218C Introduction to Manifolds and Geometry

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Contents

Introduction

This course will be about de Rham cohomology of differentiable manifolds. Topics include the Poincarè Lemma, exact sequences, Mayer-Vietoris sequence, cohomology with compact supports, Poincaré duality, Hodge theory, and some Riemannian geometry.

Main references are [Lee13, Lee97, Spi79, War83].

1 Lecture 1

1.1 Vectors, and one-forms

Let M be a smooth manifold. A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M), \tag{1.1}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \tag{1.2}$$

is the coordinate partial. We will use the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

A 1-form is a section of the cotangent bundle, $X \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad \omega_i \in C^\infty(M). \tag{1.3}$$

Remark 1.1. Note that components of vector fields have upper indices, while components of 1-forms have lower indices. However, a collection of vector fields will be indexed by lower indices, $\{Y_1, \ldots, Y_p\}$, and a collection of 1-forms will be indexed by upper indices $\{dx^1, \ldots, dx^n\}$. This is one reason why we write the coordinates with upper indices.

Note that a smooth mapping $f : M \to N$ induces a mapping

$$f_* : TM \to TN, \tag{1.4}$$

called a "push-forward", which is linear on fibers and which makes the following diagram commute

$$\begin{array}{ccc} TM & \xrightarrow{f_*} & TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{1.5}$$

This mapping is defined as follows. If $X \in T_pM$, let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve satisfying $\gamma(0) = p$, $\gamma'(0) = X$. Then

$$f_*(X) = \frac{d}{dt}(f \circ \gamma)|_{t=0}. \tag{1.6}$$

Alternatively, since a tangent vector is equivalent to a linear derivation on germs of smooth functions around a point, we can define

$$(f_* X)_{f(p)} \phi = X(\phi \circ f), \tag{1.7}$$

where ϕ is a germ of a smooth function at $f(p)$.

Next, smooth mapping $f : M \to N$, induces a mapping

$$f^* : T^*_{f(p)} N \to T^*_p M \tag{1.8}$$

for each $p \in M$, called a "pull-back", defined by the following. If $\omega \in T^*_{f(p)}N$, and $v \in T_p M$, then

$$(f^* \omega)(v) \equiv \omega(f_* v). \tag{1.9}$$

Exercise 1.2. Prove that $f^*\omega$ is smooth if ω is.

In general, there is not a mapping $f^* : T^*N \to T^*M$, however, the pull-back operation does induce a mapping on sections

$$f^* : \Gamma(T^*N) \to \Gamma(T^*M), \tag{1.10}$$

defined by the following. If $\omega \in \Gamma(T^*N)$, $p \in M$, and $X \in T_p M$, then

$$(f^*\omega)_p(X) = \omega_{f(p)}(f_* X). \tag{1.11}$$

Note that in general there is *not* a mapping

$$f_* : \Gamma(TM) \to \Gamma(TN), \tag{1.12}$$

but later we will be able to make sense of the following: if $X \in \Gamma(TM)$, then

$$f_* X \in \Gamma(f^* TN), \tag{1.13}$$

where f^*TN is called a *pull-back bundle*.

Note also the following important proposition.

Proposition 1.3 (The chain rule)**.** *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)_* = h_* \circ f_* : TM \to TM' \tag{1.14}$$

and

$$(h \circ f)^* = f^* \circ h^* : \Gamma(T^*M') \to \Gamma(T^*M). \tag{1.15}$$

1.2 Pull-back bundles

Let us consider the above in a slightly more sophisticated way. Given a smooth mapping $f : M \to N$, define

$$f^*TN = \{(p, v) \in M \times TN \mid f(p) = \pi_N(v)\}. \tag{1.16}$$

Exercise 1.4. Prove that f^*TN is a vector bundle over M, with projection given by $\pi_1(p, v) = p$, and the fiber f^*TN over $p \in M$ is identified with the fiber $T_{f(p)}N$, i.e., the following diagram commutes

$$\begin{array}{ccc} f^*(TN) & \xrightarrow{\pi_2} & TN \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{1.17}$$

Next, define $(f_*)_B : TM \to f^*TN$ by

$$(f_*)_B(v_p) = (p, f_* v). \tag{1.18}$$

(the subscript B is short for "bundle mapping"). We have the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{(f_*)_B} & f^*TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{id} & M. \end{array} \tag{1.19}$$

Then if $X \in \Gamma(TM)$, then we can define $f_*X \in \Gamma(f^*TN)$, by

$$f_*X \equiv (f_*)_B \circ X. \tag{1.20}$$

In words: under smooth mappings, vector fields push-forward to sections of the pull-back bundle.

Noting that $(f^*(TN))^*$ is isomorphic to $f^*(T^*N)$, let us dualize the diagram (23.27) to obtain

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M. \end{array} \tag{1.21}$$

Note that if $\omega \in \Gamma(T^*N)$, then we can define $\omega \circ f \in \Gamma(f^*(T^*N))$ by

$$(\omega \circ f)(p) = (p, \omega_{f(p)}). \tag{1.22}$$

In words: sections of bundles can be pulled-back to a section of the pull-back bundle. Then, if $\omega \in \Gamma(T^*N)$, we can compose with the bundle mapping in (1.21) to define $f^*\omega \equiv f_B^* \circ (\omega \circ f) \in \Gamma(T^*M)$.

Exercise 1.5. Check that this definition agrees with the previous definition.

2 Lecture 2

Today we will just discuss linear algebra.

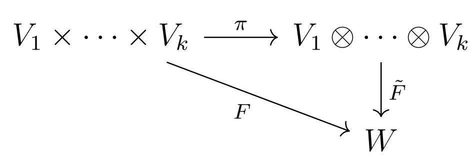
2.1 Tensor products

If $V_1, \dots, V_k$ are vector spaces over $\mathbb{R}$, then the tensor product $V_1 \otimes \cdots \otimes V_k$ is the free real vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$ modulo the subspace spanned by all elements of the form

$$(v_1, \dots, cv_i, \dots, v_k) - c(v_1, \dots, v_i, \dots, v_k) \tag{2.1}$$

$$(v_i, \dots, v_i + v_i', \dots, v_k) - (v_i, \dots, v_i, \dots, v_k) - (v_i, v_i', \dots, v_k), \tag{2.2}$$

for $c \in \mathbb{R}$. The space $V_1 \otimes \cdots \otimes V_k$ satisfies the universal mapping property as follows. Let W be any vector space, and $F : V_1 \times \cdots V_k \to W$ be a multilinear mapping, i.e., F is linear when restricted to each factor, with the other variables held fixed. Then there is a unique *linear* map $\tilde{F} : V_1 \otimes \cdots \otimes V_k$ which makes the following diagram



commutative, where π is the projection to the quotient space, which we write as

$$\pi(v_1, \dots, v_k) = v_1 \otimes \cdots \otimes v_k. \tag{2.3}$$

Exercise 2.1. Prove that

$$\dim_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k) = \dim_{\mathbb{R}}(V_1) \cdots \dim_{\mathbb{R}}(V_k). \tag{2.4}$$

Also, prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3). \tag{2.5}$$

2.2 Exterior algebra and wedge product

Let V be a real vector space, and $V^* = Hom(V, \mathbb{R})$ denote the dual vector space. The exterior algebra $\Lambda(V^*)$ is defined as

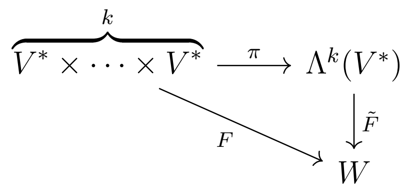
$$\Lambda(V^*) = \Big\{ \bigoplus_{k \geq 0} (V^*)^{\otimes^k} \Big\} / \mathcal{I} = \bigoplus_{k \geq 0} \Big\{ (V^*)^{\otimes^k} / \mathcal{I}_k \Big\} = \bigoplus_{k \geq 0} \Lambda^k(V^*) \tag{2.6}$$

where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $\alpha \otimes \alpha \in V^* \otimes V^*$, and $\mathcal{I}_k = (V^*)^{\otimes^k} \cap \mathcal{I}$. The wedge product of $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ is just the multiplication induced by the tensor product in this algebra, that is, lift α and β to $\tilde{\alpha} \in (V^*)^{\otimes^p}$, and $\tilde{\beta} \in (V^*)^{\otimes^q}$, and define $\alpha \wedge \beta = \pi(\tilde{\alpha} \otimes \tilde{\beta})$, where $\pi : (V^*)^{\otimes^{p+q}} \to \Lambda^{p+q}(V^*)$ is the projection. This is easily seen to be well-defined. We say that an element in $\Lambda^k(V^*)$ of the form $\alpha^1 \wedge \cdots \wedge \alpha^k$ is *decomposable.* A general element of $\Lambda^k(V^*)$ is not decomposable, but can always be written as a sum of decomposable elements.

The space $\Lambda^k(V^*)$ satisfies the universal mapping property as follows. Let W be any vector space, and let

$$F : \overbrace{V^* \times \cdots \times V^*}^{k} \to W \tag{2.7}$$

be an alternating multilinear mapping. That is, F is multilinear and $F(\alpha^1, \dots, \alpha^k) = 0$ if $\alpha^i = \alpha^j$ for some $i \neq j$. Then there is a unique linear map $\tilde{F}$ which makes the following diagram



commutative, where π is the projection, which we denote as

$$\pi(\alpha^1, \dots, \alpha^k) = \alpha^1 \wedge \cdots \wedge \alpha^k. \tag{2.8}$$

Some important properties of the wedge product:

- Bilinearity: $(\alpha^1 + \alpha^2) \wedge \beta = \alpha^1 \wedge \beta + \alpha^2 \wedge \beta$, and $(c\alpha) \wedge \beta = c(\alpha \wedge \beta)$ for $c \in \mathbb{R}$.
- If $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.
- Associativity $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

Exercise 2.2. If $\dim_{\mathbb{R}}(V) = n$, prove that $\Lambda^k(V^*) = \{0\}$ if $k > n$,

$$\dim(\Lambda^k(V^*)) = \binom{n}{k} \quad \text{if } 0 \leq k \leq n, \tag{2.9}$$

and

$$\dim(\Lambda(V^*)) = 2^n, \tag{2.10}$$

We could just stick with the above definition and try and prove all results using only this definition. However, it is very useful to think of elements of $\Lambda^k(V^*)$ as alternating multilinear maps as follows. One first has to choose a pairing

$$\Lambda^k(V^*) \cong (\Lambda^k(V))^*. \tag{2.11}$$

The pairing we will choose is as follows. If $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $v = v_1 \wedge \cdots \wedge v_k$, then

$$\alpha(v) = \det(\alpha^i(v_j)), \tag{2.12}$$

(note this is not canonical). For example,

$$\alpha^1 \wedge \alpha^2(v_1 \wedge v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{2.13}$$

We would then like to view an element of $(\Lambda^k(V))^*$ as an alternating multilinear mapping from

$$\overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}. \tag{2.14}$$

For this, we specify that if $\alpha \in (\Lambda^k(V))^*$, then

$$\alpha(v_1, \ldots, v_k) \equiv \alpha(v_1 \wedge \cdots \wedge v_k). \tag{2.15}$$

For example

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{2.16}$$

With this convention, if $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ then

$$\alpha \wedge \beta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}). \tag{2.17}$$

This then agrees with the definition of the wedge product given in [Spi79, Chapter 7].

It is convenient to have our 2 definitions of the wedge product because some proofs can be easier using one of the definitions, but harder using the other (for example, associativity of the wedge product).

3 Lecture 3

3.1 Induced mappings

Recall that if $L : V \to W$ is a linear mapping between vector spaces, then there is a mapping, $L^* : W^* \to V^*$ called the *transpose*, defined by the following. If $\omega \in W^*$, and $v \in V$, then

$$(L^*\omega)(v) = \omega(Lv). \tag{3.1}$$

This is called the transpose for the following reason. Let $dim(V) = n$, and $\dim(W) = m$. Let $e_1, \ldots, e_n$ be a basis of V and $f_1, \ldots, f_n$ be a basis of W. Let $e^1, \ldots, e^n$, and $f^1, \ldots, f^n$ denote the dual bases, that is

$$e^i(e_j) = \delta^i_j,\ 1 \le i, j \le n \tag{3.2}$$

$$f^i(f_j) = \delta^i_j,\ 1 \le i, j \le m. \tag{3.3}$$

We define the quantities L_i^j, $1 \leq i \leq n$, $1 \leq j \leq m$, by

$$Le_i = L_i^j f_j. \tag{3.4}$$

Note that if we write $v \in V$ as $v = v^i e_i$, and $w \in W$ as $w = w^i f_i$, then

$$Lv = L(v^i e_i) = v^i L(e_i) = (v^i L_i^j) f_j = \tag{3.5}$$

So the components of a vector transform like

$$\{v^i\} \mapsto \{L_i^j v^i\}, \tag{3.6}$$

which is the matrix corresponding to the transformation L.

We define the quantities $(L^*)^i_j$, $1 \leq i \leq m,\ 1 \leq j \leq n$, by

$$L^* f^i = (L^*)^i_j e^j \tag{3.7}$$

Plugging in the dual bases, we compute

$$(L^* f^i)(e_k) = (L^*)^i_j e^j(e_k) = (L^*)^i_j \delta^j_k = (L^*)^i_k. \tag{3.8}$$

However, by the definition of the transpose mapping, we have

$$(L^* f^i)(e_k) = f^i(Le_k) = f^i L_k^j f_j = L_k^j f^i(f_j) = L_k^j \delta^i_j = L_k^i \tag{3.9}$$

So if we write $\omega \in V^*$ as $\omega_i e^i$ and $\eta \in W^*$ as $\eta_j f^j$, the components of a dual vector transform like

$$\{\eta_j\} \mapsto \{L_j^i \eta_i\} \tag{3.10}$$

So the matrix corresponding to L^* in the dual basis is indeed the transpose matrix.

The mapping $L^* : W^* \to V^*$ induces a mapping

$$L^* : \overbrace{W^* \times \cdots \times W^*}^{p} \to (V^*)^{\otimes p} \tag{3.11}$$

by

$$L^*(\alpha^1, \ldots, \alpha^p) \equiv (L^* \alpha^1) \otimes \cdots \otimes (L^* \alpha^p). \tag{3.12}$$

This mapping is a multilinear mapping, so by the universal property of tensor products, this induces a unique mapping

$$L^* : (W^*)^{\otimes p} \to (V^*)^{\otimes p}. \tag{3.13}$$

By composing with the projection $\pi : (V^*)^{\otimes p} \to \Lambda^p(V^*)$, we obtain an alternating multilinear mapping

$$L^* : (W^*)^{\otimes p} \to \Lambda^p(V^*). \tag{3.14}$$

Now by the universal property of exterior products, this induces a mapping

$$L^* : \Lambda^p(W^*) \to \Lambda^p(V^*). \tag{3.15}$$

Note that by taking the direct sum on all p-s, we obtain a mapping between the full exterior algebras

$$L^* : \Lambda(W^*) \to \Lambda(V^*) \tag{3.16}$$

which is an algebra homomorphism, that is

$$L^*(\alpha \wedge \beta) = (L^* \alpha) \wedge (L^* \beta). \tag{3.17}$$

3.2 Pull-back of differential forms

A differential form is a section of $\Lambda^p(T^*M)$. I.e., a differential form is a smooth mapping $\omega : M \to \Lambda^p(T^*M)$ such that $\pi \circ \omega = Id_M$, where $\pi : \Lambda^p(T^*M) \to M$ is the bundle projection map. We will write $\omega \in \Gamma(\Lambda^p(T^*M))$, or $\omega \in \Omega^p(M)$.

If $f : M \to N$ is a smooth mapping, recall the bundle mapping (1.21) from above

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M. \end{array} \tag{3.18}$$

By the discussion in the previous section, we obtain induced mappings

$$\begin{array}{ccc} f^*(\Lambda^p(T^*N)) & \xrightarrow{f_B^*} & \Lambda^p(T^*M) \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M, \end{array} \tag{3.19}$$

which is linear on fibers, and is therefore f_B^* is a smooth mapping.

Definition 3.1 (Pull-back of a differential form)**.** If $f : M \to N$ is a smooth mapping, and $\omega \in \Lambda^p(T^*N)$, then define $\omega \circ f \in \Gamma(f^*(\Lambda^p(T^*N))$ by $\omega \circ f(p) = (p, \omega_{f(p)})$. Then define

$$f^*\omega \equiv f_B^*(\omega \circ f) \in \Gamma(\Lambda^p(T^*M)). \tag{3.20}$$

For any manifold M, define

$$\Omega(M) = \Gamma(\Lambda(T^*M)) = \Gamma\Big(\bigoplus_{p\geq 0} \Lambda^p(T^*M)\Big) = \bigoplus_{p\geq 0} \Gamma(\Lambda^p(T^*M)) = \bigoplus_{p\geq 0} \Omega^p(M). \tag{3.21}$$

By taking the direct sum of the exterior powers, we obtain a mapping

$$f^* : \Omega(N) \to \Omega(M), \tag{3.22}$$

which by (3.17) satisfies

$$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta). \tag{3.23}$$

4 Lecture 4

4.1 The exterior derivative

Given a function $f \in C^\infty(M, \mathbb{R})$ we define $df \in \Omega^1(M)$ in two ways. First, viewing vector fields as derivations on smooth functions, we can define

$$df(X) \equiv X(f). \tag{4.1}$$

Alternatively, since $f : M \to \mathbb{R}$, we have $f_* : TM \to T\mathbb{R}$. But there is a natural identification $T_p\mathbb{R} \cong \mathbb{R}$ for any $p \in \mathbb{R}$, so we can view

$$f_* : TU \to \mathbb{R}, \tag{4.2}$$

which is naturally an element in $df \in \Omega^1(U)$.

Exercise 4.1. Verify that these two definitions agree.

For a coordinate system (U, x), an element $\alpha \in \Omega^p(U)$ can be written as

$$\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{4.3}$$

where the coefficients $\alpha_{i_1 \ldots i_p} : U \to \mathbb{R}$ are well-defined functions. Note these coefficients are only defined for strictly increasing sequences $i_1 < \cdots < i_p$.

We next define the exterior derivative operator [War83, Theorem 2.20].

Proposition 4.2. *There exists an exterior derivative operator*

$$d : \Omega^p(M) \to \Omega^{p+1}(M) \tag{4.4}$$

which is the unique linear mapping satisfying

- *For $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.*
- *$d^2 = 0$.*
- *If $f \in C^\infty(M, \mathbb{R})$ then df is the differential of f defined above.*

Proof. Note that the differential of a function is given locally by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i. \tag{4.5}$$

This is obviously well-defined and independent of the coordinate system. Given a p-form α, write α locally as in (5.12), and then define

$$\begin{aligned} d\alpha &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^{n} d\alpha_{i_1 \ldots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^{n} \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{4.6}$$

The first "anti-derivation" property is easily verified by computation. The second property holds on functions, because

$$d(df) = d\Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \Big) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0, \tag{4.7}$$

since the Hessian of a smooth function is symmetric.

For existence, we need to check that this definition is independent of the coordinate system. Let d' be the operator defined with respect to another coordinate system $x' : U \to \mathbb{R}^n$. Then

$$\begin{aligned} d'(\alpha) &= d'\Big(\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \Big) \\ &= \sum_{|I|=p} (d'\alpha_{i_1 \dots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &\quad + \sum_{|I|=p} \alpha_{i_1 \dots i_p} \sum_k (-1)^{k-1} dx^{i_1} \wedge \cdots \wedge d'(dx^{i_k}) \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{|I|=p} (d\alpha_{i_1 \dots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = d(\alpha), \end{aligned} \tag{4.8}$$

since d and d' agree on functions, and since $d'dx^i = d'd'x^i = 0$.

Then for any p-form α,

$$d(d\alpha) = d\Big(\sum_{|I|=p} (d\alpha_I) \wedge dx^I \Big) = \sum_{|I|=p} (d^2\alpha_I) \wedge dx^I - d\alpha_I \wedge d(dx^I) = 0. \tag{4.9}$$

Uniqueness is left as an exercise. □

An important fact is that d commutes with pull-back.

Proposition 4.3. *If $f : M \to N$ is a smooth mapping, and $\omega \in \Omega^p(N)$, then*

$$f^*(d\omega) = d(f^*\omega). \tag{4.10}$$

Proof. If ω is a 0-form, which is a function, then $f^*\omega = \omega \circ f$. So by above, we have

$$d(f^*\omega) = d(\omega \circ f) = (\omega \circ f)_*. \tag{4.11}$$

By the chain rule, we then have

$$d(f^*\omega) = \omega_* \circ f_*. \tag{4.12}$$

On the other hand, we have that

$$f^*(d\omega)(X) = d\omega(f_*(X)) = \omega_* \circ f_*(X). \tag{4.13}$$

So the claim is true on functions. Then if ω is a p-form, write

$$\omega = \sum_{|I|=p} \omega_I dx^I. \tag{4.14}$$

Since the pull-back operation is an algebra homomorphism, we have

$$f^*\omega = \sum_{|I|=p} (f^*\omega_I) f^* dx^I = \sum_{|I|=p} (\omega_I \circ f) d(x^I \circ f). \tag{4.15}$$

Then

$$d(f^*\omega) = \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f). \tag{4.16}$$

On the other hand, we have

$$d\omega = \sum_{|I|=p} (d\omega_I) \wedge dx^I, \tag{4.17}$$

so

$$\begin{aligned} f^*(d\omega) = \sum_{|I|=p} f^*(d\omega_I) \wedge f^* dx^I &= \sum_{|I|=p} d(f^*\omega_I) \wedge d(f^* x^I) \\ &= \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f) = d(f^*\omega). \end{aligned} \tag{4.18}$$

□

5 Lecture 5

5.1 Lie bracket

Given a vector field $X \in \Gamma(TM)$, the Lie derivative of Y with respect to X is

$$\mathcal{L}_X Y = [X, Y], \tag{5.1}$$

where $[X, Y]f = X(Yf) - Y(Xf)$

Proposition 5.1. *For $X, Y \in \Gamma(TM)$, the bracket $[X, Y] \in \Gamma(TM)$.*

Proof. Recall that a vector field $X \in \Gamma(TM)$ yields a derivation on smooth functions,

$$X : C^\infty(M) \to C^\infty(M) \tag{5.2}$$

which satisfies

$$X(af + bg) = a(Xf) + b(Xg), \quad a, b \in \mathbb{R} \tag{5.3}$$

$$X(fg) = (Xf)g + f(Xg). \tag{5.4}$$

This is defined by the following. Given $p \in M$, choose a curve $\gamma : (-\epsilon, \epsilon) \to M$ such $\gamma(0) = p$, $\gamma'(0) = X_p$, and let

$$Xf(p) = \frac{d}{dt}(f \circ \gamma(t))\Big|_{t=0}. \tag{5.5}$$

From this, the properties (5.3) and (5.4) are verified directly (exercise).

