Math 222B, Complex Variables and Geometry

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Contents

Introduction

This course will be about complex manifolds and Kähler geometry.

1 Lecture 1

1.1 Connections on vector bundles

A connection is a mapping $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, with the properties

- $\nabla_X s \in \Gamma(E)$,
- $\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$,
- $\nabla_X (fs) = (Xf)s + f \nabla_X s$.

In coordinates, letting $s_i, i = 1 \dots p$, be a local basis of sections of E,

$$\nabla_{\partial_i} s_j = \Gamma_{ij}^k s_k. \tag{1.1}$$

If E carries an inner product $\langle \cdot, \cdot \rangle$, then ∇ is *compatible* if

$$X \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle. \tag{1.2}$$

For a connection in TM, ∇ is called *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM). \tag{1.3}$$

Theorem 1.1. *(Fundamental Theorem of Riemannian Geometry) There exists a unique symmetric, compatible connection in TM.*

Invariantly, the connection is defined by

$$\begin{aligned} \langle \nabla_X Y, Z \rangle = \frac{1}{2} \Big(& X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle \\ & - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \Big). \end{aligned} \tag{1.4}$$

Let E and E' be vector bundles over M, with covariant derivative operators ∇, and ∇', respectively. The covariant derivative operators in $E \otimes E'$ and $Hom(E, E')$ are

$$\nabla_X (s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s') \tag{1.5}$$

$$(\nabla_X L)(s) = \nabla'_X (L(s)) - L(\nabla_X s), \tag{1.6}$$

for $s \in \Gamma(E), s' \in \Gamma(E')$, and $L \in \Gamma(Hom(E, E'))$. Note also that the covariant derivative operator in $\Lambda(E)$ is given by

$$\nabla_X (s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^{r} s_1 \wedge \cdots \wedge (\nabla_X s_i) \wedge \cdots \wedge s_r, \tag{1.7}$$

for $s_i \in \Gamma(E)$.

These rules imply that if T is an (r, s) tensor, then the covariant derivative ∇T is an $(r, s+1)$ tensor given by

$$\nabla T(X, Y_1, \ldots, Y_s) = \nabla_X (T(Y_1, \ldots Y_s)) - \sum_{i=1}^{s} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s). \tag{1.8}$$

Recall the formula for the exterior derivative [War83, Theorem ?],

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{1.9}$$

1.2 Hermitian and Kähler metrics

We next consider (M, g, J) where J is an almost complex structure and g is a Riemannian metric.

Definition 1.2. An almost Hermitian manifold is a triple (M, g, J) such that

$$g(JX, JY) = g(X, Y). \tag{1.10}$$

The triple is called Hermitian if J is integrable.

We also say that g is J-invariant if condition (1.10) is satisfied. Extend g by complex linearity to a symmetric inner product on $T \otimes \mathbb{C}$.

To a Hermitian metric (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{1.11}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2 Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{1.12}$$

Since

$$\omega(JX, JY) = \omega(X, Y), \tag{1.13}$$

this form is a real form of type $(1, 1)$, and is called the *Kähler form* or *fundamental 2-form*.

Recall the following definition.

Proposition 1.3. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \tag{1.14}$$

vanishes if and only if J is integrable.

The following proposition gives a fundamental relation between the covariant derivative of J, the exterior derivative of ω and the Nijenhuis tensor.

Proposition 1.4. *Let (M, g, J) be an almost Hermitian manifold. Then*

$$2g((\nabla_X J)Y, Z) = -d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX). \tag{1.15}$$

Proof. Using (1.6), we have

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) - g(J(\nabla_X Y), Z). \tag{1.16}$$

Since g is J-invariant, and $J^2 = -Id$, it follows that

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) + g(\nabla_X Y, JZ). \tag{1.17}$$

Now apply formula (1.4) to both terms on the right hand side to obtain

$$\begin{aligned} 2g((\nabla_X J)Y, Z) = {} & Xg(JY, Z) + JYg(X, Z) - Zg(X, JY) \\ & - g(JY, [X, Z]) - g(Z, [JY, X]) + g(X, [Z, JY]) \\ & + Xg(Y, JZ) + Yg(JZ, X) - JZg(X, Y) \\ & - g(Y, [X, JZ]) - g(JZ, [Y, X]) + g(X, [JZ, Y]). \end{aligned} \tag{1.18}$$

Next,

$$\begin{aligned} d\omega(X, JY, JZ) &= X\omega(JY, JZ) - JY\omega(X, JZ) + JZ\omega(X, JY) \\ &\quad - \omega([X, JY], JZ) + \omega([X, JZ], JY) - \omega([JY, JZ], X]) \\ &= Xg(J^2Y, JZ) - JYg(JX, JZ) + JZg(JX, JY) \\ &\quad - g(J[X, JY], JZ) + g(J[X, JZ], JY) - g(J[JY, JZ], X]) \\ &= -Xg(Y, JZ) - JYg(X, Z) + JZg(X, Y) \\ &\quad - g([X, JY], Z) + g([X, JZ], Y) + g([JY, JZ], JX]). \end{aligned} \tag{1.19}$$

The next term is

$$\begin{aligned} d\omega(X, Y, Z) &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &\quad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X]) \\ &= Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) \\ &\quad - g(J[X, Y], Z) + g(J[X, Z], Y) - g(J[Y, Z], X]). \end{aligned} \tag{1.20}$$

The last term is

$$\begin{aligned} & \frac{1}{2}g(N(Y, Z), JX) \\ & = g([JY, JZ], JX) - g([Y, Z], JX) - g(J[Y, JZ], JX) - g(J[JY, Z], JX) \\ & = g([JY, JZ], JX) - g([Y, Z], JX) - g([Y, JZ], X) - g([JY, Z], X) \end{aligned} \tag{1.21}$$

We then obtain the right hand side of (1.15) is

$$
\begin{aligned}
&- d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2} g(N(Y, Z), JX) \\
&= Xg(Y, JZ) + JYg(X, Z) - JZg(X, Y) \\
&\quad + g([X, JY], Z) - g([X, JZ], Y) - g([JY, JZ], JX]) \\
&+ Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) \\
&\quad - g(J[X, Y], Z) + g(J[X, Z], Y) - g(J[Y, Z], X]) \\
&+ g([JY, JZ], JX) - g([Y, Z], JX) - g([Y, JZ], X) - g([JY, Z], X).
\end{aligned}
\tag{1.22}
$$

The first two terms of the last line cancel out with terms on the previous lines, so this simplifies to

$$
\begin{aligned}
&- d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2} g(N(Y, Z), JX) \\
&= Xg(Y, JZ) + JYg(X, Z) - JZg(X, Y) + g([X, JY], Z) - g([X, JZ], Y) \\
&+ Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) - g(J[X, Y], Z) + g(J[X, Z], Y) \\
&- g([Y, JZ], X) - g([JY, Z], X),
\end{aligned}
\tag{1.23}
$$

and each of these 12 terms appears exactly once in (1.18). □

Corollary 1.5. *If (M, g, J) is Hermitian, then $d\omega = 0$ if and only if J is parallel.*

Proof. Since $N = 0$, this follows immediately from (1.15). □

Corollary 1.6. *If (M, g, J) is almost Hermitian, $\nabla J = 0$ implies that $d\omega = 0$ and $N = 0$.*

Proof. If J is parallel, then ω is also. The corollary follows from the fact that the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ can be written in terms of covariant differentiation.

$$
d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j} \omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{1.24}
$$

which follows immediately from (1.24) using normal coordinates around a point. This shows that a parallel form is closed, so the corollary then follows from (1.15). □

Definition 1.7. An almost Hermitian manifold (M, g, J) is

- *Kähler* if J is integrable and $d\omega = 0$, or equivalently, if $\nabla J = 0$,
- *Calabi-Yau* if it is Kähler and the canonical bundle $K \equiv \Lambda^{n,0}$ is holomorphically trivial,
- *hyperkähler* is it is Kähler with respect to 3 complex structures I, J, and K satisfiying $IJ = K$.

Note that if (M, g, J) is Kähler, then ω is a parallel $(1, 1)$-form.

Proposition 1.8. *There are the following equivalences:*

- M^{2n} *is almost complex if and only if the structure group of the principal frame bundle can be reduced from* $GL(2n, \mathbb{R})$ *to* $GL(n, \mathbb{C})$.
- (M^{2n}, g, J) *is almost Hermitian if and only if the structure group of the bundle of orthonormal frames can be reduced from* $O(2n)$ *to* $\mathrm{U}(n)$.
- (M^{2n}, g, J) *is Kähler if and only if the holonomy group is contained in* $\mathrm{U}(n)$.
- (M^{2n}, g, J) *is Calabi-Yau if and only if the holonomy group is contained in* $\mathrm{SU}(n)$.
- (M^{4n}, g, J) *is hyperkähler if and only if the holonomy group is contained in* $\mathrm{Sp}(n)$.

2 Lecture 2

2.1 Complex tensor notation

Choosing any real basis of the form $\{X_1, JX_1, \dots, X_n, JX_n\}$, let us abbreviate

$$Z_\alpha = \frac{1}{2}\Big(X_\alpha - iJX_\alpha\Big) \tag{2.1}$$

$$Z_{\overline{\alpha}} = \frac{1}{2}\Big(X_\alpha + iJX_\alpha\Big), \tag{2.2}$$

and define

$$g_{\alpha\beta} = g(Z_\alpha, Z_\beta) \tag{2.3}$$

$$g_{\overline{\alpha}\overline{\beta}} = g(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) \tag{2.4}$$

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) \tag{2.5}$$

$$g_{\overline{\alpha}\beta} = g(Z_{\overline{\alpha}}, Z_\beta). \tag{2.6}$$

Notice that

$$\begin{aligned} g_{\alpha\beta} = g(Z_\alpha, Z_\beta) &= \frac{1}{4} g(X_\alpha - iJX_\alpha, X_\beta - iJX_\beta) \\ &= \frac{1}{4}\Big(g(X_\alpha, X_\beta) - g(JX_\alpha, JX_\beta) - i(g(X_\alpha, JX_\beta) + g(JX_\alpha, X_\beta))\Big) \\ &= 0, \end{aligned}$$

since g is J-invariant, and $J^2 = -Id$. Similarly,

$$g_{\overline{\alpha}\overline{\beta}} = 0, \tag{2.7}$$

Also, from symmetry of g, we have

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) = g(Z_{\overline{\beta}}, Z_\alpha) = g_{\overline{\beta}\alpha}. \tag{2.8}$$

However, applying conjugation, since g is real we have

$$\overline{g_{\alpha\overline{\beta}}} = \overline{g(Z_\alpha, Z_{\overline{\beta}})} = g(Z_{\overline{\alpha}}, Z_\beta) = g(Z_\beta, Z_{\overline{\alpha}}) = g_{\beta\overline{\alpha}}, \tag{2.9}$$

which says that $g_{\alpha\overline{\beta}}$ is a Hermitian matrix.

We repeat the above for the fundamental 2-form ω, and define

$$\omega_{\alpha\beta} = \omega(Z_\alpha, Z_\beta) = g(JZ_\alpha, Z_\beta) = ig_{\alpha\beta} = 0 \tag{2.10}$$
$$\omega_{\overline{\alpha}\overline{\beta}} = \omega(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = -ig_{\overline{\alpha}\overline{\beta}} = 0 \tag{2.11}$$
$$\omega_{\alpha\overline{\beta}} = \omega(Z_\alpha, Z_{\overline{\beta}}) = ig_{\alpha\overline{\beta}} \tag{2.12}$$
$$\omega_{\overline{\alpha}\beta} = \omega(Z_{\overline{\alpha}}, Z_\beta) = -ig_{\overline{\alpha}\beta}. \tag{2.13}$$

The first 2 equations are just a restatement that ω is of type $(1,1)$. Also, note that

$$\omega_{\alpha\overline{\beta}} = ig_{\alpha\overline{\beta}}, \tag{2.14}$$

defines a skew-Hermitian matrix.

On a Hermitian manifold, the fundamental 2-form in holomorphic coordinates takes the form

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta \tag{2.15}$$
$$= i \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta. \tag{2.16}$$

Remark 2.1. Note that for the Euclidean metric, we have $g_{\alpha\overline{\beta}} = \frac{1}{2}\delta_{\alpha\beta}$, so

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{2.17}$$

If (M, g, J) is Kähler, then

$$\begin{aligned}
0 = d\omega &= i \sum_{\alpha,\beta=1}^{n} (dg_{\alpha\overline{\beta}}) \wedge dz^\alpha \wedge d\overline{z}^\beta \\
&= i \sum_{\alpha,\beta=1}^{n} (\partial g_{\alpha\overline{\beta}} + \overline{\partial} g_{\alpha\overline{\beta}}) \wedge dz^\alpha \wedge d\overline{z}^\beta \\
&= i \sum_{\alpha,\beta=1}^{n} \Big\{ \sum_k \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} dz^k \Big) + \sum_k \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^k} d\overline{z}^k \Big) \Big\} \wedge dz^\alpha \wedge d\overline{z}^\beta \\
&= i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} dz^k \wedge dz^\alpha \wedge d\overline{z}^\beta + i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^\alpha \wedge d\overline{z}^\beta.
\end{aligned} \tag{2.18}$$

However, the first term is a form of type $(2,1)$, and the second term is a form of type $(1,2)$ so both sums must vanish. This is equivalent to

$$\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} = \frac{\partial g_{k\overline{\beta}}}{\partial z^\alpha}, \tag{2.19}$$

which is the Kähler condition in holomorphic coordinates.

We also see that the Kähler condition on a Hermitian manifold is equivalent to $\overline{\partial}\omega = 0$, which is also equivalent to $\partial\omega = 0$, since ω is real.

2.2 Existence of local Kähler potential

First, a special case of the $\overline{\partial}$-Poincaré lemma.

Lemma 2.2. *If α is a smooth $(0,1)$-form in a closed ball $\overline{B} \subset \mathbb{C}^n$ satisfying $\overline{\partial}\alpha = 0$, then there exists $f : B \to \mathbb{C}$ such that $\alpha = \overline{\partial} f$.*

Proof. Write $\alpha = \sum_{j=1}^n \alpha_{\overline{j}} d\overline{z}^j$. Then

$$0 = \overline{\partial}\alpha = \sum_{j,k=1}^n \frac{\partial \alpha_{\overline{j}}}{\partial \overline{z}^k} d\overline{z}^k \wedge d\overline{z}^j. \tag{2.20}$$

This implies that

$$\frac{\partial \alpha_{\overline{j}}}{\partial \overline{z}^k} = \frac{\partial \alpha_{\overline{k}}}{\partial \overline{z}^j} \tag{2.21}$$

for all $1 \leq j,k \leq n$.

We want to find f such that $\partial f = \alpha$, which in components is

$$\frac{\partial f}{\partial \overline{z}^k} = \alpha_{\overline{k}} \tag{2.22}$$

for all $1 \leq k \leq n$.

Recall from last quarter, that if $B \subset \mathbb{C}$, and $g : \overline{B} \to \mathbb{C}$ is smooth, then there exists $f : B \to \mathbb{C}$ such that $\frac{\partial}{\partial \overline{z}} f = g$. The solution can be written explicitly as

$$f(z) = \frac{1}{2\pi i} \int_B g(w) \frac{dw \wedge d\overline{w}}{w - z}. \tag{2.23}$$

So we define

$$f(z^1, \dots z^n) = \frac{1}{2\pi i} \int_B \alpha_{\overline{1}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1}. \tag{2.24}$$

By the above remark, we have $\partial_{\overline{1}} f = \alpha_{\overline{1}}$. Next, for $k > 1$,

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}^k}(z^1, \dots z^n) &= \frac{1}{2\pi i} \int_B \frac{\partial}{\partial \overline{z}^k} \alpha_{\overline{1}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1} \\ &= \frac{1}{2\pi i} \int_B \frac{\partial}{\partial \overline{z}^1} \alpha_{\overline{k}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1} \\ &= \alpha_{\overline{k}}(z^1, \dots, z^n), \end{aligned} \tag{2.25}$$

and we are done. □

We will prove the following very special property of Kähler metrics.

Proposition 2.3. *If (M, g, J) is Kähler then for each $p \in M$, there exists an open neighborhood U of p and a function $u : U \to \mathbb{R}$ such that $\omega = i\partial\overline{\partial}u$.*

Proof. Choose local homorphic coordinates z^j around p. Then in a ball B in these coordinates, since ω is a real closed 2-form, from the usual Poincaré lemma, there exists a real 1-form α such that $\omega = d\alpha$ in B. Next, write $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where $\alpha^{1,0}$ is a 1-form of type $(1, 0)$, and $\alpha^{0,1}$ is a 1-form of type $(0, 1)$. Since α is real, $\overline{\alpha^{1,0}} = \alpha^{0,1}$. Next,

$$\begin{aligned} \omega = d\alpha &= \partial\alpha + \overline{\partial}\alpha \\ &= \partial\alpha^{1,0} + \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0} + \overline{\partial}\alpha^{0,1} \end{aligned} \tag{2.26}$$

The first and last terms on the right hand side are forms of type $(2, 0)$ and $(0, 2)$, respectively. Since ω is of type $(1, 1)$, we must have $\overline{\partial}\alpha^{0,1} = 0$. Since we are in a ball in $\mathbb{C}^n$, the $\overline{\partial}-$Poincaré Lemma 2.2 says that there exists a function $f : B \to \mathbb{C}$ such that $\alpha^{0,1} = \overline{\partial}f$ in B. Substituting this into (2.26), we obtain

$$\omega = \partial\overline{\partial}f + \overline{\partial}\partial\overline{f} = i\partial\overline{\partial}(2Im(f)). \tag{2.27}$$

□

Proposition 2.4. *(M, g, J) is Kähler if and only if for each $p \in M$, there exists a holomorphic coordinate system around p such that*

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n}(\delta_{jk} + O(|z|^2)_{jk})dz^j \wedge d\overline{z}^k, \tag{2.28}$$

as $|z| \to 0$.

