \to C$

$$0 \xrightarrow{0} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{0} 0 \tag{8.6}$$

is called *exact* if the kernel of each mapping is equal to the image of the previous mapping. That is $Ker(\alpha) = \{0\}$ if and only if α is injective. Next, $Ker(\beta) = Im(\alpha)$. Finally, $Im(\beta) = C$, if and only if β is surjective.

Let C_i be a co-complex of vector spaces for $i = 1, 2, 3$.

$$\cdots \xrightarrow{d_i^{p-2}} C_i^{p-1} \xrightarrow{d_i^{p-1}} C_i^p \xrightarrow{d_i^p} C_i^{p+1} \xrightarrow{d_i^{p+1}} \cdots. \tag{8.7}$$

with $d^2 = 0$. A morphism from C_i to C_j are mappings $\alpha^k : C_i^k \to C_j^k$ such that the following diagram commutes for every p

$$\begin{array}{ccc} C_i^p & \xrightarrow{d_i^p} & C_i^{p+1} \\ \big\downarrow{\scriptstyle \alpha^p} & & \big\downarrow{\scriptstyle \alpha^{p+1}} \\ C_j^p & \xrightarrow{d_j^p} & C_j^{p+1} \end{array} \tag{8.8}$$

For co-complexes C_1, C_2, C_3, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of co-complexes is exact if

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{8.9}$$

if the sequence

$$0 \xrightarrow{0} C_1^p \xrightarrow{\alpha^p} C_2^p \xrightarrow{\beta^p} C_3^p \xrightarrow{0} 0 \tag{8.10}$$

is exact for every p.

Lemma 8.5 (The zig-zag lemma for cochain complexes)**.** *If*

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{8.11}$$

is a short exact sequence of co-complexes, then there exist connecting homomorphisms

$$\delta^p : H^p(C_3) \to H^{p+1}(C_1) \tag{8.12}$$

for every p such that the sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\delta^p} H^{p+1}(C_1) \longrightarrow \cdots \tag{8.13}$$

is exact.

Proof. We look at the huge commutative diagram

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1^{p-1} & \xrightarrow{\alpha^{p-1}} & C_2^{p-1} & \xrightarrow{\beta^{p-1}} & C_3^{p-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1^{p-1}} & & \downarrow{\scriptstyle d_2^{p-1}} & & \downarrow{\scriptstyle d_3^{p-1}} & & \\ 0 & \longrightarrow & C_1^{p} & \xrightarrow{\alpha^{p}} & C_2^{p} & \xrightarrow{\beta^{p}} & C_3^{p} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1^{p}} & & \downarrow{\scriptstyle d_2^{p}} & & \downarrow{\scriptstyle d_3^{p}} & & \\ 0 & \longrightarrow & C_1^{p+1} & \xrightarrow{\alpha^{p+1}} & C_2^{p+1} & \xrightarrow{\beta^{p+1}} & C_3^{p+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array} \tag{8.14}$$

which has all horizontal rows exact.

To define the connecting homomorphism, take $c_3^p \in C_3^p$ with $d_3^p c_3^p = 0$. By exactness of the middle row, β_p is surjective, so $c_3^p = \beta^p(c_2^p)$ for some $c_2^p \in C_2^p$. Then since the diagram commutes, we have

$$\beta^{p+1} d_2^p c_2^p = d_3^p \beta^p c_2^p = d_3^p c_3^p = 0. \tag{8.15}$$

By exactness of the bottow row, we have $d_2^p c_2^p = \alpha^{p+1} c_1^{p+1}$ for some $c_1^{p+1} \in C_1^{p+1}$. Since C_1 is a co-complex, and by commutativity of the diagram, we have

$$0 = d_2^{p+1} d_2^p c_2^p = d_2^{p+1} \alpha^{p+1} c_1^{p+1} = \alpha^{p+2} d_1^{p+1} c_1^{p+1}, \tag{8.16}$$

which implies that $d_1^{p+1}c_1^{p+1} = 0$, since α^{p+2} is injective. So we define $\delta^p(c_3^p) = [c_1^{p+1}]$, the homology class of c_1^{p+1} in $H^{p+1}(C_1)$.

To prove this mapping is well-defined, assume that we started with $c_p^3 \in C_p^3$ which was of the form $c_p^3 = d_3^{p-1}c_3^{p-1}$. Then we can write $c_3^{p-1} = \beta^{p-1}c_2^{p-1}$, and the element $\tilde{c}_2^p = d_2^{p-1}c_2^{p-1}$ satisfies $\beta^p(\tilde{c}_2^p) = c_3^p$. But this element is exact, so the next step clearly gives zero. Independence of the choice of c_2^p is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing. □

Exercise 8.6. Prove that the sequence (8.13) is exact.

8.3 Mayer-Vietoris for de Rham cohomology

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{\beta^p} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{\alpha^p} \Omega^p(U \cap V) \longrightarrow 0 \tag{8.17}$$

where

$$\beta^p(\omega) = \big((i_{U \hookrightarrow M})^*\omega, (i_{V \hookrightarrow M})^*\omega\big). \tag{8.18}$$

and

$$\alpha^p(\omega_U, \omega_V) = (i_{U\cap V \hookrightarrow U})^*\omega_U - (i_{U\cap V \hookrightarrow V})^*\omega_V \tag{8.19}$$

To see this, β^p is obviously injective. For exactness at the middle step, obviously $\alpha^p\beta^p\omega = 0$. If $\beta^p(\omega_U, \omega_V) = 0$, then $\omega_U = \omega_V$ on $U \cap V$, so then (ω_U, ω_V) is a well-defined global form on M.

To show that α is onto, let $\omega \in \Omega^p(U \cap V)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \alpha(\phi_V\omega, -\phi_U\omega)$.

By the zig-zag lemma for cohomology, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{8.20}$$

9 Lecture 9

9.1 General remarks on Mayer-Vietoris

Let us review the definition of the mapping $\delta^p : H^p(U \cap V) \to H^{p+1}(M)$. Given a cohomology class $[\omega] \in H^p_{dR}(U \cap V)$, represented by $\omega \in \Omega^p(U \cap V)$ with $d\omega = 0$, we first write $\omega = \alpha^p(\phi_V\omega, -\phi_U\omega)$, then we apply the exterior derivative to get

$$(d(\phi_V\omega), -d(\phi_U\omega)) = (d\phi_V \wedge \omega, -d\phi_U \wedge \omega) \in \Omega^p(U) \oplus \Omega^p(V). \tag{9.1}$$

Note that on $U \cap V$, we have $(\phi_U + \phi_V)\omega = \omega$, so applying d to this equation, we have that $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$ on $U \cap V$, so together these define a global form

$$\delta^p\omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{9.2}$$

and we take the cohomology class of this form.

Remark 9.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

Corollary 9.2. *If a smooth manifold M has a finite covering of open sets such that each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has finite-dimensional de Rham cohomology, then M has finite-dimensional de Rham cohomology.*

Proof. Note that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{9.3}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{9.4}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

Now we look at the following portion of the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\alpha^{p-1}} H^{p-1}_{dR}(U \cap V) \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} \cdots \tag{9.5}$$

Using induction on the number of open sets in the covering, the corollary follows. □

Remark 9.3. If M is compact, there always exists such a covering. More on this later.

The following lemma will be extremely useful.

Lemma 9.4. *If*

$$0 \longrightarrow V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \longrightarrow V_{k-1} \xrightarrow{\alpha_{k-1}} V_k \longrightarrow 0. \tag{9.6}$$

is exact, then

$$0 = \dim(V_1) - \dim(V_2) + \dim(V_3) + \cdots + (-1)^{k-1} \dim(V_k). \tag{9.7}$$

Proof. We use induction. If $k = 1$, then we have $0 \to V_1 \to 0$, which obviously implies that $\dim(V_1) = 0$. Assume the theorem is true up to $k - 1$. The mapping $\alpha_2 : V_2 \to V_3$ has kernel given by $\alpha_1(V_1)$. So there is an induced mapping $\alpha_2 : V_2/\alpha_1(V_1) \to V_2$. If $\alpha_2(v_2 + \alpha_1(V_1)) = \alpha_2(v_2) = 0$, then $v_2 = \alpha_1(w_1)$, so the induced mapping is injective. We therefore have an exact sequence

$$0 \longrightarrow V_2/\alpha_1(V_1) \xrightarrow{\alpha_2} V_3 \xrightarrow{\alpha_3} \cdots \longrightarrow V_{k-1} \xrightarrow{\alpha_{k-1}} V_k \longrightarrow 0. \tag{9.8}$$

This is an exact seqence of length $k - 1$, so by induction

$$\begin{aligned} 0 &= \dim\Big(V_2/\alpha_1(V_1)\Big) - \dim(V_3) + \cdots \\ &= -\dim(V_1) + \dim(V_2) - \dim(V_3) + \cdots, \end{aligned} \tag{9.9}$$

and we are done. □

9.2 Spheres

We consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

Corollary 9.5. *We have*

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases} \tag{9.10}$$

Proof. Cover with 2 open sets U, V, with $U \cong \mathbb{R}^n \cong V$ and $U \cap V \cong S^{n-1}$. First, consider the case of S^1. In this case, the Mayer-Vietoris sequence is

$$0 \longrightarrow H^0_{dR}(S^1) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} H^1_{dR}(S^1) \xrightarrow{\beta^1} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^1} H^1_{dR}(U \cap V) \longrightarrow 0. \tag{9.11}$$

But $U \cap V$ is contractible to 2 points, so this is

$$0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta^0} H^1_{dR}(S^1) \xrightarrow{\beta^1} 0. \tag{9.12}$$

Lemma 9.4 then says that $H^1_{dR}(S^1) \cong \mathbb{R}$.

Next, for $n > 1$, look at the beginning of the Mayer-Vietoris sequence

$$0 \longrightarrow H^0_{dR}(S^n) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots \tag{9.13}$$

But now $U \cap V$ is connected, so this is

$$0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \xrightarrow{\delta^0} \cdots. \tag{9.14}$$

Since β is injective, the kernel of α^0 is 1-dimensional. But α^0 has a 2-dimensional domain, so the image of α^0 is 1-dimensional, that is α^0 is surjective. So we can move to the next level and get

$$0 \longrightarrow H^1_{dR}(S^n) \xrightarrow{\beta^0} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^0} H^1_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots, \tag{9.15}$$

Since U and V are contractible, this says that $H^1_{dR}(S^n) = \{0\}$ for $n \geq 2$.

Next, we look at the upper portion of the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^{n-2}_{dR}(U) \oplus H^{n-2}_{dR}(V) \xrightarrow{\alpha^{n-2}} H^{n-2}_{dR}(S^{n-1}) \xrightarrow{\delta^{n-2}} H^{n-1}_{dR}(S^n) \xrightarrow{\beta^p} 0 \xrightarrow{\alpha^p} H^{n-1}_{dR}(S^{n-1}) \xrightarrow{\delta^{n-1}} H^n_{dR}(S^n) \xrightarrow{\beta^{p+1}} 0. \tag{9.16}$$

This yields

$$H^n_{dR}(S^n) \cong H^{n-1}_{dR}(S^{n-1}) \cong \mathbb{R}, \tag{9.17}$$

and

$$H^k_{dR}(S^n) \cong H^{k-1}_{dR}(S^{n-1}) = \{0\}, \tag{9.18}$$

for $2 \leq k \leq n-1$, so this finishes the proof. □

9.3 The 2-torus

Next we consider the 2-dimensional torus $T^2 = S^1 \times S^1$.

Corollary 9.6. *We have*

$$H^2_{dR}(T^2) = \begin{cases} \mathbb{R} & k = 0 \ \textit{or}\ 2 \\ \mathbb{R}^2 & k = 1 \end{cases}. \tag{9.19}$$

Proof. Writing the first S^1-factor as the union of two intervals, we can cover T^2 by $U = \mathbb{R}\times S^1$ and $V = \mathbb{R}\times S^1$ and such that the intersection is diffeomorphic to $U \cap V$ is $\mathbb{R}\times S^1 \coprod \mathbb{R}\times S^1$. The full Mayer-Vietoris sequence is

$$\begin{array}{l} 0 \longrightarrow H^0(T^2) \xrightarrow{\beta^0} H^0(U)\oplus H^0(V) \xrightarrow{\alpha^0} H^0(U\cap V) \xrightarrow{\delta^0} \\ \quad H^1(T^2) \xrightarrow{\beta^1} H^1(U)\oplus H^1(V) \xrightarrow{\alpha^1} H^1(U\cap V) \xrightarrow{\delta^1} \\ \quad H^2(T^2) \xrightarrow{\beta^2} H^2(U)\oplus H^2(V) \xrightarrow{\alpha^2} H^2(U\cap V) \xrightarrow{\delta^2} 0. \end{array} \tag{9.20}$$

Using the Poincaré Lemma and Corollary 9.5, we know this is

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\alpha^0} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\delta^0} \\ \quad H^1(T^2) \xrightarrow{\beta^1} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\alpha^1} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\delta^1} \\ \quad H^2(T^2) \xrightarrow{\beta^2} 0 \xrightarrow{\alpha^2} 0 \xrightarrow{\delta^2} 0. \end{array} \tag{9.21}$$

If we try and use Lemma 9.4, we just get that $\dim(H^1(T^2)) = \dim(H^2(T^2)) + 1$, so we really need to look at the mappings. For this, note that the α^0 mapping has a 1-dimensional image, because the difference on the intersection is the same on both components of $U \cap V$. We also claim that α^1 has a 1-dimensional image. To see this, note that $d\theta$ is a generator for $H^1(S^1)$. From the proof of homotopy invariance of de Rham cohomology, a generator of

$H^1(U) = H^1(S^1 \times \mathbb{R})$ is $\pi^* d\theta$, where π is projection onto the S^1 factor. Since the generator is independent of the $\mathbb{R}$ coordinate, the difference is the same on both components of the intersection, which proves the claim. So we can split off 2 exact sequences.