Conversely, any such derivation yields a smooth vector field as follows. Choose a local coordinate system $x : U \to \mathbb{R}^n$, and write $x = (x^1, \ldots, x^n)$. Define $X^i = X(x^i)$

for $i = 1, \ldots, n$. Define $X = X^i \partial_i$, and then one shows that this is independent of coordinates (exercise).

The property (5.3) is obvious. Next, we compute

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X((Yf)g + f(Yg)) - Y((Xf)g + f(Xg)) \\
&= X(Yf)\cdot g + (Yf)(Xg) + (Xf)(Yg) + f \cdot X(Yg) \\
&\quad - Y(Xf)\cdot g - (Xf)(Yg) - (Yf)(Xg) - f\cdot Y(Xg) \\
&= (X(Yf) - Y(Xf))\cdot g + f\cdot (X(Yg) - Y(Xg)) \\
&= ([X,Y]f)\cdot g + f \cdot ([X,Y]g),
\end{aligned} \tag{5.6}$$

which verifies (5.4). □

5.2 Tensors eating vector fields

Proposition 5.2. *A mapping* $L : \overbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}^{k} \to C^\infty(M,\mathbb{R})$ *which is multilinear over* $C^\infty(M,\mathbb{R})$ *is equivalent to a smooth section*

$$L \in \Gamma((T^*M)^{\otimes^k}) \tag{5.7}$$

by

$$L((X_1)_p, \ldots, (X_k)_p) = L(\tilde{X}_1, \ldots, \tilde{X}_k), \tag{5.8}$$

where $\tilde{X}_i$ *is any smooth extension of* $(X_1)_p$.

Proof. Let us just consider the case of $k = 1$, the higher case is similar. Let $L : \Gamma(TM) \to C^\infty(M)$ be a mapping which is linear over C^∞-functions. It sufices to show that $L(X)(p) = 0$ if $X_p = 0$. This is because if we let X' and $\tilde{X}$ be any smooth extensions of X_p, then since $X' - \tilde{X}$ vanishes at p

$$L(X - \tilde{X})(p) = 0, \tag{5.9}$$

so $L(X)(p) = L(\tilde{X})(p)$ has a well-defined value, independent of the extension of X_p. To proceed, given a coordinate system around p, choose a cutoff function which is 1 in a coordinate neighborhood of p, and 0 outside. Then

$$X = (\phi X^i)(\phi \frac{\partial}{\partial x^i}) + (1 - \phi^2)X. \tag{5.10}$$

Both terms in the above are smooth vector fields on M, so using linearity over smooth functions,

$$L(X)(p) = (\phi(p)X^i(p))L\Big(\phi\frac{\partial}{\partial x^i}\Big)(p) + (1-\phi^2)(p)L(X)(p) = 0. \tag{5.11}$$

□

5.3 d on alternating multilinear tensors

Recall from above that an element $\alpha \in \Omega^p(U)$ can be written as

$$\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{5.12}$$

where the coefficients $\alpha_{i_1 \dots i_p} : U \to \mathbb{R}$ are well-defined functions, only defined for strictly increasing sequences $i_1 < \cdots < i_p$.

Next, letting $Alt^p(TM)$ denote the alernating multilinear maps from

$$\alpha : \overbrace{TM \times \cdots \times TM}^{p} \to \mathbb{R} \tag{5.13}$$

That is, α is multilinear and $F(X_1, \dots, X_k) = 0$ if $X_i = X_j$ for some $i \neq j$.

To view $\alpha \in \Omega^p(M)$ as an element of $\Gamma(Alt^p(TM))$, then we extend the coefficients $\alpha_{i_1 \dots i_p}$, which are only defined for strictly increasing sequences $i_1 < \cdots < i_p$, to ALL indices by skew-symmetry. Then our convention adopted above is that

$$\alpha = \sum_{1 \leq i_1, i_2, \dots, i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p}. \tag{5.14}$$

This convention is slightly annoying because then the projection to the exterior algebra of this is $p!$ times the original α, but has the positive feature that coefficients depending upon p do not enter into various formulas.

Next, we define

$$d : \Gamma(Alt^p(TM)) \to \Gamma(Alt^{p+1}(TM)) \tag{5.15}$$

by the formula

$$\begin{aligned} d\omega(X_0, \dots, X_p) = & \sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \dots, \hat{X}_j, \dots, X_p) \Big) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p). \end{aligned} \tag{5.16}$$

We claim that under our identifications from before, the following diagram commutes

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{\quad d \quad} & \Omega^{p+1}(M) \\ \downarrow & & \downarrow \\ \Gamma(Alt^p(TM)) & \xrightarrow{\; d \;} & \Gamma(Alt^{p+1}(TM)), \end{array} \tag{5.17}$$

where the vertical maps are the isomorphisms on sections induced by the convention (2.12). To see this, we first show that the expression on the right hand side of (5.16) is linear over C^∞ functions, so by Proposition (5.2), the operator d' does indeed maps tensors for tensors as claimed in (5.15). For example, in the case $p = 1$, (5.16) is

$$d\omega(X_0, X_1) = X_0 \omega(X_1) - X_1 \omega(X_0) - \omega([X_0, X_1]). \tag{5.18}$$

Then

$$
\begin{aligned}
d\omega(fX_0, X_1) &= fX_0\omega(X_1) - X_1\omega(fX_0) - \omega([fX_0, X_1]) \\
&= fX_0\omega(X_1) - X_1(f\omega(X_0)) - \omega(fX_0X_1 - X_1(fX_0)) \\
&= fX_0\omega(X_1) - X_1(f)\omega(X_0) - fX_1\omega(X_0) - \omega(fX_0X_1 - (X_1f)X_0 - fX_1X_0) \\
&= fX_0\omega(X_1) - X_1(f)\omega(X_0) - fX_1\omega(X_0) - \omega(f[X_0, X_1] - (X_1f)X_0) \\
&= fX_0\omega(X_1) - X_1(f)\omega(X_0) - fX_1\omega(X_0) - f\omega([X_0, X_1]) + X_1(f)\omega(X_0) \\
&= fd\omega(X_0, X_1).
\end{aligned} \tag{5.19}
$$

We leave the general cases as an exercise.

Note that in a coordinate system, d is given by

$$
(d\alpha)_{i_0\dots i_p} = \sum_{j=0}^{p} (-1)^j \partial_{i_j} \alpha_{i_0\dots\hat{i_j}\dots i_p}. \tag{5.20}
$$

(Note this is indeed skew-symmetric in all indices.) In a local coordinate system, it agrees with the exterior derivative operator by (5.20), so by Proposition 4.2 it must be the exterior derivative operator.

For example, in the case $p = 1$, the exterior derivative on $\alpha = \sum_j \alpha_j dx^j$, is

$$
d\alpha = \sum_i (d\alpha_j) dx^j = \sum_{i,j} \partial_i \alpha_j dx^i \wedge dx^j \tag{5.21}
$$

Writing this as a sum on strictly increasing multiindices, we have

$$
d\alpha = \sum_{i<j} (\partial_i \alpha_j - \partial_j \alpha_i) dx^i \wedge dx^j. \tag{5.22}
$$

On the other hand, formula (5.20) says that

$$
(d\alpha) = (d\alpha)_{ij} dx^i \otimes dx^j = (\partial_i \alpha_j - \partial_j \alpha_i) dx^i \otimes dx^j, \tag{5.23}
$$

which agree with viewing (5.22) as a multilinear alternating 2-tensor.

6 Lecture 6

Another important fact is that we can integrate top-dimensional differential forms on a compact manifold. But we need to recall orientability. First, an orientation on a n-dimensional vector space V is a choice of ordered basis $(v_1, \dots, v_n)$ with equivalence relation if 2 ordered bases are related by a change of basis matrix with positive determinant. There are exactly 2 such equivalence classes, and if M is a manifold, the oriented double cover of M denoted by $\tilde{M}$ is the double cover obtained by replacing a point p with the 2 orientations on T_pM.

Definition 6.1. A manifold M is orientable if any of the following equivalent conditions are satisfied.

- M admits an coordinate atlas (U_α, ϕ_α) such that the overlap maps are orientation-preserving $\phi_\alpha \circ \phi_\beta^{-1}$, that is, the Jacobian $(\phi_\alpha \circ \phi_\beta^{-1})_*$ has positive determinant.
- M admits a nowhere-zero n-form.
- The oriented double cover $\tilde{M} \to M$ is trivial, i.e., it has 2 components.

If M is orientable, the choice of one of the components of $\tilde{M}$ is called an *orientation* on M.

On an oriented n-dimensional manifold, the integral of $\omega \in \Omega^n(M)$ is defined as follows. Choose an oriented coordinate atlas (U_α, ϕ_α). First, assume that $\omega \in \Omega^n(M)$ has compact support in a single coordinate system U_α. Then

$$(\phi_\alpha)_*(\omega) = f dx^1 \wedge \cdots \wedge dx^n, \tag{6.1}$$

where $f : \phi_\alpha(U_\alpha) \to \mathbb{R}$ has compact support. Define

$$\int_M \omega \equiv \int_{\phi_\alpha(U_\alpha)} f dx^1 \dots dx^n. \tag{6.2}$$

By the change-of-variables formula for integrals, this definition is independent of coordinate system containing the support of ω.

Next, if M is compact, or if ω has compact support, let χ_α be a partition of unity subordinate to U_α, and define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega. \tag{6.3}$$

Since the sum is finite, this definition is independent of the choice of coordinate atlas and choice of partition of unity.

Integration by parts on manifolds is the following.

Theorem 6.2 (Stokes' Theorem for manifolds with boundary)**.** *Let $(M, \partial M)$ be an oriented manifold with boundary of dimension n. If $\omega \in \Omega^{n-1}(M)$ has compact support, then*

$$\int_{\partial M} \omega = \int_M d\omega, \tag{6.4}$$

where the boundary has the orientation induced from the outer normal, i.e., if $v_i \in T_p(\partial M)$, then the ordered basis $(v_1, \dots v_{n-1})$ is oriented if $(v, v_1, \dots, v_{n-1})$ is positively oriented, for any outward pointing normal vector v.

Proof. A manifold with boundary, by definition, can be covered by coordinate charts (U_i, ϕ_i), where $\phi_i : U_i \to H^n$, where

$$H^n = \{(x^1, \dots x^n) \in \mathbb{R}^n \mid x^n \geq 0\}. \tag{6.5}$$

is the upper half space in $\mathbb{R}^n$.

We first consider forms compactly supported in such a coordinate chart. Then just consider an $(n-1)$-form of the form

$$\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \tag{6.6}$$

Note that

$$d\omega = (-1)^{i-1} \partial_i f dx^1 \wedge \cdots \wedge dx^n \tag{6.7}$$

If $i < n$, then ω restricted to the boundary is zero, and

$$\int_{H^n} d\omega = (-1)^{i-1} \int_{H^n} \partial_i f dx^1 \cdots dx^n = 0, \tag{6.8}$$

by Fubini's Theorem and the fundamental theorem of calculus, since f has compact support. If $i = n$, then

$$\begin{aligned} \int_{H^n} d\omega &= (-1)^{n-1} \int_{H^n} \partial_n f dx^1 \cdots dx^n \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_n f dx^1 \cdots dx^n \\ &= (-1)^n \int_{\mathbb{R}^{n-1}} \omega(x^1, \dots, x^n, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{\partial H^n} \omega, \end{aligned} \tag{6.9}$$

since the outward normal is $-e_n$, so $\{-e_n, e_1, \dots, e_n\}$ is oriented, which is equivalent to $(-1)^n$ times $\{e_1, \dots, e_n\}$. In general ω is a sum of n-terms of the above type, so this proves Stokes' Theorem for $\omega \in \Omega^{n-1}(H^n)$ with compact support.

Next, we choose a partition of unity χ_i subordinate to the cover (U_i, ϕ_i), $\phi_i : U_i \to \mathbb{R}^n$, and write $\omega = \sum_i \chi_i \omega$. Let $\omega_i = \chi_i \omega$. Then for each i in the index set, we have

$$\begin{aligned} \int_M d\omega_i = \int_{U_i} d\omega_i &= \int_{\phi_i^{-1}(U_i)} (\phi_i^{-1})^*(d\omega_i) = \int_{\phi_i^{-1}(U_i)} d(\phi_i^{-1})^*(\omega_i) \\ &= \int_{H^n} d(\phi_i^{-1})^*(\omega_i) = \int_{\partial H^n} (\phi_i^{-1})^*(\omega_i) = \int_{\partial M} \omega_i, \end{aligned} \tag{6.10}$$

where the last equality holds since $\phi_i|_{\partial M}$ is a coordinate chart on ∂M as a $(n-1)$-dimensional manifold. Finally, we have

$$\int_M d\omega = \int_M d\Big(\sum_i \omega_i\Big) = \sum_i \int_M d\omega_i = \sum_i \int_{\partial M} \omega_i = \int_{\partial M} \sum_i \omega_i = \int_{\partial M} \omega. \tag{6.11}$$

□

6.1 Manifolds with corners

Define

$$\overline{\mathbb{R}}^n_+ = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^i \geq 0, i = 1 \ldots n\}. \tag{6.12}$$

Note that

$$\partial\overline{\mathbb{R}}^n_+ = (\overline{\mathbb{R}}^n_+)_{n-1} \cup (\overline{\mathbb{R}}^n_+)_{n-2} \cup \cdots \cup \{0\} \tag{6.13}$$

where $(\overline{\mathbb{R}}^n_+)_k$ is the subset of $\overline{\mathbb{R}}^n_+$ where exactly $n-k$ of the coordinate functions vanish. Points in $(\overline{\mathbb{R}}^n_+)_k$ for $k < n-1$ are called *corner points*.

A *manifold with corners* M is a Hausdorff, second countable space which is locally homeomorphic to a relatively open subset of $\overline{\mathbb{R}}^n_+$. The set of corner points on M is well-defined, see [Lee13, Proposition 16.20].

Given an $(n-1)$-form ω compactly supported in $\overline{\mathbb{R}}^n_+$, we define

$$\int_{\partial\overline{\mathbb{R}}^n_+} \omega = \int_{(\overline{\mathbb{R}}^n_+)_{n-1}} \omega. \tag{6.14}$$

Let M be a compact n-manifold with corners. If $\omega \in \Omega^{n-1}(M)$ is supported in a single coordinate chart, we define

$$\int_{\partial M} \omega = \int_{\partial\overline{\mathbb{R}}^n_+} (\phi_i^{-1})^* \omega. \tag{6.15}$$

Finally, if $\omega \in \Omega^{n-1}(M)$, let χ_i be a partition of unity subordinate to an atlas U_i, and define

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \chi_i \omega. \tag{6.16}$$

Theorem 6.3 (Stokes' Theorem on manifolds with corners)**.** *Let M be an oriented manifold with corners, and let $\omega \in \Omega^{n-1}(M)$ have compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega \tag{6.17}$$

Proof. The reduction to a form compactly supported in $\overline{\mathbb{R}}^n_+$ is exactly the same as in the proof of Theorem 6.2. So we only consider the case that $\omega \in \Omega^{n-1}(\overline{\mathbb{R}}^n_+)$ which has compact support. We write

$$\omega = \sum_i \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \tag{6.18}$$

then

$$d\omega = \sum_i (-1)^{i-1} \partial_i \omega_i dx^1 \wedge \cdots \wedge dx^n. \tag{6.19}$$

By Fubini's Theorem and the fundamental theorem of calculus, and since f has compact support, for $R > 0$ sufficiently large, we have

$$\begin{aligned}
\int_{\overline{\mathbb{R}}^n_+} d\omega &= \sum_i (-1)^{i-1} \int_0^R \cdots \int_0^R \partial_i \omega_i dx^1 \cdots dx^n \\
&= \sum_i (-1)^{i-1} \int_0^R \cdots \int_0^R \partial_i \omega_i dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_i (-1)^{i} \int_0^R \cdots \int_0^R \omega_i (x^1, \dots, 0_i, \dots, x^n) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_i \int_{\overline{\mathbb{R}}^n_+ \cap \{x^i = 0\}} \omega = \int_{\partial \overline{\mathbb{R}}^n_+} \omega.
\end{aligned} \tag{6.20}$$

Note that we used here that the outward normal to $\overline{\mathbb{R}}^n_+ \cap \{x^i = 0\}$ is $-e_i$, and $\{-e_i, e_1, \dots, \widehat{e_i}, \dots e_n\}$ is orientation equivalent to $(-1)^i$ times $\{e_1, \dots, e_n\}$. □

7 Lecture 7

7.1 Stokes' Theorem on chains

Define the standard n-simplex to be

$$\Delta^p = \Big\{ (t_0, \dots, t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^p t_i = 1, t_i \geq 0 \Big\}. \tag{7.1}$$

We orient Δ_p with respect to the normal $\hat{n} = (1, \dots, 1)$. I.e., $(v_1, \dots, v_p) \in T_x \Delta^p$ is oriented if $(\hat{n}, v_1, \dots, v_p)$ is oriented equivalent to $(e_0, \dots, e_p)$ in $\mathbb{R}^{p+1}$. The ith face of Δ^p is the $(p-1)$-simplex

$$\Delta^p_i : \Delta^{p-1} \to \Delta^p \tag{7.2}$$

defined by

$$(t_0, \dots, t_{p-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{p-1}). \tag{7.3}$$

For a topological space X, a continuous mapping

$$c : \Delta^p \to X. \tag{7.4}$$

is called a singular p-simplex. If X is a smooth manifold, and c is smooth, then we say that c is a smooth singular p-simplex. In the smooth case, if $\omega \in \Omega^p(X)$, and c is a smooth singular p-simplex, define

$$\int_c \omega = \int_{\Delta^p} c^* \omega. \tag{7.5}$$

Definition 7.1. The pth singular chain group $C_p(X,\mathbb{R})$ is the free vector space over $\mathbb{R}$ generated by a singular p-simplices. The smooth pth singular chain group $C_p^\infty(X,\mathbb{R})$ is the free vector space over $\mathbb{R}$ generated by smooth singular p-simplices.

A singular p-chain is a finite linear combination

$$c = \sum_{i=1}^{N} a_i c_i, \tag{7.6}$$

where $a_i \in \mathbb{R}$ and c_i are singular p-simplices. In the smooth case, for $\omega \in \Omega^p(M)$, define

$$\int_c \omega = \sum_{i=1}^{N} a_i \int_{c_i} \omega. \tag{7.7}$$

Define the boundary operator

$$\partial : C_p(X,G) \to C_{p-1}(X,G) \tag{7.8}$$

by the following given a singular p-simplex $c : \Delta^p \to X$, let

$$\partial c = \sum_{i=0}^{p} (-1)^i c \circ \Delta_i^p, \tag{7.9}$$

and extend to all chains by linearity.

Theorem 7.2 (Stokes' Theorem on chains)**.** *Let M be a compact oriented manifold. If $\omega \in \Omega^{p-1}(M)$, then for any chain $c \in C_p(X,\mathbb{R})$,*

$$\int_{\partial c} \omega = \int_c d\omega. \tag{7.10}$$

Proof. The standard n-simplex is a manifold with corners, and the sign in the definition of the boundary operator gives the correct orientation on each face. □

8 Lecture 8

8.1 Singular homology

Somewhat in analogy with $d^2 = 0$, we have the following.