Proof. If this is true then $d\omega(p) = 0$ for any point p, so $d\omega \equiv 0$. Conversely, we can assume that $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$. From Proposition 2.3, we can find $u : B \to \mathbb{R}$ so that

$$u = c_0 + Re(c_{1j}z^j) + Re(c_{2ij}z^iz^j + c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3), \tag{2.29}$$

and $\omega = i\partial\overline{\partial}u$. But the first terms on the left hand side are in the kernel of the $\partial\overline{\partial}$-operator, so by subtracting these terms, we can assume that

$$u = Re(c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3). \tag{2.30}$$

Then since $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$, we have that

$$u = \frac{1}{2}|z|^2 + Re\{a_{jkl}z^jz^kz^l + b_{jkl}\overline{z}^jz^kz^l)\} + O(|z|^4). \tag{2.31}$$

Consider the coordinate change

$$z^k = w^k + \sum c_{klm} w^l w^m. \tag{2.32}$$

This will eliminate the b_{jkl} terms in the expansion of u, and the remaining cubic terms are annihilated by the $\partial\overline{\partial}$-operator, so by subtracting those terms, we can arrange that

$$u = \frac{1}{2}|w|^2 + O(|w|^4), \tag{2.33}$$

and (2.28) follows. □

3 Lecture 3

3.1 Automorphisms

If $J \in \Gamma(End(TM))$, recall the formula

$$(\mathcal{L}_X J)(Y) = \mathcal{L}_X(J(Y)) - J(\mathcal{L}_X Y) = [X, JY] - J([X, Y]). \tag{3.1}$$

Definition 3.1. An *infinitesimal automorphism* of a complex manifold is a real vector field X such that $\mathcal{L}_X J = 0$, where $\mathcal{L}$ denotes the Lie derivative operator.

It is straightforward to see that X is an infinitesimal automorphism if and only if its 1-parameter group of diffeomorphisms are holomorphic automorphisms, that is, $(\phi_s)_* \circ J = J \circ (\phi_s)_*$.

Proposition 3.2. *A vector field X is an infinitesimal automorphism if and only if*

$$J([X, Y]) = [X, JY], \tag{3.2}$$

for all vector fields Y.

Proof. We compute

$$[X, JY] = \mathcal{L}_X(JY) = \mathcal{L}_X(J)Y + J(\mathcal{L}_X Y) = \mathcal{L}_X(J)Y + J([X, Y]), \tag{3.3}$$

and the result follows. □

Proposition 3.3. *The set of infinitesimal automorphisms is a real Lie algebra under the Lie bracket. Furthermore, if $N \equiv 0$, then it is a complex Lie algebra, with complex structure given by J.*

Proof. First, recall the Jacobi identity for the Lie bracket:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{3.4}$$

Now let X and Y satisfy $\mathcal{L}_X J = 0$ and $\mathcal{L}_Y J = 0$. We need to show that $\mathcal{L}_{[X,Y]} J = 0$, so we compute

$$\begin{aligned}(\mathcal{L}_{[X,Y]}J)(Z) &= \mathcal{L}_{[X,Y]}(J(Z)) - J(\mathcal{L}_{[X,Y]}Z) \\ &= [[X,Y],J(Z)] - J([[X,Y],Z]).\end{aligned} \tag{3.5}$$

By the Jacobi identity,

$$\begin{aligned}(\mathcal{L}_{[X,Y]}J)(Z) &= -[[Y,J(Z)],X] - [[J(Z),X],Y] + J\big([[Y,Z],X] + [[Z,X],Y]\big) \\ &= -[J([Y,Z]),X] + [J([X,Z]),Y] + J\big([[Y,Z],X] + [[Z,X],Y]\big) \\ &= J\big([X,[Y,Z] - [Y,[X,Z]]) + [[Y,Z],X] + [[Z,X],Y]\big) = 0.\end{aligned} \tag{3.6}$$

For the second part, we need to show that if X is an infinitesimal automorphism, then JX is also. For this, we need to show that $\mathcal{L}_{JX} J = 0$, so we compute

$$\begin{aligned}(\mathcal{L}_{JX}J)(Z) &= \mathcal{L}_{JX}(JZ) - J(\mathcal{L}_{JX}Z) \\ &= [JX,JZ] - J([JX,Z]).\end{aligned} \tag{3.7}$$

From the definition of the Nijenhuis tensor,

$$N(X,Z) = 2\{[JX,JZ] - [X,Z] - J[X,JZ] - J[JX,Z]\} = 0, \tag{3.8}$$

so we have

$$\begin{aligned}(\mathcal{L}_{JX}J)(Z) &= [X,Z] + J([X,JZ]) \\ &= [X,Z] + J(J([X,Z])) = [X,Z] - [X,Z] = 0.\end{aligned} \tag{3.9}$$

Finally, Proposition 3.2 shows that the Lie bracket is complex linear in both arguments, so it is a complex Lie algebra. □

Next we assume that (M,J) is a complex manifold.

Definition 3.4. A *holomorphic vector field* on a complex manifold (M,J) is vector field $Z \in \Gamma(T^{1,0})$ which satisfies Zf is holomorphic for every locally defined holomorphic function f.

In complex coordinates, a holomorphic vector field can locally be written as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}, \tag{3.10}$$

where the Z^j are locally defined holomorphic functions. We extend the Lie bracket of real vector fields to complex vector fields by complex linearity.

Proposition 3.5. *If Z_1 and Z_2 are holomorphic vector fields, then $[Z_1,Z_2]$ is also a holomorphic vector field. Consequently, the space of holomorphic vector fields is a complex Lie algebra.*

Proof. If in local holomorphic coordinates,

$$Z_1 = \sum Z_1^j \frac{\partial}{\partial z^j}, \; Z_2 = \sum Z_2^k \frac{\partial}{\partial z^k}, \tag{3.11}$$

with Z_1^j and Z_2^k holomorphic functions, then

$$\begin{aligned} [Z_1, Z_2] &= \sum_{j,k} Z_1^j \frac{\partial Z_2^k}{\partial z^j} \frac{\partial}{\partial z^k} - \sum_{j,k} Z_2^k \frac{\partial Z_1^j}{\partial z^k} \frac{\partial}{\partial z^j} \\ &= \sum_{j,k} \Big(Z_1^j \frac{\partial Z_2^k}{\partial z^j} - Z_2^j \frac{\partial Z_1^k}{\partial z^j} \Big) \frac{\partial}{\partial z^k}. \end{aligned} \tag{3.12}$$

Since $\partial\overline{\partial} = -\overline{\partial}\partial$, the coefficients are holomorphic functions. □

Proposition 3.6. *For $X \in \Gamma(TM)$, associate a vector field of type $(1,0)$ by mapping $X \mapsto X^{1,0} = \frac{1}{2}(X - iJX)$. This complex linear mappping maps the subspace of infinitesimal automorphisms maps isomorphically onto the space of holomorphic vector fields. Furthermore this mapping is an isomorphism of Lie algebras, that is, for infinitesimal automorphisms X and Y,*

$$[X, Y] = [X^{1,0}, Y^{1,0}]. \tag{3.13}$$

Proof. Choose a local holomorphic coordinate system $\{z^i\}$, and for real vector fields X' and Y', write

$$X = \frac{1}{2}(X' - iJX') = \sum X^j \frac{\partial}{\partial z^j}, \tag{3.14}$$

$$Y = \frac{1}{2}(Y' - iJY') = \sum Y^j \frac{\partial}{\partial z^j}. \tag{3.15}$$

We know that X' is an infinitesimal automorphism if and only if

$$J([X', Y']) = [X', JY'], \tag{3.16}$$

for all real vector fields Y'. This condition is equivalent to

$$\sum_j \overline{Y}^j \frac{\partial X^k}{\partial \overline{z}^j} = 0, \tag{3.17}$$

for each $k = 1 \dots n$, which is equivalent to X being a holomorphic vector field.

To see this, we rewrite (3.16) in terms of complex vector fields. We have

$$\begin{aligned} X' = X + \overline{X} \quad JX' = i(X - \overline{X}) \\ Y' = Y + \overline{Y} \quad JY' = i(Y - \overline{Y}) \end{aligned}$$

The left hand side of (3.16) is

$$\begin{aligned} J([X', Y']) &= J([X + \overline{X}, Y + \overline{Y}]) \\ &= J([X, Y] + [X, \overline{Y}] + [\overline{X}, Y] + [\overline{X}, \overline{Y}]). \end{aligned}$$

But from integrability, $[X, Y]$ is also of type $(1, 0)$, and $[\overline{X}, \overline{Y}]$ is of type $(0, 1)$. So we can write this as

$$J([X', Y']) = (i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y]). \tag{3.18}$$

Next, the right hand side of (3.16) is

$$[X + \overline{X}, i(Y - \overline{Y})] = i([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{3.19}$$

Then (3.18) equals (3.19) if and only if

$$J[X, \overline{Y}] + J[\overline{X}, Y] = -i[X, \overline{Y}] + i[\overline{X}, Y]. \tag{3.20}$$

This is equivalent to

$$J(Re([X, \overline{Y}])) = Im([X, \overline{Y}]). \tag{3.21}$$

This says that $[X, \overline{Y}]$ is a vector field of type $(0, 1)$. We can write the Lie bracket as

$$\begin{aligned}
[X, \overline{Y}] &= \Big[\sum_j X^j \frac{\partial}{\partial z^j}, \sum_k \overline{Y}^k \frac{\partial}{\partial \overline{z}^k} \Big] \\
&= -\sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j} + \sum_k X^j (\frac{\partial}{\partial z^j} \overline{Y}^k) \frac{\partial}{\partial \overline{z}^k},
\end{aligned}$$

and the vanishing of the $(1, 0)$ component is exactly (3.17).

Finally, for infinitesimal automorphisms X and Y, we want to show that

$$[X, Y] \mapsto \frac{1}{4}[X - iJX, Y - iJY] = \frac{1}{4}\Big([X, Y] - [JX, JY] - i([JX, Y] + [X, JY])\Big). \tag{3.22}$$

Since X and Y are both infinitesimal automorphisms, we know that JX and JY are also. We then have

$$\begin{aligned}
&\frac{1}{4}\Big([X, Y] - [JX, JY] - i([JX, Y] + [X, JY])\Big) \\
&\qquad = \frac{1}{4}\Big([X, Y] - J([JX, Y]) - i(J([X, Y]) + J([X, Y])\Big) \\
&\qquad = \frac{1}{4}\Big([X, Y] - J^2([X, Y]) - 2iJ([X, Y])\Big) \\
&\qquad = \frac{1}{2}\Big([X, Y] - iJ([X, Y])\Big),
\end{aligned} \tag{3.23}$$

which is the indeed the image of the real Lie bracket $[X, Y]$. □

Proposition 3.7. *There is an first order differential operator*

$$\overline{\partial} : \Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}), \tag{3.24}$$

such that a vector field Z is holomorphic if and only if $\overline{\partial}(Z) = 0$.

Proof. Choose local holomorphic coordinates $\{z^j\}$, and write any section of Z of $T^{1,0}$, locally as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}. \tag{3.25}$$

Then define

$$\overline{\partial}(Z) = \sum_j (\overline{\partial} Z^j) \otimes \frac{\partial}{\partial z^j}. \tag{3.26}$$

This is in fact a well-defined global section of $\Lambda^{0,1} \otimes T^{1,0}$ since the transition functions of the bundle $T^{1,0}$ corresponding to a change of holomorphic coordinates are holomorphic.

To see this, if we have an overlapping coordinate system $\{w^j\}$ and

$$Z = \sum W^j \frac{\partial}{\partial w^j}. \tag{3.27}$$

Note that

$$\frac{\partial}{\partial z^j} = \frac{\partial w^k}{\partial z^j} \frac{\partial}{\partial w^j}, \tag{3.28}$$

which implies that

$$W^j = Z^p \frac{\partial w^j}{\partial z^p}. \tag{3.29}$$

We compute

$$\begin{aligned}
\overline{\partial}(Z) &= \sum \overline{\partial}(W^j) \otimes \frac{\partial}{\partial w^j} = \sum \overline{\partial}(Z^p \frac{\partial w^j}{\partial z^p}) \otimes \frac{\partial z^q}{\partial w^j} \frac{\partial}{\partial z^q} \\
&= \sum \frac{\partial w^j}{\partial z^p} \frac{\partial z^q}{\partial w^j} \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \delta^q_p \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \overline{\partial}(Z^j) \otimes \frac{\partial}{\partial z^j}.
\end{aligned}$$

□

4 Lecture 4

4.1 Endomorphisms

Let $End_{\mathbb{R}}(TM)$ denotes the real endomorphisms of the tangent bundle.

Proposition 4.1. *On an almost complex manifold (M, J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{\mathbb{C}}(TM) \oplus End_{\overline{\mathbb{C}}}(TM) \tag{4.1}$$

where the first factor on the left consists of endomorphisms I commuting with J,

$$IJ = JI \tag{4.2}$$

and the second factor consists of endomorphisms I anti-commuting with J,

$$IJ = -JI \tag{4.3}$$

Furthermore, $I \in End_{\mathbb{C}}(TM)$ if and only if $I \otimes \mathbb{C} : TM \otimes \mathbb{C} \mapsto TM \otimes \mathbb{C}$ lies in

$$\begin{aligned} I \otimes \mathbb{C} &\in \Big(Hom(T^{1,0}, T^{1,0}) \oplus Hom(T^{0,1}, T^{0,1}) \Big)_{\mathbb{R}} \\ &\cong \Big((\Lambda^{1,0} \otimes T^{1,0}) \oplus (\Lambda^{0,1} \otimes T^{0,1}) \Big)_{\mathbb{R}}. \end{aligned} \tag{4.4}$$

Also, $I \in End_{\overline{\mathbb{C}}}(TM)$ if and only if $I \otimes \mathbb{C} : TM \otimes \mathbb{C} \mapsto TM \otimes \mathbb{C}$

$$I \otimes \mathbb{C} \in \Big(Hom(T^{1,0}, T^{0,1}) \oplus Hom(T^{0,1}, T^{1,0}) \Big)_{\mathbb{R}} \tag{4.5}$$

$$\cong \Big((\Lambda^{1,0} \otimes T^{0,1}) \oplus (\Lambda^{0,1} \otimes T^{1,0}) \Big)_{\mathbb{R}}. \tag{4.6}$$

Proof. Given J, we define

$$I^C = \frac{1}{2}(I - JIJ) \tag{4.7}$$

$$I^A = \frac{1}{2}(I + JIJ). \tag{4.8}$$

Then

$$I^C J = \frac{1}{2}(IJ - JIJ^2) = \frac{1}{2}(IJ + JI),$$

and

$$JI^C = \frac{1}{2}(JI - J^2 IJ) = \frac{1}{2}(JI + IJ).$$

Next,

$$I^A J = \frac{1}{2}(IJ + JIJ^2) = \frac{1}{2}(IJ - JI),$$

and

$$JI^A = \frac{1}{2}(JI + J^2 IJ) = \frac{1}{2}(JI - IJ).$$

To prove uniqueness, if

$$I = I_1^C + I_1^A = I_2^C + I_2^A, \tag{4.9}$$

then

$$I_1^C - I_2^C = I_2^A - I_1^A. \tag{4.10}$$

Denote by $\tilde{I} = I_1^C - I_2^C = I_2^A - I_1^A$. Then $\tilde{I}$ both commutes and anti commutes with I, so is then easily seen to vanish identically.