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta^0} H^1(T^2) \xrightarrow{\beta^1} \mathbb{R}^2 \xrightarrow{\alpha^1} \mathbb{R} \longrightarrow 0, \tag{9.22}$$

and

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha^1} \mathbb{R}^2 \xrightarrow{\delta^1} H^2(T^2) \longrightarrow 0. \tag{9.23}$$

Using Lemma 9.4, the first sequence implies that $b^1 = \dim(H^1(T^2)) = 2$ and the second sequence implies that $b^2 = \dim(H^2(T^2)) = 1$. $\square$

Remark 9.7. The above proof also shows that $H^1(T^2)$ is spanned by $d\theta_1$ and $d\theta_2$, where θ_i is the angular coordinate on the ith factor. Also, $H^2(T^2)$ is spanned by $d\theta_1 \wedge d\theta_2$.

9.4 Real projective 2-space

Using the above, we can determine the de Rham cohomology of $\mathbb{RP}^2$.

Proposition 9.8. *We have*

$$H^k(\mathbb{RP}^2) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k = 1, 2 \end{cases} \tag{9.24}$$

Proof. We can consider $\mathbb{RP}^2$ as a disc D^2 with the boundary identified by the antipodal map. Cover by $U = D^2$, and V a tubular neighborhood of the boundary $\mathbb{RP}^1 = S^1$. The intersection is a annulus so is homotopic to S^1. The Mayer-Vietoris sequence is

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \xrightarrow{\delta^0} \\ \quad H^1(\mathbb{RP}^2) \xrightarrow{\beta^1} \mathbb{R} \xrightarrow{\alpha^1} \mathbb{R} \xrightarrow{\delta^1} \\ \quad H^2(\mathbb{RP}^2) \xrightarrow{\beta^2} 0 \xrightarrow{\alpha^2} 0 \xrightarrow{\delta^2} 0. \end{array} \tag{9.25}$$

We can split off the exact sequence

$$0 \longrightarrow H^1(\mathbb{RP}^2) \xrightarrow{\beta^1} \mathbb{R} \xrightarrow{\alpha^1} \mathbb{R} \xrightarrow{\delta^1} H^2(\mathbb{RP}^2) \longrightarrow 0. \tag{9.26}$$

If we use Lemma 9.4, we can only conclude that $H^1(\mathbb{RP}^2) \simeq H^2(\mathbb{RP}^2)$. So we need to look at the mapping $\alpha^1 : H^1(V) \to H^1(U \cap V)$. There is a natural projection $\pi : U \cap V \to \mathbb{RP}^1$, since V is a tubular neighborhood of $\mathbb{RP}^2$. Note that $U \cap V$ is homotopic to S^1, since $U \cap V$ is an annulus. Then, up the homotopy, the mapping $\iota : U \cap V \to V$ is the same as $\pi : S^1 \to \mathbb{RP}^1$, which is a double covering. Consequently the mapping $\alpha^1 : H^1(\mathbb{RP}^1) \to H^1(S^1)$ is multiplication by 2. So α is an isomorphism, and $H^1(\mathbb{RP}^2) \simeq H^2(\mathbb{RP}^2) \simeq \{0\}$.

$\square$

10 Lecture 10

10.1 Poincaré Lemma for cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega_c^{p-1}(M) \xrightarrow{d} \Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \cdots, \tag{10.1}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Lemma 10.1. *Let M be a differentiable n-manifold, then for $k \geq 1$,*

$$H^k_{c,dR}(M \times \mathbb{R}) \cong H^{k-1}_{c,dR}(M). \tag{10.2}$$

Proof. First, we define a mapping "integration over the fiber" by

$$\pi_* : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M) \tag{10.3}$$

by the following. Any k-form on $N = M \times \mathbb{R}$ can be written as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{10.4}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega_c^0(M \times \mathbb{R})$. Define

$$\pi_*(\omega) = \Big(\int_{-\infty}^{\infty} f(x,t)dt \Big) \phi_{k-1}, \tag{10.5}$$

noting that the integral is defined because ω is assumed to have compact support, and this form has compact support since f has compact support.

We claim that

$$d_M \circ \pi_* = \pi_* \circ d_{M \times \mathbb{R}}, \tag{10.6}$$

To see this, the left hand side of (10.6) is

$$\begin{aligned} d_M \circ \pi_* \omega &= d_M \Big(\Big(\int_{-\infty}^{\infty} f(x,t)dt \Big) \phi_{k-1} \Big) \\ &= \Big(\int_{-\infty}^{\infty} \sum_{i=1}^n \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(x,t)dt \Big) d_M \phi_{k-1}. \end{aligned} \tag{10.7}$$

The right hand side of (10.6) is

$$\begin{aligned} \pi_* \circ d_N \omega &= \pi_* \Big(\frac{\partial h}{\partial t} dt \wedge \pi^*\phi_k + \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge \pi^*\phi_{k-1} \wedge dt + f(x,t)\pi^*(d_M\phi_{k-1}) \wedge dt \Big) \\ &= \pi_* \Big(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge \pi^*\phi_{k-1} \wedge dt + f(x,t)\pi^*(d_M\phi_{k-1}) \wedge dt \Big) \\ &= \Big(\int_{-\infty}^{\infty} \sum_{i=1}^n \frac{\partial f}{\partial x^i} dt \Big) dx^i \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(x,t)dt) \Big) d_M \phi_{k-1}, \end{aligned} \tag{10.8}$$

since the term involving h is zero because h has compact support, and using the fundamental theorem of calculus. Therefore π_* induces a mapping

$$\pi_* : H^k_{c,dR}(M \times \mathbb{R}) \to H^{k-1}_{c,dR}(M). \tag{10.9}$$

Next, we choose $e \in \Omega^1_c(\mathbb{R})$ with $\int_R e = 1$, and define

$$e_* : \Omega^k_c(M) \to \Omega^{k+1}_c(M \times \mathbb{R}) \tag{10.10}$$

by

$$e_*(\omega) = (\pi^*\omega) \wedge e. \tag{10.11}$$

We have that

$$d_{M\times\mathbb{R}} \circ e_* = e_* \circ d_M, \tag{10.12}$$

because

$$d_N \circ e_*(\omega) = d_N(\pi^*\omega \wedge e) = (d_N\pi^*\omega) \wedge e = \pi^*(d_M\omega) \wedge e = e_* \circ d_M(\omega). \tag{10.13}$$

Therefore e_* induces a mapping

$$e_* : H^k_{c,dR}(M) \to H^{k+1}_{c,dR}(M \times \mathbb{R}). \tag{10.14}$$

Let us write $e = \chi dt$, then

$$\pi_* \circ e_*(\omega) = \pi_*\Big(\chi(t)(\pi^*\omega) \wedge dt\Big) = \Big(\int_{-\infty}^{\infty} \chi(t)dt\Big)\omega = \omega \tag{10.15}$$

Therefore, we have $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$, so $\pi_* \circ e_* = 1$ on $H^k_{c,dR}(M)$.

We next claim that $e_* \circ \pi_* = 1$ on $H^k_{c,dR}(M \times \mathbb{R})$. To see this, writing $\omega \in \Omega^k_c(N)$ as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{10.16}$$

define a mapping

$$K : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R}) \tag{10.17}$$

by

$$K(\omega) = \pi^*\phi_{k-1}\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big). \tag{10.18}$$

Note that the right hand side is indeed a $(k-1)$-form on $M \times \mathbb{R}$ with compact support, which is clear if t is sufficiently large. We claim that if $\omega \in \Omega^k_c(M \times \mathbb{R})$ then

$$(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{10.19}$$

which we will separately verify for forms of type $\omega = h(x,t)\pi^*\phi_k$, and for forms of type $\omega = f(x,t)\pi^*\phi_{k-1} \wedge dt$.

For forms of the first type, we obviously have

$$(1 - e_*\pi_*)h(x,t)\pi^*\phi_k = h(x,t)\pi^*\phi_k. \tag{10.20}$$

On the other hand, since K is zero on forms of this type,

$$\begin{aligned}
(dK - Kd)(h(x,t)\pi^*\phi_k) &= -K\Big(\Big(\frac{\partial h}{\partial x}\Big)dx \wedge \pi^*\phi_k + \Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k + h(x,t)\pi^* d\phi_k\Big)\\
&= -K\Big(\Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k\Big)\\
&= (-1)^{k-1}K\Big(\Big(\frac{\partial h}{\partial t}\Big)(\pi^*\phi_k) \wedge dt\Big)\\
&= (-1)^{k-1}\pi^*\phi_k\Big(\int_{-\infty}^{t}\frac{\partial h}{\partial t}ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty}\frac{\partial h}{\partial t}ds\Big)\\
&= (-1)^{k-1}(\pi^*\phi_k)h(x,t).
\end{aligned} \tag{10.21}$$

For forms of the second type $\omega = f(x,t)\pi^*\phi_{k-1} \wedge dt$, we have

$$\begin{aligned}
(1 - e_*\pi_*)f(x,t)\pi^*\phi_{k-1} \wedge dt &= f(x,t)\pi^*\phi_{k-1} \wedge dt - \Big(\int_{-\infty}^{\infty} f(x,t)dt\Big)(\pi^*\phi_{k-1}) \wedge e\\
&= \pi^*\phi_{k-1} \wedge \Big(f(x,t)dt - \Big(\int_{-\infty}^{\infty} f(x,t)dt\Big)e\Big)\\
&= \Big(f(x,t) - \Big(\int_{-\infty}^{\infty} f(x,t)dt\Big)\chi(t)\Big)\pi^*\phi_{k-1} \wedge dt
\end{aligned} \tag{10.22}$$

Next,

$$\begin{aligned}
d_N K\omega &= d_N\Big(\pi^*\phi_{k-1}\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big)\Big)\\
&= \pi^*(d_M\phi_{k-1})\Big(\int_{-\infty}^{t} f(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big)\\
&\quad + (-1)^{k-1}\pi^*\phi_{k-1}\sum_{i=1}^{n}\Big(\int_{-\infty}^{t}\frac{\partial f}{\partial x^i}(x,s)ds - \Big(\int_{-\infty}^{t} e\Big)\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}(x,s)ds\Big) \wedge dx^i\\
&\quad + (-1)^{k-1}\pi^*\phi_{k-1}\Big(f(x,t)dt - e\int_{-\infty}^{\infty} f(x,s)ds\Big).
\end{aligned} \tag{10.23}$$

We compute

$$
\begin{aligned}
K d_N\omega &= K\Big(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge \pi^*\phi_{k-1} + f(x,t)\pi^* d_M\phi_{k-1}\Big)\wedge dt\\
&= \sum_{i=1}^n K\Big(\frac{\partial f}{\partial x^i} dx^i \wedge \pi^*\phi_{k-1}\wedge dt\Big) + K\Big(f(x,t)\pi^* d_M\phi_{k-1}\wedge dt\Big)\\
&= \sum_{i=1}^n \pi^*(dx^i\wedge\phi_{k-1})\Big(\int_{-\infty}^t \frac{\partial f}{\partial x^i}(x,s)ds - \Big(\int_{-\infty}^t e\Big)\int_{-\infty}^\infty \frac{\partial f}{\partial x^i}(x,s)ds\Big)\\
&+ \pi^*(d_M\phi_{k-1})\Big(\int_{-\infty}^t f(x,s)ds - \Big(\int_{-\infty}^t e\Big)\int_{-\infty}^\infty f(x,s)ds\Big).
\end{aligned}
\tag{10.24}
$$

Adding together (10.23) and (10.24) and using (10.22), we obtain

$$
(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{10.25}
$$

which finishes the proof of the claim.

The claim implies that $e_* \circ \pi_* = 1$ as a mapping on $H^k_{c,dR}(M\times\mathbb{R})$, and the Poincaré Lemma for compactly supported cohomology follows. □

Corollary 10.2. *We have*

$$
H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{10.26}
$$

and a generator for $H^n_{c,dR}(\mathbb{R}^n)$ is given by any compactly supported n-form μ with $\int_{\mathbb{R}^n}\mu = 1$.