Proposition 8.1. *We have $\partial^2 = 0$.*

Proof. First, we claim that for all $0 \le j < i \le n+1$,

$$\Delta_i^n \circ \Delta_j^{n-1} = \Delta_j^n \circ \Delta_{i-1}^{n-1} \tag{8.1}$$

To see this, the left hand side of (8.1) is

$$\begin{aligned}\Delta_i^n \circ \Delta_j^{n-1}(t_0, \dots, t_{n-1}) &= \Delta_i^n(t_0, \dots, t_{j-1}, 0_j, t_j, \dots, t_{n-1}) \\ &= (t_0, \dots, t_{j-1}, 0_j, t_j, \dots, t_{i-2}, 0_i, t_{i-1}, \dots, t_{n-2}).\end{aligned} \tag{8.2}$$

The right hand side of (8.1) is

$$\begin{aligned}\Delta_j^n \circ \Delta_{i-1}^{n-1}(t_0, \dots, t_{n-1}) &= \Delta_j^n(t_0, \dots, t_{i-2}, 0_{i-1}, t_{i-1}, \dots, t_{n-1}) \\ &= (t_0, \dots, t_{j-1}, 0_j, t_j, \dots, t_{i-2}, 0_i, t_{i-1}, \dots, t_{n-2}).\end{aligned} \tag{8.3}$$

To prove the proposition, clearly we only need to consider the standard $(n+1)$-simplex. We then compute

$$\begin{aligned}&\partial_n \circ \partial_{n+1}(\Delta^{n+1}) = \partial_n\Big(\sum_{i=0}^{n+1}(-1)^i \Delta_i^{n+1}\Big) = \sum_{i=0}^{n+1}(-1)^i \partial_n\Big(\Delta_i^{n+1}\Big) = \\ &\quad = \sum_{i=0}^{n+1}(-1)^i \sum_{j=0}^{n}(-1)^j \Delta_i^{n+1} \circ \Delta_j^n \\ &\quad = \sum_{0 \le j < i \le n+1}^{n+1} (-1)^{i+j} \Delta_i^{n+1} \circ \Delta_j^n + \sum_{0 \le i \le j \le n} (-1)^{i+j} \Delta_i^{n+1} \circ \Delta_j^n \equiv I + II.\end{aligned} \tag{8.4}$$

By (8.1), we have

$$I = \sum_{0 \le j < i \le n+1}^{n+1} (-1)^{i+j} \Delta_j^{n+1} \circ \Delta_{i-1}^n \tag{8.5}$$

Reindex the sum in II by letting $j' = j + 1$, and we get

$$II = \sum_{0 \le i < j' \le n+1} (-1)^{i+j'-1} \Delta_i^{n+1} \circ \Delta_{j'-1}^n = - \sum_{0 \le i < j' \le n+1} (-1)^{i+j'} \Delta_i^{n+1} \circ \Delta_{j'-1}^n. \tag{8.6}$$

Then just by relabeling the indices as $i \to j, j' \to i$, we have

$$II = - \sum_{0 \le j < i \le n+1} (-1)^{i+j} \Delta_j^{n+1} \circ \Delta_{i-1}^n. \tag{8.7}$$

Consequently, by (8.5), we have $I + II = 0$. □

Since $\partial^2 = 0$, we have a *chain complex*

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1}(X, \mathbb{R}) \xrightarrow{\partial_{p+1}} C_p(X, \mathbb{R}) \xrightarrow{\partial_p} C_{p-1}(X, \mathbb{R}) \xrightarrow{\partial_{p-1}} \cdots. \tag{8.8}$$

Define the pth singular homology group by

$$H_p(X, \mathbb{R}) = \frac{Ker\{\partial_p : C_p(X, \mathbb{R}) \to C_{p-1}(X, \mathbb{R})\}}{Im\{\partial_{p+1} : C_{p+1}(X, \mathbb{R}) \to C_p(X, \mathbb{R})\}} \tag{8.9}$$

We next consider the functorality of homology. If $f : X \to Y$ is a continuous mapping between topological spaces, then we can push forward chains by the following. For a simplex in X, $c : \Delta^p \to X$, we define $(f_*)_p c = f \circ c$, and extend to chains by linearity. This yields mappings

$$(f_*)_p : C_p(X, \mathbb{R}) \to C_p(Y, \mathbb{R}), \tag{8.10}$$

for $p = 0, 1, 2, \ldots$.

The following says that the collections of mappings $(f_*)_p$ are a *morphism* of chain complexes.

Proposition 8.2. *The following diagram*

$$\begin{array}{ccc} C_{p+1}(X,\mathbb{R}) & \xrightarrow{\partial^X_{p+1}} & C_p(X,\mathbb{R}) \\ \big\downarrow{\scriptstyle (f_*)_{p+1}} & & \big\downarrow{\scriptstyle (f_*)_p} \\ C_{p+1}(Y,\mathbb{R}) & \xrightarrow{\partial^Y_{p+1}} & C_p(Y,\mathbb{R}) \end{array} \tag{8.11}$$

commutes.

Proof. Consider a simplex $c : \Delta^{p+1} \to X$. By definition,

$$\partial^X_{p+1} c = \sum_{i=0}^{p+1} (-1)^i c \circ \Delta_i^{p+1}, \tag{8.12}$$

so

$$(f_*)_p \circ \partial^X_{p+1} c = \sum_{i=0}^{p+1} (-1)^i f \circ c \circ \Delta_i^{p+1}. \tag{8.13}$$

On the other hand, for a simplex $c' : \Delta^{p+1} \to Y$, we have

$$\partial^Y_{p+1} c' = \sum_{i=0}^{p+1} (-1)^i c' \circ \Delta_i^{p+1}. \tag{8.14}$$

Letting $c' = (f_*)_{p+1} c = f \circ c$, we have

$$\partial^Y_{p+1} (f_*)_{p+1} c = \sum_{i=0}^{p+1} (-1)^i f \circ c \circ \Delta_i^{p+1}, \tag{8.15}$$

so we are done. □

Corollary 8.3. *If $f : X \to Y$ then there are induced mappings*

$$(f_*)_p : H_p(X, \mathbb{R}) \to H_p(Y, \mathbb{R}). \tag{8.16}$$

If $g : Y \to Z$, then

$$((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p. \tag{8.17}$$

Consequently, if X and Y are homeomorphic, then $H_p(X, \mathbb{R}) \cong H_p(Y, \mathbb{R})$ for every $p \geq 0$.

Proof. Let $[c] \in H_p(X, \mathbb{R})$ be represented by $c \in C_p(X, \mathbb{R})$ with $\partial_p^X c = 0$. By Proposition 8.11, we have

$$\partial_p^Y (f_*)_p c = (f_*)_{p-1} \partial_p^X c = 0, \tag{8.18}$$

so we can define $(f_*)_p[c] = [(f_*)_p c]$. To show this is well-defined, if $c' = c + \partial_{p+1}^X b$, where $b \in C_{p+1}(X, \mathbb{R})$, then again by Proposition 8.11, we have

$$(f_*)_p(c') = (f_*)_p(c + \partial_{p+1}^X b) = (f_*)_p c + (f_*)_p \partial_{p+1}^X b = (f_*)_p c + \partial_{p+1}^Y (f_*)_{p+1} b, \tag{8.19}$$

therefore

$$[(f_*)_p(c')] = [(f_*)_p c + \partial_{p+1}^Y (f_*)_{p+1} b] = [(f_*)_p c]. \tag{8.20}$$

Next, since $((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p$ on the level of chains, obviously the same identity holds on homology.

Finally, if $f : X \to Y$ is a homeomorphism, there exists a continuous inverse $g : Y \to X$ such that

$$g \circ f = id_X, \quad f \circ g = id_Y. \tag{8.21}$$

Since the identity map obviously induces the identity map on homology, we have

$$(g_*)_p \circ (f_*)_p = id_{H_p(X,\mathbb{R})}, \quad (f_*)_p \circ (g_*)_p = id_{H_p(Y,\mathbb{R})}. \tag{8.22}$$

□

9 Lecture 9

9.1 de Rham cohomology

Let M be a smooth manifold. Since $d^2 = 0$, we have a "cochain" complex

$$\cdots \xrightarrow{d^{p-2}} \Omega^{p-1}(M) \xrightarrow{d^{p-1}} \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \xrightarrow{d^{p+1}} \cdots. \tag{9.1}$$

which terminates at $\Omega^n(M)$, where $n = \dim(M)$. Clearly, we have that $Im(d^{p-1}) \subset Ker(d^p)$, so we can define the following vector spaces.

Definition 9.1. For $0 \leq p \leq n$, the pth de Rham cohomology group is

$$H^p_{dR}(M) = \frac{Ker\{d^p : \Omega^p(M) \to \Omega^{p+1}(M)\}}{Im\{d^{p-1} : \Omega^{p-1}(M) \to \Omega^p(M)\}}. \tag{9.2}$$

Note that $H^*_{dR}(M)$ has an algebra structure induced by the wedge product. To see this, for $[\alpha] \in H^p_{dR}(M)$ and $[\beta] \in H^q_{dR}(M)$, represented by $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, we have that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0, \tag{9.3}$$

so we define

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]. \tag{9.4}$$

To see that this is well-defined, we have

$$(\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d\gamma \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta), \tag{9.5}$$

since β is closed, so

$$[(\alpha + d\gamma) \wedge \beta] = [\alpha \wedge \beta]. \tag{9.6}$$

Well-definedness in the other factor is similar, or just use the skew-symmetry property of the wedge product. Note that from Proposition 4.2, we have

$$[\alpha] \wedge [\beta] = (-1)^{pq} [\beta] \wedge [\alpha]. \tag{9.7}$$

Next, let $f : X \to Y$ be a smooth mapping between smooth manifolds. As discussed before, we have a pullback operation on differential forms, $f^* : \Omega^*(Y) \to \Omega^*(X)$, which makes the following diagram commute

$$\begin{array}{ccc} \Omega^p(Y) & \xrightarrow{d_Y^p} & \Omega^{p+1}(Y) \\ \Big\downarrow{\scriptstyle (f^*)^p} & & \Big\downarrow{\scriptstyle (f^*)^{p+1}} \\ \Omega^p(X) & \xrightarrow{d_X^p} & \Omega^{p+1}(X). \end{array} \tag{9.8}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The de Rham cohomology algebra is a diffeomorphism invariant.

Corollary 9.2. *If $f : X \to Y$ then there are induced mappings*

$$(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X). \tag{9.9}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{9.10}$$

Consequently, if X and Y are diffeomorphic, then $H^p_{dR}(X) \cong H^p_{dR}(Y)$ for every $p \geq 0$, and moreover, the cohomology algebras are isomorphic $H^*_{dR}(X) \cong H^*_{dR}(Y)$.

Proof. We first note that any smooth mapping $f : X \to Y$ induces a well-defined mapping on cohomology $(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X)$ by the following. If $[\alpha] \in H^p_{dR}(Y)$ is represented by a form α, such that $d^p_Y \alpha = 0$, then we have

$$d^p_X (f^*)^p \alpha = (f^*)^{p+1} d^p_Y \alpha = (f^*)^{p+1} 0 = 0, \tag{9.11}$$

so we can define $f^*[\alpha] = [f^* \alpha]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$(f^*)^p (\alpha + d^{p-1}_Y \beta) = (f^*)^p \alpha + (f^*)^p d^{p-1}_Y \beta = (f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1} \beta, \tag{9.12}$$

so $[(f^*)^p(\alpha + d_Y^p\beta)] = [(f^*)^p\alpha + d_X^{p-1}(f^*)^{p-1}\beta] = [(f^*)^p\alpha]$.

If f is a diffeomorphism, then f^{-1} exists, so we have

$$f \circ f^{-1} = id_Y, \quad f^{-1} \circ f = id_X, \tag{9.13}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^*_{dR}(X)}, \quad (f^{-1})^* \circ f^* = id_{H^*_{dR}(Y)}, \tag{9.14}$$

Finally, since $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$, together these mapping form an algebra homomorphism on cohomology algebras, which will be an algebra isomorphism if X and Y are diffeomorphic. □

9.2 Singular cohomology

To define singular cohomology, let $C^p(X, \mathbb{R})$ denote the singular cochains, which are dual to singular chains, i.e.,

$$C^p(X, \mathbb{R}) = Hom(C_p(X, \mathbb{R}), \mathbb{R}), \tag{9.15}$$

and let $\delta^p : C^p(X, \mathbb{R}) \to C^{p+1}(X, \mathbb{R})$ denote the dual to the boundary operator $\partial_{p+1} : C_{p+1}(X, \mathbb{R}) \to C_p(X, \mathbb{R})$, defined as follows. For $c^p \in C^p(X, \mathbb{R})$ and $c_{p+1} \in C_p(X, \mathbb{R})$,

$$(\delta^p c^p)(c_{p+1}) = c^p(\partial_{p+1} c_{p+1}). \tag{9.16}$$

Since $\partial_p \circ \partial_{p+1} = 0$, we have $\delta^{p+1} \circ \delta^p = 0$, so we have a *cochain complex*

$$\cdots \xrightarrow{\delta^{p-2}} C^{p-1}(X, \mathbb{R}) \xrightarrow{\delta^{p-1}} C^p(X, \mathbb{R}) \xrightarrow{\delta^p} C^{p+1}(X, \mathbb{R}) \xrightarrow{\delta^{p+1}} \cdots . \tag{9.17}$$

Define the pth singular cohomology group by

$$H^p(X, \mathbb{R}) = \frac{Ker\{\delta^p : C^p(X, \mathbb{R}) \to C^{p+1}(X, \mathbb{R})\}}{Im\{\delta^{p-1} : C^{p-1}(X, \mathbb{R}) \to C^p(X, \mathbb{R})\}}. \tag{9.18}$$

Next, let $f : X \to Y$ be a smooth mapping between topological spaces. The mapping on chains $(f_*)_p : C_p(X, \mathbb{R}) \to C_p(Y, \mathbb{R})$ induces the dual mapping on cochains

$$f^* : C^p(Y, \mathbb{R}) \to C^p(X, \mathbb{R}) \tag{9.19}$$

by the following. For $c^p \in C^p(Y, \mathbb{R})$ and $c_p \in C_p(X, \mathbb{R})$, define

$$(f^* c^p)(c_p) = c^p(f_* c_p) \tag{9.20}$$

Dualizing the diagram (8.11), we have the following commutative diagram

$$\begin{array}{ccc} C^p(Y, \mathbb{R}) & \xrightarrow{\delta_Y^p} & C^{p+1}(Y, \mathbb{R}) \\ \downarrow{\scriptstyle (f^*)^p} & & \downarrow{\scriptstyle (f^*)^{p+1}} \\ C^p(X, \mathbb{R}) & \xrightarrow{\delta_X^p} & C^{p+1}(X, \mathbb{R}). \end{array} \tag{9.21}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The singular cohomology spaces are a topological invariant.

Corollary 9.3. *If $f : X \to Y$ is continuous, then there are induced mappings*

$$(f^*)^p : H^p(Y, \mathbb{R}) \to H^p(X, \mathbb{R}). \tag{9.22}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{9.23}$$

Consequently, if X and Y are homeomorphic, then $H^p(X, \mathbb{R}) \cong H^p(Y, \mathbb{R})$ for every $p \geq 0$.

Proof. Exactly the same as the proof of Corollary 9.2, with d_X, d_Y replaced by δ_X, δ_Y. □

10 Lecture 10

10.1 Cup products

Cochains have some extra ring structure: we next define the cup product

$$\cup : C^p(X, \mathbb{R}) \times C^q(X, \mathbb{R}) \to C^{p+q}(X, \mathbb{R}), \tag{10.1}$$

by the following. If c_{p+q} is a singular $(p+q)$-simplex, and c^p and c^q are singular cochains, then define

$$(c^p \cup c^q)(c_{p+q}) = c^p(c_{p+q} \circ (t_0, \cdots, t_p, 0, \ldots, 0)) \cdot c^q(c_{p+q} \circ (0, \ldots, 0, t_p, \ldots, t_{p+q})). \tag{10.2}$$

We will see below that the cup product is bilinear, so by the universal property of the tensor product, the cup product descends to a linear mapping

$$\cup : C^p(X, \mathbb{R}) \otimes C^q(X, \mathbb{R}) \to C^{p+q}(X, \mathbb{R}). \tag{10.3}$$

The cup product enjoys the following properties.

Proposition 10.1. *The cup product is associative*

$$(c^p \cup c^q) \cup c^r = c^p \cup (c^q \cup c^r). \tag{10.4}$$

The cup product is bilinear. That is, for $c \in \mathbb{R}$,

$$a^p \cup (cb^q) = c(a^p \cup b^q), \quad a^p \cup (b^q + c^q) = a^p \cup b^q + a^p \cup c^q \tag{10.5}$$

$$(ca^p) \cup b^q = c(a^p \cup b^q), \quad (a^p + b^p) \cup c^q = a^p \cup c^q + b^p \cup c^q. \tag{10.6}$$

Cup products are functorial, i.e., for $f : X \to Y$ continuous,

$$f^*(c^p \cup c^q) = f^* c^p \cup f^* c^q. \tag{10.7}$$

The coboundary operator is an anti-derivation with respect to the cup product

$$\delta(c^p \cup c^q) = (\delta c^p) \cup c^q + (-1)^p c^p \cup (\delta c^q). \tag{10.8}$$

Proof. For associativity, if c_{p+q+r} is a singular $(p+q+r)-$simplex, then

$$\begin{aligned}
((c^p \cup c^q) \cup c^r)(c_{p+q+r}) &= (c^p \cup c^q)(c_{p+q+r} \circ (t_0, \cdots, t_{p+q}, 0, \ldots, 0)) \\
&\quad \cdot c^r(c_{p+q+r} \circ (0, \ldots, 0, t_{p+q}, \ldots, t_{p+q+r})) \\
= c^p(c_{p+q+r} \circ (t_0, \cdots, t_p, 0, \ldots, 0)) &\cdot c^q(c_{p+q+r} \circ (0, \cdots, 0, t_p, \cdots, t_{p+q}, 0, \ldots, 0)) \\
&\quad \cdot c^r(c_{p+q+r} \circ (0, \ldots, 0, t_{p+q}, \ldots, t_{p+q+r})) \\
= c^p(c_{p+q+r} \circ (t_0, \cdots, t_p, 0, \ldots, 0)) &\cdot (c^q \cup c^r)(c_{p+q+r} \circ (0, \ldots 0, t_p, \ldots t_{p+q}, \ldots, t_{p+q+r}) \\
= (c^p \cup (c^q \cup c^r))(c_{p+q+r}).
\end{aligned} \tag{10.9}$$

The blinear property is easily verified. Next, if $f : X \to Y$ is continuous, and c^X_{p+q} is a singular $(p+q)-$simplex in X then

$$\begin{aligned}
f^*(c^p \cup c^q)(c) &= (c^p \cup c^q)(f_* c) = (c^p \cup c^q)(f \circ c) \\
&= c^p((f \circ c)(t_0, \ldots, t_p, 0, \ldots, 0)) \cdot c^q((f \circ c)(0, \ldots, 0, t_0, \ldots, t_{p+q})) \\
&= c^p(f \circ c(t_0, \ldots, t_p, 0, \ldots, 0)) \cdot c^q(f \circ c(0, \ldots, 0, t_0, \ldots, t_{p+q})) \\
&= (f^* c^p)(c(t_0, \ldots, t_p, 0, \ldots, 0)) \cdot (f^* c^q)(c(0, \ldots, 0, t_0, \ldots, t_{p+q})) \\
&= ((f^* c^p) \cup (f^* c^q))(c).
\end{aligned} \tag{10.10}$$

Finally, for c_{p+q+1} a $(p+q+1)-$simplex, we compute

$$\begin{aligned}
&\delta(c^p \cup c^q)(c_{p+q+1}) = (c^p \cup c^q)(\partial c_{p+q+1}) = (c^p \cup c^q)\Big(\sum_{i=0}^{p+q+1} (-1)^i c_{p+q+1} \circ \Delta_i^{p+q}\Big) \\
&= \sum_{i=0}^{p+q+1} (-1)^i (c^p \cup c^q)(c_{p+q+1}(t_0, \ldots, \widehat{t_i}, \ldots, t_{p+q+1})) \\
&= \sum_{i=0}^{p} (-1)^i (c^p \cup c^q)(c_{p+q+1}(t_0, \ldots, \widehat{t_i}, \ldots, t_{p+q+1})) \\
&\qquad + \sum_{i=p+1}^{p+q+1} (-1)^i (c^p \cup c^q)(c_{p+q+1}(t_0, \ldots, \widehat{t_i}, \ldots, t_{p+q+1})) \\
&= \sum_{i=0}^{p} (-1)^i c^p(c_{p+q+1}(t_0, \ldots, \widehat{t_i}, t_{p+1}, 0 \ldots, 0)) \cdot c^q(c_{p+q+1}(0, \ldots, 0, t_{p+1}, \ldots, t_{p+q+1})) \\
&+ \sum_{i=p+1}^{p+q+1} (-1)^i c^p(c_{p+q+1}(t_0, \ldots, t_p, 0 \ldots, 0)) \cdot c^q(c_{p+q+1}(0, \ldots, 0, t_p, \ldots, \widehat{t_i}, t_{p+q+1}))
\end{aligned} \tag{10.11}$$

Note that we can end the first sum at $p+1$, and start the second sum at p since the resulting added terms will cancel out, so we have

$$
\begin{aligned}
&\delta(c^p \cup c^q)(c_{p+q+1})\\
&= \sum_{i=0}^{p+1} (-1)^i c^p(c_{p+q+1}(t_0, \dots, \widehat{t_i}, t_{p+1}, 0 \dots, 0)) \cdot c^q(c_{p+q+1}(0, \dots, 0, t_{p+1}, \dots, t_{p+q+1}))\\
&+ \sum_{i=p}^{p+q+1} (-1)^i c^p(c_{p+q+1}(t_0, \dots, t_p, 0 \dots, 0)) \cdot c^q(c_{p+q+1}(0, \dots, 0, t_p, \dots, \widehat{t_i}, t_{p+q+1}))\\
&= c^p(\partial(c_{p+q+1}(t_0, \dots, t_{p+1}, 0, \dots 0))) \cdot c^q(c_{p+q+1}(0, \dots, 0, t_{p+1}, \dots, t_{p+q+1}))\\
&+ (-1)^p c^p(c_{p+q+1}(t_0, \dots, t_p, 0 \dots, 0)) \cdot c^q(\partial(c_{p+q+1}(0, \dots, 0, t_p, \dots, t_{p+q+1})))\\
&= ((\delta c^p) \cup c^q)(c_{p+q+1}) + (-1)^p (c^p \cup (\delta c^q))(c_{p+q+1}).
\end{aligned}
\tag{10.12}
$$

□

Corollary 10.2. *The cup product on cochains induces a cup product on cohomology*

$$\cup : H^p(X, \mathbb{R}) \otimes H^q(X, \mathbb{R}) \to H^{p+q}(X, \mathbb{R}). \tag{10.13}$$

If $f : X \to Y$ is continuous, then

$$f^* : H^*(Y, \mathbb{R}) \to H^*(X, \mathbb{R}) \tag{10.14}$$

is an algebra homeomorphism. Consequently, if X and Y are homeomorphic, then the singular cohomology algebras are isomorphic.

Proof. For $[\alpha] \in H^p(X, \mathbb{R})$ and $[\beta] \in H^q(X, \mathbb{R})$, represented by $\alpha \in C^p(M, \mathbb{R})$ and $\beta \in C^q(M, \mathbb{R})$, we have that

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^p \alpha \cup \delta\beta = 0, \tag{10.15}$$

so we define

$$[\alpha] \cup [\beta] = [\alpha \cup \beta]. \tag{10.16}$$

To see that this is well-defined, we have

$$(\alpha + \delta\gamma) \cup \beta = \alpha \cup \beta + (\delta\gamma) \cup \beta = \alpha \cup \beta + \delta(\gamma \cup \beta), \tag{10.17}$$

since β is co-closed, so

$$[(\alpha + \delta\gamma) \cup \beta] = [\alpha \cup \beta]. \tag{10.18}$$

Well-definedness in the other factor is similar, and functorality follows from (10.7). □

Remark 10.3. The wedge product satisfies $\alpha^p \wedge \beta^q = (-1)^{pq} \beta^q \wedge \alpha^p$. This does not hold for the cup product on the level of cochains, however, it can be shown that in cohomology

$$c^p \cup c^q = (-1)^{pq} c^q \cup c^p, \tag{10.19}$$

but this is quite difficult to show directly.