Next, note that

$$\begin{aligned} &Hom_{\mathbb{C}}(TM \otimes \mathbb{C}, TM \otimes \mathbb{C}) = Hom_{\mathbb{C}}(T^{1,0} \oplus T^{0,1}, T^{1,0} \oplus T^{0,1}) \\ &= \Big\{ Hom_{\mathbb{C}}(T^{1,0}, T^{1,0}) \oplus Hom_{\mathbb{C}}(T^{0,1}, T^{0,1}) \Big\} \\ &\quad \oplus \Big\{ Hom_{\mathbb{C}}(T^{1,0}, T^{0,1}) \oplus Hom_{\mathbb{C}}(T^{0,1}, T^{1,0}) \Big\} \end{aligned} \tag{4.11}$$

Any real endomorphism $I \in Hom_{\mathbb{R}}(TM, TM)$ gives a mapping $I \otimes \mathbb{C} \in Hom_{\mathbb{C}}(TM \otimes \mathbb{C}, TM \otimes \mathbb{C})$ just extending by complex linearity. Anything in the left hand side of (4.11) which is the complexification of a real mapping must be in

$$\begin{aligned} &\Big\{ Hom_{\mathbb{C}}(T^{1,0}, T^{1,0}) \oplus Hom_{\mathbb{C}}(T^{0,1}, T^{0,1}) \Big\}_{\mathbb{R}} \\ &\quad \oplus \Big\{ Hom_{\mathbb{C}}(T^{1,0}, T^{0,1}) \oplus Hom_{\mathbb{C}}(T^{0,1}, T^{1,0}) \Big\}_{\mathbb{R}} \end{aligned} \tag{4.12}$$

This decomposes $Hom_{\mathbb{R}}(TM, TM)$ into 2 pieces. But in (4.1) we have another decomposition into 2 pieces. We next show these are the same.

First, if $I \in End_{\mathbb{C}}(TM)$ then, and $X \in TM$, then

$$(I \otimes \mathbb{C})(X \pm iJX) = IX \pm iIJX = IX \pm iJIX = (IX) \pm iJ(IX), \tag{4.13}$$

which shows that $I \otimes \mathbb{C}$ lies in the first factor of (4.12). Conversely, if $I \otimes \mathbb{C}$ lies in the first factor in (4.12), and $X \in TM$, then

$$(I \otimes \mathbb{C})(X - iJX) = IX - iIJX, \tag{4.14}$$

but the right hand side must be in $T^{1,0}$ by assumption, so we must have that

$$IX - iIJX = IX - iJ(IX), \tag{4.15}$$

which proves that $IJ = JI$, so $I \in End_{\mathbb{C}}(TM)$.

Next, if $I \in End_{\overline{\mathbb{C}}}(TM)$ then, and $X \in TM$, then

$$I(X \pm iJX) = IX \pm iIJX = IX \mp iJIX = (IX) \mp iJ(IX), \tag{4.16}$$

which shows that $I \otimes \mathbb{C}$ lies in the second factor of (4.12). Conversely, if $I \otimes \mathbb{C}$ lies in the second factor in (4.12), and $X \in TM$, then

$$(I \otimes \mathbb{C})(X - iJX) = IX - iIJX, \tag{4.17}$$

but the right hand side must be in $T^{0,1}$ by assumption, so we must have that

$$IX - iIJX = IX + iJ(IX), \tag{4.18}$$

which proves that $IJ = -JI$, so $I \in End_{\overline{\mathbb{C}}}(TM)$.

□

We write down the above in a basis. Choose a real basis $\{e_1, \dots e_{2n}\}$ such that the complex structure J_0 is given by

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{4.19}$$

In matrix terms, the first part proposition is equivalent to the following decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A+D & B-C \\ C-B & A+D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A-D & B+C \\ B+C & D-A \end{pmatrix}. \tag{4.20}$$

If we choose the complex basis $Z_j = (1/2)(e_j - iJe_j)$, $j = 1 \dots n$ for $T^{1,0}M$, and $Z_{\bar{j}} = (1/2)(e_j + iJe_j)$, $j = 1 \dots n$ for $T^{0,1}M$, then we have

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2}&\begin{pmatrix} A+D+i(C-B) & 0 \\ 0 & A+D-i(C-B) \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} 0 & A-D+i(B+C) \\ A-D-i(B+C) & 0 \end{pmatrix}. \end{aligned} \tag{4.21}$$

Notice that we can refine this a bit.

Proposition 4.2. *On an almost complex manifold (M, J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{\mathbb{C},0}(TM) \oplus \mathbb{R} \oplus End_{\overline{\mathbb{C}}}(TM), \tag{4.22}$$

where the first factor consists of traceless endomorphisms, and the middle factor consists of multiples of the identity transformation.

Remark 4.3. Note that the complex anti-linear endomorphisms are necessarily traceless.

4.2 Some linear algebra

First, recall the following. Recall that $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$, where $T^{1,0}$ is the $+i$-eigenspace of J, and $T^{0,1}$ is the $-i$ eigenspace of J. For $X \in TM \otimes \mathbb{C}$, we have $Re(X) = \frac{1}{2}(X + \mathcal{C}(X))$, where $\mathcal{C} : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ is complex conjugation. We claim that the following diagram is commutative

$$\begin{array}{ccc} T^{0,1} & \xrightarrow{-i} & T^{0,1} \\ {\scriptstyle Re}\downarrow & & {\scriptstyle Re}\downarrow \\ TM & \xrightarrow{J} & TM. \end{array} \tag{4.23}$$

To see this, if $X \in T^{0,1}$, then

$$J(Re(X)) = \frac{1}{2}(J(X + \mathcal{C}X)) = \frac{1}{2}(-iX + i\mathcal{C}(X)) = \frac{i}{2}(-X + \mathcal{C}(X)). \tag{4.24}$$

On the other hand,

$$Re(-iX) = -\frac{1}{2}(iX + \mathcal{C}(iX)) = -\frac{1}{2}(iX - i\mathcal{C}X) = \frac{i}{2}(-X + \mathcal{C}(X)). \tag{4.25}$$

This says that $Re : T^{0,1} \to TM$ is a complex anti-linear mapping. Furthermore, we claim that Re is an isomorphism, with inverse mapping given by $2\Pi_{T^{0,1}}$. To see this, for $X' \in TM$,

$$Re \circ \Pi_{T^{0,1}}(X') = Re\Big(\frac{1}{2}(X' + iJX')\Big) = \frac{1}{2}X', \tag{4.26}$$

and for $X \in T^{0,1}$,

$$\begin{aligned} \Pi_{T^{0,1}} \circ Re(X) &= \Pi_{T^{0,1}}\Big(\frac{1}{2}(X + \mathcal{C}(X)\Big) \\ &= \frac{1}{4}(X + \mathcal{C}X + iJ(X + \mathcal{C}X)) \\ &= \frac{1}{4}(X + \mathcal{C}X + i(-iX + i\mathcal{C}X)) \\ &= \frac{1}{4}(X + X + \mathcal{C}X - \mathcal{C}X) = \frac{1}{2}X. \end{aligned} \tag{4.27}$$

The above discussion works for any real vector space V with complex structure $J : V \to V$. That is $V \otimes \mathbb{C} = V_i \oplus V_{-i}$ where $V_{\pm i}$ are the $\pm i$-eigenspaces of $J \otimes \mathbb{C}$. Note there is always a mapping $\mathcal{C} : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ satisfying $\mathcal{C}^2 = Id$, which is complex *antilinear*, and such that $\mathcal{C} : V_{\pm i} \to V_{\mp i}$. Such a mapping $\mathcal{C}$ is called a *real structure* on a complex vector space. In this more general setting, we have the commutative diagram

$$\begin{array}{ccc} V_{-i} & \xrightarrow{-i} & V_{-i} \\ {\scriptstyle Re}\downarrow & & {\scriptstyle Re}\downarrow \\ V & \xrightarrow{J} & V, \end{array} \tag{4.28}$$

with $Re = \frac{1}{2}(Id + \mathcal{C})$ a complex anti-linear isomorphism with inverse given by $2\Pi_{V_{-i}} = Id + iJ$. To see this, for $V \in V_{-i}$, we have

$$\begin{aligned} \Pi_{V_{-i}} \circ Re(V) &= \Pi_{V_{-i}}\frac{1}{2}(V + \mathcal{C}V) = \frac{1}{4}(V + \mathcal{C}V + iJ(V + \mathcal{C}V)) \\ &= \frac{1}{4}(V + \mathcal{C}V + i(-iV + i\mathcal{C}V) = \frac{1}{4}(V + V + \mathcal{C}V - \mathcal{C}V) = \frac{1}{2}V, \end{aligned} \tag{4.29}$$

and for $v \in V$,

$$Re \circ \Pi_{V_i}(v) = Re(\frac{1}{2}(v + iJ(v))) = \frac{1}{2}v. \tag{4.30}$$

5 Lecture 5

5.1 Endomorphisms

We next apply the previous discussion to the space $V = End_{\overline{\mathbb{C}}}(TM)$.

Proposition 5.1. *The bundle $End_{\overline{\mathbb{C}}}(TM)$ has an almost complex structure $J_1 : End_{\overline{\mathbb{C}}}(TM) \to End_{\overline{\mathbb{C}}}(TM)$ given by $I \mapsto IJ$. Furthermore, we have the decomposition*

$$End_{\overline{\mathbb{C}}}TM \otimes \mathbb{C} = V_i \oplus V_{-i}, \tag{5.1}$$

where $V_i = \Lambda^{1,0} \otimes T^{0,1}$ and $V_{-i} = \Lambda^{0,1} \otimes T^{1,0}$ are the i and $-i$ eigenspaces of J_1, respectively.

Proof. We first check that $IJ \in End_{\overline{\mathbb{C}}}(TM)$,

$$(IJ)J + J(IJ) = IJ^2 - J^2 I = -I + I = 0. \tag{5.2}$$

Clearly, $J_1^2 : End_{\overline{\mathbb{C}}}(TM) \to End_{\overline{\mathbb{C}}}(TM)$ is given by $J_1^2(I) = (IJ)J = IJ^2 = (-Id)I$. To see the second statement, for $I \in End_{\overline{\mathbb{C}}}(TM)$,

$$J_1(I) = J_1(I_+ + I_-) = iI_+ - iI_-, \tag{5.3}$$

where $I_\pm$ are the projections of I to $V_{\pm i}$. Note that since I is real, we have $I_- = \mathcal{C}I_+$. Then for $X' \in TM$,

$$\begin{aligned} J_1(I)(X') &= \frac{1}{2}(iI_+ - iI_-)\Big(X' - iJX' + X' + iJX'\Big) \\ &= \frac{i}{2}\big(I_+(X' - iJX') - I_-(X' + iJX')\big) \\ &= \frac{i}{2}\big(I_+(X' - iJX') - \mathcal{C}(I_+)(X' + iJX')\big) \\ &= \frac{i}{2}\big(I_+(X' - iJX') - \mathcal{C}(I_+(X' - iJX'))\big) \\ &= \frac{i}{2}\big(2iIm(I_+(X' - iJX'))\big) \\ &= -Im\big(I_+(X' - iJX')\big) \\ &= -\frac{1}{2}Im\Big((I - iJ_1(I))(X' - iJX')\Big) \\ &= -\frac{1}{2}(-J_1(I)X' - I(JX')). \end{aligned} \tag{5.4}$$

This yields

$$J_1(I)(X') = (IJ)X'. \tag{5.5}$$

□

Note that we have a commutative diagram

$$\begin{array}{ccc} \Lambda^{0,1} \otimes T^{1,0} & \xrightarrow{-i} & \Lambda^{0,1} \otimes T^{1,0} \\ {\scriptstyle Re}\downarrow & & {\scriptstyle Re}\downarrow \\ End_{\overline{\mathbb{C}}}(TM) & \xrightarrow{J_1} & End_{\overline{\mathbb{C}}}(TM). \end{array} \tag{5.6}$$

Furthermore, the mapping $Re : \Lambda^{0,1} \otimes T^{1,0} \to End_{\overline{\mathbb{C}}}(TM)$ is a complex anti-linear isomorphism, with the inverse given by $2\Pi_{\Lambda^{0,1}\otimes T^{1,0}}$. Note also that if $\mathcal{I} \in \Lambda^{0,1} \otimes T^{1,0}$, and $X' \in TM$, then

$$\begin{aligned}(Re(\mathcal{I}))(X') &= \frac{1}{2}(\mathcal{I} + \mathcal{C}(\mathcal{I}))(X') \\ &= \frac{1}{2}(\mathcal{I} + \mathcal{C}(\mathcal{I}))\Big(\frac{1}{2}(X' + iJX') + \frac{1}{2}(X' - iJX')\Big) \\ &= \frac{1}{4}\Big(\mathcal{I}(X' + iJX') + \mathcal{C}(\mathcal{I})(X' - iJX')\Big) \\ &= \frac{1}{4}\Big(\mathcal{I}(X' + iJX') + \mathcal{C}(\mathcal{I}(X' + iJX'))\Big) \\ &= \frac{1}{2}Re(\mathcal{I}(X' + iJX')) = Re(\mathcal{I}(\Pi_{T^{0,1}}X')).\end{aligned} \tag{5.7}$$

Also, for $\mathcal{I} \in \Lambda^{1,0} \otimes T^{0,1} \oplus \Lambda^{0,1} \otimes T^{1,0}$, and $X \in T^{0,1}$, we have

$$\begin{aligned}\Pi_{\Lambda^{0,1}\otimes T^{1,0}}(\mathcal{I})(X) &= \Pi_{\Lambda^{0,1}\otimes T^{1,0}}(\mathcal{I})(X) + 0 \\ &= \Pi_{\Lambda^{0,1}\otimes T^{1,0}}(\mathcal{I})(X) + \Pi_{\Lambda^{1,0}\otimes T^{0,1}}(\mathcal{I})(X) \\ &= (\Pi_{\Lambda^{0,1}\otimes T^{1,0}} + \Pi_{\Lambda^{1,0}\otimes T^{0,1}})(\mathcal{I})(X) \\ &= \mathcal{I}(X).\end{aligned} \tag{5.8}$$

5.2 The Lie derivative as a $\overline{\partial}$-operator

Next, we want to identify the lower mapping in the following diagram

$$\begin{array}{ccc} \Gamma(T^{1,0}) & \xrightarrow{\ \overline{\partial}\ } & \Gamma(\Lambda^{0,1} \otimes T^{1,0}) \\ {\scriptstyle Re}\big\downarrow & & {\scriptstyle Re}\big\downarrow \\ \Gamma(TM) & \xrightarrow{\ ?\ } & \Gamma(End_{\overline{\mathbb{C}}}(TM)). \end{array} \tag{5.9}$$

There is a natural operator mapping from

$$\Gamma(TM) \to \Gamma(End_{\overline{\mathbb{C}}}(TM)) \tag{5.10}$$

defined as follows. If $X \in \Gamma(TM)$, then consider $\mathcal{L}_X J$. Since $J^2 = -Id$, applying the Lie derivative, we have

$$(\mathcal{L}_X J) \circ J + J \circ (\mathcal{L}_X J) = 0, \tag{5.11}$$

that is, $\mathcal{L}_X J$ anti-commutes with J, so $\mathcal{L}_X J \in \Gamma(End_{\overline{\mathbb{C}}}(TM))$. Up to a factor, this is the mapping we are looking for.