Proof. We start with $M = \{p\}$ a single point. From above, we have an isomorphism

$$
\mathbb{R} \cong H^0_{c,dR}(\{p\}) \cong H^1_{c,dR}(\mathbb{R}). \tag{10.27}
$$

Furthermore, since the isomorphism is given by e_*, the proof shows that a generator of the left hand side is $\chi(x^1)dx^1$. Next, we have

$$
H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}, \tag{10.28}
$$

and a generator of the left hand side is $\chi(x^1)dx^1\wedge\chi(x^2)dx^2$. In general, a generator is given by

$$
\chi(x^1)\cdots\chi(x^n)dx^1\wedge\cdots\wedge dx^n. \tag{10.29}
$$

Next, we use the fact that π_* is an isomorphism. The isomorphism

$$
H^1_{c,dR}(\mathbb{R}) \cong H^0_{c,dR}(\{p\}) \cong \mathbb{R} \tag{10.30}
$$

is given by

$$
\phi_1 \mapsto \int_{\mathbb{R}} \phi_1 dx^1. \tag{10.31}
$$

Then the isomorphism

$$H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}, \tag{10.32}$$

is given by

$$f(x^1, x^2) dx^1 \wedge dx^2 \mapsto \left(\int_{\mathbb{R}} f(x^1, x^2) dx^2 \right) dx^1. \tag{10.33}$$

Composing these isomorphisms and using Fubini's Theorem, we get

$$f(x^1, x^2) dx^1 \wedge dx^2 \mapsto \int_{\mathbb{R}^2} f(x^1, x^2) dx^1 \wedge dx^2. \tag{10.34}$$

In general, the isomorphism is given by

$$f(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n \mapsto \int_{\mathbb{R}^n} f(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n. \tag{10.35}$$

□

Remark 10.3. This shows that $H^*_{c,dR}(M)$ is not a homotopy invariant, since (10.26) is not the same as the cohomology of a point. But of course, $H^*_{c,dR}(M)$ *is* a diffeomorphism invariant.

11 Lecture 11

11.1 Top degree cohomology

We can now determine the top degree compactly supported cohomology of any manifold.

Theorem 11.1. *Let M be a connected smooth n-dimensional manifold. Then*

$$H^n_{c,dR}(M) = \begin{cases} \mathbb{R} & M \text{ is orientable} \\ 0 & M \text{ is non-orientable} \end{cases}. \tag{11.1}$$

In particular, if M is compact, then

$$H^n_{dR}(M) = \begin{cases} \mathbb{R} & M \text{ is orientable} \\ 0 & M \text{ is non-orientable} \end{cases} \tag{11.2}$$

Proof. Assume that M is orientable. From the Poincaré Lemma for cohomology with compact support, we know that $H^n_c(\mathbb{R}^n) = \mathbb{R}$. Cover M by open sets U_i diffeomorphic to $\mathbb{R}^n$. Given $\omega \in \Omega^n_c(M)$, write

$$\omega = \sum_i \omega_i = \sum_i \chi_i \omega, \tag{11.3}$$

where χ_i is a partition of unity subordinate to $\{U_i\}$. Since the result is true for $\mathbb{R}^n$, we know that $\omega_i = d\alpha_i + c_i\beta_i$, where β_i is a bump form, which we can take to be supported near any point $p_i \in U_i$. So we can assume that ω is cohomologous to a sum of bump forms. We claim that any 2 bump forms are cohomologous to a multiple of each other. To see this, given $p, p' \in M$, we can connect by a sequence of coordinate patches V_i and sequence of points chosen as follows. Let $p_0 = p$, contained in V_0. Then choose $p_1 \in V_1 \cap V_2$, $p_2 \in V_2 \cap V_3$, etc, with $p' \in V_N$. This shows that $H^n_c(M)$ is at most 1-dimensional. By Stokes' theorem, we already knew that $H^n_c(M)$ was a least 1-dimensional. So M orientable implies $\dim(H^n_c(M)) = 1$.

If M is non-orientable, then there must be a path $\gamma : S^1 \to M$ such that $\Lambda^n(\gamma^* TM)$ is the non-orientable bundle on S^1. This is because the orientation double cover $\pi : \tilde{M} \to M$ is a non-trivial double covering, so there must be a closed path γ which lifts to an open path in $\tilde{M}$ connecting different points in a single fiber. Consequently, there exists a sequence of coordinate neighborhoods $V_1, \ldots, V_N$ with $V_1 = V_N$ and such that there are an odd number of orientation-reversing transition functions. The above argument then shows that β_1, a bump form supported in V_1, is cohomologous to $-\beta_1$, and therefore must be trivial in cohomology. □

Next, we prove the following, which will complete our understanding of the top degree de Rham cohomology of any smooth manifold.

Proposition 11.2. *If M is a non-compact smooth manifold of dimension n, then*

$$H^n_{dR}(M) \cong \{0\}. \tag{11.4}$$

Proof. Assume that $\omega \in \Omega^n$ has compact support in a coordinate neighborhood $U = U_1$. We then find a "ray" of coordinate neighborhoods, that is, U_i such that $U_i \cap U_{i+1} \neq \emptyset$ and is connected, but the sequence leaves any compact subset of M. Choose $\omega_i \in \Omega^n_c(U_i \cap U_{i+1})$ which generates $H^n_c(U_i)$. Then

$$\begin{aligned} \omega &= d\eta_1 + c_1\omega_1 \\ &= d\eta_1 + c_1(c_2\omega_2 + d\eta_2) \\ &= d\eta_1 + c_1 d\eta_2 + c_1c_2(d\eta_3 + c_3\omega_3) = \cdots . \end{aligned} \tag{11.5}$$

So we have

$$\begin{aligned} \omega &= d\eta_1 + c_1 d\eta_2 + c_1c_2 d\eta_3 + c_1c_2c_3 d\eta_4 + \cdots \\ &= d\Big(\eta_1 + c_1 d\eta_2 + c_1c_2 d\eta_3 + c_1c_2c_3 d\eta_4 + \cdots\Big). \end{aligned} \tag{11.6}$$

Since the open sets U_i leave any compact subset, this is a finite sum at any point, so this shows that $\omega = d\eta$, where $\eta \in \Omega^{n-1}(M)$.

Next, given any ω, we can find a sequence of coordinate neighborhoods U_i such that $\{U_i\}$ covers M and which is locally finite, and which leaves any compact subset. Let ϕ_i be a partition of unity subordinate to $\{U_i\}$. Then by the above, we can write $\omega_i = \phi_i\omega = d\eta_i$, where the support of η_i is contained in $U_i \cup U_{i+1} \cup \cdots$. Finally,

$$\omega = \sum \omega_i = \sum d\eta_i = d\Big(\sum \eta_i\Big), \tag{11.7}$$

and the form $\eta = \sum \eta_i$ makes sense since the U_i-s leave any compact subset. □

Let us summarize our results on the top-dimensional cohomology of any connected smooth n-dimensional manifold.

Theorem 11.3. *Let M be a connected smooth n-dimensional manifold. Then*

$$H^n_{dR}(M) = \begin{cases} \mathbb{R} & M \text{ compact and orientable} \\ 0 & M \text{ otherwise} \end{cases} \tag{11.8}$$

and

$$H^n_{c,dR}(M) = \begin{cases} \mathbb{R} & M \text{ orientable} \\ 0 & M \text{ otherwise} \end{cases}. \tag{11.9}$$

11.2 Degree of a smooth mapping

Definition 11.4. A mapping $f : X \to Y$ between topological spaces is proper if the inverse image of any compact set is compact.

Exercise 11.5. Prove the following statements about proper mappings.

(i) Let X and Y be metric spaces. For a sequence of points $x_i \in X$, we say that $\lim_{i\to\infty} x_i = \infty$ if given any compact subset $K \subset X$, then there exists an integer N so that $x_i \in X \setminus K$ for $i > N$. Show that $f : X \to Y$ is proper iff for any sequence $x_i \in X$ such that $\lim_{i\to\infty} x_i = \infty$, then $\lim_{i\to\infty} f(x_i) = \infty$.

(ii) If Y is a manifold and $f : X \to Y$ is proper and continuous, then f is a closed mapping, that is, f maps closed sets to closed sets.

Let $f : M \to N$ be a proper smooth mapping between n-dimensional connected and oriented smooth manifolds. Since f is proper, $f^* : \Omega^n_c(N) \to \Omega^n_c(M)$, and therefore there is an induced mapping $f^* : H^n_{c,dR}(N) \to H^n_{c,dR}(M)$. From Theorem 11.1, we know that $H^n_{c,dR}(M) \cong \mathbb{R}$, with isomorphism given by $[\omega] \mapsto \int_M \omega$, and similarly for N. Therefore, we can make the following definition.

Definition 11.6. The degree of f is the real number $\deg(f)$ so that

$$\int_M f^*\omega = \deg(f) \int_N \omega \tag{11.10}$$

for all $\omega \in \Omega^n_c(N)$.

Proposition 11.7. *If $f : M \to N$ and $g : N \to \tilde{M}$ are both proper then $g \circ f : N \to \tilde{M}$ is proper and*

$$\deg(g \circ f) = \deg(g) \cdot \deg(f) \tag{11.11}$$

Proof. The composition of proper maps is obviously proper. Given $\omega \in \Omega^n_c(\tilde{M})$ then

$$\int_M (g \circ f)^*\omega = \int_M f^* g^*\omega = \deg(f) \int_N g^*\omega = \deg(f)\deg(g) \int_{\tilde{M}} \omega. \tag{11.12}$$

□

Proposition 11.8. *Let $f : M \to N$ be a diffeomorphism. Then* $\deg(f) = 1$ *is f is orientation preserving, and* $\deg(f) = -1$ *if f is orientation reversing.*

Proof. This follows from the change of variables formula, the definition of the integral is clearly invariant under diffeomorphisms, but only up to sign. □

Proposition 11.9. *If M and N are compact, and $f : M \to N$ is smoothly homotopic to $g : M \to N$, then* $\deg(f) = \deg(g)$.

Proof. If M and N are compact, we know that $H^n_{c,dR}(M) = H^n_{dR}(M)$ and $H^n_{c,dR}(N) = H^n_{dR}(N)$. By the Proposition 7.4 $f^* = g^* : H^n_{c,dR}(N) \to H^n_{c,dR}(M)$. □

Remark 11.10. This is not true in the noncompact case. The functions z and z^2 as mappings from $\mathbb{C}$ to itself are properly homotopic, yet have different degrees.

Proposition 11.11. *If $f : M \to N$ is proper and not surjective, then* $\deg(f) = 0$.

Proof. If f is not surjective, then there exists $q \in N$ which is not in the image of f. Furthermore, since f is a closed mapping, there exists a neighborhood U of q which contains no points in the image of f. Let χ be an n-form supported in U with $\int_U \chi = 1$. But $f^*\chi \equiv 0$, so $\int_M f^*\chi = 0$, and thus $\deg(f) = 0$. □

Proposition 11.12. *If $f : M \to N$ is proper, then* $\deg(f) \in \mathbb{Z}$. *Furthermore, if $q \in N$ be a regular value of f, and let $f^{-1}(q) = \{p_1, \ldots, p_k\}$. Then*

$$\deg(f) = \sum_i \mathrm{sgn}(f_*|_{p_i}), \tag{11.13}$$

where

$$\mathrm{sgn}(f_*|_{p_i}) = \det(f_*|_{p_i})/|\det(f_*|_{p_i})|, \tag{11.14}$$

Proof. By Sard's Theorem, there exists a regular value q. Since f is proper, $f^{-1}(q)$ is compact, and since it consists of isolated points (by the inverse function theorem), it must be a finite set. We can choose a neighborhood U of q so that $f^{-1}(U) = U_1 \coprod \cdots \coprod U_k$ and such that $f : U_i \to U$ is a diffeomorphism. Let $\chi \in \Omega^n_c(U)$ satisfy $\int_U \chi = 1$. Then $f^*\chi$ is supported in $U_1 \cup \cdots \cup U_k$, and we have

$$\int_M f^*\chi = \sum_{i=1}^k \int_{U_i} f^*\chi = \sum_{i=1}^k \mathrm{sgn}(f_*|_{p_i}) \int_U \chi = \sum_{i=1}^k \mathrm{sgn}(f_*|_{p_i}). \tag{11.15}$$

□

12 Lecture 12

12.1 Applications of degree

In this subsection, we will give several applications of the degree theory developed above.

Corollary 12.1 (Fundamental theorem of algebra)**.** *Any nonconstant polynomial in* $\mathbb{C}$ *has a zero.*

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Recall that $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$ where $(z^1, z^2) \sim \lambda(z^1, z^2)$ for $\lambda \neq 0$. It is easy to see that we can extend $P(x) : S^2 \to S^2$, as a holomorphic map (it is a meromorphic function on $\mathbb{C}$ with a single pole at infinity of order n). Since P is holomorphic, it is orientation preserving. Since P is a polynomial, the set of critical values is a finite set. Since P is non-constant, P attains some value $q \in S^2$ which is not critical. Since P is orientation preserving at all regular points, the degree must then be non-zero (in fact, the degree is n). Proposition 11.11 then implies that P is surjective. $\square$

Corollary 12.2. *There does not exist any non-zero vector field on* S^n *for* n *even.*

Proof. Let $A : S^n \to S^n$ be the antipodal map. We first claim that $\deg(A) = -1$ for n even. Clearly A is a diffeomorphism, so we just need to check if it is orientation preserving or not. The standard orientation of S^n is given by

$$\omega = \left(x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \cdots \wedge dx^{n+1}). \tag{12.1}$$

Clearly

$$A^*(\omega) = (-1)^{n+1}\omega. \tag{12.2}$$

So if n is even, we have $\deg(A) = -1$.