Remark 10.4. The cohomology ring contains more information that the homology groups. For example, the 4-manifolds $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ have the same homology and cohomology groups, but their cohomology rings are not isomorphic. Another example is $\mathbb{CP}^2$ and $S^2 \vee S^4$.

11 Lecture 11

11.1 De Rham's Theorem

We are now in a position to state the theorem of de Rham relating de Rham cohomology and singular cohomology with real coefficients of a smooth manifold M. Morever, we can write the explicit mapping. Consider the following diagram:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d^{p-2}} & \Omega^{p-1}(M) & \xrightarrow{d^{p-1}} & \Omega^p(M) & \xrightarrow{d^p} & \Omega^{p+1}(M) & \xrightarrow{d^{p+1}} & \cdots \\ & & \downarrow{\scriptstyle \mathcal{F}^{p-1}} & & \downarrow{\scriptstyle \mathcal{F}^{p}} & & \downarrow{\scriptstyle \mathcal{F}^{p+1}} & & \\ \cdots & \xrightarrow{\delta^{p-2}} & C^{p-1}(M,\mathbb{R}) & \xrightarrow{\delta^{p-1}} & C^p(M,\mathbb{R}) & \xrightarrow{\delta^p} & C^{p+1}(M,\mathbb{R}) & \xrightarrow{\delta^{p+1}} & \cdots , \end{array} \tag{11.1}$$

where the vertical maps are defined as follows. If $\omega \in \Omega^p(M)$, and c_p is a p-chain, then let

$$(\mathcal{F}^p\omega)(c_p) = \int_{c_p} \omega. \tag{11.2}$$

Proposition 11.1. *The diagram* (11.1) *commutes. Consequently, there are induced mappings*

$$H^p\mathcal{F}^p : H^p_{dR}(M) \to H^p(M,\mathbb{R}). \tag{11.3}$$

Proof. Commutativity says that

$$\delta^p\mathcal{F}^p = \mathcal{F}^{p+1}d^p \tag{11.4}$$

Given $\omega \in \Omega^p(M)$, and $(p+1)$-chain c_{p+1}, the left hand side of (22.3) evaluates to

$$\delta^p(\mathcal{F}^p(\omega))(c_{p+1}) = \mathcal{F}^p(\omega)(\partial_{p+1}c_{p+1}) = \int_{\partial_{p+1}c_{p+1}} \omega. \tag{11.5}$$

The right hand side of (22.3) evaluates to

$$\mathcal{F}^{p+1}d^p\omega(c_{p+1}) = \int_{c_{p+1}} d^p\omega, \tag{11.6}$$

These are equal by Theorem 7.2, Stokes' Theorem on chains.

Next, if $[\omega] \in H^p_{dR}(M)$ is represented by $\omega \in \Omega^p(M)$, then

$$\delta^p\mathcal{F}^p\omega = \mathcal{F}^{p+1}d^p\omega = 0, \tag{11.7}$$

so we can define $\mathcal{F}^p[\omega] = [\mathcal{F}^p\omega]$. To see that this is well-defined,

$$\begin{aligned}\mathcal{F}^p[\omega + d^{p-1}\alpha] &= [\mathcal{F}^p\omega + \mathcal{F}^p d^{p-1}\alpha] = [\mathcal{F}^p\omega + \mathcal{F}^p d^{p-1}\alpha] \\ &= [\mathcal{F}^p\omega + \delta^{p-1}\mathcal{F}^{p-1}\alpha] = [\mathcal{F}^p\omega].\end{aligned} \tag{11.8}$$

□

Theorem 11.2 (de Rham). *The mapping $\mathcal{F}^p$ are isomorphisms. Moreover, $\mathcal{F}^* : H^*_{dR}(X) \to H^*(X, \mathbb{R})$ is an isomorphism of algebras, with respect to the wedge product and cup product.*

Remark 11.3. There is still a slight problem because $H^p(X, \mathbb{R})$ is defined using continuous chains, but we can only integrate over smooth chains. It turns out that $H^p(X, \mathbb{R}) \cong H^p_{C^\infty}(X, \mathbb{R})$. The basic idea is that any continuous chain can be well-approximated by a smooth chain if the space X is a smooth manifold.

12 Lecture 12

12.1 Chain complexes

A collection A_p of vector spaces for $p \geq 0$ and operators $\partial^A_p : A_p \to A_{p-1}$ for $p \geq 1$ satisfying $\partial^A_p \partial^A_{p+1} = 0$ is called a *chain complex.*

$$\cdots \xrightarrow{\partial^A_{p+2}} A_{p+1} \xrightarrow{\partial^A_{p+1}} A_p \xrightarrow{\partial^A_p} A_{p-1} \xrightarrow{\partial^A_{p-1}} \cdots . \tag{12.1}$$

Definition 12.1. The pth homology of a chain complex is the vector space

$$H_p(A) = \frac{Ker\{\partial^A_p : A_p \to A_{p-1}\}}{Im\{\partial^A_{p+1} : A_{p+1} \to A_p\}} \tag{12.2}$$

Definition 12.2. A morphism $\alpha : A \to B$ of chain complexes is a collection of mappings $\alpha_p : A_p \to B_p$ such that $\partial^B_{p+1}\alpha_{p+1} = \alpha_p \partial^A_{p+1}$ for $p \geq 0$. In other words, $\alpha : A \to B$ is a morphism if the following diagram commutes

$$\begin{array}{ccc} A_{p+1} & \xrightarrow{\partial^A_{p+1}} & A_p \\ \Big\downarrow{\scriptstyle \alpha_{p+1}} & & \Big\downarrow{\scriptstyle \alpha_p} \\ B_{p+1} & \xrightarrow{\partial^B_{p+1}} & B_p. \end{array} \tag{12.3}$$

Proposition 12.3. *Morphisms satisfy the following properties:*

- *Composition of morphisms: If $\alpha : A \to B$ and $\beta : B \to C$ are morphisms of chain complexes, then $\beta \circ \alpha : A \to C$ is a morphism.*
- *Associativity: If $\gamma : C \to D$ is another morphism, then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.*

Proof. The diagram looks like

$$\begin{array}{ccc} A_{p+1} & \xrightarrow{\partial^A_{p+1}} & A_p \\ \downarrow{\scriptstyle \alpha_{p+1}} & & \downarrow{\scriptstyle \alpha_p} \\ B_{p+1} & \xrightarrow{\partial^B_{p+1}} & B_p. \\ \downarrow{\scriptstyle \beta_{p+1}} & & \downarrow{\scriptstyle \beta_p} \\ C_{p+1} & \xrightarrow{\partial^C_{p+1}} & C_p. \end{array} \tag{12.4}$$

We want to show that

$$\beta_p \circ \alpha_p \circ \partial^A_{p+1} = \partial^C_{p+1} \circ \beta_{p+1} \circ \alpha_{p+1} \tag{12.5}$$

Using commutativity of the top square, the left hand side of (12.5) is

$$\beta_p \circ \alpha_p \circ \partial^A_{p+1} = \beta_p \circ \partial^B_{p+1} \circ \alpha_{p+1}. \tag{12.6}$$

Using commutativity of the bottom square, the right hand side of (12.5) is

$$\beta_p \partial^B_{p+1} \circ \alpha_{p+1}, \tag{12.7}$$

which proves (12.5).

Associtivity is clear: $\gamma_p \circ (\beta_p \circ \alpha_p) = (\gamma_p \circ \beta_p) \circ \alpha_p$ holds for every $p \geq 0$ since composition of mappings is associative. □

Proposition 12.4. *A morphism of chain complexes $\alpha : A \to B$ induces mappings $H_p\alpha : H_p(A) \to H_p(B)$. Furthermore, if $\beta : B \to C$ is another morphism of chain complexes, then*

$$H_p(\beta \circ \alpha) = H_p\beta \circ H_p\alpha. \tag{12.8}$$

Proof. Given $[a_p] \in H_p(A)$ represented by $a_p \in A_p$ satisfying $\partial^A_p a_p = 0$, we have

$$\partial^B_p \alpha_p a_p = \alpha_{p-1} \partial^A_p a_p = 0, \tag{12.9}$$

therefore we can define $(H_p\alpha_p)[a_p] = [\alpha_p a_p]$. To check that this is well-defined,

$$[\alpha_p(a_p + \partial^A_{p+1} a_{p+1})] = [\alpha_p a_p + \alpha_p \partial^A_{p+1} a_{p+1}] = [\alpha_p a_p + \partial^B_{p+1} \alpha_{p+1} a_{p+1}] = [\alpha_p a_p]. \tag{12.10}$$

Next,for $[a_p] \in H_p(A)$ represented by $a_p \in A_p$, we have

$$H_p(\beta \circ \alpha)[a_p] = [(\beta_p \circ \alpha_p) a_p] = [\beta_p(\alpha_p(a_p))] = H_p\beta_p[\alpha_p(a_p)] = H_p\beta(H_p\alpha[a_p]). \tag{12.11}$$

□

Proposition 12.5 (Functorality)**.** *For $p \geq 0$, the pth homology functor is the covariant functor mapping between the category of chain complexes to the category of vector spaces (with morphisms being linear mappings).*

Proof. The functor H_p maps objects to objects, just by mapping the chain complex C to the vector space $H_p(C)$. Also for each morphism $\alpha : A \to B$ between chain complexes, we associate the morphism $H_p\alpha : H_p(A) \to H_p(B)$. The covariant property is (12.8). □

Definition 12.6. The topological chain functor is the functor from the category of topological spaces and continuous mappings to the category of chain complexes mapping X to $\{C_p(X, \mathbb{R}), \partial_p : C_p(X, \mathbb{R}) \to C_{p-1}(X, \mathbb{R})\}$ and $C_p(X, \mathbb{R})$ is the vector space of continuous p-chains. A continuous mapping $f : X \to Y$ maps to $f_* : C_p(X, \mathbb{R}) \to C_p(Y, \mathbb{R})$.

The topological pth homology functor is the functor from the category of topological spaces and continuous mapping to vector spaces the category of vector spaces and linear mappings given by $X \mapsto H_p(X, \mathbb{R})$ and $f : X \to Y$ maps to $H_p f = (f_*)_p : H_p(X) \to H_p(Y)$.

The smooth chain functor is the functor from the category of smooth manifolds and smooth mappings to the category of chain complexes mapping X to the complex of smooth p-chains, $\{C_p^\infty(X, \mathbb{R}), \partial_p : C_p(X, \mathbb{R}) \to C_{p-1}(X, \mathbb{R})\}$. A smooth mapping $f : X \to Y$ maps to $f_* : C_p^\infty(X, \mathbb{R}) \to C_p^\infty(Y, \mathbb{R})$.

The smooth pth homology functor is the functor from the category of smooth manifolds and smooth mappings to vector spaces the category of vector spaces and linear mappings given by $M \mapsto H_p(M, \mathbb{R})$ and $f : X \to Y$ maps to $H_p f = (f_*)_p : H_p(X) \to H_p(Y)$.

Proposition 12.7. *All of the above functors are covariant functors.*

Proof. This follows from $(g \circ f)_* = g_* \circ f_*$, and the fact that the composition of covariant functors is a covariant functor. □

Theorem 12.8. *The smooth pth homology functor is isomorphic to the topological pth homology functor restricted to the category of smooth manifolds. The natural inclusion $\iota : C_p^\infty(X, \mathbb{R}) \to C_p(X, \mathbb{R})$ induces a natural isomorphism $H_p\iota$ between these functors. That is, the mapping $H_p\iota : H_p^\infty(X, \mathbb{R}) \to H_p(X, \mathbb{R})$ is an isomorphism of vector spaces, and for any smooth mapping $f : X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} H_p^\infty(X, \mathbb{R}) & \xrightarrow{H_p\iota} & H_p(X, \mathbb{R}) \\ \downarrow{\scriptstyle H_p f} & & \downarrow{\scriptstyle H_p f} \\ H_p^\infty & \xrightarrow{H_p\iota} & H_p(Y, \mathbb{R}). \end{array} \tag{12.12}$$

Proof. Note that the mapping ι is morphism between chain complexes, so it induces a mapping $H_p\iota$ between the homology groups.

Lee proves this by finding an inverse between the functors $H_p s : H_p(X, \mathbb{R}) \to H_p^\infty(X, \mathbb{R})$. This is achieved by finding a homotopy operator from continuous chains to smooth chains using the Whitney approximation theorem. We will discuss a general outline of Lee's proof, but we will be giving a different proof. □

13 Lecture 13

13.1 Chain homotopy between morphisms of chain complexes

Definition 13.1. Let $f : A \to B$, and $g : A \to B$ be two morphisms of chain complexes. We say that f is chain homotopic to g if there exists mappings $S_p : A_p \to B_{p+1}$ such that

$$f_p - g_p = \partial^B_{p+1} S_p + S_{p-1} \partial^A_p. \tag{13.1}$$

Proposition 13.2. *If f is chain homotopic to g then $H_p f = H_p g : H_p(A) \to H_p(B)$.*

Proof. Consider the mapping $H_p f - H_p g$, and take $[a_p] \in H_p(A)$ represented by $a_p \in A_p$ satisfying $\partial^A_p a_p = 0$. Then

$$\begin{aligned}(H_p f - H_p g)[a_p] &= (H_p(f - g))[a_p] = [(H_p(f - g))a_p] \\ &= [\partial^B_{p+1} S_P a_p + S_{p-1} \partial^A_p a_p] = [\partial^B_{p+1} S_P a_p] = 0.\end{aligned} \tag{13.2}$$

□

13.2 Comparison of smooth and continuous singular homology

If M is a smooth manifold, then we have the singular chain complex $\{C^0_p, \partial\}$ consisting of continuous chains, but we also have a complex $\{C^\infty_p, \partial\}$ consisting of smooth chains. There is an obvious morphism $\iota : C^\infty \to C^0$ since any smooth mapping is continuous, so there is an induced mapping between the homologies

$$H_p \iota : H_p(C^\infty) \to H_p(C^0). \tag{13.3}$$

To get a mapping in the other direction, one needs a smoothing operator to map a continuous chain to a smooth chain. So we need a morphism

$$s : C^0 \to C^\infty \tag{13.4}$$

We want that $s|_{C^\infty} = id$, i.e., if a chain is already smooth, nothing happens. So then $s \circ \iota = id$, which implies that

$$H_p s \circ H_p \iota = id_{H_p(C^\infty)}. \tag{13.5}$$

We also want to prove that

$$H_p \iota \circ H_p s = id_{H_p(C^0)}. \tag{13.6}$$

For this, we want to find a chain homotopy from $\iota \circ s$ to the identity map on continuous chains. I.e., we want mappings

$$S_p : C^0_p(X, \mathbb{R}) \to C^0_{p+1}(X, \mathbb{R}) \tag{13.7}$$

such that

$$\iota_p \circ s_p - id_p = \partial_{p+1} S_p + S_{p-1} \partial_p. \tag{13.8}$$

We only need to define s_p and S_p for for singular p-simplices.

First, we discuss $p = 0$. Recall that

$$\Delta^0 = \{1\} \subset \mathbb{R} \tag{13.9}$$
$$\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1,\ t_i \geq 0, i = 0, 1\}. \tag{13.10}$$

Define

$$p^0_0 : \Delta^1 \to \Delta^0 \times [0, 1] \tag{13.11}$$

by

$$p^0_0((t_0, t_1)) = (t_0 + t_1, t_1) = (1, t_1). \tag{13.12}$$

A 0-simplex $c_0 : \Delta^0 \to M$. Let $H : \Delta^0 \times [0, 1] \to M$ by $H(t, s) = c_0(t)$ (think of this as a constant homotopy). Define $s_0 c_0 = c_0$, and $S_0 c_0 = c_1$, where $c_1 : \Delta^1 \to M$ is $c_1 = H \circ p^0_0$. I.e., c_1 is the 1-simplex which is a constant mapping to this point. We just need to verify that

$$(\iota_0 \circ s_0) c_0 - c_0 = \partial_1 S_0 c_0. \tag{13.13}$$

The left hand side obviously vanishes. The right hand side is

$$\partial_1 S_0 c_0 = \partial_1 c_1 = \partial_1 (H \circ p^0_0) = H \circ p^0_0 \circ \Delta^1_0 - H \circ p^0_0 \circ \Delta^1_1 \tag{13.14}$$

The first term is

$$H \circ p^0_0 \circ \Delta^1_0(1) = H \circ p^0_0(0, 1) = H(1, 1) = c_0(1). \tag{13.15}$$

and the second term is

$$H \circ p^0_0 \circ \Delta^1_1(1) = H \circ p^0_0(1, 0) = H(1, 0) = c_0(1). \tag{13.16}$$

So the right hand side of (13.14) also vanishes.

We next define s_1 and S_1. Let c_1 be 1-simplex $c_1 : \Delta^1 \to M$. We let $e_0 = (1, 0)$, and $e_1 = (0, 1)$ be the unit basis vectors in $\mathbb{R}^2$. If c_1 is not smooth, then do the following.

Proposition 13.3. *There exists a homotopy $H : \Delta^1 \times [0, 1] \to M$ such that*

$$H(t, 0) = c_1(t), \quad H(e_0, s) = c_1(e_0), \quad H(e_1, s) = c_1(e_1), \tag{13.17}$$

and such that $H(t, 1) : \Delta^1 \to M$ is smooth.

Proof. If c_1 is already smooth, then choose $H(t,s) = c_1(t)$.

Otherwise, there exists a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $\gamma(t) = c_1((1-t)e_0 + te_1))$ restricted to $[t_i, t_{i+1}]$ maps into a coordinate chart.

So consider the case that $\gamma : [0,1] \to \mathbb{R}^n$. Let $L(t)$ be the straight line path between $\gamma(0)$ and $\gamma(1)$. Define $H(t,s) = (1-s)\gamma(t) + sL(t)$.

We do this for each subinterval to find a homotopy between $\gamma(t)$ and a piecewise smooth path.

Then is is easy to see that a piecwise linear mapping from $[0,1]$ to $\mathbb{R}^n$ can locally be homotopied to a smooth path in just a small neighborhood of the corner points (exercise).

□

We define $s_1(c_1) = H(t,1) : \Delta^1 \to M$, which is a smooth 1-simplex. Next, we define $S_1 : C_1^0 \to C_2^0$. Recall that

$$\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 \mid t_0 + t_1 + t_2 = 1,\ t_i \geq 0, i = 0, 1, 2\}. \tag{13.18}$$

We will divide $\Delta^1 \times [0,1]$ into 2 2-simplices. For $i = 0, 1$, define

$$p_i^1 : \Delta^2 \to \Delta^1 \times [0,1] \tag{13.19}$$

by

$$p_0^1 : (t_0, t_1, t_2) \to (t_0 + t_1, t_2, t_1 + t_2) \tag{13.20}$$
$$p_1^1 : (t_0, t_1, t_2) \to (t_0, t_1 + t_2, t_2). \tag{13.21}$$

(SHOW FIGURE HERE)

Define

$$S_1(c_1) = H \circ p_0^1 - H \circ p_1^1. \tag{13.22}$$

We want to verify the formula

$$(\iota_1 \circ s_1)c_1 - c_1 = \partial_2 S_1 c_1 + S_0 \partial_1 c_1 \tag{13.23}$$

The left hand side is $H(t,1) - c_1(t) = s_1 c_1(t) - c_1(t)$.

Next, we have

$$\begin{aligned}
\partial_2 S_1 c_1 &= \partial_2(H \circ p_0^1) - \partial_2(H \circ p_1^1)\\
&= \partial_2(H_* p_0^1) - \partial_2(H_* p_1^1)\\
&= H_*(\partial_2 p_0^1) - H_*(\partial_2 p_1^1)\\
&= H_*\Big(\sum_{j=0}^{2}(-1)^j p_0^1 \circ \Delta_j^2\Big) - H_*\Big(\sum_{j=0}^{2}(-1)^j p_1^1 \circ \Delta_j^2\Big)\\
&= H_*(p_0^1 \circ \Delta_0^2 - p_0^1 \circ \Delta_1^2 + p_0^1 \circ \Delta_2^2) - H_*(p_1^1 \circ \Delta_0^2 - p_1^1 \circ \Delta_1^2 + p_1^1 \circ \Delta_2^2).
\end{aligned} \tag{13.24}$$

We next compute terms inside the parenthesis. First,

$$p_0^1 \circ \Delta_0^2(t_0, t_1) = p_0^1(0, t_0, t_1) = (t_0, t_1, t_0 + t_1) \tag{13.25}$$
$$p_0^1 \circ \Delta_1^2(t_0, t_1) = p_0^1(t_0, 0, t_1) = (t_0, t_1, t_1) \tag{13.26}$$
$$p_0^1 \circ \Delta_2^2(t_0, t_1) = p_0^1(t_0, t_1, 0) = (t_0 + t_1, 0, t_1). \tag{13.27}$$

Next,

$$p_1^1 \circ \Delta_0^2(t_0, t_1) = p_0^1(0, t_0, t_1) = (0, t_0 + t_1, t_1) \tag{13.28}$$
$$p_1^1 \circ \Delta_1^2(t_0, t_1) = p_0^1(t_0, 0, t_1) = (t_0, t_1, t_1) \tag{13.29}$$
$$p_1^1 \circ \Delta_2^2(t_0, t_1) = p_0^1(t_0, t_1, 0) = (t_0, t_1, 0). \tag{13.30}$$

So we have

$$\begin{aligned} \partial_2 S_1 c_1 = H_*\Big(&(t_0, t_1, t_0 + t_1) - (t_0, t_1, t_1) + (t_0 + t_1, 0, t_1) \\ &- (0, t_0 + t_1, t_1) + (t_0, t_1, t_1) - (t_0, t_1, 0)\Big) \\ = H&(t, 1) + H(e_0, t_1) - H(e_1, t_1) - H(t, 0) \\ = s_1&c_1 + c_1(e_0) - c_1(e_1) - c_1. \end{aligned} \tag{13.31}$$

The term $S_0\partial_1 c_1$ is

$$\begin{aligned} S_0\partial_1 c_1 &= S_0(c_1 \circ \Delta_0^1 - c_1 \circ \Delta_1^1) \\ &= S_0(c_1(e_1) - c_1(e_0)) = c_1(e_1) - c_1(e_0). \end{aligned} \tag{13.32}$$

So summing everything up gives

$$\begin{aligned} \partial_2 S_1 c_1 + S_0\partial_1 c_1 &= s_1 c_1 + c_1(e_0) - c_1(e_1) - c_1 + c_1(e_1) - c_1(e_0) \\ &= s_1 c_1 - c_1 = (\iota_1 \circ s_1)c_1 - c_1. \end{aligned} \tag{13.33}$$

For the higher case, one proceeds by induction, where the homotopy also smooths out the boundary faces using the Whitney approximation theorem. Then one can cut $\Delta^p \times [0, 1]$ into $p + 1$ $(p + 1)$-simplices. For details, see [Lee13, page 473-480].