Proposition 5.2. *For $X \in \Gamma(T^{1,0})$,*

$$Re(\overline{\partial}(X)) = -\frac{1}{2}J \circ \mathcal{L}_{Re(X)}J = \frac{1}{2}(\mathcal{L}_{Re(X)}J) \circ J. \tag{5.12}$$

Equivalently,

$$Re(i \cdot \overline{\partial}(X)) = \frac{1}{2}\mathcal{L}_{Re(X)}J. \tag{5.13}$$

Written in terms of a real vector field: if $X' \in \Gamma(TM)$*, then*

$$i \cdot \overline{\partial}(\Pi_{T^{1,0}}X') = \frac{1}{2}\Pi_{\Lambda^{0,1}\otimes T^{1,0}}(\mathcal{L}_{X'}J). \tag{5.14}$$

Proof. The proof is similar to the proof of Proposition 3.6 above. For real vector fields X' and Y', we let

$$\begin{aligned} X &= \frac{1}{2}(X' - iJX') = \sum X^j \frac{\partial}{\partial z^j}, \\ Y &= \frac{1}{2}(Y' - iJY') = \sum Y^j \frac{\partial}{\partial z^j}, \end{aligned}$$

and we have the formulas

$$\begin{aligned} X' &= X + \overline{X} \quad JX' = i(X - \overline{X}) \\ Y' &= Y + \overline{Y} \quad JY' = i(Y - \overline{Y}) \end{aligned}$$

Expanding the Lie derivative,

$$(\mathcal{L}_{X'}J)(Y') = \mathcal{L}_{X'}(J(Y')) - J(\mathcal{L}_{X'}Y') = [X', JY'] - J[X', Y']. \tag{5.15}$$

In the proof of Proposition 3.6, it was shown that

$$J([X', Y']) = i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y], \tag{5.16}$$

and

$$[X', JY'] = i([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{5.17}$$

So we have

$$\begin{aligned} [X', JY'] - J[X', Y'] &= -i[X, \overline{Y}] + i[\overline{X}, Y] - J[X, \overline{Y}] - J[\overline{X}, Y] \\ &= -\Big(i(Z - \overline{Z}) + J(Z + \overline{Z})\Big), \end{aligned}$$

where $Z = [X, \overline{Y}]$. We have that

$$Z = [X, \overline{Y}] = -\sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j} + \sum_k X^j (\frac{\partial}{\partial z^j} \overline{Y}^k) \frac{\partial}{\partial \overline{z}^k},$$

which we write as

$$Z = \sum Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j}. \tag{5.18}$$

We next compute

$$\begin{aligned}
i(Z-\overline{Z})+J(Z+\overline{Z}) &= i\Big(Z^j\frac{\partial}{\partial z^j}+W^j\frac{\partial}{\partial \overline{z}^j}-\overline{Z}^j\frac{\partial}{\partial \overline{z}^j}-\overline{W}^j\frac{\partial}{\partial z^j}\Big)\\
&+J\Big(\sum Z^j\frac{\partial}{\partial z^j}+W^j\frac{\partial}{\partial \overline{z}^j}+\overline{Z}^j\frac{\partial}{\partial \overline{z}^j}+\overline{W}^j\frac{\partial}{\partial z^j})\\
&= i\Big(Z^j\frac{\partial}{\partial z^j}+W^j\frac{\partial}{\partial \overline{z}^j}-\overline{Z}^j\frac{\partial}{\partial \overline{z}^j}-\overline{W}^j\frac{\partial}{\partial z^j}\Big)\\
&+i\Big(\sum Z^j\frac{\partial}{\partial z^j}-W^j\frac{\partial}{\partial \overline{z}^j}-\overline{Z}^j\frac{\partial}{\partial \overline{z}^j}+\overline{W}^j\frac{\partial}{\partial z^j})\\
&= 2i\Big(\sum Z^j\frac{\partial}{\partial z^j}-\overline{Z}^j\frac{\partial}{\partial \overline{z}^j}\Big).
\end{aligned}$$

We have obtained the formula

$$(\mathcal{L}_{X'}J)(Y') = -2i\Big(\sum Z^j\frac{\partial}{\partial z^j}-\overline{Z}^j\frac{\partial}{\partial \overline{z}^j}\Big) = 4Im(Z^{1,0}), \tag{5.19}$$

where $Z^{1,0}$ is the $(1,0)$ part of Z, which is

$$Z^{1,0} = -\sum_j \overline{Y}^k(\frac{\partial}{\partial \overline{z}^k}X^j)\frac{\partial}{\partial z^j}. \tag{5.20}$$

Next, we need to view $\overline{\partial}X$ as a real endomorphism, which from (7.8) is given by

$$\begin{aligned}
(\overline{\partial}X)(Y') &= \frac{1}{2}Re\big((\overline{\partial}X)(Y'+iJY')\big)\\
&= \frac{1}{2}Re\Big\{\Big(\sum_j \overline{\partial}X^j\otimes\frac{\partial}{\partial z^j}\Big)(Y'+iJY')\Big\}\\
&= \frac{1}{2}Re\Big\{\Big(\sum_j \overline{\partial}X^j\Big)(Y'+iJY')\frac{\partial}{\partial z^j}\Big\}.
\end{aligned}$$

But note that

$$Y'+iJY' = \overline{Y'-iJY'} = 2\overline{Y} = 2\sum_j \overline{Y}^j\frac{\partial}{\partial \overline{z}^j}. \tag{5.21}$$

So we have

$$\begin{aligned}
(\overline{\partial}X)(Y') &= Re\Big\{\Big(\sum_j \overline{\partial}X^j\Big)(\overline{Y})\frac{\partial}{\partial z^j}\Big\}\\
&= Re\Big\{\sum_{p,j}\overline{Y}^p\Big(\frac{\partial}{\partial \overline{z}^p}X^j\Big)\frac{\partial}{\partial z^j}\Big\} = -Re(Z^{1,0}).
\end{aligned}$$

But since $Z^{1,0}$ is of type $(1,0)$,

$$Im(Z^{1,0}) = -J(Re(Z^{1,0}). \tag{5.22}$$

We then have

$$(\overline{\partial}X)(Y') = -Re(Z^{1,0}) = -J(Im(Z^{1,0})) = -\frac{1}{4}J((\mathcal{L}_{X'}J)(Y')), \tag{5.23}$$

and since $Re(X) = \frac{1}{2}X'$, (5.12) follows.

Next, note that we can write (5.12) as

$$Re(\overline{\partial}(X)) = \frac{1}{2}J_1(\mathcal{L}_{Re(X)}J). \tag{5.24}$$

Applying J_1 to both sides gives

$$J_1 Re(\overline{\partial}(X)) = -\frac{1}{2}(\mathcal{L}_{Re(X)}J). \tag{5.25}$$

Using the diagram (5.6), this becomes

$$-Re(i \cdot \overline{\partial}(X)) = -\frac{1}{2}(\mathcal{L}_{Re(X)}J), \tag{5.26}$$

which is (5.13).

Next, letting $X = \Pi_{T^{0,1}}X'$, for $X' \in \Gamma(TM)$, (5.13) is

$$Re(i \cdot \overline{\partial}(\Pi_{T^{0,1}}X')) = \frac{1}{2}(\mathcal{L}_{Re(\Pi_{T^{0,1}}X')}J) = \frac{1}{4}(\mathcal{L}_{X'}J) \tag{5.27}$$

Applying $\Pi_{T^{0,1}}$ to both sides of this gives

$$\frac{1}{2}(i \cdot \overline{\partial}(\Pi_{T^{0,1}}X')) = \frac{1}{4}\Pi_{T^{0,1}}(\mathcal{L}_{X'}J), \tag{5.28}$$

which yields (5.14). □

6 Lecture 6

6.1 The $\overline{\partial}$ operator on holomorphic vector bundles

The above $\overline{\partial}$ operator on vector fields is a special case of a general construction on holomorphic vector bundles. Recall that the transition functions of a complex vector bundle are locally defined functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$, satisfying

$$\phi_{\alpha\beta} = \phi_{\alpha\gamma}\phi_{\gamma\beta}. \tag{6.1}$$

Notice the main property we used in the proof of Proposition 3.7 is that the transition functions of the bundle are holomorphic. Thus we make the following definition.

Definition 6.1. *A vector bundle $\pi : E \to M$ is a* holomorphic vector bundle *if in complex coordinates the transitition functions $\phi_{\alpha\beta}$ are holomorphic.*

Recall that a section of a vector bundle is a mapping $\sigma : M \to E$ satisfying $\pi \circ \sigma = Id_M$. In local coordinates, a section satisfies

$$\sigma_\alpha = \phi_{\alpha\beta}\sigma_\beta, \tag{6.2}$$

and conversely any locally defined collection of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}^m$ satisfying (6.2) defines a global section. A section is *holomorphic* if in complex coordinates, the σ_α are holomorphic.

We next have the generalization of Proposition 3.7.

Proposition 6.2. *If $\pi : E \to M$ is a holomorphic vector bundle, then there is are first order differential operators*

$$\overline{\partial} : \Gamma(\Lambda^{p,q} \otimes E) \to \Gamma(\Lambda^{p,q+1} \otimes E), \tag{6.3}$$

satisfying the following properties:

- *A section $\sigma \in \Gamma(E)$ is holomorphic if and only if $\overline{\partial}(\sigma) = 0$.*
- *For a function $f : M \to \mathbb{C}$, and a section $\sigma \in \Gamma(\Lambda^{p,q} \otimes E)$,*
$$\overline{\partial}(f \cdot \sigma) = (\overline{\partial} f) \wedge \sigma + f \cdot \overline{\partial}\sigma. \tag{6.4}$$
- $\overline{\partial} \circ \overline{\partial} = 0$.

Proof. Let σ_j be a local basis of holomorphic sections of E in U_α, and write any section $\sigma \in \Gamma(U_\alpha, \Lambda^{p,q} \otimes E)$ as

$$\sigma = \sum s_j \otimes \sigma_j, \tag{6.5}$$

where $s_j \in \Gamma(U_\alpha, \Lambda^{p,q})$. Then define

$$\overline{\partial}\sigma = \sum (\overline{\partial} s_j) \otimes \sigma_j. \tag{6.6}$$

We claim this is a global section of $\Gamma(\Lambda^{p,q+1} \otimes E)$. Choose a local basis σ'_j of holomorphic sections of E in U_β, and write σ as

$$\sigma = \sum s'_j \otimes \sigma'_j. \tag{6.7}$$

Since $\sigma'_j = (\phi^{-1}_{\alpha\beta})_{jl}\sigma_l$, we have that

$$s'_j = (\phi_{\alpha\beta})_{jl} s_l, \tag{6.8}$$

so we can write

$$\sigma = \sum (\phi_{\alpha\beta})_{jl} s_l \otimes \sigma'_j. \tag{6.9}$$

Consequently

$$\begin{aligned} \overline{\partial}\sigma &= \sum (\overline{\partial} s'_j) \otimes \sigma'_j = \sum \overline{\partial}((\phi_{\alpha\beta})_{jk} s_k) \otimes \sigma'_j \\ &= \sum (\phi_{\alpha\beta})_{jk} \overline{\partial}(s_k) \otimes \sigma'_j = \sum (\overline{\partial} s_k) \otimes (\phi_{\alpha\beta})_{jk}\sigma'_j = \sum (\overline{\partial} s_k) \otimes \sigma_k. \end{aligned}$$

The other properties follow immediately from the definition. □

Definition 6.3. The (p,q) Dolbeault cohomology group with coefficients in E is

$$H^{p,q}_{\overline{\partial}}(M,E) = \frac{\{\alpha \in \Lambda^{p,q}(M,E) \mid \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M,E))}. \tag{6.10}$$

The Dolbeault Theorem says that if M is compact, then

$$H^{p,q}_{\overline{\partial}}(M,E) \cong H^q(M, \Omega^p(E)). \tag{6.11}$$

where the right hand side is a sheaf cohomology group.

Letting Θ denote $T^{1,0}$, there is a complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,3} \otimes \Theta) \xrightarrow{\overline{\partial}} \cdots . \tag{6.12}$$

The holomorphic vector fields (equivalently, the automorphisms of the complex structure) are identified with $H^0(M,\Theta)$. The higher cohomology groups $H^1(M,\Theta)$ and $H^2(M,\Theta)$ of this complex play a central role in the theory of deformations of complex structures.

6.2 The linearized Nijenhuis tensor

Recall the following.

Proposition 6.4. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} \tag{6.13}$$

is a tensor of type $(1,2)$ *and satisfies*

$$N(Y,X) = -N(X,Y), \tag{6.14}$$
$$N(JX,JY) = -N(X,Y), \tag{6.15}$$
$$N(X,JY) = N(JX,Y) = -J(N(X,Y)). \tag{6.16}$$

Proof. The first two are easy. For the last one,

$$N(X,JY) = -N(JX,J^2Y) = N(JX,Y), \tag{6.17}$$

and

$$\begin{aligned} N(X,JY) &= 2\{[JX,J^2Y] - [X,JY] - J[X,J^2Y] - J[JX,JY]\} \\ &= 2\{-[JX,Y] - [X,JY] + J[X,Y] - J[JX,JY]\} \\ &= 2J\{J[JX,Y] + J[X,JY] + [X,Y] - [JX,JY]\} \\ &= -2J\{N(X,Y)\}. \end{aligned} \tag{6.18}$$

□

Let $J(t)$ be any differentiable 1-parameter family of almost complex structures satisying $J(t)^2 = -Id$ with $J(0) = J$. Differentiating this at $t = 0$ yields

$$J(0)J'(0) + J'(0)J(0) = 0, \tag{6.19}$$

which says that $J'(0) \in \Gamma(End_{\overline{\mathbb{C}}}(TM))$. Thus we can view the space of sections of $End_{\overline{\mathbb{C}}}(TM)$ as the "tangent space" to the space of almost complex structures on M, which we call $\mathcal{A}(M)$. We can view the Nijenhuis tensor as a mapping

$$N : \mathcal{A}(M) \to \Gamma(\Lambda^2 \otimes TM). \tag{6.20}$$

Gven a complex structure J, we define the linearized Nijenhuis operator at J in the direction of $I \in \Gamma(End_{\overline{\mathbb{C}}}(TM))$ by

$$N'_J(I) = \frac{d}{dt} N_{J(t)}\Big|_{t=0} \tag{6.21}$$

where $J(t)$ is any path of almost complex structures satisfying $J(0) = J$ and $J'(0) = I$. This is a well-defined operator, independent of the choice of path $J(t)$.

Proposition 6.5. *We have*

$$N'_J(I) \in \Gamma\Big(\big\{\big(\Lambda^{2,0} \otimes T^{0,1}\big) \oplus \big(\Lambda^{0,2} \otimes T^{1,0}\big)\big\}_{\mathbb{R}}\Big). \tag{6.22}$$

Proof. First, notice that properties (6.14)-(6.16) clearly also hold for the linearized Nijenhuis tensor. If we complexify, just using (6.14), we have

$$\begin{aligned} N'_J(I) &\in \Gamma((\Lambda^2 \otimes TM) \otimes \mathbb{C})) \\ &= \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}\big) \oplus \big(T^{1,0} \oplus T^{0,1}\big)\Big). \end{aligned} \tag{6.23}$$

But (6.15) says that that $\Lambda^{1,1}$ component vanishes. So we have

$$N'_J(I) \in \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2}\big) \oplus \big(T^{1,0} \oplus T^{0,1}\big). \tag{6.24}$$

Using (6.16), for $X', Y' \in \Gamma(TM)$, we have

$$\begin{aligned} &N'_J(I)(X' - iJX', Y' - iJY') \\ &\quad = N'_J(I)(X', Y') - N'_J(I)(JX', JY') - iN'_J(I)(JX', Y') - iN'_J(I)(X', JY') \\ &\quad = N'_J(I)(X', Y') + N'_J(I)(X', Y') + iJN'_J(I)(X', Y') + iJN'_J(I)(X', Y') \\ &\quad = 2N'_J(I)(X', Y') + 2iJN'_J(I)(X', Y'), \end{aligned} \tag{6.25}$$

which lies in $T^{0,1}$. This shows that the $\Lambda^{2,0} \otimes T^{1,0}$ component vanishes, so the $\Lambda^{0,2} \otimes T^{0,1}$ component also vanishes, and (6.22) follows. □

7 Lecture 7

7.1 Some linear algebra

We next apply the linear algebra discussion from above to the bundle

$$V_{\mathbb{R}} = \big\{ \big(\Lambda^{2,0} \otimes T^{0,1}\big) \oplus \big(\Lambda^{0,2} \otimes T^{1,0}\big) \big\}_{\mathbb{R}}. \tag{7.1}$$

Proposition 7.1. *The bundle $V_{\mathbb{R}}$ has an almost complex structure $J_2 : V_{\mathbb{R}} \to V_{\mathbb{R}}$ given by $N \mapsto -JN$. Furthermore, we have the decomposition*

$$V_{\mathbb{R}} \otimes \mathbb{C} = V_i \oplus V_{-i}, \tag{7.2}$$

where $V_i = \Lambda^{2,0} \otimes T^{0,1}$ and $V_{-i} = \Lambda^{0,2} \otimes T^{1,0}$ are the i and $-i$ eigenspaces of J_2, respectively.