But if X is a non-zero vector field on S^n, let $\gamma_p(t)$ be the portion of the great circle such that $\gamma_p(0) = p$, $\gamma_p(1) = -p$ and such that $\gamma_p'(0)$ points in the direction of X_p. Then $H(p,t) = \gamma_p(t)$ is a homotopy between Id and A. This is a contradiction since $\deg(Id) = 1$. $\square$

Remark 12.3. Odd-dimensional spheres always have a non-zero vector field:

$$X = (-x_2, x_1, -x_4, x_3, \cdots) \tag{12.3}$$

We also have the following:

Proposition 12.4. *If* $f : S^n \to S^n$ *is smooth and* $\deg(f) \neq (-1)^{n+1}$*, then* f *has a fixed point.*

Proof. If no fixed point, then for $p \in S^n$, the line segment from $f(p)$ to $-p$ does not hit the origin. Then we can define

$$H(p,t) = \frac{(1-t)f(p) - tp}{|(1-t)f(p) - tp|}, \tag{12.4}$$

which is a homotopy between f and the antipodal map. $\square$

Next, we have

Proposition 12.5. *Let M be a smooth n-dimensional oriented compact manifold with connected boundary ∂M. Let N be a compact connected oriented $(n-1)$ manifold. Let $g : \partial M \to N$ be a smooth mapping which extends to a smooth mapping $G : M \to N$. Then* $\deg(g) = 0$.

Proof. Let ω be any smooth $(n-1)$ form on N such that $\int_N \omega = 1$. Obviously $d_N\omega = 0$ since N is of dimension $n-1$. Then by Stokes' Theorem

$$\deg(g) = \int_{\partial M} g^*\omega = \int_{\partial M} G^*\omega = \int_M d_M(G^*\omega) = \int_M G^*(d_N\omega) = 0. \tag{12.5}$$

□

Corollary 12.6 (Brouwer fixed point theorem)**.** *If $f : \overline{B^n} \to \overline{B^n}$ is smooth, then f has a fixed point.*

Proof. Assume by contradiction that f has no fixed point. Then define $G : \overline{B^n} \to S^{n-1}$ by

$$G(x) = \frac{x - f(x)}{|x - f(x)|}. \tag{12.6}$$

Letting $g = G|_{S^{n-1}}$, by the previous proposition, we have $\deg(g) = 0$. However, define $H : S^{n-1} \times [0,1] \to S^{n-1}$ by

$$H(x,t) = \frac{x - tf(x)}{|x - tf(x)|}. \tag{12.7}$$

Clearly, the denominator never vanishes, so H is a smooth homotopy between g and the identity map. Since $\deg(Id) = 1$, this is a contradiction. □

12.2 Real projective spaces

Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}), \;\; c \neq 0. \tag{12.8}$$

Since every line through the origin hits the unit sphere in exactly two points, we can desribe $\mathbb{RP}^n$ as a quotient space. That is,

$$\mathbb{RP}^n = S^n/\mathbb{Z}_2, \tag{12.9}$$

where $\mathbb{Z}_2$ acts by $p \mapsto A(p) = -p$. Let $\pi : S^n \to \mathbb{RP}^n$ denote the projection mapping.

Proposition 12.7. $\mathbb{RP}^n$ *is orientable if n is odd, and non-orientable if n is even.*

Proof. Above, we saw that

$$\omega = \left(x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \dots \wedge dx^{n+1}) \tag{12.10}$$

is a nowhere-vanishing n-form on $S^n \subset \mathbb{R}^{n+1}$. Clearly,

$$A^*\omega = (-1)^{n+1}\omega. \tag{12.11}$$

So if n is odd, ω is invariant under the above $\mathbb{Z}_2$ action and thus descends to be nowhere-zero n-form on $\mathbb{RP}^n$.

In n is even, then $A^*\omega = -\omega$. This says that A is orientation-reversing. If $\mathbb{RP}^n$ were orientable, then it would have a non-zero n-form $\omega \in \Omega^n(\mathbb{RP}^n)$, and the pull back form $\pi^*\omega$ would be a non-zero n-form on S^n which is invariant under A:

$$A^*\pi^*\omega = (\pi \circ A)^*\omega = \pi^*\omega, \tag{12.12}$$

since $\pi \circ A = \pi$. This says that A is orientation-preserving, which is a contradiction. □

We next compute the de Rham cohomology of $\mathbb{RP}^n$.

Theorem 12.8. *We have*

$$H^k_{dR}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k < n \\ \mathbb{R} & k = n \text{ odd} \\ 0 & k = n \text{ even} \end{cases} \tag{12.13}$$

Proof. Since $A : S^n \to S^n$ satisfies $A^2 = Id_{S^n}$, for each $0 \leq k \leq n$, we have that

$$\Omega^k(S^n) = \Omega^k_+(S^n) \oplus \Omega^k_-(S^n) \tag{12.14}$$

where

$$\Omega^k_\pm(S^n) = \{\omega \in \Omega^k(S^n) \mid A^*\omega = \pm\omega\}, \tag{12.15}$$

because we can write

$$\omega = \frac{1}{2}(\omega + A^*\omega) + \frac{1}{2}(\omega - A^*\omega) \tag{12.16}$$

We claim that

$$\pi^* : \Omega^k(\mathbb{RP}^n) \to \Omega^k_+(S^n) \subset \Omega^k(S^n), \tag{12.17}$$

and is an isomorphism. Just as above $\pi \circ A = \pi$ implies that $A^*\pi^*\omega = \pi^*\omega$, so clearly the image of the pull-back lies in the space of invariant forms. Next, we need to show that if $\omega \in \Omega^k_+(S^n)$, then ω is the pull-back of a form $\alpha \in \Omega^k(\mathbb{RP}^n)$. That is, if $A^*\omega = \omega$, then for $p \in S^n$, and $X_1, \dots, X_k \in T_pS^n$,

$$\omega_p(X_1, \dots, X_k) = (\pi^*\alpha)_p(X_1, \dots, X_k) = \alpha_{\pi(p)}(\pi_*X_1, \dots, \pi_*X_k). \tag{12.18}$$

Let us use this equation to define $\alpha_{\pi(p)}$. We need to prove this is well-defined. Given $[p] \in \mathbb{RP}^n$, there are exactly 2 preimages p and $A(p) = -p$. The mappings $(\pi_*)_p : T_pS^n \to$

$T_{[p]}\mathbb{RP}^n$, $(\pi_*)_{A(p)} : T_{A(p)}S^n \to T_{[p]}\mathbb{RP}^n$, and $(A_*)_p : T_pS^n \to T_{A(p)}S^n$ are isomorphisms. Given $[p] \in \mathbb{RP}^n$ and $Y_1, \dots Y_k \in T_{[p]}\mathbb{RP}^n$, there are exactly 2 choices:

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_p((\pi_*)_p^{-1}Y_1, \dots, (\pi_*)_p^{-1}Y_k), \tag{12.19}$$

or

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A(p)}((\pi_*)_{A(p)}^{-1}Y_1, \dots, (\pi_*)_{A(p)}^{-1}Y_k). \tag{12.20}$$

Since $\pi \circ A = \pi$, we have

$$(\pi_*)_{A(p)}(A_*)_p = (\pi_*)_p. \tag{12.21}$$

Since all of the mappings are isomorphisms, this implies that

$$(\pi_*)_{A(p)}^{-1} = (A_*)_p(\pi_*)_p^{-1}, \tag{12.22}$$

so (12.20) can be rewritten as

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A(p)}((A_*)_p(\pi_*)_p^{-1}Y_1, \dots, (A_*)_p(\pi_*)_p^{-1}Y_k). \tag{12.23}$$

The condition that ω is invariant under A, $A^*\omega = \omega$ says that

$$(A^*\omega)_p(X_1, \dots, X_k) = \omega_{A(p)}(A_*X_1, \dots, A_*X_k) \tag{12.24}$$

Choosing $X_i = (\pi_*)_p^{-1}Y_i$, we see that (12.19) = (12.20), therefore α is well-defined.

We next note that if $A^*\omega = \omega$ then

$$A^*d\omega = dA^*\omega = d\omega, \tag{12.25}$$

so the exterior derivative maps

$$d : \Omega_+^k(S^n) \to \Omega_+^{k+1}(S^n). \tag{12.26}$$

We therefore have the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(\mathbb{RP}^n) & \xrightarrow{d} & \Omega^k(\mathbb{RP}^n) & \xrightarrow{d} & \Omega^{k+1}(\mathbb{RP}^n) & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\ \cdots & \xrightarrow{d} & \Omega_+^{k-1}(S^n) & \xrightarrow{d} & \Omega_+^k(S^n) & \xrightarrow{d} & \Omega_+^{k+1}(S^n) & \xrightarrow{d} & \cdots \end{array} \tag{12.27}$$

Since π^* is an isomorphism, we have

$$H_{dR}^k(\mathbb{RP}^n) = H^k(\Omega_+^*(S^n)) \tag{12.28}$$

We next note that if $A^*\omega = -\omega$ then

$$A^*d\omega = dA^*\omega = -d\omega, \tag{12.29}$$

so the exterior derivative maps

$$d : \Omega^k_-(S^n) \to \Omega^{k+1}_-(S^n). \tag{12.30}$$

We can also decompose the de Rham complex on S^n by

$$\Omega^{k-1}_+(S^n) \oplus \Omega^{k-1}_-(S^n) \xrightarrow{d \oplus d} \Omega^k_+(S^n) \oplus \Omega^k_-(S^n) \xrightarrow{d \oplus d} \Omega^{k+1}_+(S^n) \oplus \Omega^{k+1}_-(S^n). \tag{12.31}$$

This implies that

$$H^k_{dR}(S^n) = H^k(\Omega^*_+(S^n)) \oplus H^k(\Omega^*_-(S^n)). \tag{12.32}$$

Next, note that since A is a diffeomorphism satisfying $A^2 = Id_{S^n}$, we have that A induces a mapping on cohomology

$$A^* : H^k_{dR}(S^n) \to H^k_{dR}(S^n), \tag{12.33}$$

which also satisfies $(A^*)^2 = Id_{H^k(S^n)}$. Consequently, we can decompose

$$H^k_{dR}(S^n) = H^k_+(S^n) \oplus H^k_-(S^n), \tag{12.34}$$

where $H^k_+(S^n), H^k_-(S^n)$ are the invariant and anti-invariant cohomology classes, respectively. Equivalently, these are the $+1$ and -1 eigenspaces of A^*. We next claim that

$$H^k(\Omega^*_\pm(S^n)) = H^k_\pm(S^n). \tag{12.35}$$

This follows because we have two decompositions

$$\begin{aligned} H^k_{dR}(S^n) &= H^k(\Omega^*_+(S^n)) \oplus H^k(\Omega^*_-(S^n)) \\ &= H^k_+(S^n) \oplus H^k_-(S^n), \end{aligned} \tag{12.36}$$

the first factors are the $+1$ eigenspace, and the second factors are the -1 eigenspace, so they must be equal.

To finish the proof, we clearly have $H^k_{dR}(\mathbb{RP}^n) = \{0\}$ for $0 < k < n$. For $k = n$, we know that $A^* = (-1)^{n+1}$ acting on $H^n_{dR}(S^n)$, so we have that $H^n_{dR}(\mathbb{RP}^n) = \mathbb{R}$ if n is odd, and $H^n_{dR}(\mathbb{RP}^n) = \{0\}$ if n is even.

□

13 Lecture 13

13.1 Finite group quotients

Let M be a smooth manifold, and Γ be a finite group acting freely on M. That is, we have is a smooth mapping

$$A : \Gamma \times M \to M \tag{13.1}$$

satisfying

$$A(g_1 g_2, p) = A(g_1, A(g_2, p)) \tag{13.2}$$

and $A(e, p) = p$ for all $p \in M$, where e is the identity element of Γ. The action A is *free* if $A(g, p) = p$ for some $p \in M$ implies that $g = e$. Let $A_g : M \to M$ denote the diffeomorphism $A_g(p) = A(g, p)$.

Recall that the quotient space M/Γ is a manifold. Furthermore, $\pi : M \to M/\Gamma$ is a covering space of order $|\Gamma|$ with deck transformation group Γ.