We will also discuss a different proof than Lee, which avoids invoking the Whitney approximation theorem. Both proofs use the decomposition of $\Delta^p \times [0, 1]$ into $p + 1$ $(p + 1)$-simplices for $p > 1$, this is called the "prism construction".

14 Lecture 14

14.1 The prism operator

Given a topological space X, we will define an operator

$$S_p : C_p^0(X, \mathbb{R}) \to C_{p+1}^0(X \times [0, 1], \mathbb{R}) \tag{14.1}$$

such that

$$(\iota_1)_* - (\iota_0)_* = \partial_{p+1} S_p + S_{p-1}\partial_p, \tag{14.2}$$

where $\iota_t : X \to X \times [0,1]$ is the inclusion $\iota_t(x) = (x,t)$.

In other words, S is a chain homotopy between the morphisms $(\iota_0)_*$ and $(\iota_1)_*$ from the singular chain complex on X and the singular chain complex on $X \times [0,1]$.

We only need to define S_p for for singular p-simplices, and extend to all chain by linearity.

14.2 $p = 0$

First, we discuss $p = 0$. Recall that

$$\Delta^0 = \{1\} \subset \mathbb{R} \tag{14.3}$$

$$\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1,\ t_i \geq 0, i = 0,1\}. \tag{14.4}$$

Define

$$p_0^0 : \Delta^1 \to \Delta^0 \times [0,1] \tag{14.5}$$

by

$$p_0^0((t_0,t_1)) = (t_0 + t_1, t_1) = (1, t_1). \tag{14.6}$$

A 0-simplex is a mapping $c_0 : \Delta^0 \to M$. Consider the mapping

$$c_0 \times id : \Delta^0 \times [0,1] \to M \times [0,1] \tag{14.7}$$

given by $(c_0 \times id)(x,s) = (c(x), s)$. Define $S_0 c_0 = c_1$, where $c_1 : \Delta^1 \to M$ is $c_1 = (c_0 \times id) \circ p_0^0$. That is,

$$(S_0 c_0)(t_0, t_1) = (c_0 \times id)(1, t_1) = (c_0(e_0), t_1). \tag{14.8}$$

We need to verify that

$$(\iota_1)_* c_0 - (\iota_0)_* c_0 = \partial_1 S_0 c_0. \tag{14.9}$$

For $e_0 \in \Delta^0$, the left hand side evaluated on e_0 is

$$(\iota_1)_* c_0(e_0) - (\iota_0)_* c_0(e_0) = \iota_1(c_0(e_0)) - \iota_0(c_0(e_0)) = (c_0(e_0), 1) - (c_0(e_0), 0). \tag{14.10}$$

The right hand side is

$$\begin{aligned} \partial_1 S_0 c_0 = \partial_1 c_1 &= \partial_1((c_0 \times id)_* \circ p_0^0) \\ &= (c_0 \times id)_* \partial_1 p_0^0 = (c_0 \times id)_*(p_0^0 \circ \Delta_0^1 - p_0^0 \circ \Delta_1^1). \end{aligned} \tag{14.11}$$

We have

$$p_0^0 \circ \Delta_0^1(1) = p_0^0(0,1) = (1,1) \tag{14.12}$$

and

$$p_0^0 \circ \Delta_1^1(1) = p_0^0(1,0) = (1,0). \tag{14.13}$$

So we have

$$\partial_1 S_0 c_0(e_0) = (c_0 \times id)_*(1,1) - (c_0 \times id)_*(1,0) = (c_0(e_0),1) - (c_0(e_0),0), \tag{14.14}$$

and we are done with $p = 0$.

14.3 $p = 1$

We next consider $p = 1$. Let c_1 be a 1-simplex $c_1 : \Delta^1 \to X$. We let $e_0 = (1, 0)$, and $e_1 = (0, 1)$ be the unit basis vectors in $\mathbb{R}^2$. Recall that

$$\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 \mid t_0 + t_1 + t_2 = 1,\ t_i \geq 0, i = 0, 1, 2\}. \tag{14.15}$$

We will divide $\Delta^1 \times [0, 1]$ into 2 2-simplices. For $i = 0, 1$, define

$$p^1_i : \Delta^2 \to \Delta^1 \times [0, 1] \tag{14.16}$$

by

$$p^1_0 : (t_0, t_1, t_2) \mapsto (t_0 + t_1, t_2, t_1 + t_2) \tag{14.17}$$
$$p^1_1 : (t_0, t_1, t_2) \mapsto (t_0, t_1 + t_2, t_2). \tag{14.18}$$

Note that p^1_0 maps the following

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_0, 1), \quad e_2 \mapsto (e_1, 1), \tag{14.19}$$

so the image of p^1_0 is the "uppermost" triangle. The mapping p^1_1 maps

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_1, 0), \quad e_2 \mapsto (e_1, 1) \tag{14.20}$$

so the image of p^1_1 is the "lower" triangle. (SHOW FIGURE HERE)

Define

$$S_1(c_1) = (c_1 \times id) \circ p^1_0 - (c_1 \times id) \circ p^1_1. \tag{14.21}$$

We want to verify the formula

$$(\iota_1)_* c_1 - (\iota_0)_* c_1 = \partial_2 S_1 c_1 + S_0 \partial_1 c_1 \tag{14.22}$$

The left hand side is

$$\iota_1 \circ c_1 - \iota_0 \circ c_1 = (c_1, 1) - (c_1, 0). \tag{14.23}$$

Next, we have

$$\begin{aligned}
\partial_2 S_1 c_1 &= \partial_2((c_1 \times id) \circ p^1_0) - \partial_2((c_1 \times id) \circ p^1_1) \\
&= \partial_2((c_1 \times id)_* p^1_0) - \partial_2((c_1 \times id)_* p^1_1) \\
&= (c_1 \times id)_* (\partial_2 p^1_0 - \partial_2 p^1_1) \\
&= (c_1 \times id)_* \Big(\sum_{j=0}^{2} (-1)^j p^1_0 \circ \Delta^2_j - \sum_{j=0}^{2} (-1)^j p^1_1 \circ \Delta^2_j \Big) \\
&= (c_1 \times id)_* (p^1_0 \circ \Delta^2_0 - p^1_0 \circ \Delta^2_1 + p^1_0 \circ \Delta^2_2 - p^1_1 \circ \Delta^2_0 + p^1_1 \circ \Delta^2_1 - p^1_1 \circ \Delta^2_2).
\end{aligned} \tag{14.24}$$

We next compute terms inside the parenthesis. First,

$$p_0^1 \circ \Delta_0^2(t_0, t_1) = p_0^1(0, t_0, t_1) = (t_0, t_1, t_0 + t_1) \tag{14.25}$$
$$p_0^1 \circ \Delta_1^2(t_0, t_1) = p_0^1(t_0, 0, t_1) = (t_0, t_1, t_1) \tag{14.26}$$
$$p_0^1 \circ \Delta_2^2(t_0, t_1) = p_0^1(t_0, t_1, 0) = (t_0 + t_1, 0, t_1). \tag{14.27}$$

Next,

$$p_1^1 \circ \Delta_0^2(t_0, t_1) = p_0^1(0, t_0, t_1) = (0, t_0 + t_1, t_1) \tag{14.28}$$
$$p_1^1 \circ \Delta_1^2(t_0, t_1) = p_0^1(t_0, 0, t_1) = (t_0, t_1, t_1) \tag{14.29}$$
$$p_1^1 \circ \Delta_2^2(t_0, t_1) = p_0^1(t_0, t_1, 0) = (t_0, t_1, 0). \tag{14.30}$$

So we have (with a slight abuse of notation)

$$\begin{aligned}\partial_2 S_1 c_1 = (c_1 \times id)_* \Big(&(t_0, t_1, t_0 + t_1) - (t_0, t_1, t_1) + (t_0 + t_1, 0, t_1) \\ &- (0, t_0 + t_1, t_1) + (t_0, t_1, t_1) - (t_0, t_1, 0) \Big) \\ = (c_1, 1) &+ (c_1(e_0), t_1) - (c_1(e_1), t_1) - (c_1, 0).\end{aligned} \tag{14.31}$$

The term $S_0 \partial_1 c_1$ is

$$\begin{aligned}S_0 \partial_1 c_1 &= S_0(c_1 \circ \Delta_0^1 - c_1 \circ \Delta_1^1) \\ &= S_0(c_1(e_1) - c_1(e_0)) = (c_1(e_1), t_1) - (c_1(e_0), t_1).\end{aligned} \tag{14.32}$$

So summing everything up gives

$$\begin{aligned}\partial_2 S_1 c_1 + S_0 \partial_1 c_1 &= (c_1, 1) + (c(e_0), t_1) - (c(e_1), t_1) - (c_1, 0) + (c_1(e_1), t_1) - (c_1(e_0), t_1) \\ &= (c_1, 1) - (c_1, 0) = \iota_1 \circ c_1 - \iota_0 \circ c_1.\end{aligned} \tag{14.33}$$

15 Lecture 15

In this lecture, we will do the case $p = 2$ for illustration before proving the general case.

15.1 p=2

Let c_2 be a 2-simplex $c_2 : \Delta^2 \to X$. Recall that

$$\Delta^3 = \{(t_0, t_1, t_2, t_2) \in \mathbb{R}^4 \mid t_0 + t_1 + t_2 + t_3 = 1, \; t_i \geq 0, i = 0, 1, 2, 3\}. \tag{15.1}$$

We will divide $\Delta^2 \times [0, 1]$ into 3 3-simplices. For $i = 0, 1, 2$, define

$$p_i^2 : \Delta^3 \to \Delta^2 \times [0, 1] \tag{15.2}$$

by

$$p_0^2 : (t_0, t_1, t_2, t_3) \mapsto (t_0 + t_1, t_2, t_3, t_1 + t_2 + t_3) \tag{15.3}$$
$$p_1^2 : (t_0, t_1, t_2, t_3) \mapsto (t_0, t_1 + t_2, t_3, t_2 + t_3) \tag{15.4}$$
$$p_2^2 : (t_0, t_1, t_2, t_3) \mapsto (t_0, t_1, t_2 + t_3, t_3). \tag{15.5}$$

Note that p_0^2 maps the following

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_0, 1), \quad e_2 \mapsto (e_1, 1), \quad e_3 \mapsto (e_2, 1), \tag{15.6}$$

so the image of p_0^2 is the "uppermost" tetrahedron in the "prism" $\Delta^2 \times [0, 1]$. The mapping p_1^2 maps

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_1, 0), \quad e_2 \mapsto (e_1, 1), \quad e_3 \mapsto (e_2, 1), \tag{15.7}$$

so the image of p_1^2 is the "middle" tetrahedron. The mapping p_2^2 maps

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_1, 0), \quad e_2 \mapsto (e_2, 0), \quad e_3 \mapsto (e_2, 1), \tag{15.8}$$

so the image of p_2^2 is the "lower" tetrahedron. (SHOW FIGURE HERE)

Define $S_2 : C_2(X, \mathbb{R}) \to C_3(X \times [0, 1], \mathbb{R})$ by

$$S_2(c_2) = \sum_{i=0}^{2} (-1)^i (c_2 \times id) \circ p_i^2, \tag{15.9}$$

and extend to all chains by linearity.

We want to verify the formula

$$(\iota_1)_* c_2 - (\iota_0)_* c_2 = \partial_3 S_2 c_2 + S_1 \partial_2 c_2 \tag{15.10}$$

The left hand side is

$$\iota_1 \circ c_2 - \iota_0 \circ c_2 = (c_2, 1) - (c_2, 0). \tag{15.11}$$

Next, we have

$$\begin{aligned}
\partial_3 S_2 c_2 &= \partial_3 \Big(\sum_{i=0}^{2} (-1)^i (c_2 \times id) \circ p_i^2 \Big) \\
&= \sum_{i=0}^{2} (-1)^i \partial_3 \Big((c_2 \times id)_* p_i^2 \Big) \\
&= (c_2 \times id)_* \sum_{i=0}^{2} (-1)^i \partial_3 (p_i^2) \\
&= (c_2 \times id)_* \sum_{i=0}^{2} (-1)^i \sum_{j=0}^{3} (-1)^j p_i^2 \circ \Delta_j^3 \\
&= (c_2 \times id)_* \sum_{i=0}^{2} \sum_{j=0}^{3} (-1)^{i+j} p_i^2 \circ \Delta_j^3.
\end{aligned} \tag{15.12}$$

We next compute the above double sum,

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=0}^{3}(-1)^{i+j}p_i^2\circ\Delta_j^3 &= p_0^2\circ\Delta_0^3 - p_0^2\circ\Delta_1^3 + p_0^2\circ\Delta_2^3 - p_0^2\circ\Delta_3^3\\ &\quad - p_1^2\circ\Delta_0^3 + p_1^2\circ\Delta_1^3 - p_1^2\circ\Delta_2^3 + p_1^2\circ\Delta_3^3\\ &\quad - p_2^2\circ\Delta_0^3 - p_2^2\circ\Delta_1^3 + p_2^2\circ\Delta_2^3 - p_2^2\circ\Delta_3^3\\ &= p_0^2(0,t_0,t_1,t_2) - p_0^2(t_0,0,t_1,t_2) + p_0^2(t_0,t_1,0,t_2) - p_0^2(t_0,t_1,t_2,0)\\ &\quad - p_1^2(0,t_0,t_1,t_2) + p_1^2(t_0,0,t_1,t_2) - p_1^2(t_0,t_1,0,t_2) + p_1^2(t_0,t_1,t_2,0)\\ &\quad + p_2^2(0,t_0,t_1,t_2) - p_2^2(t_0,0,t_1,t_2) + p_2^2(t_0,t_1,0,t_2) - p_2^2(t_0,t_1,t_2,0)\\ &= (t_0,t_1,t_2,1) - (t_0,t_1,t_2,t_1+t_2) + (t_0+t_1,0,t_2,t_1+t_2) - (t_0+t_1,t_2,0,t_1+t_2)\\ &\quad - (0,t_0+t_1,t_2,t_1+t_2) + (t_0,t_1,t_2,t_1+t_2) - (t_0,t_1,t_2,t_2) + (t_0,t_1+t_2,0,t_2)\\ &\quad + (0,t_0,t_1+t_2,t_2) - (t_0,0,t_1+t_2,t_2) + (t_0,t_1,t_2,t_2) - (t_0,t_1,t_2,0).\end{aligned} \tag{15.13}$$

Note that the second term and sixth term cancel, and the seventh and eleventh terms cancel (these correspond to the overlapping faces in the interior of the prism), so we have

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=0}^{3}(-1)^{i+j}p_i^2\circ\Delta_j^3 &= (t_0,t_1,t_2,1) + (t_0+t_1,0,t_2,t_1+t_2) - (t_0+t_1,t_2,0,t_1+t_2)\\ &\quad - (0,t_0+t_1,t_2,t_1+t_2) + (t_0,t_1+t_2,0,t_2)\\ &\quad + (0,t_0,t_1+t_2,t_2) - (t_0,0,t_1+t_2,t_2) - (t_0,t_1,t_2,0).\end{aligned} \tag{15.14}$$

The first and last terms are the ones we want. We want the remaining 6 terms to cancel out with the prism construction applied to the boundary of c_2 (since ∂c_2 consists of 3 1-simplices, the prism construction applied to these will yield $6 = 2\cdot 3$ 2-simplices). Next, we compute the term $S_1\partial_2 c_2$,

$$\begin{aligned}S_1\partial_2 c_2 &= S_1\Big(\sum_{i=0}^{2}(-1)^i c_2\circ\Delta_i^2\Big)\\ &= \sum_{i=0}^{2}(-1)^i S_1(c_2\circ\Delta_i^2)\\ &= \sum_{i=0}^{2}(-1)^i\sum_{j=0}^{1}(-1)^j\Big((c_2\circ\Delta_i^2)\times id\Big)\circ p_j^1\\ &= \sum_{i=0}^{2}\sum_{j=0}^{1}(-1)^{i+j}(c_2\times id)\circ(\Delta_i^2\times id)\circ p_j^1\\ &= (c_2\times id)_*\sum_{i=0}^{2}\sum_{j=0}^{1}(-1)^{i+j}(\Delta_i^2\times id)\circ p_j^1.\end{aligned} \tag{15.15}$$

We expand the sum

$$\sum_{i=0}^{2}\sum_{j=0}^{1}(-1)^{i+j}(\Delta_i^2\times id)\circ p_j^1 = (\Delta_0^2\times id)\circ p_0^1 - (\Delta_1^2\times id)\circ p_0^1 + (\Delta_2^2\times id)\circ p_0^1$$
$$- (\Delta_0^2\times id)\circ p_1^1 + (\Delta_1^2\times id)\circ p_1^1 - (\Delta_2^2\times id)\circ p_1^1$$
$$= (\Delta_0^2\times id)(t_0+t_1,t_2,t_1+t_2) - (\Delta_1^2\times id)(t_0+t_1,t_2,t_1+t_2) + (\Delta_2^2\times id)(t_0+t_1,t_2,t_1+t_2)$$
$$- (\Delta_0^2\times id)(t_0,t_1+t_2,t_2) + (\Delta_1^2\times id)(t_0,t_1+t_2,t_2) - (\Delta_2^2\times id)(t_0,t_1+t_2,t_2)$$
$$= (0,t_0+t_1,t_2,t_1+t_2) - (t_0+t_1,0,t_2,t_1+t_2) + (t_0+t_1,t_2,0,t_1+t_2)$$
$$- (0,t_0,t_1+t_2,t_2) + (t_0,0,t_1+t_2,t_2) - (t_0,t_1+t_2,0,t_2) \tag{15.16}$$

So summing everything up gives

$$\partial_3 S_2 c_2 + S_1\partial_2 c_2 = (c_2\times id)_*\Big(\sum_{i=0}^{2}\sum_{j=0}^{3}(-1)^{i+j}p_i^2\circ\Delta_j^3 + \sum_{i=0}^{2}\sum_{j=0}^{1}(-1)^{i+j}(\Delta_j^2\times id)\circ p_j^1\Big)$$
$$= (c_2\times id)_*\Big((t_0,t_1,t_2,1) - (t_0,t_1,t_2,0)\Big)$$
$$= (c_2,1) - (c_2,0). \tag{15.17}$$

16 Lecture 16

16.1 General case

We will divide $\Delta^n\times[0,1]$ into $(n+1)$ $(n+1)$-simplices. For $i=0,\dots,n$, define

$$p_i^n : \Delta^{n+1}\to\Delta^n\times[0,1] \tag{16.1}$$

by

$$(t_0,\dots,t_{n+1})\mapsto((t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_{n+1}),t_{i+1}+\cdots+t_{n+1}) \tag{16.2}$$

We will view $\Delta^n\times[0,1]\subset\mathbb{R}^{n+2}$, so will henceforth omit the inner parenthesis.

Define $S_n : C_n(X,\mathbb{R})\to C_{n+1}(X\times[0,1],\mathbb{R})$ by

$$S_n(c_n) = \sum_{i=0}^{n}(-1)^i(c_n\times id)\circ p_i^n, \tag{16.3}$$

and extend to all chains by linearity.

We want to verify the formula

$$(\iota_1)_*c_n - (\iota_0)_*c_n = \partial_{n+1}S_nc_n + S_{n-1}\partial_nc_n \tag{16.4}$$

The left hand side of (16.4) is

$$\iota_1\circ c_n - \iota_0\circ c_n = (c_n,1) - (c_n,0). \tag{16.5}$$

The first term on the right hand side of (16.4) is

$$\begin{aligned}\partial_{n+1} S_n c_n &= \partial_{n+1}\Big(\sum_{i=0}^{n}(-1)^i (c_n \times id)\circ p_i^n\Big)\\ &= \sum_{i=0}^{n}(-1)^i \partial_{n+1}\Big((c_n \times id)_* p_i^n\Big)\\ &= (c_n \times id)_* \sum_{i=0}^{n}(-1)^i \partial_{n+1}(p_i^n)\\ &= (c_n \times id)_* \sum_{i=0}^{n}(-1)^i \sum_{j=0}^{n+1}(-1)^j p_i^n \circ \Delta_j^{n+1}\\ &= (c_n \times id)_* \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j} p_i^n \circ \Delta_j^{n+1}.\end{aligned} \tag{16.6}$$

The second term on the right hand side of (16.4) is

$$\begin{aligned}S_{n-1}\partial_n c_n &= S_{n-1}\Big(\sum_{i=0}^{n}(-1)^i c_n \circ \Delta_i^n\Big)\\ &= \sum_{i=0}^{n}(-1)^i S_{n-1}(c_n \circ \Delta_i^n)\\ &= \sum_{i=0}^{n}(-1)^i \sum_{j=0}^{n-1}(-1)^j \Big((c_n \circ \Delta_i^n)\times id\Big)\circ p_j^{n-1}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(c_n \times id)\circ(\Delta_i^n \times id)\circ p_j^{n-1}\\ &= (c_n \times id)_* \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(\Delta_i^n \times id)\circ p_j^{n-1}.\end{aligned} \tag{16.7}$$

Combining these yields

$$\begin{aligned}&\partial_{n+1} S_n c_n + S_{n-1}\partial_n c_n\\ &= (c_n \times id)_*\Big(\sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j} p_i^n \circ \Delta_j^{n+1} + \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(\Delta_i^n \times id)\circ p_j^{n-1}\Big).\end{aligned} \tag{16.8}$$

To analyze these sums, we need a few lemmas. Let $\iota_s : \Delta^n \to \Delta^n \times [0,1]$ be the mapping $\iota_s(x) = (x, s)$.