Proof. Clearly, $-JN \in \Gamma(V_{\mathbb{R}})$, and $J_2^2 = -Id$. To see the second statement, for $N \in V_{\mathbb{R}}$,

$$J_2(N) = J_2(N_+ + N_-) = iN_+ - iN_-, \tag{7.3}$$

where $N_{\pm}$ are the projections of I to $V_{\pm i}$. Note that since N is real, we have $N_- = \mathcal{C}N_+$. Then for $X', Y' \in TM$,

$$\begin{aligned}
J_2(N)(X',Y') &= \frac{1}{4}(iN_+ - iN_-)\Big(X' - iJX' + X' + iJX', Y' - iJY' + Y' + iJY'\Big) \\
&= \frac{i}{4}\big(N_+(X' - iJX', Y' - iJY') - I_-(X' + iJX', Y' + iJY')\big) \\
&= \frac{i}{4}\big(N_+(X' - iJX', Y' - iJY') - \mathcal{C}(N_+)(X' + iJX', Y' + iJY')\big) \\
&= \frac{i}{4}\big(N_+(X' - iJX', Y' - iJY') - \mathcal{C}(N_+(X' - iJX', Y' - iJY'))\big) \\
&= \frac{i}{4}\big(2iIm(N_+(X' - iJX', Y' - iJY'))\big) \\
&= -\frac{1}{2}Im\big(N_+(X' - iJX', Y' - iJY')\big) \\
&= -\frac{1}{4}Im\Big((N - iJ_1(N))(X' - iJX', Y' - iJY')\Big) \\
&= -\frac{1}{4}\big(- J_1(N)(X',Y') + J_1(N)(JX', JY') - N(X', JY') - N(JX', Y')\big).
\end{aligned} \tag{7.4}$$

Using the properties (6.14)-(6.16), we have

$$\begin{aligned}
J_2(N)(X',Y') &= -\frac{1}{4}(-2J_1(N)(X',Y') - 2N(X', JY') - N(JX', Y')) \\
&= -\frac{1}{2}(-J_1(N)(X',Y') + JN(X',Y')).
\end{aligned} \tag{7.5}$$

This yields

$$J_2(N)(X',Y') = -J(N(X',Y')). \tag{7.6}$$

□

Note that we have a commutative diagram

$$\begin{array}{ccc} \Lambda^{0,2}\otimes T^{1,0} & \xrightarrow{-i} & \Lambda^{0,2}\otimes T^{1,0} \\ {\scriptstyle Re}\downarrow & & {\scriptstyle Re}\downarrow \\ V & \xrightarrow{J_2} & V. \end{array} \tag{7.7}$$

Furthermore, the mapping $Re : \Lambda^{0,2}\otimes T^{1,0} \to V$ is a complex anti-linear isomorphism, with the inverse given by $2\Pi_{\Lambda^{0,2}\otimes T^{1,0}}$. Note also that if $\mathcal{N} \in \Lambda^{0,2}\otimes T^{1,0}$, and $X', Y' \in TM$, then

$$\begin{aligned} (Re(\mathcal{N}))(X',Y') &= \frac{1}{2}(\mathcal{N}+\mathcal{C}(\mathcal{N}))(X',Y') \\ &= \frac{1}{8}\Big(\mathcal{N}(X'+iJX',Y'+iJY') + \mathcal{C}(\mathcal{N})(X'-iJX',Y'-iJY')\Big) \\ &= \frac{1}{8}\Big(\mathcal{N}(X'+iJX',Y'+iJY') + \mathcal{C}(\mathcal{N}(X'+iJX',Y'+iJY'))\Big) \\ &= \frac{1}{4}Re(\mathcal{N}(X'+iJX',Y'+iJY')) = Re(\mathcal{N}(\Pi_{T^{0,1}}X, \Pi_{T^{0,1}}Y)). \end{aligned} \tag{7.8}$$

Also, for $\mathcal{N} \in \Lambda^{2,0}\otimes T^{0,1} \oplus \Lambda^{0,2}\otimes T^{1,0}$, and $X, Y \in T^{0,1}$, we have

$$\begin{aligned} \Pi_{\Lambda^{0,2}\otimes T^{1,0}}(\mathcal{N})(X,Y) &= \Pi_{\Lambda^{0,2}\otimes T^{1,0}}(\mathcal{N})(X,Y) + 0 \\ &= \Pi_{\Lambda^{0,2}\otimes T^{1,0}}(\mathcal{N})(X) + \Pi_{\Lambda^{2,0}\otimes T^{0,1}}(\mathcal{N})(X,Y) \\ &= (\Pi_{\Lambda^{0,2}\otimes T^{1,0}} + \Pi_{\Lambda^{2,0}\otimes T^{0,1}})(\mathcal{N})(X,Y) \\ &= \mathcal{N}(X,Y). \end{aligned} \tag{7.9}$$

7.2 The linearized Nijenhuis tensor as a $\overline{\partial}$-operator

Let us continue the above diagram:

$$\begin{array}{ccccc} \Gamma(T^{1,0}) & \xrightarrow{\overline{\partial}} & \Gamma(\Lambda^{0,1}\otimes T^{1,0}) & \xrightarrow{\overline{\partial}} & \Gamma(\Lambda^{0,2}\otimes T^{1,0}) \\ {\scriptstyle Re}\downarrow & & {\scriptstyle Re}\downarrow & & {\scriptstyle Re}\downarrow \\ \Gamma(TM) & \xrightarrow{-\frac{1}{2}J\circ\mathcal{L}_X J} & \Gamma(End_{\overline{\mathbb{C}}}(TM)) & \xrightarrow{?} & \Gamma(V_{\mathbb{R}}) \end{array} \tag{7.10}$$

Another natural operator mapping between the spaces in question is given by the following.

Proposition 7.2. *The operator* $N'_J : \Gamma(End_{\overline{\mathbb{C}}}(TM)) \to \Gamma(V_{\mathbb{R}})$ *is given by the following. For* $X, Y \in \Gamma(TM)$

$$\begin{aligned} N'_J(I)(X,Y) = 2\{[IX,JY] + [JX,IY] &- I([X,JY]) - J([X,IY]) \\ &- I([JX,Y]) - J([IX,Y])\} \end{aligned} \tag{7.11}$$

Proof. The proof is straightforward. □

The main result of the lecture is the following.

Proposition 7.3. *For* $\mathcal{I} \in \Gamma(\Lambda^{0,2} \otimes T^{1,0})$,

$$Re\{\overline{\partial}(\mathcal{I})\} = \frac{1}{4} J \circ N'_J(Re(\mathcal{I})). \tag{7.12}$$

Equivalently,

$$Re\{i \cdot \overline{\partial}(\mathcal{I})\} = -\frac{1}{4} N'_J(Re(\mathcal{I})). \tag{7.13}$$

In terms of $I \in End_{\overline{\mathbb{C}}}(TM)$,

$$i \cdot \overline{\partial}(\Pi_{\Lambda^{0,1}\otimes T^{1,0}} I) = -\frac{1}{4} \Pi_{\Lambda^{0,2}\otimes T^{1,0}}(N'_J(I)). \tag{7.14}$$

Proof. For real vector fields X' and Y', we let

$$X = \frac{1}{2}(X' - iJX') = \sum X^j \frac{\partial}{\partial z^j},$$
$$Y = \frac{1}{2}(Y' - iJY') = \sum Y^j \frac{\partial}{\partial z^j},$$

and we have the formulas

$$X' = X + \overline{X} \quad JX' = i(X - \overline{X})$$
$$Y' = Y + \overline{Y} \quad JY' = i(Y - \overline{Y})$$

First, for $\mathcal{I} \in \Gamma(\Lambda^{0,1} \otimes T^{1,0})$,

$$\begin{aligned} Re(\overline{\partial}\mathcal{I})(X', Y') &= \frac{1}{4} Re\big\{\overline{\partial}\mathcal{I}(X' + iJX', Y' + iJY')\big\} \\ &= Re\big\{\overline{\partial}\mathcal{I}(\overline{X}, \overline{Y})\big\}. \end{aligned} \tag{7.15}$$

In holomorphic coordinates, write

$$\mathcal{I} = \sum_{j,k} a^j_{\overline{k}} d\overline{z}^k \otimes \frac{\partial}{\partial z^j}, \tag{7.16}$$

and

$$X = \sum_j X^j \frac{\partial}{\partial z^j}, \quad Y = \sum_j Y^j \frac{\partial}{\partial z^j}, \tag{7.17}$$

Then

$$\begin{aligned} Re(\overline{\partial}\mathcal{I})(X', Y') &= Re\Big\{\overline{\partial}\Big(\sum_{j,k} a^j_{\overline{k}} d\overline{z}^k \otimes \frac{\partial}{\partial z^j}\Big)(\overline{X}, \overline{Y})\Big\} \\ &= Re\Big\{\Big(\sum_{j,k} \overline{\partial}(a^j_{\overline{k}}) d\overline{z}^k \otimes \frac{\partial}{\partial z^j}\Big)(\overline{X}, \overline{Y})\Big\} \\ &= Re\Big\{\Big(\sum_{j,k} \Big(\frac{\partial}{\partial \overline{z}^p} a^j_{\overline{k}}\Big) d\overline{z}^p \wedge d\overline{z}^k \otimes \frac{\partial}{\partial z^j}\Big)(\overline{X}, \overline{Y})\Big\} \\ &= Re\Big\{\Big(\sum_{j,k} \Big(\frac{\partial}{\partial \overline{z}^p} a^j_{\overline{k}}\Big)(\overline{X}^p \overline{Y}^k - \overline{Y}^p \overline{X}^k) \frac{\partial}{\partial z^k}\Big\}. \end{aligned} \tag{7.18}$$

Next, let $I = Re(\mathcal{I})$, and we compute

$$\begin{aligned}
JN'_J(I)(X',Y') &= JN'_J(I)(X+\overline{X}, Y+\overline{Y}) \\
&= JN'_J(I)(X,Y) + JN'_J(I)(\overline{X},\overline{Y}) \\
&= 2Re\{JN'_J(I)(\overline{X},\overline{Y})\} \\
&= 2Re\big\{J \circ 2\{[I\overline{X}, J\overline{Y}] + [J\overline{X}, I\overline{Y}] - I([\overline{X}, J\overline{Y}]) - J([\overline{X}, I\overline{Y}]) \\
&\qquad\qquad - I([J\overline{X},\overline{Y}]) - J([I\overline{X},\overline{Y}])\}\big\} \\
&= 2Re\big\{J \circ 2\{[I\overline{X}, (-i)\overline{Y}] + [(-i)\overline{X}, I\overline{Y}] - I([\overline{X}, (-i)\overline{Y}]) - J([\overline{X}, I\overline{Y}]) \\
&\qquad\qquad - I([(-i)\overline{X},\overline{Y}]) - J([I\overline{X},\overline{Y}])\}\big\} \\
&= 4Re\big\{J \circ \{-i[I\overline{X},\overline{Y}] - i[\overline{X}, I\overline{Y}] + iI([\overline{X},\overline{Y}]) - J([\overline{X}, I\overline{Y}]) \\
&\qquad\qquad + iI([\overline{X},\overline{Y}]) - J([I\overline{X},\overline{Y}])\}\big\} \\
&= 4Re\big\{[\overline{X}, I\overline{Y}] - iJ[\overline{X}, I\overline{Y}] + [I\overline{X},\overline{Y}] - iJ[I\overline{X},\overline{Y}] - 2I([\overline{X},\overline{Y}])\big\}.
\end{aligned} \tag{7.19}$$

Next,

$$\begin{aligned}
[\overline{X}, I\overline{Y}] - iJ[\overline{X}, I\overline{Y}] &= 2\Pi_{T^{1,0}}[\overline{X}, I\overline{Y}] \\
&= 2\Pi_{T^{1,0}}\Big[\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}, I\Big(\overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big)\Big] \\
&= 2\Pi_{T^{1,0}}\Big[\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}, \frac{1}{2}(\mathcal{I}+\overline{\mathcal{I}})\Big(\overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big)\Big] \\
&= \Pi_{T^{1,0}}\Big[\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}, \mathcal{I}\Big(\overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big)\Big] \\
&= \Pi_{T^{1,0}}\Big[\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}, \sum_k a^k_{\overline{p}} \overline{Y}^p \frac{\partial}{\partial z^k}\Big] \\
&= \sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}(a^k_{\overline{p}} \overline{Y}^p) \frac{\partial}{\partial z^k} \\
&= \sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}(a^k_{\overline{p}}) \overline{Y}^p \frac{\partial}{\partial z^k} + \sum \overline{X}^j a^k_{\overline{p}} \frac{\partial}{\partial \overline{z}^j}(\overline{Y}^p) \frac{\partial}{\partial z^k}
\end{aligned} \tag{7.20}$$

Similarly,

$$\begin{aligned}
[I\overline{X}, \overline{Y}] - iJ[I\overline{X}, \overline{Y}] &= 2\Pi_{T^{1,0}}[I\overline{X}, \overline{Y}] \\
&= 2\Pi_{T^{1,0}}\Big[I\Big(\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}\Big), \overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big] \\
&= \Pi_{T^{1,0}}\Big[\mathcal{I}\Big(\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}\Big), \overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big] \\
&= \Pi_{T^{1,0}}\Big[\sum_k a^k_{\overline{p}} \overline{X}^p \frac{\partial}{\partial z^k}, \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}\Big] \\
&= -\sum \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}(a^k_{\overline{p}} \overline{X}^p) \frac{\partial}{\partial z^k} \\
&= -\sum \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}(a^k_{\overline{p}}) \overline{X}^p \frac{\partial}{\partial z^k} - \sum \overline{Y}^j a^k_{\overline{p}} \frac{\partial}{\partial \overline{z}^j}(\overline{X}^p) \frac{\partial}{\partial z^k}
\end{aligned} \tag{7.21}$$

The last term is

$$
\begin{aligned}
I([\overline{X},\overline{Y}]) = &\\
&= I\Big(\Big[\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}, \sum \overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big]\Big)\\
&= \frac{1}{2}(\mathcal{I} + \overline{\mathcal{I}})\Big(\sum \Big(\overline{X}^j \frac{\partial}{\partial \overline{z}^j}(\overline{Y}^k) - \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}(\overline{X}^k)\Big)\frac{\partial}{\partial \overline{z}^k}\Big)\\
&= \frac{1}{2}\sum a_{\overline{k}}^{p}\Big(\overline{X}^j \frac{\partial}{\partial \overline{z}^j}(\overline{Y}^k) - \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}(\overline{X}^k)\Big)\frac{\partial}{\partial \overline{z}^p}.
\end{aligned}
\tag{7.22}
$$

Putting everything together, we obtain

$$
\begin{aligned}
JN'_J(I)(X',Y') = 4Re\Big\{&\sum \overline{X}^j \frac{\partial}{\partial \overline{z}^j}(a_{\overline{p}}^{k})\overline{Y}^p \frac{\partial}{\partial z^k} + \sum \overline{X}^j a_{\overline{p}}^{k} \frac{\partial}{\partial \overline{z}^j}(\overline{Y}^p)\frac{\partial}{\partial z^k}\\
&- \sum \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}(a_{\overline{p}}^{k})\overline{X}^p \frac{\partial}{\partial z^k} - \sum \overline{Y}^j a_{\overline{p}}^{k} \frac{\partial}{\partial \overline{z}^j}(\overline{X}^p)\frac{\partial}{\partial z^k}\\
&- \sum a_{\overline{k}}^{p}\Big(\overline{X}^j \frac{\partial}{\partial \overline{z}^j}(\overline{Y}^k) - \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}(\overline{X}^k)\Big)\frac{\partial}{\partial \overline{z}^p}\Big\}\\
= 4Re&(\overline{\partial}\mathcal{I})(X',Y')
\end{aligned}
\tag{7.23}
$$

This proves the first formula, and the other two formulas follow from this, using the diagram (7.7). □

8 Lecture 8

8.1 The space of almost complex structures

We define

$$
\mathcal{J}(\mathbb{R}^{2n}) \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n,\mathbb{R}), J^2 = -I_{2n}\}
\tag{8.1}
$$

We next give some alternative descriptions of this space.

Proposition 8.1. *The space $\mathcal{J}(\mathbb{R}^{2n})$ is the homogeneous space $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.*

Proof. We note that $GL(2n,\mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n,\mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$
\Phi_A : J \mapsto AJA^{-1}.
\tag{8.2}
$$

Obviously,

$$
(AJA^{-1})^2 = AJA^{-1}AJA^{-1} = AJ^2A^{-1} = -I,
\tag{8.3}
$$

and

$$
\Phi_{AB}(J) = (AB)J(AB)^{-1} = ABJB^{-1}A^{-1} = \Phi_A\Phi_B(J),
\tag{8.4}
$$

so is indeed a group action. Given J and J', there exists bases

$$\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \text{ and } \{e'_1, \ldots, e'_n, J'e'_1, \ldots, J'e'_n\}. \tag{8.5}$$

Define $S \in GL(2n, \mathbb{R})$ by $Se_k = e'_k$ and $S(Je_k) = J'e'_k$. Then $J' = SJS^{-1}$, and the action is therefore transitive. The stabilizer subgroup of J_0 is

$$Stab(J_0) = \{A \in GL(2n, \mathbb{R}) : AJ_0A^{-1} = J_0\}, \tag{8.6}$$

that is, A commutes with J_0. We have seen previously that this can be identified with $GL(n, \mathbb{C})$. □

We next give yet another description of this space. Define

$$\begin{aligned}\mathcal{P}(\mathbb{R}^{2n}) = \{P \subset \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n} \mid dim_{\mathbb{C}}(P) = n,\\ P \text{ is a complex subspace satisfying } P \cap \overline{P} = \{0\}\}.\end{aligned}$$

If we consider $\mathbb{R}^{2n} \otimes \mathbb{C}$, we note that complex conjugation is a well defined complex anti-linear map $\mathbb{R}^{2n} \otimes \mathbb{C} \to \mathbb{R}^{2n} \otimes \mathbb{C}$.

Proposition 8.2. *The space $\mathcal{P}(\mathbb{R}^{2n})$ can be explicitly identified with $\mathcal{J}(\mathbb{R}^{2n})$ by the following. If $J \in \mathcal{J}(\mathbb{R}^{2n})$ then let*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = T^{1,0}(J) \oplus T^{0,1}(J), \tag{8.7}$$

where

$$T^{0,1}(J) = \{X + iJX, X \in \mathbb{R}^{2n}\} = \{-i\}\text{-eigenspace of } J. \tag{8.8}$$

This an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and letting $T^{1,0}(J) = \overline{T^{0,1}(J)}$, we have $T^{1,0} \cap T^{0,1} = \{0\}$.