Definition 13.1. The space of invariant k-forms

$$\Omega^k_+(M) = \{\omega \in \Omega^k(M) \mid A_g^* \omega = \omega \text{ for all } g \in \Gamma\}. \tag{13.3}$$

Proposition 13.2. *The mapping $\pi^* : \Omega^k(M/\Gamma) \to \Omega^k_+(M)$ is an isomorphism.*

Proof. For each $g \in \Gamma$, we have $\pi \circ A_g = \pi$ which implies that $A^* \pi^* \omega = \pi^* \omega$, so clearly the image of the pull-back lies in the space of invariant forms. Next, we need to show that if $\omega \in \Omega^k_+(M)$, then ω is the pull-back of a form $\alpha \in \Omega^k(M/\Gamma)$. That is, if $A_g^* \omega = \omega$ for all $g \in \Gamma$, then for $p \in M$, and $X_1, \dots, X_k \in T_p M$,

$$\omega_p(X_1, \dots, X_k) = (\pi^* \alpha)_p(X_1, \dots, X_k) = \alpha_{\pi(p)}(\pi_* X_1, \dots, \pi_* X_k). \tag{13.4}$$

Let p be any preimage of $[p]$ under the projection π. The mapping $(\pi_*)_p : T_p M \to T_{[p]}(M/\Gamma)$ is an isomorphism. Given $Y_1, \dots Y_k \in T_{[p]}(M/\Gamma)$, we define

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_p((\pi_*)_p^{-1} Y_1, \dots, (\pi_*)_p^{-1} Y_k). \tag{13.5}$$

We need to show this is well-defined. Let $\tilde{p}$ be any other preimage. Then there exists $g \in \Gamma$ such that $\tilde{p} = A_g p$. Using $A_g p$ instead of p in the definition yields

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A_g(p)}((\pi_*)^{-1}_{A_g(p)} Y_1, \dots, (\pi_*)^{-1}_{A_g(p)} Y_k). \tag{13.6}$$

Since $\pi \circ A_g = \pi$, we have

$$(\pi_*)_{A_g(p)}((A_g)_*)_p = (\pi_*)_p. \tag{13.7}$$

Since all of these mappings are isomorphisms, this implies that

$$(\pi_*)^{-1}_{A_g(p)} = ((A_g)_*)_p (\pi_*)_p^{-1}, \tag{13.8}$$

so (13.6) can be rewritten as

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A_g(p)}(((A_g)_*)_p (\pi_*)_p^{-1} Y_1, \dots, (A_*)_p (\pi_*)_p^{-1} Y_k). \tag{13.9}$$

The condition that ω is invariant under A_g, $A_g^* \omega = \omega$ says that

$$\omega_p(X_1, \dots, X_k) = (A_g^* \omega)_p(X_1, \dots, X_k) = \omega_{A_g(p)}((A_g)_* X_1, \dots, (A_g)_* X_k). \tag{13.10}$$

Choosing $X_i = (\pi_*)_p^{-1} Y_i$, we see that (13.5) = (13.6), therefore α is well-defined. □

Proposition 13.3. *We have*

$$H^k_{dR}(M/\Gamma) = H^k(\Omega^*_+(M)) \tag{13.11}$$

Proof. If $A^*\omega = \omega$ then

$$A^* d\omega = dA^*\omega = d\omega, \tag{13.12}$$

so the exterior derivative maps

$$d : \Omega^k_+(M) \to \Omega^{k+1}_+(M). \tag{13.13}$$

We therefore have the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(M/\Gamma) & \xrightarrow{d} & \Omega^k(M/\Gamma) & \xrightarrow{d} & \Omega^{k+1}(M/\Gamma) & \xrightarrow{d} & \cdots \\ & & \downarrow{\pi^*} & & \downarrow{\pi^*} & & \downarrow{\pi^*} & & \\ \cdots & \xrightarrow{d} & \Omega^{k-1}_+(M) & \xrightarrow{d} & \Omega^k_+(M) & \xrightarrow{d} & \Omega^{k+1}_+(M) & \xrightarrow{d} & \cdots \end{array} \tag{13.14}$$

Since π^* is an isomorphism, this finishes the proof. □

Next, we have the following

Proposition 13.4. *The induced mapping*

$$\pi^* : H^k_{dR}(M/\Gamma) \to H^k_{dR}(M) \tag{13.15}$$

is injective.

Proof. We have that $\Omega^*_+(M) \subset \Omega^*(M)$ is a subcocomplex. This induces a mapping

$$H^k(\Omega^*_+(M)) \to H^k_{dR}(M) \tag{13.16}$$

by the following. Take an equivalence class $[\omega] \in H^k(\Omega^*_+(M))$ represented by $\omega \in \Omega^k_+(M)$, and map this to the cohomology class $[\omega] \in H^k_{dR}(M)$. This is well-defined, since if $\omega = d\alpha$ where $\alpha \in \Omega^{k-1}_+(M)$ then obviously $[\omega] = 0$ in $H^k_{dR}(M)$ also.

By the previous proposition, we just need to show that the mapping (13.16) is an injection. For this, we need to show that if $\omega \in \Omega^k_+(M)$ satisfies $\omega = d\alpha$ for $\alpha \in \Omega^{k-1}(M)$, then $\omega = d\beta$, where $\beta \in \Omega^{k-1}_+(M)$. For this, simply define

$$\beta = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} A^*_g \alpha. \tag{13.17}$$

For any $g' \in G$, this satisfies

$$A^*_{g'}\beta = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} A^*_{g'} A^*_g \alpha = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} A^*_{gg'} \alpha = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} A^*_g \alpha = \beta \tag{13.18}$$

so $\beta \in \Omega^{k-1}_+(M)$. Then

$$d\beta = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} dA^*_g \alpha = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} A^*_g d\alpha = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} d\alpha = d\alpha. \tag{13.19}$$

□

Definition 13.5. The kth Betti number of M is

$$b^k(M) = \dim(H^k_{dR}(M)). \tag{13.20}$$

We can phrase the above result as follows.

Theorem 13.6. *If $\pi : M \to M/\Gamma$ is as above, then*

$$b^k(M) \geq b^k(M/\Gamma). \tag{13.21}$$

Example 13.7. The example of $\pi : S^2 \to \mathbb{RP}^2$ shows that (13.21) can be strict.

Example 13.8 (Lens spaces)**.** Choose relatively prime integers $1 \leq q < p$. Consider $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}. \tag{13.22}$$

Let $\Gamma = \mathbb{Z}/p\mathbb{Z}$ act on S^3 generated by

$$(z_1, z_2) \mapsto (\zeta_p z_1, \zeta_p^q z_2), \tag{13.23}$$

where ζ_p is a primitive pth root of unity. It is easy to see this is a free action, so $S^3/\Gamma \equiv L(p, q)$ is a smooth 3-manifold. The inequalities (13.21) show that $b^k(S^3/\Gamma) = b^k(S^3)$, so the de Rham theory is unable to distinguish these spaces. However, since $\pi_1(S^3/\Gamma) = \Gamma$, if $p \neq p'$ then $L(q, p)$ cannot be homeomorphic to $L(q', p')$, so the fundamental group *can* distinguish these. An interesting question is: when exactly are $L(q, p)$ and $L(q', p)$ diffeomorphic?

13.2 Triangulations of smooth manifolds

Define the standard p-simplex to be

$$\Delta^p = \Big\{(t_0, \ldots, t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\Big\}. \tag{13.24}$$

For $0 \leq i \leq p$, the ith face of Δ^p is the $(p-1)$-simplex

$$\Delta^p_i : \Delta^{p-1} \to \Delta^p \tag{13.25}$$

defined by

$$(t_0, \ldots, t_{p-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots t_{p-1}). \tag{13.26}$$

More generally, for $k < p - 1$, a k-face of Δ^p is a simplex obtained from Δ_p obtained by setting $p - k$ of the coordinates equal to 0.

Definition 13.9. If M is a smooth compact n-dimensional manifold, a triangulation of M is collection of diffeomorphisms

$$c^n_i : \Delta^n \to M \tag{13.27}$$

for $i = 1 \ldots N$ whose images cover M and such that if

$$c^n_i(\Delta^p) \cap c^n_j(\Delta^p) \neq \emptyset, \tag{13.28}$$

for $i \neq j$, then the intersection is exactly a k-face of both simplices for $0 < k \leq p - 1$.

We will refer to image of c_i^n as an n-simplex of the triangulation, and the image of any k-face of a simplex will be called a k-simplex of the triangulation.

Remark 13.10. Using his embedding theorem, Whitney proved that every smooth manifold M admits a triangulation. The basic idea is to embed M into $\mathbb{R}^n$. Taking a very fine cubical lattice in general position, he constructs a simpicial complex in a tubular neighborhood of M which projects to a triangulation on M. We will not give details of this, but refer the interested student to Cairns' 1961 paper "A simple triangulation method for smooth manifolds", which is just 2 pages!

14 Lecture 14

14.1 Euler's polyhedral formula

If M^n is a smooth compact n-dimensional manifold with a triangulation, then let α_k be the number of k-simplices in a triangulation. For surfaces, these are also notated as $V = \alpha_0, E = \alpha_1, F = \alpha_2$ since V is the number of vertices, E is the number of edges, and F is the number of faces in the triangulation. We also define Euler characteristic

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b^i(M), \tag{14.1}$$

where $b^i(M) = \dim(H^i(M))$ is called the ith Betti number of M.

Theorem 14.1. *If a smooth compact surface M admits a triangulation, then*

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2 = V - E + F. \tag{14.2}$$

Consequently, the sum $V - E + F$ is independent of the triangulation.

Proof. Clearly, we can assume that M is connected. Let U be the union of small balls around the barycenters (center of mass) of the 2-simplices p_i. Let $V_1 = M \setminus \cup_{i=1}^{\alpha_2}\{p_i\}$. Then $U \cap V$ is homotopic to the disjoint union of α_2 copies of S^1.

Applying the Mayer-Vietoris sequence to U and V yields that

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}^{\alpha_2} \oplus H^0_{dR}(V_1) & \longrightarrow & \mathbb{R}^{\alpha_2} \\ & \hookrightarrow & H^1_{dR}(M) & \longrightarrow & H^1_{dR}(V_1) & \longrightarrow & \mathbb{R}^{\alpha_2} \\ & \hookrightarrow & H^2_{dR}(M) & \longrightarrow & H^2_{dR}(V_1) & \longrightarrow & 0. \end{array} \tag{14.3}$$

Lemma 9.4 implies that

$$0 = 1 - \alpha_2 - b^0(V_1) + \alpha_2 - b^1(M) + b^1(V_1) - \alpha_2 + b^2(M) - b^2(V_1), \tag{14.4}$$

or

$$\chi(M) = \chi(V_1) + \alpha_2. \tag{14.5}$$

We next apply the Mayer-Vietoris sequence on V_1, with a new U and V. For this, if 2 2-simplices intersect along a 1-face, then we can connect the barycenters by a curve which intersects the 1-face in the barycenter of the 1-face. Then let U be the disjoint union of sets diffeomorphic to balls which are slight "fattenings" of slight shrinkings of the curves (so that they are disjoint near the endpoints). Let V_0 be the complement in V_1 of the union of the curves joining the barycenters of the faces. Then $V_1 = U \cup V_0$, U is the union of α_1 balls, and the set V_0 deformation retracts onto the set of 0-faces. Also, the intersection $U \cap V_0$ consists of $2\alpha_1$ sets diffeomorphic to balls, since each curve cuts the fattenings into 2 pieces, so we have the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^{\alpha_1} \oplus \mathbb{R}^{\alpha_0} \longrightarrow \mathbb{R}^{2\alpha_1} \longrightarrow H^1_{dR}(V_1) \longrightarrow 0 \tag{14.6}$$

Lemma 9.4 implies that

$$\chi(V_1) = \alpha_0 - \alpha_1. \tag{14.7}$$

Combining with (14.5), we have

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2. \tag{14.8}$$

□

Remark 14.2. In dimension n, it is true that

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \alpha_k, \tag{14.9}$$

but we leave the details as an exercise for the interested student. (The above proof extends to the higher-dimensional case with only a litte extra work; see [Spi79, Theorem 11.5].)

Corollary 14.3. *If a finite group Γ acts freely on a compact manifold M, then*

$$\chi(M) = |\Gamma| \cdot \chi(M/\Gamma). \tag{14.10}$$

In particular, if $\pi : \tilde{M} \to M$ is a d-fold covering space, then

$$\chi(\tilde{M}) = d \cdot \chi(M). \tag{14.11}$$

Proof. If M/Γ has a triangulation with α_k k-simplices, then we can pull-back the triangulation to M, which is a triangulation with $\tilde{\alpha}_k = |\Gamma|\alpha_k$ k-simplices. □

Remark 14.4. This is a special case of a much more general formula. If $\pi : E \to B$ is any fiber bundle with fiber F then $\chi(E) = \chi(F) \cdot \chi(B)$. This follows from the Serre spectral sequence; see also [BT82, Exercise 14.37].