Lemma 16.1. *We have*

$$p_0^n \circ \Delta_0^{n+1} = \iota_1 \tag{16.9}$$
$$p_n^n \circ \Delta_{n+1}^{n+1} = \iota_0 \tag{16.10}$$
$$p_i^n \circ \Delta_i^{n+1} = p_{i-1}^n \circ \Delta_i^{n+1} \quad 1 \le i \le n. \tag{16.11}$$

Proof. The mapping p^n_0 is given by

$$p^n_0(t_0,\dots,t_{n+1}) = (t_0+t_1, t_2,\dots,t_{n+1}, t_1+\cdots+t_{n+1}), \tag{16.12}$$

so we have

$$\begin{aligned} p^n_0 \circ \Delta^{n+1}_0(t_0,\dots,t_n) &= p^n_0(0,t_0,\dots,t_n) = (t_0,t_1,\dots,t_n,t_0+\cdots+t_n) \\ &= (t_0,\dots,t_n,1), \end{aligned} \tag{16.13}$$

which proves (16.9). Next, the mapping p^n_n is given by

$$p^n_n(t_0,\dots,t_{n+1}) = (t_0,\dots,t_n+t_{n+1},t_{n+1}), \tag{16.14}$$

so we have

$$p^n_n \circ \Delta^{n+1}_{n+1}(t_0,\dots,t_n) = p^n_n(t_0,\dots,t_n,0) = (t_0,\dots,t_n,0), \tag{16.15}$$

which proves (16.10).

Next, the left hand side of (16.11) is the mapping

$$\begin{aligned} p^n_i \circ \Delta^{n+1}_i(t_0,\dots,t_n) &= p^n_i(t_0,\dots,t_{i-1},0_i,t_i,\dots,t_n) \\ &= (t_0,\dots,t_{i-1},0_i+t_i,t_{i+1},\dots,t_n,t_i+\cdots+t_n) \\ &= \Big(t_0,\dots,t_n,\sum_{j=i}^n t_j\Big). \end{aligned} \tag{16.16}$$

Next, the right hand side of (16.11) is the mapping

$$\begin{aligned} p^n_{i-1} \circ \Delta^{n+1}_i(t_0,\dots,t_n) &= p^n_{i-1}(t_0,\dots,t_{i-1},0_i,t_i,\dots,t_n) \\ &= (t_0,\dots,t_{i-2},t_{i-1}+0_i,t_i,\dots,t_n,t_i+\cdots+t_n) \\ &= \Big(t_0,\dots,t_n,\sum_{j=i}^n t_j\Big). \end{aligned} \tag{16.17}$$

□

Next, we have the following.

Lemma 16.2. *For $n \geq 0$, and $0 \leq i \leq n$ the mapping*

$$p^n_i : \Delta^{n+1} \to \Delta^n \times [0,1] \tag{16.18}$$

is the unique affine mapping satisfying

$$p^n_i(e_k) = \begin{cases} (e_k,0) & 0 \leq k \leq i \\ (e_{k-1},1) & i < k \leq n+1. \end{cases} \tag{16.19}$$

Also, for $0 \leq i \leq n+1$, the i-th face of the standard $(n+1)$-simplex is the unique affine mapping $\Delta^{n+1}_i : \Delta^n \to \Delta^{n+1}$ satisfying

$$\Delta^{n+1}_i(e_k) = \begin{cases} e_k & 0 \leq k < i \\ e_{k+1} & i \leq k \leq n. \end{cases} \tag{16.20}$$

Proof. The mappings p_i^n is uniquely determined by its action on the vertices for the following reason. The mapping $p_i^n : \Delta^{n+1} \to \Delta^n \times [0,1] \subset \mathbb{R}^{n+2}$ is the restriction of an affine mapping $p_i^n : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$, which is of the form

$$p_i^n = L_i^n + c_i^n, \tag{16.21}$$

where L_i^n is a linear mapping, and c_i^n is a constant vector. Any $t \in \Delta^{n+1}$ can be written as a linear combination

$$t = \sum_{j=0}^{n+1} t_j e_j, \tag{16.22}$$

where $t_j \geq 0$ and $\sum_{j=0}^{n} t_j = 1$, so we have

$$p_i^n(t) = L_i^n\Big(\sum_{j=0}^{n+1} t_j e_j\Big) + c_i^n = \sum_{j=0}^{n+1} t_i L_i^n(e_j) + c_i^n, \tag{16.23}$$

so p_i^n is determined by its action on the vertices, as claimed.

Since

$$p_i^n : (t_0, \ldots, t_{n+1}) \mapsto ((t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), t_{i+1} + \cdots + t_{n+1}), \tag{16.24}$$

formula (16.19) follows immediately.

Similar to above, the mapping Δ_i^{n+1} is uniquely determined by its action on the vertices. The face mapping is $\Delta_i^{n+1} : \Delta^n \to \Delta^{n+1}$ defined by

$$\Delta_i^{n+1}(t_0, \ldots, t_{n1}) = (t_0, \ldots, t_{i-1}, 0_i, t_i, \ldots, t_n), \tag{16.25}$$

and formula (16.20) follows from this. Finally, similar to above, the mapping Δ_i^{n+1} is uniquely determined by its action on the vertices.

□

The next lemma is crucial.

Lemma 16.3. *We have*

$$p_{j+1}^n \circ \Delta_i^{n+1} = (\Delta_i^n \times id) \circ p_j^{n-1} \quad j \geq i \tag{16.26}$$

$$p_j^n \circ \Delta_{i+1}^{n+1} = (\Delta_i^n \times id) \circ p_j^{n-1} \quad j < i. \tag{16.27}$$

Proof. Since both sides of each equation are affine maps with domain trhe standard n-simplex, we just need to check the equation on the vertices e_k, for $0 \leq k \leq n$. For this, we use (16.19) and (16.20).

For (16.26), we assume that $j \geq i$. If $k < i$, then

$$p_{j+1}^n \circ \Delta_i^{n+1}(e_k) = p_{j+1}^n(e_k) = (e_k, 0), \tag{16.28}$$

and

$$(\Delta_i^n \times id) \circ p_j^{n-1}(e_k) = (\Delta_i^n \times id)(e_k, 0) = (e_k, 0). \tag{16.29}$$

Next, if $i \leq k \leq j$, then

$$p^n_{j+1} \circ \Delta^{n+1}_i(e_k) = p^n_{j+1}(e_{k+1}) = (e_{k+1}, 0), \tag{16.30}$$

and

$$(\Delta^n_i \times id) \circ p^{n-1}_j(e_k) = (\Delta^n_i \times id)(e_k, 0) = (e_{k+1}, 0). \tag{16.31}$$

Next, if $k \geq j$, then

$$p^n_{j+1} \circ \Delta^{n+1}_i(e_k) = p^n_{j+1}(e_{k+1}) = (e_k, 1), \tag{16.32}$$

and

$$(\Delta^n_i \times id) \circ p^{n-1}_j(e_k) = (\Delta^n_i \times id)(e_{k-1}, 1) = (e_k, 1). \tag{16.33}$$

The formula (16.27) is proved similarly, the details are left as an exercise. □

Exercise 16.4. Prove formula (16.27).

Now we return to analyzing the formula

$$\begin{aligned} &\partial_{n+1} S_n c_n + S_{n-1}\partial_n c_n \\ &= (c_n \times id)_* \Big(\sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} p^n_i \circ \Delta^{n+1}_j + \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big). \end{aligned} \tag{16.34}$$

We split up the second sum on the right hand side as follows

$$\begin{aligned} \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big) &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big) \\ &+ \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big) \\ &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} p^n_j \circ \Delta^{n+1}_{i+1} \\ &+ \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} p^n_{j+1} \circ \Delta^{n+1}_i, \end{aligned} \tag{16.35}$$

where we used (16.26) and (16.27) on the last line.

Flipping i and j, we write the other sum as

$$\begin{aligned} \sum_{j=0}^{n} \sum_{i=0}^{n+1} (-1)^{i+j} p^n_j \circ \Delta^{n+1}_i &= \Big(\sum_{0 \leq i < j \leq n} + \sum_{i=j=0}^{n} + \sum_{i=j+1=1}^{n+1} + \sum_{1 \leq j+1 < i \leq n+1} \Big) (-1)^{i+j} p^n_j \circ \Delta^{n+1}_i \\ &= \sum_{0 \leq i < j \leq n} (-1)^{i+j} p^n_j \circ \Delta^{n+1}_i + \sum_{i=0}^{n} p^n_i \circ \Delta^{n+1}_i - \sum_{i=1}^{n+1} p^n_{i-1} \circ \Delta^{n+1}_i \\ &+ \sum_{1 \leq j+1 \leq i \leq n+1} (-1)^{i+j} p^n_j \circ \Delta^{n+1}_i. \end{aligned} \tag{16.36}$$

Using (16.9), (16.10), and (16.11), the middle two sums are

$$\sum_{i=0}^{n} p_i^n \circ \Delta_i^{n+1} - \sum_{i=1}^{n+1} p_{i-1}^n \circ \Delta_i^{n+1} = p_0^n \circ \Delta_o^{n+1} - p_n^n \circ \Delta_i^{n+1} = \iota_1 - \iota_0. \tag{16.37}$$

(This is the cancellation of interior overlapping faces, leaving only to top face minus the bottom face).

The other two sums are

$$\begin{aligned} &\sum_{0\le i<j\le n} (-1)^{i+j} p_j^n \circ \Delta_i^{n+1} + \sum_{1\le j+1<i\le n+1} (-1)^{i+j} p_j^n \circ \Delta_i^{n+1} \\ &\qquad = -\sum_{0\le i\le j\le n} (-1)^{i+j} p_{j+1}^n \circ \Delta_i^{n+1} - \sum_{0\le j<i\le n} (-1)^{i+j} p_j^n \circ \Delta_{i+1}^{n+1} \end{aligned} \tag{16.38}$$

which follows from making the reindexing $j' = j-1$ in the first sum, and $i' = i-1$ in the second sum. These terms cancel out those in (16.35).

17 Lecture 17

17.1 Homotopy invariance of homology

Definition 17.1. Let X and Y be topological spaces. Continuous mappings $f, g : X \to Y$ are said to be continuously homotopic if there exists a continuous mapping $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

Let X and Y be smooth manifolds. Smooth mappings $f, g : X \to Y$ are said to be smoothly homotopic if there exists a smooth mapping $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

Proposition 17.2. *If $f, g : X \to Y$ are continuously homotopic then*

$$H_k f = H_k g : H_k(X, \mathbb{R}) \to H_k(Y, \mathbb{R}) \tag{17.1}$$

If $f, g : X \to Y$ are smoothly homotopic then

$$H_k f = H_k g : H_k^\infty(X, \mathbb{R}) \to H_k^\infty(Y, \mathbb{R}) \tag{17.2}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x,t)$. In the previous lecture, we constructed mappings

$$S_k : C_k(X, \mathbb{R}) \to C_{k+1}(X \times [0,1], \mathbb{R}) \tag{17.3}$$

such that

$$(\iota_1)_* - (\iota_0)_* = \partial_{k+1} S_k + S_{k-1} \partial_k, \tag{17.4}$$

which implies that

$$H_k \iota_0 = H_k \iota_1 : H_k(X, \mathbb{R}) \to H_k(X \times [0,1], \mathbb{R}). \tag{17.5}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, and H_k is a covariant functor, we have

$$H_k f = H_k F \circ H_k \iota_0, \quad H_k g = H_k F \circ H_k \iota_1, \tag{17.6}$$

therefore $H_k f = H_k g : H_k(X, \mathbb{R}) \to H_k(Y, \mathbb{R})$.

Note that in the smooth case, the operator S_k when restricted to smooth chains, maps to smooth chains, that is

$$S_k : C_k^{\infty}(X, \mathbb{R}) \to C_{k+1}^{\infty}(X \times [0, 1]), \tag{17.7}$$

so the same argument is valid in the smooth case. □

Corollary 17.3. *The homology groups of* $\mathbb{R}^n$ *are given by*

$$H_k(\mathbb{R}^n, \mathbb{R}) = H_k^{\infty}(\mathbb{R}^n, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{17.8}$$

Proof. Consider first $n = 0$, in which case $\mathbb{R}^0$ is a single point. Any continuous (smooth) k-simplex is $c_k : \Delta^k \to \mathbb{R}^0$ is a constant mapping, so $C_k(\mathbb{R}^0, \mathbb{R}) = \mathbb{R}$ for all $k \geq 0$. Also, we have

$$\partial_1 = 0, \quad \partial_2 c_2 = c_2, \quad \partial_3 = 0, \quad \partial_4 c_4 = c_4, \ldots. \tag{17.9}$$

Therefore, the chain complex looks like

$$\cdots \xrightarrow{\partial_4 = 1} \mathbb{R} \xrightarrow{\partial_3 = 0} \mathbb{R} \xrightarrow{\partial_2 = 1} \mathbb{R} \xrightarrow{\partial_1 = 0} \mathbb{R} \xrightarrow{\partial_0 = 0} 0, \tag{17.10}$$

and the claim follows for $n = 0$.

The mapping $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ defined by

$$F(x, t) = tx. \tag{17.11}$$

is a homotopy from the zero mapping O to the identity map $Id : \mathbb{R}^n \to \mathbb{R}^n$. Proposition 19.2 says that

$$H_k O = H_k Id : H_k(\mathbb{R}^n, \mathbb{R}) \to H_k(\mathbb{R}^n, \mathbb{R}). \tag{17.12}$$

But note that O factors as

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{o} & \mathbb{R}^0 = \{0\} \\ & \searrow^{O} & \downarrow i \\ & & \mathbb{R}^n \end{array},$$

that is $O = i \circ o$, which implies that

$$Id = H_k O = H_k i \circ H_k o, \tag{17.13}$$

in a diagram, this is

$$\begin{array}{ccc} H_k(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{H_k o} & H_k(\mathbb{R}^0,\mathbb{R}) \\ & \overset{Id}{\searrow} & \big\downarrow H_k i \\ & & H_k(\mathbb{R}^n,\mathbb{R}) \end{array}.$$

This implies that $H_k i$ is surjective and $H_k o$ is injective.

For $k = 0$, the mapping $H_0 o : H_0(\mathbb{R}^n, \mathbb{R}) \to H_0(\mathbb{R}^0, \mathbb{R}) = \mathbb{R}$ is surjective since the constant zero-simplex is a generator of $H_0(\mathbb{R}^0, \mathbb{R})$ and is the image of a constant simplex in $\mathbb{R}^n$. Therefore both $H_0 o$ and $H_0 i$ are isomorphisms.

For $k > 0$, we have that $H_k O$ must be the zero mapping, but this is equal to the identity mapping only if the vector space is 0-dimensional. □

17.2 Homotopy type

Definition 17.4. Topological spaces X and Y have the same homotopy type if there exist continuous mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to id_Y.

Smooth manifolds X and Y have the same smooth homotopy type if there exist smooth mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is smoothly homotopic to Id_X and $f \circ g$ is smoothly homotopic to id_Y.

Corollary 17.5. *If X and Y have the same homotopy type, then $H_*(X, \mathbb{R}) \cong H_*(Y, \mathbb{R})$. If X and Y have the same smooth homotopy type, then $H_*^\infty(X, \mathbb{R}) \cong H_*^\infty(Y, \mathbb{R})$.*

Proof. From Proposition 19.2, we have

$$H_* g \circ H_* f = Id_{H_*(X,\mathbb{R})} \tag{17.14}$$

$$H_* f \circ H_* g = Id_{H_*(Y,\mathbb{R})}, \tag{17.15}$$

so $H_* f$ and $H_* g$ are isomorphisms. □

Some special cases of this are the following.

Definition 17.6. A space X is contractible if X has the same homotopy type as a point. A smooth manifold X is smoothly contractible if X has the same smooth homotopy type as a point.

Corollary 17.7. *If X is contractible, then*

$$H_k(X, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{17.16}$$

If X is smoothly contractible, then

$$H_k^\infty(X, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{17.17}$$

Example 17.8. A domain $A \subset \mathbb{R}^n$ is star-shaped if there exists a $p \in A$ such that for any $x \in A$, the line segment between p and x is contained in A. In this case, let $F : A \times [0,1] \to \mathbb{R}^n$ be the mapping $F(x,t) = (1-t)x + tp$. This shows that A is (smoothly) contractible, so A has the same homology groups as a point.

Definition 17.9. A subset (submanifold) $i : A \hookrightarrow X$ is a (smooth) deformation retraction of X if there exists a (smooth) mapping $r : X \to X$ such that

$$r \circ i = id_A, \tag{17.18}$$

and $i \circ r$ is (smoothly) homotopic to Id_X.

Corollary 17.10. *If A is a (smooth) deformation retraction of X then the (smooth) singular homology groups are isomorphic.*

Example 17.11. Consider $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \subset \mathbb{R}^n$ given by $r(x) = x/|x|$. The mapping $F(x,t) = (1-t)x + t(x/|x|)$ is a smooth homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, so S^{n-1} is a smooth deformation retraction of $\mathbb{R}^n \setminus \{0\}$ and we therefore have

$$H_k(S^{n-1}, \mathbb{R}) = H_k(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \tag{17.19}$$

$$H_k^{\infty}(S^{n-1}, \mathbb{R}) = H_k^{\infty}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}). \tag{17.20}$$

18 Lecture 18

18.1 Cochain complexes

A collection A^p of vector spaces for $p \geq 0$ and operators $\delta_A^p : A^p \to A^{p+1}$ for $p \geq 0$ satisfying $\delta_A^{p+1}\delta_A^p = 0$ is called a *cochain complex.*

$$\cdots \xrightarrow{\delta_A^{p-2}} A^{p-1} \xrightarrow{\delta_A^{p-1}} A^p \xrightarrow{\delta_A^{p}} A^{p+1} \xrightarrow{\delta_A^{p+1}} \cdots . \tag{18.1}$$

Definition 18.1. The pth cohomology of a chain complex is the vector space

$$H^p(A) = \frac{Ker\{\delta_A^p : A^p \to A^{p+1}\}}{Im\{\delta_A^{p-1} : A^{p-1} \to A^p\}} \tag{18.2}$$

Definition 18.2. A morphism $\alpha : A \to B$ of cochain complexes is a collection of mappings $\alpha^p : A^p \to B^p$ such that $\delta_B^p \alpha^p = \alpha^{p+1}\delta_A^p$ for $p \geq 0$. In other words, $\alpha : A \to B$ is a morphism if the following diagram commutes

$$\begin{array}{ccc} A^p & \xrightarrow{\delta_A^p} & A_{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ B^p & \xrightarrow{\delta_B^p} & B^{p+1}. \end{array} \tag{18.3}$$

Proposition 18.3. *A morphism of cochain complexes $\alpha : A \to B$ induces mappings $H^p\alpha^p : H^p(A) \to H^p(B)$.*

Proof. Given $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$, we have

$$\delta_B^p \alpha^p a^p = \alpha^{p+1} \delta_A^p a^p = 0, \tag{18.4}$$

therefore we can define $(H^p \alpha^p)[a^p] = [\alpha^p a^p]$. To check that this is well-defined,

$$[\alpha^p(a^p + \delta_A^{p-1} a^{p-1})] = [\alpha^p a^p + \alpha^p \delta_A^{p-1} a^{p-1}] = [\alpha^p a^p + \delta_B^{p-1} \alpha^{p-1} a^{p-1}] = [\alpha^p a^p]. \tag{18.5}$$

□

18.2 Cochain homotopy between morphisms of cochain complexes

Definition 18.4. Let $f : A \to B$, and $g : A \to B$ be two morphisms of cochain complexes. We say that f is cochain homotopic to g if there exists mappings $S^p : A^p \to B^{p-1}$ such that

$$f^p - g^p = \delta_B^{p-1} S^p + S^{p+1} \delta_A^p. \tag{18.6}$$

Proposition 18.5. *If f is cochain homotopic to g then $H^p f = H^p g : H^p(A) \to H^p(B)$.*

Proof. Consider the mapping $H^p f - H^p g$, and take $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$. Then

$$\begin{aligned}(H^p f - H^p g)[a^p] &= (H^p(f-g))[a^p] = [(H^p(f-g))a^p] \\ &= [\delta_B^{p-1} S^p a^p + S^{p+1} \delta_A^p a^p = [\delta_B^{p-1} S^p a^p] = 0.\end{aligned} \tag{18.7}$$

□

18.3 Homotopy invariance of singular cohomology

Let us recall the definition of singular cohomology. Let $C^p(X, \mathbb{R})$ denote the singular cochains, which are dual to singular chains, i.e.,

$$C^p(X, \mathbb{R}) = Hom(C_p(X, \mathbb{R}), \mathbb{R}), \tag{18.8}$$

and let $\delta^p : C^p(X, \mathbb{R}) \to C^{p+1}(X, \mathbb{R})$ denote the dual to the boundary operator $\partial_{p+1} : C_{p+1}(X, \mathbb{R}) \to C_p(X, \mathbb{R})$, defined as follows. For $c^p \in C^p(X, \mathbb{R})$ and $c_{p+1} \in C_p(X, \mathbb{R})$,

$$(\delta^p c^p)(c_{p+1}) = c^p(\partial_{p+1} c_{p+1}). \tag{18.9}$$

Since $\partial_p \circ \partial_{p+1} = 0$, we have $\delta^{p+1} \circ \delta^p = 0$, so we have *the cochain complex of singular cochains*

$$\cdots \xrightarrow{\delta^{p-2}} C^{p-1}(X, \mathbb{R}) \xrightarrow{\delta^{p-1}} C^p(X, \mathbb{R}) \xrightarrow{\delta^p} C^{p+1}(X, \mathbb{R}) \xrightarrow{\delta^{p+1}} \cdots . \tag{18.10}$$

The cohomology groups of this complex are the singular cohomology groups (with real coefficients)

$$H^p(X,\mathbb{R}) = \frac{Ker\{\delta^p : C^p(X,\mathbb{R}) \to C^{p+1}(X,\mathbb{R})\}}{Im\{\delta^{p-1} : C^{p-1}(X,\mathbb{R}) \to C^p(X,\mathbb{R})\}}. \tag{18.11}$$

If X happens to be a smooth manifold, then recall we have the smooth singular chains $C_p^\infty(X,\mathbb{R})$ and we can define

$$C^p_\infty(X,\mathbb{R}) = Hom(C_p^\infty(X,\mathbb{R}),\mathbb{R}). \tag{18.12}$$

Since $\partial_{p+1} : C^\infty_{p+1}(X,\mathbb{R}) \to C^\infty_p(X,\mathbb{R})$, we obtain the *cochain complex of smooth singular cochains*

$$\cdots \xrightarrow{\delta^{p-2}} C^{p-1}_\infty(X,\mathbb{R}) \xrightarrow{\delta^{p-1}} C^p_\infty(X,\mathbb{R}) \xrightarrow{\delta^p} C^{p+1}_\infty(X,\mathbb{R}) \xrightarrow{\delta^{p+1}} \cdots . \tag{18.13}$$

The cohomology groups of this complex are the smooth singular cohomology groups (with real coefficients)

$$H^p_\infty(X,\mathbb{R}) = \frac{Ker\{\delta^p : C^p_\infty(X,\mathbb{R}) \to C^{p+1}_\infty(X,\mathbb{R})\}}{Im\{\delta^{p-1} : C^{p-1}_\infty(X,\mathbb{R}) \to C^p_\infty(X,\mathbb{R})\}}. \tag{18.14}$$

Recall that a continuous (smooth) mapping $f : X \to Y$ induces a mapping $(f_*)_p : C_p(X,\mathbb{R}) \to C_p(Y,\mathbb{R})$ by $(f_*)_p(c_p) = f \circ c_p$ for a simplex c_p, and extend to all chains by linearity. These yield a morphism of chain complexes, $f_* : C_*(X,\mathbb{R}) \to C_*(Y,\mathbb{R})$, i.e.,

$$\partial^Y_{p+1}(f_*)_{p+1} = (f_*)_p\partial^X_{p+1} \tag{18.15}$$

The dual mapping of $(f_*)_p$ is

$$(f^*)^p : C^p(Y,\mathbb{R}) \to C^p(X,\mathbb{R}). \tag{18.16}$$

Taking the dual of (18.15) yields

$$(f^*)^{p+1}\delta^p_Y = \delta^p_X(f^*)^p, \tag{18.17}$$

so these dual mappings yield a morphism of chain complexes, $f^* : C^*(Y,\mathbb{R}) \to C^*(X,\mathbb{R})$, and by Proposition 18.3, we have induced mappings

$$H^pf : H^p(Y,\mathbb{R}) \to H^p(X,\mathbb{R}) \tag{18.18}$$

in the continuous case, and

$$H^pf : H^p_\infty(Y,\mathbb{R}) \to H^p_\infty(X,\mathbb{R}) \tag{18.19}$$

in the smooth case.