For the converse, given $P \in \mathcal{P}(\mathbb{R}^{2n})$, then P may be written as a graph over $\mathbb{R}^{2n} \otimes 1$, that is

$$P = \{X' + iJX' \mid X' \in \mathbb{R}^{2n} \subset \mathbb{C}^{2n}\}, \tag{8.9}$$

with $J \in \mathcal{J}(\mathbb{R}^{2n})$, and

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \overline{P} \oplus P = T^{1,0}(J) \oplus T^{0,1}(J). \tag{8.10}$$

Proof. For the forward direction, we already know this. To see the other direction, consider the projection map Re restricted to P

$$\pi = Re : P \to \mathbb{R}^{2n}. \tag{8.11}$$

We claim this is a real linear isomorphism. Obviously, it is linear over the reals. Let $X \in P$ satisfy $\pi(X) = 0$. Then $Re(X) = 0$, so $X = iX'$ for some real $X' \in \mathbb{R}^{2n}$. But $\overline{X} = -iX' \in P \cap \overline{P}$, so by assumption $X = 0$. Since these spaces are of the same real dimension, π has an inverse, which we denote by J. Clearly then, (8.9) is satisfied. Since P is a complex subspace, given any $X = X' + iJX' \in P$, the vector $iX' = (-JX') + iX'$ must also lie in P, so

$$(-JX') + iX' = X'' + iJX'', \tag{8.12}$$

for some real X'', which yields the two equations

$$JX' = -X'' \tag{8.13}$$

$$X' = JX''. \tag{8.14}$$

applying J to the first equation yields

$$J^2X' = -JX'' = -X'. \tag{8.15}$$

Since this is true for any X', we have $J^2 = -I_{2n}$. □

Remark 8.3. We note that $J \mapsto -J$ corresponds to interchanging $T^{0,1}$ and $T^{1,0}$.

Remark 8.4. The above propositions embed $\mathcal{J}(\mathbb{R}^{2n})$ as a subset of the complex Grassmannian $G(n, 2n, \mathbb{C})$. These spaces have the same dimension, so it is an *open* subset. Furthermore, the condition that the projection to the real part is an isomorphism is generic, so it is also dense.

8.2 Deformations of complex structure

We next let $J(t)$ be a path of almost complex structures through $J = J(0)$. Such a $J(t)$ is equivalent to a decomposition

$$TM \otimes \mathbb{C} = T^{1,0}(J_t) \oplus T^{0,1}(J_t). \tag{8.16}$$

Note that, for t sufficiently small, this determines an element $\phi(t) \in \Lambda^{0,1}(J) \otimes T^{1,0}(J)$ which we view as a mapping

$$\phi(t) : T^{0,1}(J) \to T^{1,0}(J), \tag{8.17}$$

by writing

$$T^{0,1}(J_t) = \{v + \phi(t)v \mid v \in T^{0,1}(J_0)\}. \tag{8.18}$$

That is, we write $T^{0,1}(J_t)$ as a graph over $T^{0,1}(J_0)$. Conversely, a path $\phi(t)$ in (8.17), corresponds to a path $J(t)$ of almost complex structures.

Corollary 8.5. *Let M be compact, and J an almost complex structure. Then there is a canonical correspondence almost complex structures near J and sections of $\Lambda^{0,1}(J) \otimes T^{1,0}(J)$ near 0.*

Proof. We choose any compatible almost-Hermitian metric to measure nearness. Call the base complex structure J_0. Given

$$\phi \in \Gamma(\Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0)) = \Gamma(Hom_{\mathbb{C}}(T^{0,1}(J_0), T^{1,0}(J_0))), \tag{8.19}$$

then

$$T^{0,1}(J) = \{v + \phi v, v \in T^{0,1}(J_0)\}. \tag{8.20}$$

For any point p, this is always an n-dimensional complex subspace of the complexified tangent space at p. If $X \in T^{0,1}(J) \cap \overline{T^{0,1}(J)}$, then

$$v + \phi v = w + \overline{\phi} w, \tag{8.21}$$

where $v \in T^{0,1}(J_0)$ and $w \in T^{1,0}(J_0)$. This yields the equations

$$\overline{\phi} w = v \tag{8.22}$$

$$\phi v = w. \tag{8.23}$$

This says that $\phi\overline{\phi}$ has 1 as an eigenvalue. But if ϕ is sufficiently small, this cannot happen.

Conversely, given any J near J_0, we obtain a ϕ by choosing $P = T^{0,1}(J)$ and writing P as a graph over $P_0 = T^{0,1}(J_0)$. This can be done because projection from P to P_0 is an isomorphism if $P(t)$ is sufficiently close to $P(0)$ in the Grassmanian.

□

Remark 8.6. We need to restrict to small complex structures because if we choose $P = T^{0,1}(J) = i\mathbb{R}^n$, then Re restricted to P is not an isomorphism, for example.

Next, we will write out the correspondence explicitly.

Proposition 8.7. *Let $\phi \in \Lambda^{0,1}(J) \otimes T^{1,0}(J)$, and let $I = Re(\phi)$ denote the corresponding element in $End_{\overline{\mathbb{C}}}(TM)$. If $Id + 2I$ is invertible, then the corresponding almost complex structure is given by*

$$J_\phi = (Id + 2I)J(Id + 2I)^{-1}. \tag{8.24}$$

Proof. Note first the following. If $X' \in TM$, $\phi \in \Lambda^{0,1}(J) \otimes T^{1,0}(J)$, then

$$\phi(J(X')) = -J(\phi(X')). \tag{8.25}$$

To see this, write $X' = X^{1,0} + X^{0,1}$, then

$$\phi(J(X')) = \phi(J(X^{1,0} + X^{0,1})) = \phi(iX^{1,0} - iX^{0,1}) = -i\phi(X^{0,1}). \tag{8.26}$$

On the other hand,

$$J(\phi(X')) = J(\phi(X^{1,0} + X^{0,1})) = J(\phi(X^{0,1})) = i\phi(X^{0,1}). \tag{8.27}$$

Next, we write

$$T^{0,1}(J_\phi) = \{v + \phi v, v \in T^{0,1}(J_0)\}. \tag{8.28}$$

Any element $v \in T^{0,1}(J_0)$ can be written as

$$v = X' + iJX', \tag{8.29}$$

for $X' \in TM$, so any $\tilde{v} \in T^{0,1}(J_\phi)$ is written as

$$\begin{aligned} \tilde{v} &= X' + iJX' + \phi(X' + iJX') \\ &= X' + \phi(X') + i(JX' + \phi JX') \\ &= X' + \phi(X') + i(JX' - J\phi X') \\ &= X' + \phi(X') + iJ(X' - \phi X'), \end{aligned} \tag{8.30}$$

where we used (8.25). However, $\tilde{v} \in T^{0,1}(J_\phi)$ can also be written as

$$\tilde{v} = Re(\tilde{v}) + iJ_\phi Re(\tilde{v}). \tag{8.31}$$

We compute

$$\begin{aligned} Re(\tilde{v}) &= Re\{X' + \phi(X') + iJ(X' - \phi X')\} \\ &= X' + Re\{\phi(X') - iJ\phi(X')\} \\ &= X' + 2Re\{\phi(X')\} \\ &= X' + 2Re\{\phi\}(X') \\ &= X' + 2I(X') = (Id + 2I)(X'), \end{aligned} \tag{8.32}$$

and

$$\begin{aligned} Im(\tilde{v}) &= Im\{X' + \phi(X') + iJ(X' - \phi X')\} \\ &= JX' + Im\{\phi(X') - iJ\phi(X')\} \\ &= JX' + 2Im\{\phi(X')\} \\ &= JX' - 2JRe\{\phi(X')\} \\ &= J(X' - 2Re\{\phi(X')\}) \\ &= J(X' - 2Re\{\phi\}(X')) \\ &= J(X' - 2I(X')) = J(Id - 2I)(X'), \end{aligned} \tag{8.33}$$

since $\phi(X')$ is in $T^{1,0}$. So we must have

$$J(Id - 2I)(X') = J_\phi(Id + 2I)(X'), \tag{8.34}$$

or equivalently,

$$J_\phi = J(Id - 2I)(Id + 2I)^{-1} = (Id + 2I)J(Id + 2I)^{-1}, \tag{8.35}$$

since I anti-commutes with J. □

Finally, we have the following correspondence. The path $\phi_t \in \Gamma(\Lambda^{0,1}(J) \otimes T^{1,0}(J))$, is associated to the path of almost complex structures

$$J_t = \Big(J + \frac{1}{2}Re(\phi_t)\Big) J \Big(J + \frac{1}{2}Re(\phi_t)\Big)^{-1}. \tag{8.36}$$

Notice that this correspondence has the nice property that

$$(J_t)'(0) = Re\{\phi_t'(0)\}. \tag{8.37}$$

9 Lecture 9

9.1 Maurer-Cartan equation for integrability

From above we have a correspondence $\phi \mapsto J_\phi$ between sections of $\Lambda^{0,1} \otimes T^{1,0}$ and almost complex structures near J.

Proposition 9.1. *The complex structure J_ϕ is integrable if and only if*

$$\overline{\partial}\phi(t) + [\phi(t), \phi(t)] = 0, \tag{9.1}$$

where $[\phi(t), \phi(t)] \in \Lambda^{0,2} \otimes T^{1,0}$ is a term which is quadratic in the $\phi(t)$ and its first derivatives, that is,

$$\|[\phi(t), \phi(t)]\| \leq \|\phi\| \cdot \|\nabla\phi\|, \tag{9.2}$$

in any local coordinate system.

Proof. By Proposition **??**, the integrability equation is equivalent to $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$. Writing

$$\phi = \sum \phi_{ij} d\overline{z}_i \otimes \frac{\partial}{\partial z^j}, \tag{9.3}$$

if $J(t)$ is integrable, then we must have

$$\Big[\frac{\partial}{\partial \overline{z}^i} + \phi\Big(\frac{\partial}{\partial \overline{z}^i}\Big), \frac{\partial}{\partial \overline{z}^k} + \phi\Big(\frac{\partial}{\partial \overline{z}^k}\Big)\Big] \in T_t^{0,1}. \tag{9.4}$$

This yields

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] \in T_t^{0,1} \tag{9.5}$$

The first two terms are

$$\begin{aligned}\Big[\frac{\partial}{\partial \overline{z}^i}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] &= \sum_j \Big(\frac{\partial \phi_{kj}}{\partial \overline{z}^i} - \frac{\partial \phi_{ij}}{\partial \overline{z}^k}\Big)\frac{\partial}{\partial z^j} \\ &= (\overline{\partial}\phi)\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial z^j}\Big).\end{aligned}$$

The third term is

$$\Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] = \phi_{ij}\Big(\frac{\partial}{\partial z^j}\phi_{kl}\Big)\frac{\partial}{\partial z^l} - \phi_{kl}\Big(\frac{\partial}{\partial z^l}\phi_{ij}\Big)\frac{\partial}{\partial z^j}$$
$$= [\phi,\phi]\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\Big),$$

where $[\phi,\phi]$ is defined by

$$[\phi,\phi] = \sum (d\overline{z}^i \wedge d\overline{z}^k)\Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big], \tag{9.6}$$

and is easily seen to be a well-defined global section of $\Lambda^{0,2} \otimes T^{1,0}$. We have shown that

$$(\overline{\partial}\phi(t) + [\phi(t),\phi(t)])\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\Big) \in T^{0,1}_t. \tag{9.7}$$

But the left hand side is also in $T^{1,0}$. For sufficiently small t however, $T^{0,1}_t \cap T^{1,0} = \{0\}$, and therefore (9.1) holds.

For the converse, if (9.1) is satisfied, then the above argument in reverse shows that the integrability of $T^{0,1}_t$ holds as a distribution, which by Proposition **??** is equivalent to integrability of the complex structure $J(t)$. □

9.2 A fixed point theorem

The following is a cruial tool in the analytic study of moduli spaces and gluing theorems, see for example [Biq13, Lemma 7.3].

Lemma 9.2. *Let $H : E \to F$ be a differentiable mapping between Banach spaces. Define $Q = H - H(0) - H'(0)$. Assume that there are positive constants C_1, s_0, C_2 so that the following are satisfied:*

- (1) *The nonlinear term Q satisfies*

$$\|Q(x) - Q(y)\|_F \leq C_1(\|x\|_E + \|y\|_E)\|x - y\|_E$$

for every $x, y \in B_E(0, s_0)$.

- (2) *The linearized operator at 0, $H'(0) : E \to F$ is an isomorphism with inverse bounded by C_2.*

If s and $\|H(0)\|_F$ are sufficiently small (depending upon C_1, s_0, C_2), then there is a unique solution $x \in B_E(0, s)$ of the equation $H(x) = 0$.

Outline of Proof. The equation $H(x) = 0$ expands to

$$H(0) + H'(0)(x) + Q(x) = 0. \tag{9.8}$$

If we let $x = Gy$, where G is the inverse of $H'(0)$, then we have

$$H(0) + y + Q(Gy) = 0, \tag{9.9}$$

or

$$y = -H(0) - Q(Gy). \tag{9.10}$$

In other words, y is a fixed point of the mapping

$$T : y \mapsto -H(0) - Q(Gy). \tag{9.11}$$

With the assumptions in the lemma, it follows that T is a contraction mapping, so a fixed point exists by the standard fixed point theorem ($T^n y_0$ converges to a unique fixed point for any y_0 sufficiently small). □

Next, we have

Proposition 9.3. *If $H'(0)$ is Fredholm, (finite-dimensional kernel and cokernel and closed range), and there exists a complement of the cokernel on which $H'(0)$ has a bounded right inverse, then there exists a map*

$$\Psi : Ker(H'(0)) \to Coker(H'(0)), \tag{9.12}$$

whose zero set is locally isomorphic to the zero set of H.

Proof. Consider $P = \Pi \circ H$, where Π is projection to a complement of $Coker(H'(0))$. The differential of the map P, $P'(0)$ is now surjective. Choose any complement K to the space $Ker(H'(0))$, and restrict the mapping to this complement. Equivalently, let G be any right inverse, i.e., $H'(0)G = Id$, and let K be the image of G. Given a kernel element $x_0 \in H^1_E$, the equation $H(x_0 + Gy) = 0$ expands to

$$H(0) + H'(0)(x_0 + Gy) + Q(x_0 + Gy) = 0. \tag{9.13}$$

We therefore need to find a fixed point of the map

$$T_{x_0} : y \mapsto -H(0) - Q(x_0 + Gy), \tag{9.14}$$

and the proof is the same as before. □

9.3 Infinitesimal slice theorem for the moduli space of almost complex structures

We want a local model for the space of almost complex structures near a complex structure J modulo diffeomorphism. For this, we first need a way to parametrize diffeomorphisms close to the identity.

The notation $C^{k,\alpha}$ will denote the space of Hölder continuous mappings (or tensors) with $0 < \alpha < 1$. If (M, g) is any Riemannian metric, and Y is a vector field on M, the Riemannian exponential mapping $\exp_p : T_pM \to M$ induces a mapping

$$\Phi_Y : M \to M \tag{9.15}$$

by

$$\Phi_Y(p) = \exp_p(Y). \tag{9.16}$$

If $Y \in C^{k,\alpha}(TX)$ has sufficiently small norm, then Φ_Y is a diffeomorphism. We will use the correspondence $Y \mapsto \Phi_Y$ to parametrize a neighborhood of the identity, analogous to [**?**].

Definition 9.4. We say that $\Phi : X \to X$ is a *small diffeomorphism* if Φ is of the form $\Phi = \Phi_Y$ for some vector field Y satisfying

$$\|Y\|_{C^{k+1,\alpha}} < \epsilon \tag{9.17}$$

for some $\epsilon > 0$ sufficiently small.

Remark 9.5. One might think of using the time 1 flow of a vector field to parametrize diffeomorphisms near the identity. However, this mapping is not so nice, in particular, is not even a local homeomorphism!

Recall from above, that we have an "exponential map" for the space of almost complex structures on M

$$Exp_J : End_{\overline{\mathbb{C}}}(TM, J) \to \mathcal{A}(M) \tag{9.18}$$

defined by

$$Exp_J(I) = \Big(J + \frac{1}{2}I\Big)J\Big(J + \frac{1}{2}I\Big)^{-1} \tag{9.19}$$

We will use this mapping to parametrize almost complex structures near J.

Definition 9.6. We say that $\tilde{J} \in C^{k,\alpha}(End(TM))$ is a is a *small almost complex structure* if $\tilde{J}$ is of the form $\tilde{J} = Exp_J(I)$ for a section $I \in C^{k,\alpha}(End_{\overline{\mathbb{C}}}(TM))$ satisfying

$$\|I\|_{C^{k,\alpha}} < \epsilon \tag{9.20}$$

for some $\epsilon > 0$ sufficiently small.

We note the following. There is an action of diffeomorphisms of class $C^{k+1,\alpha}$ on the space of almost complex structures of class $C^{k,\alpha}$

$$\Phi : \mathcal{D}^{k+1,\alpha} \times \mathcal{A}^{k,\alpha} \to \mathcal{A}^{k,\alpha} \tag{9.21}$$

given by $(\phi, \tilde{J}) \mapsto \phi^* \tilde{J}$, where

$$\phi^* \tilde{J}(X) = (\phi_*)^{-1} \tilde{J}(\phi_* X). \tag{9.22}$$

Note that this mapping does map between the above spaces for the following reason. Choose local coordinates x^j so that

$$J(\partial_i) = J_i^j \partial_j, \tag{9.23}$$

and

$$\phi_*(\partial_i) = (\partial_i \phi^j)\partial_j \tag{9.24}$$

Then

$$\begin{aligned} \phi^* \tilde{J}(\partial_i)_p = (\phi_*)^{-1} J_{\phi(p)}(\phi_* \partial_i) &= (\phi_*)^{-1} J_{\phi(p)}((\partial_i \phi^j)\partial_j) \\ &= (\phi_*)^{-1}(J_j^k(\phi(p))\partial_k)(\partial_i \phi^j) \\ &= \left((\partial_\cdot \phi^\cdot)^{-1}\right)_k^l (J_j^k(\phi(p))(\partial_i \phi^j)\partial_l, \end{aligned} \tag{9.25}$$

and we see that J is not differentiated, only the diffeomorphism is differentiated.