14.2 Compactly supported cohomology

If M is noncompact, the mapping $\pi : M \to M/\Gamma$ is proper. Therefore we have

$$\pi^* : \Omega^k_c(M/\Gamma) \to \Omega^k_{c,+}(M). \tag{14.12}$$

The above arguments holds verbatim for compactly supported cohomology, so we have:

Proposition 14.5. *The mapping $\pi^* : \Omega^k_c(M/\Gamma) \to \Omega^k_{c,+}(M)$ is an isomorphism, and*

$$H^k_{c,dR}(M/\Gamma) = H^k(\Omega^*_{c,+}(M)). \tag{14.13}$$

Furthermore, the induced mapping

$$\pi^* : H^k_{c,dR}(M/\Gamma) \to H^k_{c,dR}(M) \tag{14.14}$$

is injective. In other words, ,

$$b^k_c(M) \equiv \dim_{\mathbb{R}} H^k_c(M) \geq \dim_{\mathbb{R}} H^k_c(M/\Gamma) = b^k_c(M/\Gamma). \tag{14.15}$$

As an application, we can give another proof of the following.

Theorem 14.6. *If M is a smooth manifold of dimension n which is non-orientable and connected then $H^n_{c,dR}(M) = \{0\}$.*

Proof. Recall the construction of the orientable double cover $\pi : \tilde{M} \to M$: the bundle $\Lambda^n(M)$ is a real line bundle. Endow this bundle with a Riemannian metric, and then $\tilde{M}$ is the unit sphere bundle. Since M is non-orientable, $\tilde{M}$ is connected. The mapping $A : \omega_p \mapsto -\omega_p$ is clearly a free $\mathbb{Z}_2$-action on $\tilde{M}$, and $M = \tilde{M}/\mathbb{Z}_2$.

We claim that $\tilde{M}$ is orientable. Given $p \in M$, there are precisely 2 preimages $\tilde{p}$ and $A\tilde{p}$ under π. The point $\tilde{p} = \omega_p$ is, by definition, a non-zero n-form on T_pM, so determines an orientation on T_pM. The mapping $\pi_* : T_{\tilde{p}}\tilde{M} \to T_pM$ is an isomorphism, so we give $T_{\tilde{p}}\tilde{M}$ the induced orientation. Similarly, we give $T_{A\tilde{p}}$ the induced orientation. This clearly gives a smooth orientation on $\tilde{M}$, called the *tautological* orientation. Note that the mapping A is orientation-reversing (otherwise, the quotient space would also be orientable).

For compactly supported cohomology, we have

$$\pi^* : H^k_{c,dR}(M/\Gamma) \to H^k_{c,dR}(M) \tag{14.16}$$

is injective. Given $\omega \in \Omega^n_c(M)$, we have $\tilde{\omega} = \pi^*\omega$ satisfies $A^*\tilde{\omega} = \tilde{\omega}$. But

$$\int_{\tilde{M}} \tilde{\omega} = -\int_{\tilde{M}} A^*\tilde{\omega} = -\int_{\tilde{M}} \tilde{\omega}, \tag{14.17}$$

since A is orientation-reversing. Consequently,

$$\int_{\tilde{M}} \tilde{\omega} = 0. \tag{14.18}$$

By the first part of the proof of Theorem 11.1, this implies that $[\tilde{\omega}] = 0 \in H^n_{c,dR}(\tilde{M})$. But since π^* is injective, this implies that $[\omega] = 0 \in H^n_{c,dR}(M)$. ☐

15 Lecture 15

15.1 Mayer-Vietoris for cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega^{p-1}_c(M) \xrightarrow{d} \Omega^p_c(M) \xrightarrow{d} \Omega^{p+1}_c(M) \xrightarrow{d} \cdots, \tag{15.1}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Write $M = U \cup V$ as the union of two open sets in M. Note that if $U_1 \subset U_2$ and $\omega \in \Omega^k_c(U_1)$ then ω extends to be a compactly supported form in U_2. Letting $\iota : U_1 \hookrightarrow U_2$ denote the inclusion mapping, we denote by $i_*\omega$ this extension map on forms. We claim that the following sequence is exact:

$$0 \longrightarrow \Omega^p_c(U \cap V) \xrightarrow{\tilde{\alpha}^p} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{\tilde{\beta}^p} \Omega^p_c(U \cup V) \longrightarrow 0 \tag{15.2}$$

where

$$\tilde{\alpha}^p(\omega_{U\cap V}) = \big((i_{U\cap V\hookrightarrow U})_*\omega_{U\cap V}, -(i_{U\cap V\hookrightarrow V})_*\omega_{U\cap V}\big) \tag{15.3}$$

and

$$\tilde{\beta}^p(\omega_U, \omega_V) = (i_{U\hookrightarrow M})_*\omega_U + (i_{V\hookrightarrow M})_*\omega_V. \tag{15.4}$$

To see this, $\tilde{\alpha}^p$ is obviously injective. For exactness at the middle step, obviously $\tilde{\beta}^p\tilde{\alpha}^p\omega = 0$. If $\tilde{\beta}^p(\omega_U, \omega_V) = 0$, then $\omega_U = -\omega_V$. This implies that the support of both forms is contained in $U \cap V$, and since they are equal there, take $\omega_{U\cap V} = \omega_U$, and then $(\omega_U, \omega_V) = \tilde{\alpha}^p(\omega_U)$.

To show that $\tilde{\beta}$ is onto, let $\omega \in \Omega^p_c(M)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$.

Consequently, from the ziz-zag Lemma, we obtain a long exact sequence

$$\cdots \xrightarrow{\tilde{\delta}^{p-1}} H^p_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} \cdots \tag{15.5}$$

Let us review the definition of the mapping $\tilde{\delta}^p$. Given a cohomology class $[\omega] \in H^p_{c,dR}(U \cup V)$, represented by $\omega \in \Omega^p_c(U \cup V)$ with $d\omega = 0$, we first write $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$, then we apply the exterior derivative to get

$$(d(\phi_U\omega), d(\phi_V\omega)) = (d\phi_U \wedge \omega, d\phi_V \wedge \omega) \in \Omega^p_c(U) \oplus \Omega^p_c(V) \tag{15.6}$$

Either of these elements is supported in $U \cap V$ and then since $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$,

$$\tilde{\delta}^p\omega = [d\phi_U \wedge \omega] = [-d\phi_V \wedge \omega] \in H^{p+1}_{c,dR}(U \cap V). \tag{15.7}$$

Remark 15.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

15.2 Good covers

Recall the Poincaré lemma for compactly supported cohomology, Lemma 10.1, showed that

$$H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{15.8}$$

Remark 15.2. This shows that $H^*_{c,dR}(M)$ is not a homotopy invariant, since (10.26) is not the same as the cohomology of a point. But of course, $H^*_{c,dR}(M)$ *is* a diffeomorphism invariant.

We have the following definition.

Definition 15.3. We say that a manifold M has a good cover U_i each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has the same de Rham cohomology as $\mathbb{R}^n$, and the same compactly supported de Rham cohomology as $\mathbb{R}^n$.

Recall we proved ealier that if M has a finite good cover, then the de Rham cohomology is finite-dimensional. We next extend this to compactly supported cohomology.

Corollary 15.4. *If M has a finite good cover, then the compactly supported de Rham cohomology is finite-dimensional.*

Proof. Recall that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{15.9}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{15.10}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

We prove the corollary using induction on the number of open sets in a finite good cover. To see this, let k be the number of sets in a good cover. For $k = 1$, we know the corollary is true. Assume the corollary is true up to k, and let $\{U_1, \dots, U_{k+1}\}$ be a good cover of a manifold M. Let $U = U_1 \cup \cdots \cup U_k$, and let $V = U_{k+1}$. Then U and V have good covers with fewer that $k+1$ open sets, so their compactly supported de Rham cohomology is finite-dimensional. Also, $U_1 \cap U_{k+1}, \dots, U_k \cap U_{k+1}$ is a good cover of $U \cap V$, so the theorem is true for $U \cap V$ as well.

Now we look at the following portion of the compactly supported Mayer-Vietoris sequence

$$\cdots \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} H^{p+1}_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^{p+1}} \cdots \tag{15.11}$$

The above observation then implies that $H^p_{c,dR}(U \cup V)$ is finite-dimensional. ☐

Next, we need the following technical lemma.

Lemma 15.5. *If U is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{c,dR}(U) \cong H^k_{c,dR}(\mathbb{R}^n)$ for all $0 \leq k \leq n$. Furthermore, an isomorphism of $H^n_{c,dR}(U)$ and $\mathbb{R}$ is given by integration.*

Proof. The proof is not too difficult, but we only give an outline. The main idea is to first show it is true for a star-shaped open set U whose boundary is a smooth graph over a small sphere around the star point; it is easy to show such a set is diffeomorphic to the unit ball in $\mathbb{R}^n$ by a diffeomorphism which is maps lines through the star point to lines through the origin. Then one shows that an arbitary star-shaped open set U can be approximated from the inside by star-shaped open sets U_i with smooth boundary such that $U_i \subset U_{i+1}$ and $U = \cup_i U_i$. Then if $\omega \in \Omega^k(U)$ has compact support K, then there exists i such that $K \subset U_i$, which reduces to the case with smooth boundary.

For $k = n$, we already determined the top degree compactly supported cohomology of any oriented manifold; the isomorphism with $\mathbb{R}$ is given by integration; see Theorem 11.1. □

Remark 15.6. It is actually true that U is diffeomorphic to $\mathbb{R}^n$, but this is more difficult to show, and we do not need such a strong result.

Corollary 15.7. *If M is compact, then M admits a finite good cover.*

Proof. Using a Riemannian metric, there exists a covering of M by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map at any point, a geodesically convex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$. This is contractible, so from the Poincaré Lemma, it has the same de Rham cohomology as $\mathbb{R}^n$. Lemma 15.5 tells us that it also has the same compactly supported de Rham cohomology as $\mathbb{R}^n$, so we are done. □

Remark 15.8. For those of you not familiar with Riemannian geometry, it is also possible to construct a good cover using a triangulation using the "open stars" of a triangulation, but we won't go into this.

16 Lecture 16

16.1 Some homological algebra

We next discuss some homological algebra, which we will use next time to prove Poincaré duality and the Künneth formula.

Lemma 16.1 (The five lemma)**.** *Assume the diagram*

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{\alpha_1} & V_2 & \xrightarrow{\alpha_2} & V_3 & \xrightarrow{\alpha_3} & V_4 & \xrightarrow{\alpha_4} & V_5 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\ W_1 & \xrightarrow{\beta_1} & W_2 & \xrightarrow{\beta_2} & W_3 & \xrightarrow{\beta_3} & W_4 & \xrightarrow{\beta_4} & W_5 \end{array} \tag{16.1}$$

commutes, and has exact rows. If $\phi_1, \phi_2, \phi_4, \phi_5$ are isomorphisms, then ϕ_3 is also an isomorphism.

Proof. Injectivity of ϕ_3: If $\phi_3(v_3) = 0$, then $\beta_3(\phi_3(v_3) = 0 = \phi_4\alpha_3(v_3)$. Since ϕ_4 is injective, $\alpha_3(v_3) = 0$. By exactness, $v_3 = \alpha_2(v_2)$. Then $\phi_3\alpha_2(v_2) = 0 = \beta_2\phi_2(v_2)$. By exactness, $\phi_2(v_2) = \beta_1(w_1)$. By surjectivity of ϕ_1, $w_1 = \phi_1(v_1)$. Then

$$\phi_2(v_2) = \beta_1\phi_1(v_1) = \phi_2\alpha_1(v_1), \tag{16.2}$$

but since ϕ_2 is injective, this implies that $v_2 = \alpha_1(v_1)$. Finally, $v_2 = \alpha_2(v_2) = \alpha_2\alpha_1(v_1) = 0$, by exactness.

The proof of surjectivity is similar, and left to the student. □

Exercise 16.2. Prove the surjectivity of ϕ_3. Also show that for the conclusion of the five lemma to hold, the assumptions can be weakened to ϕ_1 is surjective, and ϕ_5 is injective.

Another useful lemma is the following.

Lemma 16.3. *If the sequence*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{16.3}$$

is exact at W_2*, then the dual sequence*

$$W_3^* \xrightarrow{\beta^*} W_2^* \xrightarrow{\alpha^*} W_1^* \tag{16.4}$$

is exact at W_2^**.*

Proof. First, if $w_3^* \in W_3^*$, and $w_1 \in W_1$, then

$$\alpha^*(\beta^* w_3^*)(w_1) = (\beta^* w_3^*)(\alpha(w_1)) = w_3^*(\beta\alpha(w_1)) = 0, \tag{16.5}$$

since $\beta \circ \alpha = 0$ by assumption. This proves that $Im(\beta^*) \subset Ker(\alpha^*)$. For the other direction, if $w_2^* \in Ker(\alpha^*)$, then for all $w_1 \in W_1$, $\alpha^*(w_2^*)(w_1) = w_2^*(\alpha(w_1))$. So the element $0 = w_2^* \circ \alpha \in W_1^*$. We want to find $w_3^* \in W_3^*$ such that $w_2^* = \beta^* w_3^*$. For all $w_2 \in W_2$, this is $w_2^*(w_2) = w_3^*\beta w_2$, which is just $w_2^* = w_3^* \circ \beta$. So if $w_3 \in W_3$ is of the form $\beta(w_2)$ then define

$$w_3^*(w_3) \equiv w_2^*(w_2). \tag{16.6}$$

If $w_3 = \beta(w_2')$, then $\beta(w_2 - w_2') = 0$, so $w_2 - w_2' = \alpha(w_1)$. Then

$$w_2^*(w_2 - w_2') = w_2^*(\alpha(w_1)) = (w_2^*\alpha)(w_1) = 0. \tag{16.7}$$

So we have defined w_3^* on the subspace $Im(\beta) \subset W_3$. To extend to a linear mapping on all of W_3, just take any subspace so that $W_3 = Im(\beta) \oplus W$, and define w_3^* to vanish on W. Then the condition $w_2^* = w_3^* \circ \beta$ is obviously satisifed. □

Remark 16.4. The previous lemma holds in the category of vector spaces, but fails in general for modules over a ring R. If

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0 \tag{16.8}$$

is a short exact sequence of R-modules, then there is a long exact sequence

$$0 \longrightarrow Hom_R(M_3, R) \longrightarrow Hom_R(M_2, R) \longrightarrow Hom_R(M_1, R) \longrightarrow Ext^1_R(M_3, R) \longrightarrow \cdots \tag{16.9}$$

That is, the dualization functor is only left exact, and the $Ext^j_R(\cdot, R)$ are the corresponding right derived functors.