Proposition 18.6. *If $f, g : X \to Y$ are continuously homotopic then*

$$H^k f = H^k g : H^k(Y, \mathbb{R}) \to H^k(X, \mathbb{R}) \tag{18.20}$$

If $f, g : X \to Y$ are smoothly homotopic then

$$H^k f = H^k g : H^k_\infty(Y, \mathbb{R}) \to H^k_\infty(X, \mathbb{R}) \tag{18.21}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x, t)$, and note that

$$(\iota_t)^* : C^*(X \times [0,1], \mathbb{R}) \to C^*(X, \mathbb{R}). \tag{18.22}$$

Previously, we constructed mappings

$$S_k : C_k(X, \mathbb{R}) \to C_{k+1}(X \times [0,1], \mathbb{R}) \tag{18.23}$$

such that

$$(\iota_1)_* - (\iota_0)_* = \partial_{k+1} S_k + S_{k-1} \partial_k, \tag{18.24}$$

Let us define

$$S^k = S^*_{k-1} : C^k(X \times [0,1], \mathbb{R}) \to C^{k-1}(X, \mathbb{R}). \tag{18.25}$$

Dualizing the formula (18.24) yields

$$(\iota_1)^* - (\iota_0)^* = S^{k+1}\delta^k + \delta^{k-1} S^k. \tag{18.26}$$

Therefore S^k is a chain homotopy between ι_1^* and ι_0^*. By Proposition 18.5, we have that

$$H^k \iota_0 = H^k \iota_1 : H^k(X \times [0,1], \mathbb{R}) \to H^k(X, \mathbb{R}). \tag{18.27}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, and H^k is a contravariant functor, we have

$$H^k f = H^k \iota_0 \circ H^k F, \quad H^k g = H^k \iota_1 \circ H^k F, \tag{18.28}$$

therefore $H^k f = H^k g : H^k(Y, \mathbb{R}) \to H^k(X, \mathbb{R})$.

Note that in the smooth case, the operator S_k when restricted to smooth chains, maps to smooth chains, that is

$$S_k : C_k^\infty(X, \mathbb{R}) \to C_{k+1}^\infty(X \times [0,1]), \tag{18.29}$$

so we have

$$S^k = S^*_{k-1} : C^k_\infty(X \times [0,1], \mathbb{R}) \to C^{k-1}_\infty(X, \mathbb{R}), \tag{18.30}$$

so the same argument is valid in the smooth case. □

18.4 Homotopy type and singular cohomology

All of the results in Subsection 17.2 hold for cohomology.

Corollary 18.7. *If X and Y have the same homotopy type, then $H^*(X, \mathbb{R}) \cong H^*(Y, \mathbb{R})$. If X and Y have the same smooth homotopy type, then $H^*_\infty(X, \mathbb{R}) \cong H^*_\infty(Y, \mathbb{R})$. In both cases, the cohomology algebras are isomorphic as algebras.*

The cohomology of a point follows easily from dualizing (17.10), so we have the following corollary.

Corollary 18.8. *If X is contractible, then*

$$H^k(X, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{18.31}$$

If X is smoothly contractible, then

$$H^k_\infty(X, \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{18.32}$$

Example 18.9. A star-shaped domain $A \subset \mathbb{R}^n$ is contractible, so has the same cohomology groups as a point.

Corollary 18.10. *If A is a (smooth) deformation retraction of X then the (smooth) singular cohomology groups are isomorphic.*

Example 18.11. For $S^{n-1} \subset \mathbb{R}^n$, we have

$$H^k(S^{n-1}, \mathbb{R}) = H^k(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \tag{18.33}$$

$$H^k_\infty(S^{n-1}, \mathbb{R}) = H^k_\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}). \tag{18.34}$$

19 Lecture 19

19.1 Homotopy invariance of de Rham cohomology

Let M be a smooth manifold, possibly noncompact. Let $\Omega^p(M)$ denote the smooth p-forms on M. Recall that we have a cocomplex

$$\cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots, \tag{19.1}$$

and $H^p_{dR}(M)$ is defined to be the cohomology of this complex.

Let M be a differentiable n-manifold, and consider $N = M \times [0,1]$. Let $\pi : N \to M$ be the projection $\pi(x,t) = x$. Also, let $\iota_t : M \to M \times [0,1]$ be the inclusion $\iota_t(x) = (x,t)$.

We next define a mapping

$$I^k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M) \tag{19.2}$$

by the following. Any k-form on N can be written as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)dt \wedge (\pi^*\phi_{k-1}), \tag{19.3}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0(M \times \mathbb{R})$. Define

$$I^k(\omega) = \Big(\int_0^1 f(p,t)dt\Big)\phi_{k-1}. \tag{19.4}$$

Proposition 19.1. *For $\omega \in \Omega^k(N)$, we have*

$$(\iota_1)^*\omega - (\iota_0)^*\omega = d_M I^k\omega + I^{k+1}d_N\omega. \tag{19.5}$$

In other words, I^k is a cochain homotopy between $(\iota_0)^*$ and $(\iota_1)^*$.

Proof. Writing ω in the form (19.3), since $\iota_t^* dt = 0$, and $\pi \circ \iota_t = id_M$, the left hand side of (19.5) is

$$\begin{aligned}(\iota_1)^*\omega - (\iota_0)^*\omega &= (\iota_1)^*h(x,t)\pi^*\phi_k - (\iota_0)^*h(x,t)\pi^*\phi_k \\ &= (h(x,1) - h(x,0))\phi_k\end{aligned} \tag{19.6}$$

Next, assume that ω is just of the form

$$\omega = h(x,t)\pi^*\phi_k. \tag{19.7}$$

Then

$$d_N\omega = \Big(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt\Big) \wedge \pi^*\phi_k + h(x,t)\pi^* d_M\phi_k. \tag{19.8}$$

By defintion of I^*,

$$d_M I^k\omega = 0, \tag{19.9}$$

and

$$\begin{aligned}I^{k+1}d_N\omega &= I^{k+1}\Big\{\Big(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt\Big) \wedge \pi^*\phi_k + h(x,t)\pi^*d_M\phi_k\Big\} \\ &= I^{k+1}\Big\{\frac{\partial h}{\partial t}dt \wedge \pi^*\phi_k\Big\} = \Big(\int_0^1 \frac{\partial h}{\partial t}dt\Big)\phi_k = (h(x,1) - h(x,0))\phi_k.\end{aligned} \tag{19.10}$$

So the proposition holds for forms of this type.

Next, assume that ω is just of the form

$$\omega = f(x,t)dt \wedge (\pi^*\phi_{k-1}). \tag{19.11}$$

From (19.6) above, we have

$$(\iota_1)^*\omega - (\iota_0)^*\omega = 0. \tag{19.12}$$

Note that

$$d_N\omega = \frac{\partial f}{\partial x} dx \wedge dt \wedge (\pi^*\phi_{k-1}) - f(x,t) dt \wedge \pi^*(d_M\phi_{k-1})$$
$$= -\frac{\partial f}{\partial x} dt \wedge \pi^*\Big(dx \wedge \phi_{k-1}\Big) - f dt \wedge \pi^*\Big(d_M\phi_{k-1}\Big). \tag{19.13}$$

By defintion of I^k,

$$d_M I^k\omega = d_M\Big\{\Big(\int_0^1 f(x,t)dt\Big)\phi_{k-1}\Big\}$$
$$= \Big(\int_0^1 \frac{\partial f}{\partial x} dt\Big) dx \wedge \phi_{k-1} + \Big(\int_0^1 f dt\Big) d_M\phi_{k-1}. \tag{19.14}$$

Next, by definition of I^{k+1} and (19.13), we have

$$I^{k+1} d_N\omega = -\Big(\int_0^1 \frac{\partial f}{\partial x} dt\Big) dx \wedge \phi_{k-1} - \Big(\int_0^1 f dt\Big) d_M\phi_{k-1}. \tag{19.15}$$

So on forms of this type, we have

$$d_M I^k\omega + I^{k+1} d_N\omega = 0. \tag{19.16}$$

So the proposition is true for forms of the second type. By linearity, the proposition holds for all forms, and we are done. □

Proposition 19.2. *Let X and Y be smooth manifolds. If $f, g : X \to Y$ are smoothly homotopic then*

$$f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X) \tag{19.17}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x,t)$, and note that

$$(\iota_t)^* : \Omega^*(X \times [0,1]) \to \Omega^*(X). \tag{19.18}$$

In Proposition 19.1, we constructed a cochain homotopy between ι_1^* and ι_0^*,

$$I^k : \Omega^k(X \times [0,1]) \to \Omega^{k-1}(X) \tag{19.19}$$

satisfying

$$(\iota_1)^* - (\iota_0)^* = I^{k+1} d_{X\times[0,1]} + d_X I^k. \tag{19.20}$$

By Proposition 18.5, we have that

$$(\iota_0)^* = (\iota_1)^* : H^k_{dR}(X \times [0,1]) \to H^k_{dR}(X). \tag{19.21}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, and H^k_{dR} is a contravariant functor, we have

$$f^* = (\iota_0)^* \circ F^*, \quad g^* = (\iota_1)^* \circ F^*, \tag{19.22}$$

therefore $f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X)$. □

Then all of the results in Subsection 17.2 also hold for de Rham cohomology.

Corollary 19.3. *If smooth manifolds X and Y have the same smooth homotopy type, then $H^*_{dR}(X) \cong H^*_{dR}(Y)$.*

The de Rham cohomology of a point is trivial to compute, so we have the following corollary.

Corollary 19.4. *If M is contractible, then*

$$H^k_{dR}(M) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{19.23}$$

Example 19.5. A star-shaped domain $A \subset \mathbb{R}^n$ is contractible, so has the same de Rham cohomology groups as a point.

Corollary 19.6. *If a submanifold A is a (smooth) deformation retraction of M then the (smooth) singular cohomology groups are isomorphic.*

Example 19.7. For $S^{n-1} \subset \mathbb{R}^n$, we have

$$H^k_{dR}(S^{n-1}) = H^k_{dR}(\mathbb{R}^n \setminus \{0\}). \tag{19.24}$$

20 Lecture 20

20.1 Exact sequences of chain complexes

Let C^i be a complex of vector spaces for $i = 1, 2, 3$.

$$\cdots \xrightarrow{\partial^i_{p+2}} C^i_{p+1} \xrightarrow{\partial^i_{p+1}} C^i_p \xrightarrow{\partial^i_p} C^i_{p-1} \xrightarrow{\partial^i_{p-1}} \cdots. \tag{20.1}$$

with $\partial^2 = 0$. A morphism from C^i to C^j are mappings $\alpha_k : C^i_k \to C^j_k$ such that the following diagram commutes for every p

$$\begin{array}{ccc} C^i_{p+1} & \xrightarrow{\partial^i_{p+1}} & C^i_p \\ \downarrow{\alpha_{p+1}} & & \downarrow{\alpha_p} \\ C^j_{p+1} & \xrightarrow{\partial^j_{p+1}} & C^j_p \end{array} \tag{20.2}$$

Definition 20.1. A sequence of vector spaces A, B, C, with linear mappings $\alpha : A \to B$, $\beta : B \to C$

$$0 \xrightarrow{0} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{0} 0 \tag{20.3}$$

is called *exact* if the kernel of each mapping is equal to the image of the previous mapping. That is $Ker(\alpha) = \{0\}$ if and only if α is injective. Next, $Ker(\beta) = Im(\alpha)$. Finally, $Im(\beta) = C$, if and only if β is surjective.

For complexes C^1, C^2, C^3, and morphisms $\alpha : C^1 \to C^2$ and $\beta : C^2 \to C^3$. We say that a sequence of complexes is exact if

$$0 \xrightarrow{0} C^1 \xrightarrow{\alpha} C^2 \xrightarrow{\beta} C^3 \xrightarrow{0} 0 \tag{20.4}$$

if the sequence

$$0 \xrightarrow{0} C^1_p \xrightarrow{\alpha_p} C^2_p \xrightarrow{\beta_p} C^3_p \xrightarrow{0} 0 \tag{20.5}$$

is exact for every p.

Lemma 20.2 (The zig-zag lemma for chain complexes)**.** *If*

$$0 \xrightarrow{0} C^1 \xrightarrow{\alpha} C^2 \xrightarrow{\beta} C^3 \xrightarrow{0} 0 \tag{20.6}$$

is a short exact sequence of complexes, then there exists mappings

$$\partial_p : H_p(C^3) \to H_{p-1}(C^1) \tag{20.7}$$

for every p such that the sequence

$$\cdots \xrightarrow{\partial_{p+1}} H_p(C_1) \xrightarrow{\alpha_p} H_p(C_2) \xrightarrow{\beta_p} H_p(C_3) \xrightarrow{\partial_p} H_{p-1}(C_1) \longrightarrow \cdots \tag{20.8}$$

is exact.

Proof. We look at the huge commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1_{p+1} & \xrightarrow{\alpha_{p+1}} & C^2_{p+1} & \xrightarrow{\beta_{p+1}} & C^3_{p+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^1_{p+1}} & & \downarrow{\scriptstyle \partial^2_{p+1}} & & \downarrow{\scriptstyle \partial^3_{p+1}} & & \\
0 & \longrightarrow & C^1_p & \xrightarrow{\alpha_p} & C^2_p & \xrightarrow{\beta_p} & C^3_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^1_p} & & \downarrow{\scriptstyle \partial^2_p} & & \downarrow{\scriptstyle \partial^3_p} & & \\
0 & \longrightarrow & C^1_{p-1} & \xrightarrow{\alpha_{p-1}} & C^2_{p-1} & \xrightarrow{\beta_{p-1}} & C^3_{p-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{20.9}$$

which has all horizontal rows exact.

To define the boundary operator, take $c^3_p \in C^3_p$ with $\partial^3_p c^3_p = 0$. By exactness of the middle row, β_p is surjective, so $c^3_p = \beta_p(c^2_p)$ for some $c^2_p \in C^2_p$. Then since the diagram commutes, we have

$$\beta_{p-1}\partial^2_p c^2_p = \partial^3_p \beta_p c^2_p = \partial^3_p c^3_p = 0. \tag{20.10}$$

By exactness of the bottow row, we have $\partial^2_p c^2_p = \alpha_{p-1} c^1_{p-1}$ for some $c^1_{p-1} \in C^1_{p-1}$. Since C^1 is a complex, and by commutativity of the diagram, we have

$$0 = \partial^2_{p-1}\partial^2_p c^2_p = \partial^2_{p-1}\alpha_{p-1}c^1_{p-1} = \alpha_{p-2}\partial^1_{p-1}c^1_{p-1}, \tag{20.11}$$

which implies that $\partial^1_{p-1}c^1_{p-1} = 0$, since α_{p-2} is injective. So we define $\partial_p(c^3_p) = [c^1_{p-1}]$, the homology class of c^1_{p-1} in $H^{p-1}(C^1)$.

To prove this mapping is well-defined, assume that we started with $c^3_p \in C^3_p$ which was of the form $c^3_p = \partial^3_{p+1}c^3_{p+1}$. Then we can write $c^3_{p+1} = \beta_{p+1}c^2_{p+1}$, and the element $\tilde{c}^2_p = \partial^2_{p+1}c^2_{p+1}$ satisfies $\beta_p(\tilde{c}^2_p) = c^3_p$. But this element is exact, so the next step clearly gives zero. Independence of the choice of c^2_p is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing.

□

20.2 Exact sequences of cochain complexes

Let C_i be a co-complex of vector spaces for $i = 1, 2, 3$.

$$\cdots \xrightarrow{d_i^{p-2}} C_i^{p-1} \xrightarrow{d_i^{p-1}} C_i^p \xrightarrow{d_i^p} C_i^{p+1} \xrightarrow{d_i^{p+1}} \cdots . \tag{20.12}$$

with $d^2 = 0$. A morphism from C^i to C^j are mappings $\alpha^k : C^k_i \to C^k_j$ such that the following diagram commutes for every p

$$\begin{array}{ccc} C_i^p & \xrightarrow{d_i^p} & C_i^{p+1} \\ \downarrow{\alpha^p} & & \downarrow{\alpha^{p+1}} \\ C_j^p & \xrightarrow{d_j^p} & C_j^{p+1} \end{array} \tag{20.13}$$

For co-complexes C_1, C_2, C_3, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of co-complexes is exact if

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{20.14}$$

if the sequence

$$0 \xrightarrow{0} C_1^p \xrightarrow{\alpha_p} C_2^p \xrightarrow{\beta_p} C_3^p \xrightarrow{0} 0 \tag{20.15}$$

is exact for every p.

Lemma 20.3 (The zig-zag lemma for cochain complexes)**.** *If*

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{20.16}$$

is a short exact sequence of co-complexes, then there exist mappings

$$\delta^p : H^p(C_3) \to H^{p+1}(C_1) \tag{20.17}$$

for every p such that the sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\delta^p} H^{p+1}(C_1) \longrightarrow \cdots \tag{20.18}$$

is exact.

Proof. Same as before, with arrows reversed. □

21 Lecture 21

21.1 Mayer-Vietoris for de Rham cohomology

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{\beta^p} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{\alpha^p} \Omega^p(U \cap V) \longrightarrow 0 \tag{21.1}$$

where

$$\beta^p(\omega) = \big((i_{U \hookrightarrow M})^* \omega, (i_{V \hookrightarrow M})^* \omega\big). \tag{21.2}$$

and

$$\alpha^p(\omega_U, \omega_V) = (i_{U \cap V \hookrightarrow U})^* \omega_U - (i_{U \cap V \hookrightarrow V})^* \omega_V \tag{21.3}$$

To see this, β^p is obviously injective. For exactness at the middle step, obviously $\alpha^p \beta^p \omega = 0$. If $\beta^p(\omega_U, \omega_V) = 0$, then $\omega_U = \omega_V$ on $U \cap V$, so then (ω_U, ω_V) is a well-defined global form on M.

To show that α is onto, let $\omega \in \Omega^p(U \cap V)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \alpha(\phi_V \omega, -\phi_U \omega)$.

By the zig-zag lemma for cohomology, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{DR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{21.4}$$

Let us review the definition of the mapping δ^p. Given a cohomology class $[\omega] \in H^p_{dR}(U \cap V)$, represented by $\omega \in \Omega^p(U \cap V)$ with $d\omega = 0$, we first write $\omega = \alpha^p(\phi_V \omega, -\phi_U \omega)$, then we apply the exterior derivative to get

$$(d(\phi_V \omega), -d(\phi_U \omega)) = (d\phi_V \wedge \omega, -d\phi_U \wedge \omega) \in \Omega^p(U) \oplus \Omega^p(V). \tag{21.5}$$

Note that on $U \cap V$, we have $(\phi_U + \phi_V)\omega = \omega$, so applying d to this equation, we have that $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$ on $U \cap V$, so together these define a global form

$$\delta^p \omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{21.6}$$

and we take the cohomology class of this form.