This mapping is continuous, however, it is not differentiable! If it were differentiable, then the differential at (Id, J) would map

$$\Phi_* : C^{k+1,\alpha}(TM) \times C^{k,\alpha}(End_{\overline{\mathbb{C}}}(TM)) \to C^{k,\alpha}(End_{\overline{\mathbb{C}}}(TM)), \tag{9.26}$$

and would be given by

$$(X, I) \mapsto \mathcal{L}_X J + I. \tag{9.27}$$

But the operator $\mathcal{L}_X J$ differentiates J, so the linearized operator would not map into $C^{k,\alpha}$, since J was assumed to be of class $C^{k,\alpha}$. However, we do have the following:

Lemma 9.7. *Consider (X, g, J) with $g \in C^\infty(\mathcal{M})$ and $J \in C^\infty(\mathcal{A})$. Then the mapping from $\mathcal{D}^{k+1,\alpha} \to C^{k,\alpha}(\mathcal{M})$ given by*

$$\phi \mapsto \phi_* g \tag{9.28}$$

and the mapping from $\mathcal{D}^{k+1,\alpha} \to C^{k,\alpha}(\mathcal{A})$ given by

$$\phi \mapsto \phi_* J \tag{9.29}$$

are smooth.

Proof. if we write $g = g_{ij} dx^i \otimes dx^j$ in coordinates, then

$$(\phi^* g)_{ij}(p) = g_{kl}(\phi(p))(\partial_i \phi^k)(\partial_j \phi^l), \tag{9.30}$$

Since g is fixed and smooth, this mapping is smmoth. A simliar argument holds for the other mapping. □

The main result of the section is the following infinitesimal version of a "slice" theorem due to Ebin-Palais, adapted to the complex case by Koiso [Koi83].

Fix a hermitian metric g compatible with J. Define the operator

$$\nabla_g^* : \Gamma(End_{\overline{\mathbb{C}}}(TM)) \to \Gamma(TM) \tag{9.31}$$

by

$$(\nabla_g^* I)^j = -\sum_{k,l} g^{kl} \nabla_k I_l^j. \tag{9.32}$$

The main infinitesimal slicing result is the following:

Theorem 9.8. *For each ACS J_1 in a sufficiently small $C^{k,\alpha}$-neighborhood of J ($k \geq 2$), there is a $C^{k+1,\alpha}$-diffeomorphism $\varphi : M \to M$ such that*

$$\tilde{I} = Exp_J^{-1} \varphi^* J_1 \tag{9.33}$$

satisfies

$$\nabla_g^*(\tilde{I}) = 0. \tag{9.34}$$

Proof. Let $\{X_1, \ldots, X_\kappa\}$ denote a basis of the space of infinitesimal automorphisms (which are real parts of holomorphic vector fields). Consider the map

$$\mathcal{N} : C^{k+1,\alpha}(TM) \times \mathbb{R}^\kappa \times C^{k,\alpha}(End_{\overline{\mathbb{C}}}(TM)) \to C^{k-1,\alpha}(TM) \tag{9.35}$$

given by

$$\mathcal{N}(X, v, I) = \mathcal{N}_I(X, v) = \big(\nabla^*_{(\Phi_X)_* g} Exp^{-1}_{(\Phi_X)_* J}\big[Exp_J(I)\big] + \sum_i v_i (\Phi_X)_* X_i\big). \tag{9.36}$$

Using the expansion

$$\nabla_{g_0 + h} = \nabla_{g_0} + (g_0 + h)^{-1} * \nabla_{g_0} h, \tag{9.37}$$

and the previous Lemma, it follows that this mapping is smooth. Linearizing in (X, v) at $(X, v, \theta) = (0, 0, 0)$, we find

$$\begin{aligned} \mathcal{N}_0'(Y, a) &= \frac{d}{d\epsilon} \big(\nabla_g^*\big[\varphi_{\epsilon Y}^*(J)\big] + \sum_i (\epsilon a_i) X_i\big)\Big|_{\epsilon=0} \\ &= \big(\nabla_g^*[\mathcal{L}_Y J] + \sum_i a_i X_i\big) \\ &= \big(\Box Y + \sum_i a_i X_i\big), \end{aligned}$$

where $\Box Y = \nabla_g^* \mathcal{L}_Y(J)$. Notice that from above, we can identify

$$\Box = \overline{\partial}^* \overline{\partial}, \tag{9.38}$$

so $\Box$ is a self-adjoint operator.

The adjoint map $(\mathcal{N}_0')^* : C^{m+1,\alpha}(TM) \to C^{m-1,\alpha}(TM) \times \mathbb{R}^\kappa$ is given by

$$(\mathcal{N}_0')^*(\eta) = \Big((\Box \eta), \int_M \langle \eta, X_i \rangle \; dV_g\Big). \tag{9.39}$$

If η is in the kernel of the adjoint, the first equation implies that η is a holomorphic vector field, while the second implies that η is orthogonal (in L^2) to the space of holomorphic vector fields. It follows that $\eta = 0$, so the map $\mathcal{N}_0'$ is surjective.

Applying the fixed point theorem from above, given $I_1 \in C^{k,\alpha}(End_{\overline{\mathbb{C}}}(TM))$ small enough, we can solve the equation $\mathcal{N}_{I_1} = 0$; i.e., there is a vector field $X \in C^{k+1,\alpha}(TM)$, and a $v \in \mathbb{R}^\kappa$, such that

$$\nabla^*_{(\Phi_X)_* g} Exp^{-1}_{(\Phi_X)_* J}\big[Exp_J(I_1)\big] + \sum_i v_i (\Phi_X)_* X_i = 0. \tag{9.40}$$

This is equivalent to

$$\nabla^*_g Exp_J^{-1}\big[(\Phi_X)^* Exp_J(I_1)\big] + \sum_i v_i \omega_i = 0. \tag{9.41}$$

Letting $\tilde{I} = Exp_J^{-1}\big[(\Phi_X)^* Exp_J(I_1)\big]$, then $\tilde{I}$ satisfies

$$\nabla^*[\tilde{I}] + \sum_i v_i X_i = 0, \tag{9.42}$$

Pairing with X_j, for $j = 1 \ldots \kappa$, and integrating by parts, we see that $v_j = 0$, and we are done. □

Remark 9.9. The above is just an "infinitesimal" version of the Slice Theorem. The full Ebin-Palais Slice Theorem for Riemannian metrics constructs a local slice for the action of the diffeomorphism group, see [Ebi68]. The main difficulty is that the natural action of the diffeomorphism group on the space of Riemannian metrics is not differentiable as a mapping of Banach spaces (with say Sobolev or Hölder norms). It is however differentiable as a mapping of ILH spaces, see [Omo70, Koi78].

Remark 9.10. It is actually not possible to have a global slice theorem for almost complex structures, some extra structure is needed [**?**, **?**].

10 Lecture 10

10.1 Outline of Kuranishi Theory

The main theorem of Kuranishi is the following.

Theorem 10.1. *Let (M, J) be a complex surface. The space $H^1(M, \Theta)$ is identified with*

$$H^1(M, \Theta) \simeq \frac{Ker(N_J)'}{Im(X \to \mathcal{L}_X J)}, \tag{10.1}$$

and therefore consists of essential infinitesimal deformations of the complex structure.

Furthermore, there is a map

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta) \tag{10.2}$$

called the Kuranishi map *such that the moduli space of complex structures near J is given by the orbit space*

$$\Psi^{-1}(0)/H^0(M, \Theta). \tag{10.3}$$

Instead, we will outline Kuranishi's method following Kodaira-Morrow [MK71]. Let $\Box$ denote the Laplacian

$$\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \tag{10.4}$$

where $\overline{\partial}^*$ is the L^2-adjoint of $\overline{\partial}$ (we have fixed a Hermitian metric compatible with J).

Let $\mathbb{H}^k$ denote the space of harmonic forms in $\Lambda^{0,k} \otimes \Theta$, that is

$$\mathbb{H}^k = \{\phi \in \Gamma(\Lambda^{0,k} \otimes \Theta) | \Box\phi = 0\}. \tag{10.5}$$

Hodge theory tells us that $H^{0,k}(M, \Theta) \cong \mathbb{H}^k$ and that

$$\Gamma(\Lambda^{0,k} \otimes \Theta) = \mathbb{H}^k \oplus Im(\Box) \tag{10.6}$$

$$= \mathbb{H}^k \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^{\#}), \tag{10.7}$$

where these are orthogonal direct sums in L^2. Given any $\phi \in \Gamma(\Lambda^{0,k} \otimes \Theta)$, we have

$$\phi = h + \Box\psi \tag{10.8}$$

where h is harmonic. Applying the same decomposition to ψ

$$\psi = h_1 + \psi_1 \tag{10.9}$$

where h_1 is harmonic and $\psi_1 \in Im(\Box)$, enables us to write

$$\phi = h + \Box\psi_1, \tag{10.10}$$

where ψ_1 is orthogonal to $\mathbb{H}^k$. It is straightforward to show that ψ_1 is unique.

Definition 10.2. *The Green's operator is defined as*

$$G\phi = \psi_1, \tag{10.11}$$

so that any ϕ can be written as

$$\phi = \mathbf{H}\phi + \Box G\phi, \tag{10.12}$$

where $\mathbf{H}$ is harmonic projection onto $\mathbb{H}^k$.

Proposition 10.3. *We have that*

$$\overline{\partial}G = G\overline{\partial} \tag{10.13}$$

Proof. We have

$$\phi = \phi_h + \Box G\phi, \tag{10.14}$$

so

$$\overline{\partial}\phi = \overline{\partial}\Box G\phi = \Box\overline{\partial}G\phi, \tag{10.15}$$

since obviously $\Box\overline{\partial} = \overline{\partial}\Box$. On the other hand, we have

$$\overline{\partial}\phi = (\overline{\partial}\phi)_h + \Box G(\overline{\partial}\phi) = \Box G(\overline{\partial}\phi). \tag{10.16}$$

So we have

$$\Box\overline{\partial}G\phi = \Box G(\overline{\partial}\phi). \tag{10.17}$$

Which says that $\overline{\partial}G\phi - G(\overline{\partial}\phi)$ is harmonic, but this expression is orthogonal to the space of harmonic forms, so must vanish. $\Box$

Proposition 10.4. *If $\gamma \in \Gamma(\Lambda^{0,2} \otimes T^{1,0})$ satisfies $\overline{\partial}\gamma = 0$ then*

$$\phi = \overline{\partial}^* G\gamma \tag{10.18}$$

is the unique solution of

$$\overline{\partial}\phi = \gamma - \gamma_h, \tag{10.19}$$

where γ_h is the L^2 harmonic projection of γ, satisfying $\overline{\partial}^*\phi = 0$.

Proof. We have

$$\begin{aligned}\overline{\partial}\phi &= \overline{\partial}\overline{\partial}^* G\gamma \\ &= (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})G\gamma - \overline{\partial}^*\overline{\partial}G\gamma \\ &= \Box G\gamma - \overline{\partial}^* G\overline{\partial}\gamma \\ &= \gamma - \gamma_h.\end{aligned} \tag{10.20}$$

For uniqueness, if we have any solution ϕ with $\overline{\partial}^*\phi = 0$ then

$$\begin{aligned}\phi &= \phi_h + \Box G\phi \\ &= \phi_h + \overline{\partial}\overline{\partial}^* G\phi + \overline{\partial}^*\overline{\partial}G\phi \\ &= \phi_h + \overline{\partial}\overline{\partial}^* G\phi + \overline{\partial}^* G\overline{\partial}\phi \\ &= \phi_h + \overline{\partial}\overline{\partial}^* G\phi + \overline{\partial}^* G(\gamma - \gamma_h).\end{aligned} \tag{10.21}$$

Rearranging terms,

$$\phi - \overline{\partial}^* G(\gamma - \gamma_h) = \phi_h + \overline{\partial}\overline{\partial}^* G\phi. \tag{10.22}$$

The left hand side is $\overline{\partial}^*$-closed by assumption, and the right hand side is $\overline{\partial}$-closed. Since these spaces are orthogonal, both sides vanish, so we necessarily have

$$\phi = \overline{\partial}^* G(\gamma - \gamma_h) = \overline{\partial}^* G\gamma. \tag{10.23}$$

$\Box$

Next, define a mapping

$$\Psi : \mathbb{H}^1 \to \Gamma((\Lambda^{0,1} \otimes \Theta)) \tag{10.24}$$

as follows. Given $\phi_1 \in \mathbb{H}^1$, we want to solve the following equation for ϕ:

$$\phi = \phi_1 - \overline{\partial}^* G[\phi, \phi], \tag{10.25}$$

with $\overline{\partial}^* \phi = 0$ necessarily. Defining $\tilde{\phi} = \phi - \phi_1$, this is equivalent to finding a fixed point of the mapping

$$\tilde{\phi} \mapsto -\overline{\partial}^* G[\tilde{\phi} + \phi_1, \tilde{\phi} + \phi_1]. \tag{10.26}$$

If ϕ_1 has sufficiently small norm, then this admits a unique solution using an iteration procedure similar to the above process using Hölder norms, but working in the Banach space of $\overline{\partial}^*$-closed sections (details omitted). Next,

Proposition 10.5. *If ϕ_1 is in a sufficiently small ball around the origin in $\mathbb{H}^1$, then the solution ϕ of* (10.25) *solves*

$$\overline{\partial}\phi + [\phi, \phi] = 0 \tag{10.27}$$

if and only if

$$\mathbf{H}[\phi, \phi] = 0. \tag{10.28}$$

Proof. One way is easy: if $\overline{\partial}\phi + [\phi, \phi] = 0$ then obviously $\overline{\partial}[\phi, \phi] = 0$. Then by Proposition 10.4,

$$\begin{aligned} \overline{\partial}\phi &= \overline{\partial}\phi_1 - \overline{\partial}\overline{\partial}^* G[\phi, \phi] \\ &= 0 - [\phi, \phi] + H[\phi, \phi] = -[\phi, \phi], \end{aligned} \tag{10.29}$$

so $H[\phi, \phi] = 0$. For the other way, let

$$\psi = \overline{\partial}\phi + [\phi, \phi]. \tag{10.30}$$

Then

$$\begin{aligned} \psi &= -\overline{\partial}\overline{\partial}^* G[\phi, \phi] + [\phi, \phi] \\ &= -\overline{\partial}\overline{\partial}^* G[\phi, \phi] + [\phi, \phi] - H[\phi, \phi] \\ &= -\overline{\partial}\overline{\partial}^* G[\phi, \phi] + \Box G[\phi, \phi] \\ &= \overline{\partial}^* \overline{\partial} G[\phi, \phi] \\ &= \overline{\partial}^* G \overline{\partial}[\phi, \phi] \\ &= 2\overline{\partial}^* G[\overline{\partial}\phi, \phi] \\ &= 2\overline{\partial}^* G[\psi - [\phi, \phi], \phi] \\ &= 2\overline{\partial}^* G[\psi, \phi]. \end{aligned} \tag{10.31}$$

Then since $\overline{\partial}^* G$ is a bounded operator as a mapping between Hölder spaces,

$$\|\psi\|_{C^{k,\alpha}} \leq C \|\psi\|_{C^{k,\alpha}} \|\phi\|_{C^{k,\alpha}}. \tag{10.32}$$

If ϕ is sufficiently small, this implies that $\psi \equiv 0$. □

The Kuranishi map is the corresponding mapping

$$\Phi : \mathbb{H}^1 \to \mathbb{H}^2, \tag{10.33}$$

and the zeroes of Φ parametrize the integrable complex structures near J satisfying the condition $\overline{\partial}^* \phi = 0$:

Proposition 10.6. *If ψ is sufficiently small and solves $\overline{\partial}\psi + [\psi, \psi]$ with $\overline{\partial}^* \psi = 0$, then ψ is in the Kuranishi family.*

Proof. We have

$$\Box\psi = \overline{\partial}^* \overline{\partial}\psi + \overline{\partial}\overline{\partial}^* \psi = \overline{\partial}^* [\psi, \psi]. \tag{10.34}$$

Also,

$$\psi - H\psi = G\Box\psi = G\overline{\partial}^* [\psi, \psi], \tag{10.35}$$

or

$$\psi = \phi_1 + G\overline{\partial}^* [\psi, \psi], \tag{10.36}$$

where $\phi_1 = H\psi$. From the uniqeness part of the fixed point theorem, this equation has a unique solution in a small ball for ϕ_1 sufficiently small, so the solution must be in the Kuranishi family. □

Finally, by the gauging in Theorem 9.8, any complex structure sufficiently near J can be put into this divergence-free gauge after diffeomorphism. So the Kuranishi map indeed parametrizes all complex structues near J modulo diffeomorphism. Kuranishi also shows that that Φ is holomorphic, so that the moduli space is an analytic space.