Next, a lemma about the dual of a direct sum.

Lemma 16.5. *Let B and C be vector spaces. Then*

$$(B \oplus C)^* \cong B^* \oplus C^* \tag{16.10}$$

Proof. Let $\iota_B : B \to B \oplus C$ and $\iota_C : C \to B \oplus C$ denote the inclusion mappings. Define $f : (B \oplus C)^* \to B^* \oplus C^*$ by

$$f(m^*) = (\iota_B^* m^*, \iota_C^* m^*). \tag{16.11}$$

Define $g : B^* \oplus C^* \to (B \oplus C)^*$ by

$$g(b^*, c^*)(b, c) = b^*(b) + c^*(c). \tag{16.12}$$

Then

$$\pi_{B^*}(f \circ g)(b^*, c^*)(b) = \iota_B^* g(b^*, c^*)(b) = g(b^*, c^*)(b, 0) = b^*(b). \tag{16.13}$$

Similarly,

$$\pi_{C^*}(f \circ g)(b^*, c^*)(c) = \iota_C^* g(b^*, c^*)(c) = g(b^*, c^*)(0, c) = c^*(c). \tag{16.14}$$

This implies that $f \circ g = Id_{B^* \oplus C^*}$. Next,

$$\begin{aligned} g \circ f(m^*)(b, c) &= g(\iota_B^* m^*, \iota_C^* m^*)(b, c) = \iota_B^* m^*(b) + \iota_C^* m^*(c) \\ &= m^*(\iota_B(b)) + m^*(\iota_C(c)) = m^*(b, 0) + m^*(0, c) = m^*(b, c), \end{aligned} \tag{16.15}$$

so $g \circ f = Id_{(B \oplus C)^*}$. □

Next, a lemma about the dual of a mapping into a direct sum.

Lemma 16.6. *Let $f : A \to B \oplus C$ be a linear map between finite dimensional vector spaces. Write $f = (f_B, f_C)$, where $f_B : A \to B$ and $f_C : A \to C$. Then $f^* : (B \oplus C)^* \to A^*$ is given by*

$$f^*(m^*)(a) = (\iota_B^* m^*)(f_B(a)) + (\iota_C^* m^*)(f_C(a)), \tag{16.16}$$

for $m^* \in (B \oplus C)^*$, where $\iota_B : B \to B \oplus C$ and $\iota_C : C \to B \oplus C$ are the inclusion mappings. Consequently, if we use the the isomorphism $(B \oplus C)^* \cong B^* \oplus C^*$ from the previous lemma, then $f^* : B^* \oplus C^* \to A^*$ is given by

$$f^*(b^*, c^*)(a) = b^*(f_B(a)) + c^*(f_C(a)). \tag{16.17}$$

Proof. We have

$$m^*(f(a)) = m^*(f_B(a), f_C(a)) = m^*(f_B(a), 0) + m^*(0, f_C(a)). \tag{16.18}$$

We show that

$$\iota_B^* m^*(f_B(a)) = m^*(f_B(a), 0). \tag{16.19}$$

To see this, we have

$$\iota_B^* m^*(f_B(a)) = m^*(\iota_B(f_B(a))) = m^*(f_B(a), 0). \tag{16.20}$$

The same calculation hold for the other term, so we are done. □

16.2 Poincaré duality

If M is any oriented manifold of dimension n, then we have a pairing

$$\Omega^k(M) \times \Omega_c^{n-k}(M) \to \mathbb{R}, \tag{16.21}$$

given by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta. \tag{16.22}$$

By Stokes' Theorem, this mapping descends to cohomology, and since this mapping is bilinear, we obtain a pairing

$$PD : H^k_{dR}(M) \otimes H_c^{n-k}(M) \to \mathbb{R}. \tag{16.23}$$

In the case M^n has the same de Rham cohomology and the same compactly supported de Rham cohomology as $\mathbb{R}^m$, then we have $H^k_{c,dR}(M) \cong H^{n-k}_{dR}(M)$. Furthermore, we have an isomorphism

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{16.24}$$

given by $PD(\alpha)(\beta) = \int_M \alpha \wedge \beta$. This follows from Theorem 11.1.

Theorem 16.7. *If M^n is orientable and has a finite good cover, then*

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{16.25}$$

is an isomorphism for all $0 \leq k \leq n$.

Proof. Let $m = n - k$, and consider the diagram

$$\begin{array}{ccccccccc}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_{dR}(U \cap V) & \xrightarrow{\delta^{k-1}} & H^k_{dR}(U \cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U) \oplus H^k_{dR}(V) & \xrightarrow{\alpha^k} & H^k_{dR}(U \cap V) \\
\downarrow{\scriptstyle PD \oplus PD} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD \oplus PD} & & \downarrow{\scriptstyle PD} \\
(H^{m+1}_{c,dR}(U) \oplus H^{m+1}_{d,dR}(V))^* & \xrightarrow{(\tilde{\alpha}^{m+1})^*} & H^{m+1}_{c,dR}(U \cap V)^* & \xrightarrow{(\tilde{\delta}^m)^*} & H^m_{c,dR}(U \cup V)^* & \xrightarrow{(\tilde{\beta}^m)^*} & (H^m_{c,dR}(U) \oplus H^m_{c,dR}(V))^* & \xrightarrow{(\tilde{\alpha}^m)^*} & H^m_{c,dR}(U \cap V)^*
\end{array}$$

The top horizontal row is exact since it is the usual Mayer-Vietoris sequence. The bottom horizontal row is exact since is the dual exact sequence of the Mayer-Vietoris sequence with compact support. We next claim that this diagram commutes up to sign, so by changing some of the vertical maps to their negatives if necessary, we obtain a commutative diagram.

Consider the square

$$\begin{array}{ccc}
H^{k-1}_{dR}(U \cap V) & \xrightarrow{\delta^{k-1}} & H^k_{dR}(U \cup V) \\
\downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} \\
H^{m+1}_{c,dR}(U \cap V)^* & \xrightarrow{(\tilde{\delta}^m)^*} & H^m_{c,dR}(U \cup V)^*
\end{array} \tag{16.26}$$

For the mapping

$$PD \circ \delta^{k-1} : H^{k-1}_{dR}(U \cap V) \to H^m_{c,dR}(U \cup V)^* \tag{16.27}$$

let's take an element $[\omega] \in H^{k-1}_{dR}(U \cap V)$, and an element $[\tau] \in H^m_{c,dR}(U \cup V)$. Then

$$(PD \circ \delta^{k-1}[\omega])[\tau] = PD(\delta^{k-1}[\omega])[\tau] = \int_M (\delta^{k-1}\omega) \wedge \tau. \tag{16.28}$$

Recall from our discussion of the Mayer-Vietoris sequence that

$$\delta^{k-1}\omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{16.29}$$

This form is supported in $U \cap V$, so we have

$$(PD \circ \delta^{k-1}[\omega])[\tau] = \int_{U\cap V} (\delta^{k-1}\omega) \wedge \tau = \int_{U\cap V} (-d\phi_U \wedge \omega) \wedge \tau. \tag{16.30}$$

Next, we look at the mapping

$$(\tilde{\delta}^m)^* \circ PD : H^{k-1}_{dR}(U \cap V) \to H^m_{c,dR}(U \cup V)^*. \tag{16.31}$$

We then have

$$\big((\tilde{\delta}^m)^* \circ PD[\omega]\big)[\tau] = PD[\omega](\tilde{\delta}^m[\tau]) = \int_{U\cap V} \omega \wedge \tilde{\delta}^m \tau. \tag{16.32}$$

Recall from our discussion of the compactly supported Mayer-Vietoris sequence that

$$\tilde{\delta}^m \tau = [d\phi_U \wedge \tau] = [-d\phi_V \wedge \tau] \in H^{m+1}_{c,dR}(U \cap V). \tag{16.33}$$

So we have

$$\big((\tilde{\delta}^m)^* \circ PD[\omega]\big)[\tau] = \int_{U\cap V} \omega \wedge d\phi_U \wedge \tau = (-1)^{k-1} \int_{U\cap V} d\phi_U \wedge \omega \wedge \tau. \tag{16.34}$$

So we see that

$$(\tilde{\delta}^m)^* \circ PD = (-1)^k PD \circ \delta^{k-1} \tag{16.35}$$

Next, we look at the square

$$\begin{array}{ccc} H^k_{dR}(U \cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U) \oplus H^k_{dR}(V) \\ \Big\downarrow{\scriptstyle PD} & & \Big\downarrow{\scriptstyle PD \oplus PD} \\ H^m_{c,dR}(U \cup V)^* & \xrightarrow{(\tilde{\beta}^m)^*} & \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^* \end{array} \tag{16.36}$$

For the mapping

$$(PD \oplus PD) \circ \beta^k : H^k_{dR}(U \cup V) \to \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^*, \tag{16.37}$$

choose $[\omega] \in H^k_{dR}(U \cup V)$, $[\tau_1] \in H^m_{c,dR}(U)$ and $[\tau_2] \in H^m_{c,dR}(V)$, and we have

$$\begin{aligned}\big((PD \oplus PD) \circ \beta^k[\omega]\big)([\tau_1],[\tau_2]) &= \big(PD_U \circ \beta^k_U[\omega]\big)([\tau_1]) + PD_V \circ \beta^k_V[\omega]\big)([\tau_2]) \\ &= \int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2.\end{aligned} \tag{16.38}$$

Next, we look at the mapping

$$(\tilde{\beta}^m)^* \circ PD : H^k_{dR}(U \cup V) \to \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^*, \tag{16.39}$$

for which we have

$$\begin{aligned}\Big((\tilde{\beta}^m)^* \circ PD[\omega]\Big)([\tau_1],[\tau_2]) &= PD[\omega](\tilde{\beta}^m(([\tau_1],[\tau_2])) = PD[\omega](\tau_1 + \tau_2) \\ &= \int_M \omega \wedge (\tau_1 + \tau_2) = \int_M \omega \wedge \tau_1 + \int_M \omega \wedge \tau_2 \\ &= \int_U \omega \wedge \tau_1 + \int_V \omega \wedge \tau_2,\end{aligned} \tag{16.40}$$

since τ_1 has compact support on U and τ_2 has compact support on V. So we have that

$$(PD \oplus PD) \circ \beta^k = (\tilde{\beta}^m)^* \circ PD. \tag{16.41}$$

We leave the remaining α square(s) as an exercise.

By the five lemma, if the outer 4 vertical maps are isomorphisms, then so is the central vertical map. The proof is completed by induction on the number of open sets in the good cover, since we know it is true for $\mathbb{R}^n$ from the previous lecture. □

Exercise 16.8. Show that the α squares commute up to sign.

17 Lecture 17

17.1 Some consequences of Poincaré duality

One immediate corollary is to the top-dimensional cohomology, which we proved by hand ealier.

Corollary 17.1. *If M^n is a connected and orientable n-manifold with a finite good cover, then $H^n_{c,dR}(M) \cong \mathbb{R}$. If M is compact, then $H^n_{dR}(M) \cong \mathbb{R}$. If M is noncompact, then $H^n_{dR}(M) = \{0\}$.*

Another immediate corollary of Poincaré duality is the following.

Corollary 17.2. *If M^n is a connected and orientable n-manifold with a finite good cover then $H^k_{dR}(M)$ and $H^{n-k}_{c,dR}(M)$ have the same dimension. If M is moreover compact, then $H^k_{dR}(M)$ and $H^{n-k}_{dR}(M)$ have the same dimension.*

We also have the following corollary.