Remark 21.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

Example 21.2. S^n: Cover with 2 open sets U, V, with $U \cong \mathbb{R}^n \cong V$ and $U \cap V \cong S^{n-1}$, use the Mayer-Vietoris sequence and induction to get

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases} \tag{21.7}$$

Corollary 21.3. *Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are diffeomorphic only if $m = n$.*

Proof. If $\mathbb{R}^n$ and $\mathbb{R}^m$ are diffeomorphic, then $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{0\}$ are diffeomorphic. From the deformation retraction argument, this would imply that S^{n-1} and S^{m-1} are homotopic. By the previous example, this can only be true if $m = n$. □

Definition 21.4. We say that a manifold M has a good cover U_i each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is diffeomorphic to a star-shaped open set in $\mathbb{R}^n$.

Corollary 21.5. *If M has a finite good cover, then the de Rham cohomology of M is finite-dimensional.*

Proof. Note that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{21.8}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{21.9}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

Now we look at the following portion of the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\alpha^{p-1}} H^{p-1}_{dR}(U \cap V) \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} \cdots \tag{21.10}$$

Using induction on the number of open sets in a finite good cover, the corollary follows. □

Corollary 21.6. *If M is compact, then the de Rham cohomology of M is finite-dimensional.*

Proof. Using a Riemannian metric, there exists a covering of M by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map, it is not hard to see that a geodesically covex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$. It follows that every compact manifold admits a finite good cover. □

21.2 Mayer-Vietoris for singular chains

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow C_p(U \cap V) \xrightarrow{\alpha_p} C_p(U) \oplus C_p(V) \xrightarrow{\beta_p} C_p(U) + C_p(V) \longrightarrow 0 \tag{21.11}$$

where

$$\alpha(c_p) = \big((i_{U\cap V \hookrightarrow U})_* c_p, (i_{U\cap V \hookrightarrow V})_* c_p\big) \tag{21.12}$$

and

$$\beta(a_p, b_p) = (i_{U \hookrightarrow M})_* a_p - (i_{V \hookrightarrow M})_* b_p. \tag{21.13}$$

It is not hard to see this sequence is exact. Furthermore, by a barycentric subdivision argument, the homology $H_*(C_p(U) + C_p(V))$ is isomorphic to $H_*(U \cup V)$. (Roughly, keep subdividing simplices until their images are contained in U or V.) Consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\partial_{p+1}} H_p(U \cap V) \xrightarrow{\alpha_p} H_p(U) \oplus H_p(V) \xrightarrow{\beta_p} H_p(U \cup V) \xrightarrow{\partial_p} \cdots \tag{21.14}$$

We can now improve "diffeomorphism" to "homemorphism" in Corollary 21.3.

Corollary 21.7. *Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic only if $m = n$.*

Proof. The arugment is the same as in the de Rham cohomology case, using singular homology instead. □

Remark 21.8. In the case that M is a smooth manifold, a similar argument gives a Mayer-Vietoris sequence for smooth singular chains.

21.3 Mayer-Vietoris for singular co-chains

First, we have the following lemma.

Lemma 21.9. *Let W_1, W_2, W_3 be vector spaces. If the sequence*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{21.15}$$

is exact at W_2, then the dual sequence

$$W_3^* \xrightarrow{\beta^*} W_2^* \xrightarrow{\alpha^*} W_1^* \tag{21.16}$$

is exact at W_2^*.

Proof. First, if $w_3^* \in W_3^*$, and $w_1 \in W_1$, then

$$\alpha^*(\beta^* w_3^*)(w_1) = (\beta^* w_3^*)(\alpha(w_1)) = w_3^*(\beta\alpha(w_1)) = 0, \tag{21.17}$$

since $\beta \circ \alpha = 1$ by assumption. This proves that $Im(\beta^*) \subset Ker(\alpha^*)$. For the other direction, if $w_2^* \in Ker(\alpha^*)$, then for all $w_1 \in W_1$, $\alpha^*(w_2^*)(w_1) = w_2^*(\alpha(w_1))$. So the element $0 = w_2^* \circ \alpha \in W_1^*$. We want to find $w_3^* \in W_3^*$ such that $w_2^* = \beta^* w_3^*$. For all $w_2 \in W_2$, this is $w_2^*(w_2) = w_3^* \beta w_2$, which is just $w_2^* = w_3^* \circ \beta$. So if $w_3 \in W_3$ is of the form $\beta(w_2)$ then define

$$w_3^*(w_3) \equiv w_2^*(w_2). \tag{21.18}$$

If $w_3 = \beta(w_2')$, then $\beta(w_2 - w_2') = 0$, so $w_2 - w_2' = \alpha(w_1)$. Then

$$w_2^*(w_2 - w_2') = w_2^*(\alpha(w_1)) = (w_2^* \alpha)(w_1) = 0. \tag{21.19}$$

So we have defined w_3^* on the subspace $Im(\beta) \subset W_3$. To extend to a linear mapping on all of W_3, just take any subspace so that $W_3 = Im(\beta) \oplus W$, and define w_3^* to vanish on W. Then the condition $w_2^* = w_3^* \circ \beta$ is obviously satisifed. $\square$

Write $M = U \cup V$ as the union of two open sets in M. By Lemma 21.9, the following sequence is exact:

$$0 \longrightarrow (C_p(U) + C_p(V))^* \xrightarrow{\beta^p} C^p(U) \oplus C^p(V) \xrightarrow{\alpha^p} C^p(U \cap V) \longrightarrow 0 \tag{21.20}$$

where $\beta^p = (\beta_p)^*$ and $\alpha^p = (\alpha_p)^*$.

By the barycentric subdivision argument, one can show that

$$H^p\Big((C_p(U) + C_p(V))^*\Big) \cong H^p(U \cup V, \mathbb{R}), \tag{21.21}$$

consequently, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(U \cup V) \xrightarrow{\beta^p} H^p(U) \oplus H^p(V) \xrightarrow{\alpha^p} H^p(U \cap V) \xrightarrow{\delta^p} \cdots \tag{21.22}$$

Remark 21.10. In the case that M is a smooth manifold, a similar argument gives a Mayer-Vietoris sequence for smooth singular cochains.

22 Lecture 22

22.1 Proof of de Rham's Theorem

The following lemma is crucial for the proof.

Lemma 22.1 (The Five Lemma)**.** *Assume the diagram*

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{\alpha_1} & V_2 & \xrightarrow{\alpha_2} & V_3 & \xrightarrow{\alpha_3} & V_4 & \xrightarrow{\alpha_4} & V_5 \\ \downarrow{\phi_1} & & \downarrow{\phi_2} & & \downarrow{\phi_3} & & \downarrow{\phi_4} & & \downarrow{\phi_5} \\ W_1 & \xrightarrow{\beta_1} & W_2 & \xrightarrow{\beta_2} & W_3 & \xrightarrow{\beta_3} & W_4 & \xrightarrow{\beta_4} & W_5 \end{array} \tag{22.1}$$

commutes, and has exact rows. If $\phi_1, \phi_2, \phi_4, \phi_5$ are isomorphisms, then ϕ_3 is also an isomorphism.

Proof. Injectivity of ϕ_3: If $\phi_3(v_3) = 0$, then $\beta_3(\phi_3(v_3) = 0 = \phi_4\alpha_3(v_3)$. Since ϕ_4 is injective, $\alpha_3(v_3) = 0$. By exactness, $v_3 = \alpha_2(v_2)$. Then $\phi_3\alpha_2(v_2) = 0 = \beta_2\phi_2(v_2)$. By exactness, $\phi_2(v_2) = \beta_1(w_1)$. By surjectivity of ϕ_1, $w_1 = \phi_1(v_1)$. Then

$$\phi_2(v_2) = \beta_1\phi_1(v_1) = \phi_2\alpha_1(v_1), \tag{22.2}$$

but since ϕ_2 is injective, this implies that $v_2 = \alpha_1(v_1)$. Finally, $v_2 = \alpha_2(v_2) = \alpha_2\alpha_1(v_1) = 0$, by exactness.

The proof of surjectivity is similar, and left to the reader. □

Recall that if $\omega \in \Omega^p(M)$, and c_p is a p-chain, then

$$(\mathcal{F}^p\omega)(c_p) = \int_{c_p} \omega. \tag{22.3}$$

Previously, we showed that $\mathcal{F}^p$ induces a mapping

$$\mathcal{F}^p : H^p_{dR}(M) \to H^p_\infty(M, \mathbb{R}). \tag{22.4}$$

Theorem 22.2 (de Rham)**.** *If M has a finite good cover then the mappings*

$$\mathcal{F}^p : H^p_{dR}(M) \to H^p_\infty(M, \mathbb{R}), \tag{22.5}$$

are isomorphisms for all $p \geq 0$.

Proof. If there is only 1 element in the covering, then we are done by the above results. Next, consider the following diagram

$$\begin{array}{ccccccccc} H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_{dR}(U \cap V) & \xrightarrow{\delta^k} & H^k_{dR}(U \cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U) \oplus H^k_{dR}(V) & \xrightarrow{\alpha^k} & H^k_{dR}(U \cap V) \\ \downarrow{\mathcal{F}^{k-1} \oplus \mathcal{F}^{k-1}} & & \downarrow{\mathcal{F}^{k-1}} & & \downarrow{\mathcal{F}^k} & & \downarrow{\mathcal{F}^k \oplus \mathcal{F}^k} & & \downarrow{\mathcal{F}^k} \\ H^{k-1}_s(U) \oplus H^{k-1}_s(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_s(U \cap V) & \xrightarrow{\delta^k} & H^k_s(U \cup V) & \xrightarrow{\beta^k} & H^k_s(U) \oplus H^k_s(V) & \xrightarrow{\alpha^k} & H^k_s(U \cap V) \end{array} \tag{22.6}$$

By the Five Lemma, if the result is true for U, V and $U \cap V$, then it is also true for $U \cup V$. By induction on the number of elements in a finite good cover, the theorem is then true for any manifold which admits a finite good cover. □

One can show also the following:

$$\mathcal{F}^{k+l}[\alpha^k \wedge \beta^l] = \mathcal{F}^k[\alpha^k] \cup \mathcal{F}^l[\beta^l], \tag{22.7}$$

so the mapping between cohomology rings

$$H^*_{DR}(M) \xrightarrow{\mathcal{F}} H^*_\infty(M, \mathbb{R}), \tag{22.8}$$

is moreover a ring isomorphism.

22.2 Smooth singular cohomology

Next, we discuss the comparison of singular cohomology and smooth singular cohomology on a smooth manifold.

Proposition 22.3. *If M is a smooth manifold which has a finite good cover, then*

$$H^p(M, \mathbb{R}) \cong H^p_\infty(M, \mathbb{R}) \tag{22.9}$$

Proof. Both cohomology theories agree on contractible spaces, and both satisfy a Mayer-Vietoris sequence. By the same argument as above using the five lemma and induction on the number of elements in a good cover, the cohomology groups are isomorphic in any degree. □

Corollary 22.4. *If M is a smooth manifold of dimension n, then $H^p(M, \mathbb{R}) = 0$ for $p > n$.*

Proof. There are no p-forms for $p > n$, so obviously $H^p_{dR}(M) = 0$. By the de Rham isomorphism, the same is true for $H^p(M, \mathbb{R})$. □

23 Lecture 23

23.1 Cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega^{p-1}_c(M) \xrightarrow{d} \Omega^p_c(M) \xrightarrow{d} \Omega^{p+1}_c(M) \xrightarrow{d} \cdots, \tag{23.1}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Theorem 23.1 (Poincaré Lemma for compact supported cohomology)**.** *Let M be a differentiable n-manifold, then*

$$H^k_{c,dR}(M \times \mathbb{R}) \cong H^{k-1}_{c,dR}(M). \tag{23.2}$$

The details are somewhat similar to the regular Poincaré Lemma, in the following we just give an outline of the proof. First, we define a mapping "integration over the fiber" by

$$\pi_* : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M) \tag{23.3}$$

by the following. Any k-form on N can be written as

$$\omega = h(p,t)\pi^*\phi_k + f(p,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{23.4}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0_c(M \times \mathbb{R})$. Define

$$\pi_*(\omega) = \Big(\int_{-\infty}^{\infty} f(p,t) dt \Big) \phi_{k-1}, \tag{23.5}$$

noting that the integral is defined because ω is assumed to have compact support, and this form has compact support since f has compact support.

It is not hard to show that

$$d_M \circ \pi_* = \pi_* \circ d_{M \times \mathbb{R}}, \tag{23.6}$$

therefore π_* induces a mapping

$$\pi_* : H^k_{c,dR}(M \times \mathbb{R}) \to H^{k-1}_{c,dR}(M). \tag{23.7}$$

Next, we choose $e \in \Omega^1_c(\mathbb{R})$ with $\int_R e = 1$, and define

$$e_* : \Omega^k_c(M) \to \Omega^{k+1}_c(M \times \mathbb{R}) \tag{23.8}$$

by

$$e_*(\omega) = (\pi^* \omega) \wedge e. \tag{23.9}$$

It is not hard to see that

$$d_{M \times \mathbb{R}} \circ e_* = e_* \circ d_M. \tag{23.10}$$

Therefore e_* induces a mapping

$$e_* : H^k_{c,dR}(M) \to H^{k+1}_{c,dR}(M \times \mathbb{R}). \tag{23.11}$$

Let us write $e = \chi dt$, then

$$\pi_* \circ e_*(\omega) = \pi_* \Big(\chi(t) (\pi^* \omega) \wedge dt \Big) = \Big(\int_{-\infty}^{\infty} \chi(t) dt \Big) \omega = \omega \tag{23.12}$$

Therefore, we have $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$, so $\pi_* \circ e_* = 1$ on $H^k_{c,dR}(M)$.

Proposition 23.2. *We have $e_* \circ \pi_* = 1$ on $H^k_{c,dR}(M \times \mathbb{R})$. Consequently, π_* and e_* are isomorphisms on compactly supported cohomology.*

Proof. Again writing

$$\omega = h(p,t) \pi^* \phi_k + f(p,t)(\pi^* \phi_{k-1}) \wedge dt, \tag{23.13}$$

define a mapping

$$K : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R}) \tag{23.14}$$

by

$$K(\omega) = \pi^*\phi_{k-1}\Big(\int_{-\infty}^t f(x,s)ds - \Big(\int_{-\infty}^t e\Big)\int_{-\infty}^{\infty} f(x,s)ds\Big). \tag{23.15}$$

We claim that if $\omega \in \Omega^k_c(M \times \mathbb{R})$ then

$$(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{23.16}$$

which can be separately verified for $\omega = h(p,t)\pi^*\phi_k$, and for forms of type $\omega = f(p,t)dt \wedge \pi^*\phi_{k-1}$. The proof is left as an exercise.

This formula then implies that $e_* \circ \pi_* = 1$ as a mapping on $H^k_{c,dR}(M \times \mathbb{R})$, and the proposition follows. □

Corollary 23.3. *We have*

$$H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{23.17}$$

and a generator for $H^n_{c,dR}(\mathbb{R}^n)$ is given by any compactly supported n-form μ with $\int_{\mathbb{R}^n} \mu = 1$.

Notice that $H^k_{c,dR}(\mathbb{R}^n) \cong H^{n-k}_{dR}(\mathbb{R}^n)$. Furthermore, we have an isomorphism

$$PD : H^k_{dR}(\mathbb{R}^n) \to (H^{n-k}_{c,dR}(\mathbb{R}^n))^* \tag{23.18}$$

given by $PD(\alpha)(\beta) = \int_{\mathbb{R}^n} \alpha \wedge \beta$.

Remark 23.4. This shows that $H^*_{c,dR}(M)$ is not a homotopy invariant, since (23.17) is not the same as the cohomology of a point.

However, we do have the following.

Corollary 23.5. *If U is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{c,dR}(U) \cong H^k_{c,dR}(\mathbb{R}^n)$ for all $0 \leq k \leq n$.*

Proof. For $0 \leq k < n$, a compactly supported form in U is compactly supported in a set which is diffeomorphic to $\mathbb{R}^n$ (just by smoothing the boundary of U, since the support of the form stays away from the boundary). For $k = n$, if U is diffeomorphic to $\mathbb{R}^n$, then we are done. If U is not diffeomorphic to $\mathbb{R}^n$, we will leave this as an exercise for the interested reader. □

23.2 Mayer-Vietoris for cohomology with compact supports

Write $M = U \cup V$ as the union of two open sets in M. Note that if $U_1 \subset U_2$ and $\omega \in \Omega^k_c(U_1)$ then ω extends to be a compactly supported form in U_2. Letting

$\iota : U_1 \hookrightarrow U_2$ denote the inclusion mapping, we denote by $i_* \omega$ this extension map on forms. We claim that the following sequence is exact:

$$0 \longrightarrow \Omega^p_c(U \cap V) \xrightarrow{\tilde{\alpha}^p} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{\tilde{\beta}^p} \Omega^p_c(U \cup V) \longrightarrow 0 \tag{23.19}$$

where

$$\tilde{\alpha}^p(\omega_{U\cap V}) = \big((i_{U\cap V \hookrightarrow U})_* \omega_{U \cap V}, -(i_{U \cap V \hookrightarrow V})_* \omega_{U\cap V}\big) \tag{23.20}$$

and

$$\tilde{\beta}^p(\omega_U, \omega_V) = (i_{U \hookrightarrow M})_* \omega_U + (i_{V \hookrightarrow M})_* \omega_V. \tag{23.21}$$

To see this, $\tilde{\alpha}^p$ is obviously injective. For exactness at the middle step, obviously $\tilde{\beta}^p \tilde{\alpha}^p \omega = 0$. If $\tilde{\beta}^p(\omega_U, \omega_V) = 0$, then $\omega_U = -\omega_V$. This implies that the support of both forms is contained in $U \cap V$, and since they are equal there, take $\omega_{U \cap V} = \omega_U$, and then $(\omega_U, \omega_V) = \tilde{\alpha}^p(\omega_U)$.

To show that $\tilde{\beta}$ is onto, let $\omega \in \Omega^p_c(M)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \tilde{\beta}^p(\phi_U \omega, \phi_V \omega)$.

Consequently, from the ziz-zag Lemma, we obtain a long exact sequence

$$\cdots \xrightarrow{\tilde{\delta}^{p-1}} H^p_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} \cdots \tag{23.22}$$

Let us review the definition of the mapping $\tilde{\delta}^p$. Given a cohomology class $[\omega] \in H^p_{c,dR}(U \cup V)$, represented by $\omega \in \Omega^p_c(U \cup V)$ with $d\omega = 0$, we first write $\omega = \tilde{\beta}^p(\phi_U \omega, \phi_V \omega)$, then we apply the exterior derivative to get

$$(d(\phi_U \omega), d(\phi_V \omega)) = (d\phi_U \wedge \omega, d\phi_V \wedge \omega) \in \Omega^p_c(U) \oplus \Omega^p_c(V) \tag{23.23}$$

Either of these elements is supported in $U \cap V$ and then since $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$,

$$\tilde{\delta}^p \omega = [d\phi_U \wedge \omega] = [-d\phi_V \wedge \omega] \in H^{p+1}_{c,dR}(U \cap V). \tag{23.24}$$

Remark 23.6. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

23.3 Poincaré Duality

If M is orientable, we define the mapping

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{23.25}$$

by $PD(\alpha)(\beta) = \int_M \alpha \wedge \beta$.

Theorem 23.7. *If M^n is orientable and has a finite good cover, then*

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{23.26}$$

is an isomorphism for all $0 \leq k \leq n$.

Proof. Let $m = n - k$, and consider the diagram

$$\begin{array}{ccccccccc}
H_{dR}^{k-1}(U) \oplus H_{dR}^{k-1}(V) & \xrightarrow{\alpha^{k-1}} & H_{dR}^{k-1}(U \cap V) & \xrightarrow{\delta^k} & H_{dR}^{k}(U \cup V) & \xrightarrow{\beta^k} & H_{dR}^{k}(U) \oplus H_{dR}^{k}(V) & \xrightarrow{\alpha^k} & H_{dR}^{k}(U \cap V) \\
\big\downarrow{\scriptstyle PD \oplus PD} & & \big\downarrow{\scriptstyle PD} & & \big\downarrow{\scriptstyle PD} & & \big\downarrow{\scriptstyle PD \oplus PD} & & \big\downarrow{\scriptstyle PD} \\
\big(H_{c,dR}^{m+1}(U) \oplus H_{d,dR}^{m+1}(V)\big)^* & \xrightarrow{(\tilde{\alpha}^{m+1})^*} & H_{c,dR}^{m+1}(U \cap V)^* & \xrightarrow{(\tilde{\delta}^m)^*} & H_{c,dR}^{m}(U \cup V)^* & \xrightarrow{(\tilde{\beta}^m)^*} & \big(H_{c,dR}^{m}(U) \oplus H_{c,dR}^{m}(V)\big)^* & \xrightarrow{(\tilde{\alpha}^m)^*} & H_{c,dR}^{m}(U \cap V)^*
\end{array} \tag{23.27}$$

The top horizontal row is exact since it is the usual Mayer-Vietoris sequence. The bottom horizontal row is exact since is the dual exact sequence of the Mayer-Vietoris sequence with compact support. We next claim that this diagram commutes up to sign, so by changing some of the vertical maps to their negatives if necessary, we obtain a commutative diagram. (Proof left as an exercise).

By the five lemma, if the outer 4 vertical maps are isomorphisms, then so is the central vertical map. The proof is completed by induction on the number of open sets in the good cover, since we know it is true for $\mathbb{R}^n$ from the previous lecture. ☐

Corollary 23.8. *If M^n is a connected and orientable n-manifold with a finite good cover, then $H_{c,dR}^n(M) \cong \mathbb{R}$. If M is moreover compact, then $H_{dR}^n(M) \cong \mathbb{R}$.*

Corollary 23.9. *If M^n is a connected and orientable n-manifold with a finite good cover then $H_{dR}^k(M)$ and $H_{c,dR}^{n-k}(M)$ have the same dimension. If M is moreover compact, then $H_{dR}^k(M)$ and $H_{dR}^{n-k}(M)$ have the same dimension.*

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