Corollary 17.3. *If M^n is a compact odd-dimensional manifold, then the Euler characteristic $\chi(M) = 0$.*

Proof. If n is odd and M is orientable, then by Poincaré duality,

$$\begin{aligned}\chi(M) = \sum_{i=0}^{n}(-1)^i b^i(M) &= \sum_{i=0}^{[n/2]}(-1)^i b^i(M) + \sum_{i=[n/2]+1}^{n}(-1)^i b^i(M) \\ &= \sum_{i=0}^{[n/2]}(-1)^i b^i(M) + \sum_{j=0}^{[n/2]}(-1)^{[n/2]+1+j} b^{[n/2]+1+j}(M) \\ &= \sum_{i=0}^{[n/2]}(-1)^i b^i(M) - \sum_{j=0}^{[n/2]}(-1)^j b^j(M) = 0.\end{aligned} \tag{17.1}$$

If M is not orientable, then let $\tilde{M}$ be the orientable double cover. Then $0 = \chi(\tilde{M}) = 2\chi(M)$ implies that $\chi(M) = 0$. □

Remark 17.4. By the Poincaré-Hopf Theorem, this implies that every odd-dimensional manifold admits a nowhere-vanishing vector field.

Let M^n be a compact even-dimensional oriented manifold, and write $n = 2m$. Poincaré duality says that

$$PD : H^m_{dR}(M) \otimes H^m(M) \to \mathbb{R} \tag{17.2}$$

is a non-degenerate pairing. We have that

$$PD(\alpha, \beta) = \int_M \alpha \wedge \beta = (-1)^{m^2} \int_M \beta \wedge \alpha = (-1)^{m^2} PD(\beta, \alpha), \tag{17.3}$$

therefore if m is even, PD is symmetric, while if m is odd, PD is skew-symmetric.

Corollary 17.5. *If M^n is a compact oriented manifold of dimension $n = 4k+2$, then $\chi(M)$ is even.*

Proof. Using Poincaré duality, we have that

$$\begin{aligned}\chi(M) = \sum_{i=0}^{4k+2}(-1)^i b^i(M) &= \sum_{i=0}^{2k+1}(-1)^i b^i(M) + \sum_{i=2k+2}^{4k+2}(-1)^i b^i(M) \\ &= -b^{2k+1}(M) + \sum_{i=0}^{2k}(-1)^i b^i(M) + \sum_{j=0}^{2k}(-1)^{2k+2+j} b^{2k+2+j}(M) \\ &= -b^{2k+1}(M) + 2\sum_{i=0}^{2k}(-1)^i b^i(M).\end{aligned} \tag{17.4}$$

So the claim is equivalent to $b^{2k+1}(M)$ being even. However, as observed above, the intersection form is a non-degenerate skew-symmetric form on $H^{2k+1}(M)$, so it must be even dimensional. □

In case $n = 4k$, then PD is a nondegenerate symmetric bilinear form on $H^{2k}(M)$. By Sylvester's Theorem, we can make the following definition.

Definition 17.6. If M^{4k} is a compact oriented manifold of dimension a multiple of 4, let $b^{2k}_+(M)$ denote the number of positive eigenvalues and $b^{2k}_-(M)$ denote the number of negative eigenvalues of PD on $H^{2k}(M)$. The *signature* of M^{4k} is

$$\sigma(M^{4k}) \equiv b^{2k}_+(M) - b^{2k}_-(M). \tag{17.5}$$

Since PD is non-degenerate, it cannot have a zero eigenvalue, so we must have

$$b^{2k}(M) = b^{2k}_+(M) + b^{2k}_-(M). \tag{17.6}$$

Note that $b^{2k}_+(M), b^{2k}_-(M)$, and $\sigma(M)$ are oriented diffeomorphism invariants of a compact oriented manifold M^{4k} of dimension a multiple of four.

17.2 Some more homological algebra

We begin with an algbraic lemma.

Lemma 17.7. *If the sequence of vector spaces*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{17.7}$$

is exact at W_2, and V is any finite dimensional vector space, then the sequence

$$W_1 \otimes V \xrightarrow{\alpha\otimes 1_V} W_2 \otimes V \xrightarrow{\beta\otimes 1_V} W_3 \otimes V \tag{17.8}$$

is exact at $W_2 \otimes V$.

Proof. Clearly, $(\beta \otimes 1_V) \circ (\alpha \otimes 1_V) = 0$, which implies that $\mathrm{Image}(\alpha \otimes 1_V) \subset \mathrm{Ker}(\beta \otimes 1_V)$. For the reverse inclusion, choose a basis $e_i, 1 \leq i \leq \dim(V)$, of V. Any element $v \in W_2 \otimes V$ may be written as a linear combination

$$v = \sum_{i=1}^{\dim(V)} b_i \otimes e_i, \tag{17.9}$$

where $b_i \in W_2$. If $\beta \otimes 1_V(v) = 0$, then

$$0 = \sum_{i=1}^{\dim(V)} \beta(b_i) \otimes e_i. \tag{17.10}$$

The only way this element can vanish in the tensor product is that $\beta(b_i) = 0$ for all i. By exactness of the original sequence, this implies that $b_i = \alpha(a_i)$ for elements $a_i \in W_1$. Then

$$v = \sum_{i=1}^{\dim(V)} \alpha(a_i) \otimes e_i = (\alpha \otimes 1_V)\Big(\sum_{i=1}^{\dim(V)} a_i \otimes e_i \Big). \tag{17.11}$$

□

Remark 17.8. Similar to the *Hom* functor, the previous lemma holds in the category of vector spaces, but fails in general for modules over a ring R. If

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0 \tag{17.12}$$

is a short exact sequence of left R-modules, and A is any right R-module, then there is a long exact sequence

$$\cdots \longrightarrow Tor_R^1(A, M_3) \longrightarrow A \otimes_R M_1 \longrightarrow A \otimes_R M_2 \longrightarrow A \otimes_R M_3 \longrightarrow 0. \tag{17.13}$$

That is, the tensor functor is only right-exact, and the $Tor_R^j(A, \cdot)$ are the corresponding left derived functors.

17.3 Künneth formula

Let M and N be smooth manifolds. Let $\pi : M \times N \to M$ denote the projection onto the first factor, and $\rho : M \times N \to N$ be projection onto the second factor. There is a mapping from

$$K : \Omega^p(M) \times \Omega^q(N) \to \Omega^{p+q}(M \times N) \tag{17.14}$$

given by $K : (\omega, \phi) \mapsto \pi^*\omega \wedge \rho^*\phi$. Since K is bilinear, there is an induced mapping

$$K : \Omega^p(M) \otimes \Omega^q(N) \to \Omega^{p+q}(M \times N) \tag{17.15}$$

Note that if $d\omega = 0$ and $d\phi = 0$ then

$$\begin{aligned} K((\omega + d\alpha), \phi) &= \pi^*(\omega + d\alpha) \wedge \rho^*\phi \\ &= \pi^*\omega \wedge \rho^*\phi + d\pi^*\alpha \wedge \rho^*\phi \\ &= \pi^*\omega \wedge \rho^*\phi + d(\pi^*\alpha \wedge \rho^*\phi). \end{aligned} \tag{17.16}$$

Consequently, there is an induced mapping

$$K : H^p_{dR}(M) \otimes H^q_{dR}(N) \to H^{p+q}(M \times N). \tag{17.17}$$

By taking direct sums, we obtain a mapping

$$\psi : \bigoplus_{p+q=k} H^p(M) \otimes H^q(N) \to H^k(M \times N). \tag{17.18}$$

The next theorem says that ψ is an isomorphism for each k.

Theorem 17.9 (Künneth formula)**.** *Let M and N be smooth manifolds, and assume that M has a finite good cover. Then for any $k \in \mathbb{Z}, k \geq 0$, we have*

$$H^k_{dR}(M \times N) \cong \bigoplus_{p+q=k} H^p_{dR}(M) \otimes H^q_{dR}(N). \tag{17.19}$$

Proof. If $M = U \cup V$, then consider the Mayer-Vietoris sequence on M:

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{17.20}$$

In the category of vector spaces, tensor products preserve exact sequences, so we have an exact sequence

$$\cdots \longrightarrow H^p(U \cup V) \otimes H^{k-p}(N) \longrightarrow (H^p(U) \otimes H^{k-p}(N)) \oplus (H^p(V) \otimes H^{k-p}(N)) \longrightarrow H^p(U \cap V) \otimes H^{k-p}(N) \longrightarrow \cdots \tag{17.21}$$

Next, take the direct sum on p from 0 to k, and we have a long exact sequence. Consider the following diagram.

$$\begin{array}{ccccc} \oplus_{p=0}^k H^p(U \cup V) \otimes H^{k-p}(N) & \longrightarrow & \oplus_{p=0}^k (H^p(U) \otimes H^{k-p}(N)) \oplus (H^p(V) \otimes H^{k-p}(N)) & \longrightarrow & \oplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(N) \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\ H^k((U \cup V) \times N) & \longrightarrow & H^k(U \times N) \oplus H^k(V \times N) & \longrightarrow & H^k((U \cap V) \times N), \end{array} \tag{17.22}$$

where the lower row is the usual Mayer-Vietoris sequence with respect to the open covering $\{U \times N, V \times N\}$ of $M \times N$. This is straightforward to check commutativity. If we continue the diagram to the right, we see the following square

$$\begin{array}{ccc} \oplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(N) & \xrightarrow{\delta_1} & \oplus_{p=0}^k H^{p+1}(U \cup V) \otimes H^{k-p}(N) \\ \downarrow \psi & & \downarrow \psi \\ H^k((U \cap V) \times N) & \xrightarrow{\delta_2} & H^{k+1}((U \cup V) \times N). \end{array} \tag{17.23}$$

If $\omega \otimes \phi \in H^p(U \cap V) \otimes H^{k-p}(N)$. Then

$$\psi\delta_1(\omega \otimes \phi) = \psi((\delta\omega) \otimes \phi) = \pi^*(\delta\omega) \wedge \rho^*\phi \tag{17.24}$$

$$\delta_2\psi(\omega \otimes \phi) = \delta_2(\pi^*\omega \wedge \rho^*\phi). \tag{17.25}$$

Recall the definition of δ: if ρ_U, ρ_V is a partition of unity with respect to the covering $\{U, V\}$ of M, then

$$\delta(\omega) = \begin{cases} d(\rho_V\omega) & \text{in } U \\ -d(\rho_U\omega) & \text{in } V. \end{cases} \tag{17.26}$$

Note also that $\pi^*\rho_U, \pi^*\rho_V$ is a partition of unity with respect to the above open covering $\{U \times N, V \times N\}$ of $M \times N$. Therefore

$$\delta_2(\gamma) = \begin{cases} d(\pi^*\rho_V\gamma) & \text{in } U \times N \\ -d(\pi^*\rho_U\gamma) & \text{in } V \times N. \end{cases} \tag{17.27}$$

On the set $U \times N$, we have

$$\begin{aligned} \psi\delta_1(\omega \otimes \phi) &= \pi^*(d(\rho_V\omega)) \wedge \rho^*\phi \\ &= d(\pi^*\rho_V) \wedge \pi^*\omega \wedge \rho^*\phi, \end{aligned} \tag{17.28}$$

and

$$\begin{aligned}\delta_2\psi(\omega\otimes\phi) &= \delta_2(\pi^*\omega\wedge\rho^*\phi)\\ &= d\big(\pi^*\rho_V(\pi^*\omega\wedge\rho^*\phi)\big)\\ &= d(\pi^*\rho_V)\wedge\pi^*\omega\wedge\rho^*\phi.\end{aligned}\tag{17.29}$$

A similar computation holds on $V\times N$, so the above square commutes.

If $M=\mathbb{R}^n$, this is the Poincaré Lemma, see Proposition 7.1. The result then follows by the five lemma and the usual argument of induction on the number of sets in a good cover of M. $\square$

There is also a Künneth formula for cohomology with compact support.

Theorem 17.10. *Let M and N be orientable. Then for any $k\in\mathbb{Z}, k\geq 0$, we have*

$$H^k_{c,dR}(M\times N)\cong\bigoplus_{p+q=k}H^p_{c,dR}(M)\otimes H^q_{c,dR}(N).\tag{17.30}$$

Proof. If M and N are orientable, then $M\times N$ is orientable. The result then follows from the Künneth formula for ordinary de Rham cohomology, and Poincaré duality. $\square$

Remark 17.11. The above result is true without any orientability assumption. For this, use the Mayer-Vietoris sequence for compactly supported cohomology, and imitate the above proof of Künneth for ordinary de Rham cohomology.

We end this lecture with some corollaries. The first is the cohomology of higher-dimensional tori.

Corollary 17.12. *Let*

$$T^n=\overbrace{S^1\times\cdots\times S^1}^{n},\tag{17.31}$$

then

$$\dim(H^k(T^n))=\binom{n}{k}.\tag{17.32}$$

The next corollary is the cohomology of the product of spheres.

Corollary 17.13. *Let $m,n\in\mathbb{Z}_+$, then*

$$H^k_{dR}(S^n\times S^m)=\begin{cases}\mathbb{R} & k=0,m+n\\ \mathbb{R} & k=m \text{ or } n \text{ if } m\neq n\\ \mathbb{R}^2 & k=m \text{ if } m=n\\ 0 & \text{otherwise}\end{cases}.\tag{17.33}$$

In a special case of the above, we furthermore have the following.

Corollary 17.14. *Let $m\in\mathbb{Z}_+$. Then*

$$b^{2m}_{\pm}(S^{2m}\times S^{2m})=1\tag{17.34}$$

and $\sigma(S^{2m}\times S^{2m})=0$.

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