In the case of non-trivial automorphisms, note that Φ is equivariant under automorphisms of J, so if (M, J) admits non-trivial holomorphic vector fields, then the moduli space of complex structure modulo diffeomorphic is isomorphic to

$$\Psi^{-1}(0)/\mathbb{H}^0, \tag{10.37}$$

but the full proof of this identification requires a more elaborate slice theorem.

Corollary 10.7. *If $H^1(M, \Theta) = 0$ then (M, J) is rigid as a complex manifold.*

Example 10.8. Complex projective space $\mathbb{P}^n$ is rigid.

Corollary 10.9. *If $H^0(M, \Theta) = 0$ and $H^2(M, \Theta) = 0$, then the moduli space of complex structures near J is a smooth manifold of real dimension $2dim_{\mathbb{C}}(H^1(M, \Theta)$.*

Remark 10.10. There are examples of complex manifold for which there exists non-zero elements of $H^1(M, \Theta)$ which do not arise from actual deformations of complex structure, these elements are *obstructed* [**?**, **?**]

A special case where the Kuranishi mapping has been computed is the case of Calabi-Yau metrics. In this case, the following is known (we will not discuss the proof, and refer the reader to [Huy05]):

Theorem 10.11 (Tian-Todorov)**.** *For a Calabi-Yau metric* (X, g)*, the Kuranishi map* $\Psi \equiv 0$*. That is, every infintesimal Einstein deformation integrates to an actual deformation.*

Note that there are many example of Calabi-Yaus for which $H^2(M, \Theta)$ is non-zero, so this theorem is remarkable.

11 Lecture 11

11.1 Inner products

Let Z_1 and Z_2 be sections of $T^{1,0}$. Then we define the inner product of Z_1 and Z_2 to be

$$(Z_1, Z_2) = \langle Z_1, \overline{Z_2} \rangle, \tag{11.1}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian inner product extended by complex linearity. We will need the following formula.

Proposition 11.1. *For* $Z_1, Z_2 \in \Gamma(T^{1,0})$*,*

$$Re(Z_1, Z_2) = 2\langle Re(Z_1), Re(Z_2) \rangle \tag{11.2}$$

Proof. We compute

$$\begin{aligned} Re(Z_1, Z_2) &= Re\big\{ \big\langle Re(Z_1) - iJRe(Z_2), Re(Z_2) + iJRe(Z_2) \big\rangle \big\} \\ &= \big\langle Re(Z_1), Re(Z_2) \big\rangle + \big\langle JRe(Z_1), JRe(Z_2) \big\rangle \\ &= 2\big\langle Re(Z_1), Re(Z_2) \big\rangle, \end{aligned} \tag{11.3}$$

since the inner product is J-invariant. □

Next, consider φ_1 and φ_2 sections of $\Lambda^{0,1} \otimes T^{1,0}$. Define

$$(\varphi_1, \varphi_2) = \langle \varphi_1, \overline{\varphi_2} \rangle, \tag{11.4}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian inner product extended by complex linearity. Similarly, we have

Proposition 11.2. *For* $\varphi_1, \varphi_2 \in \Gamma(\Lambda^{0,1} \otimes T^{1,0})$*,*

$$Re(\varphi_1, \varphi_2) = 2\langle Re(\varphi_1), Re(\varphi_2) \rangle \tag{11.5}$$

Proof. We compute

$$\begin{aligned} Re(\varphi_1, \varphi_2) &= Re\Big\{\Big\langle Re(\varphi_1) - iJ_1 Re(\varphi_2), Re(\varphi_2) + iJ_1 Re(\varphi_2)\Big\rangle\Big\} \\ &= \Big\langle Re(\varphi_1), Re(\varphi_2)\Big\rangle + \Big\langle J_1 Re(\varphi_1), J_1 Re(\varphi_2)\Big\rangle \\ &= 2\Big\langle Re(\varphi_1), Re(\varphi_2)\Big\rangle, \end{aligned} \tag{11.6}$$

since the inner product is J_1-invariant. To see this, from Proposition 5.1, for any $I_1, I_2 \in \Gamma(End_{\overline{C}}TM)$, we have

$$\begin{aligned} \langle J_1 I_1, J_1 I_2\rangle &= g_{aa'} g^{cc'} (I_1)^a_b J^b_c (I_2)^{a'}_{b'} J^{b'}_{c'} \\ &= g_{aa'} (I_1)^a_b (I_2)^{a'}_{b'} g^{cc'} J^b_c J^{b'}_{c'}. \end{aligned} \tag{11.7}$$

But since g is J-invariant, we have

$$g_{cc'} J^c_b J^{c'}_{b'} = g_{bb'}. \tag{11.8}$$

Since $J^{-1} = -J$, taking an inverse of this equation yields

$$g^{cc'} J^b_c J^{b'}_{c'} = g^{bb'}. \tag{11.9}$$

So we have

$$\langle J_1 I_1, J_1 I_2\rangle = g_{aa'} (I_1)^a_b (I_2)^{a'}_{b'} g^{bb'} = \langle I_1, I_2\rangle. \tag{11.10}$$

□

11.2 L^2-adjoints

We next want to compute the formal L^2 adjoints of our above operators. For

$$\Gamma(T^{1,0}) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes T^{1,0}), \tag{11.11}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{0,1} \otimes T^{1,0}) \xrightarrow{\overline{\partial}^*} \Gamma(T^{1,0}), \tag{11.12}$$

is defined as follows. For $Z \in \Gamma(T^{1,0})$ and $\varphi \in \Gamma(\Lambda^{0,1} \otimes T^{1,0})$, we have

$$\int_M (\varphi, \overline{\partial} Z) dV = \int_M (\overline{\partial}^* \varphi, Z) dV, \tag{11.13}$$

where dV denotes the Riemannian volume element.

For

$$\Gamma(TM) \xrightarrow{-\frac{1}{2} J \circ L_{X'} J} \Gamma(End_{\overline{\mathbb{C}}}(TM)), \tag{11.14}$$

the Riemannian adjoint

$$\Gamma(End_{\overline{\mathbb{C}}}(TM)) \xrightarrow{(-\frac{1}{2} J \circ L_{X'} J)^*} \Gamma(TM), \tag{11.15}$$

is defined as follows. For $X' \in \Gamma(TM)$ and $I \in \Gamma(End_{\overline{\mathbb{C}}}(TM))$, we have

$$\int_M \Big\langle I, -\frac{1}{2} J \circ L_{X'} J\Big\rangle dV = \int_M \Big\langle (-\frac{1}{2} J \circ L_{X'} J)^* I, X'\Big\rangle dV. \tag{11.16}$$

Proposition 11.3. *The following diagram commutes.*

$$\begin{array}{ccc} \Gamma(\Lambda^{0,1}\otimes T^{1,0}) & \xrightarrow{\overline{\partial}^*} & \Gamma(T^{1,0}) \\ \Big\downarrow{\scriptstyle Re} & & \Big\downarrow{\scriptstyle Re} \\ \Gamma(End_{\overline{\mathbb{C}}}(TM)) & \xrightarrow{(-\frac{1}{2}J\circ L_{X'}J)^*} & \Gamma(TM), \end{array} \tag{11.17}$$

Proof. For $Z \in \Gamma(T^{1,0})$, and $\varphi \in \Gamma(\Lambda^{0,1}\otimes T^{1,0})$,

$$\begin{aligned} \int_M \Big\langle (-\frac{1}{2}J\circ L_{X'}J)^* Re(\varphi), Re(Z)\Big\rangle dV &= \int_M \Big\langle Re(\varphi), -\frac{1}{2}J\circ L_{Re(Z)}J\Big\rangle dV \\ &= \int_M \Big\langle Re(\varphi), Re(\overline{\partial}Z)\Big\rangle dV \\ &= \frac{1}{2}\int_M Re(\varphi,\overline{\partial}Z)dV \\ &= \frac{1}{2}Re\int_M (\varphi,\overline{\partial}Z)dV \\ &= \frac{1}{2}Re\int_M (\overline{\partial}^*\varphi, Z)dV \\ &= \frac{1}{2}\int_M Re(\overline{\partial}^*\varphi, Z)dV \\ &= \int_M \langle Re(\overline{\partial}^*\varphi), Re(Z)\rangle dV. \end{aligned} \tag{11.18}$$

Since this is true for any Z, we must have

$$(-\frac{1}{2}J\circ L_{X'}J)^* Re(\varphi) = Re(\overline{\partial}^*\varphi). \tag{11.19}$$

□

11.3 The divergence operator

We define the real operator

$$\nabla^* : \Gamma(End(TM)) \to \Gamma(TM) \tag{11.20}$$

by

$$(\nabla^* I)^j = -\sum_{k,l} g^{kl}\nabla_k I_l^j. \tag{11.21}$$

Proposition 11.4. *We have*

$$(\frac{1}{2}J\circ L_{X'}J)^* = \nabla^* \tag{11.22}$$

Proof. First, we want to write a formula for the operator $X \mapsto \mathcal{L}_X J$ using covariant differentiation. Given $Y \in \Gamma(TM)$,

$$\begin{aligned} \mathcal{L}_X J(Y) &= \mathcal{L}_X(J(Y)) - J(\mathcal{L}_X Y) \\ &= [X, JY] - J([X,Y]) \\ &= \nabla_X(JY) - \nabla_{JY}X - J(\nabla_X Y - \nabla_Y X), \end{aligned} \tag{11.23}$$

since the Riemannian connection is symmetric (torsion-free). Then since g is Kähler, $\nabla J \equiv 0$, so

$$\begin{aligned} \mathcal{L}_X J(Y) &= (\nabla_X(J))Y + J(\nabla_X Y) - \nabla_{JY}X - J(\nabla_X Y - \nabla_Y X) \\ &= -\nabla_{JY}X + J(\nabla_Y X). \end{aligned} \tag{11.24}$$

Let us write this formula in coordinates using,

$$(\mathcal{L}_X J)(\partial_i) = (\mathcal{L}_X J)_i^l \partial_l. \tag{11.25}$$

The left hand side of this is

$$\begin{aligned} (\mathcal{L}_X J)(\partial_i) &= -\nabla_{J\partial_i}X + J(\nabla_{\partial_i}X) \\ &= -\nabla_{J_i^j \partial_j}X + J(\nabla_{\partial_i}(X^l \partial_l))) \\ &= -J_i^j \nabla_j X + J((\nabla_i X^l)\partial_l) \\ &= -J_i^j(\nabla_j X^l)\partial_l + (\nabla_i X^l)J(\partial_l) \\ &= -J_i^j(\nabla_j X^l)\partial_l + (\nabla_i X^l)J_l^k(\partial_k). \end{aligned} \tag{11.26}$$

From this we obtain

$$(\mathcal{L}_X J)_i^l = -J_i^j \nabla_j X^l + J_k^l \nabla_i X^k. \tag{11.27}$$

It follows that

$$\begin{aligned} (J \circ \mathcal{L}_X J)_i^k &= -J_i^p J_p^j \nabla_j X^k + J_i^p J_l^k \nabla_p X^l \\ &= \nabla_i X^k + J_i^p \nabla_p X^l J_l^k. \end{aligned} \tag{11.28}$$

Recalling that the projection $\Pi : End(TM) \to End_{\overline{\mathbb{C}}}(TM)$ is given by $I \mapsto \frac{1}{2}(I + JIJ)$, we obtain the formula

$$J \circ \mathcal{L}_X J = 2\Pi_{End_{\overline{\mathbb{C}}}(TM)}(\nabla X). \tag{11.29}$$

We next claim that the decomposition

$$End(TM) = End_{\mathbb{C}}(TM) \oplus End_{\overline{\mathbb{C}}}(TM) \tag{11.30}$$

is orthogonal. To see this, take $I \in End_{\mathbb{C}}(TM)$ and $\tilde{I} \in End_{\overline{\mathbb{C}}}(TM)$. Then using the fact that the inner product is J_1-invariant (see the proof of Proposition (11.2)),

$$\begin{aligned} \langle I, \tilde{I} \rangle &= \frac{1}{4}\langle I - JIJ, \tilde{I} + J\tilde{I}J \rangle \\ &= \frac{1}{4}\Big(\langle I, \tilde{I} \rangle + \langle I, J\tilde{I}J \rangle - \langle JIJ, \tilde{I} \rangle - \langle JIJ, J\tilde{I}J \rangle\Big) \\ &= \frac{1}{4}\Big(\langle I, \tilde{I} \rangle - \langle IJ, J\tilde{I} \rangle + \langle IJ, J\tilde{I} \rangle - \langle I, \tilde{I} \rangle\Big) = 0. \end{aligned} \tag{11.31}$$

Using this, we compute the adjoint

$$\begin{aligned}\int_M \langle \frac{1}{2} J \circ \mathcal{L}_X J, I \rangle dV &= \int_M \langle \Pi_{End_{\overline{\mathbb{C}}}(TM)}(\nabla X), I \rangle dV \\ &= \int_M \langle \nabla X, I \rangle dV, \\ &= \int_M \langle X, \nabla^* I \rangle dV,\end{aligned} \tag{11.32}$$

since I is already in $End_{\overline{\mathbb{C}}}(TM)$, and the orthogonal decomposition just shown above.

Last, we derive the coordinate formula for the ∇^* operator, which is defined as

$$\int_M \langle \nabla X, I \rangle dV = \int_M \langle X, \nabla^* I \rangle dV, \tag{11.33}$$

for any $X \in \Gamma(TM)$ and $I \in \Gamma(End_{\overline{\mathbb{C}}}(TM))$. If these are supported in a coordinate ball, then

$$\begin{aligned}\int_M \langle \nabla X, I \rangle dV &= \int_M \nabla_i X^j g^{ip} g_{jq} I_p^q dV \\ &= -\int_M X^j \nabla_i (g^{ip} g_{jq} I_p^q) dV \\ &= -\int_M X^j g^{ip} g_{jq} \nabla_i I_p^q) dV \\ &= -\int_X g^{jq} X^j g^{ip} \nabla_i I_p^q dV.\end{aligned} \tag{11.34}$$

which proves the formula (11.21). □

11.4 Laplacian-type operators

Define

$$\Box_{\overline{\partial}} : \Gamma(T^{1,0}) \to \Gamma(T^{1,0}) \tag{11.35}$$

as $\Box_{\overline{\partial}} X = \overline{\partial}^* \overline{\partial} X$, and

$$\Box_{\mathbb{R}} : \Gamma(TM) \to \Gamma(TM) \tag{11.36}$$

by $\Box_{\mathbb{R}}(X') = \nabla^*(\frac{1}{2} J \circ \mathcal{L}_{X'} J)$. By combining Propositions 5.2 and 11.3, we obtain the commutative diagram

$$\begin{array}{ccc} \Gamma(T^{1,0}) & \xrightarrow{\Box_{\overline{\partial}}} & \Gamma(T^{1,0}) \\ \downarrow{\scriptstyle Re} & & \downarrow{\scriptstyle Re} \\ \Gamma(TM) & \xrightarrow{\Box_{\mathbb{R}}} & \Gamma(TM), \end{array} \tag{11.37}$$

that is $\Box_{\mathbb{R}}(ReX) = Re(\Box_{\overline{\partial}} X)$.

Proposition 11.5. *If M is compact, the kernel of $\Box_{\overline{\partial}}$ consists of precisely the holomorphic vector fields, and the kernel of $\Box_{\mathbb{R}}$ consists of precisely the infinitesimal automorphisms.*

Proof. Let X satisfy $\Box_{\overline{\partial}} X = 0$. Using integration by parts,

$$\begin{aligned} 0 &= \int_M (\Box_{\overline{\partial}} X, X) dV \\ &= \int_M (\overline{\partial}^* \overline{\partial} X, X) dV \\ &= \int_M (\overline{\partial} X, \overline{\partial} X) dV \\ &= \int_M \langle \overline{\partial} X, \overline{\overline{\partial} X} \rangle dV. \end{aligned} \tag{11.38}$$

The last integrand is a real quantity, so we have

$$0 = \int_M Re(\overline{\partial} X, \overline{\partial} X) dV = 2 \int_M \langle Re(\overline{\partial} X), Re(\overline{\partial} X) \rangle dV. \tag{11.39}$$

This shows that $Re(\overline{\partial} X) = 0$, and consequently, $\overline{\partial} X = 0$.

Next, if $\Box_{\mathbb{R}}(X') = 0$, then let $X = X' - iJX'$. Then

$$0 = \Box_{\mathbb{R}}(X') = \Box_{\mathbb{R}}(ReX) = Re(\Box_{\overline{\partial}} X), \tag{11.40}$$

But if $X \in \Gamma(T^{1,0})$, then $\Box_{\overline{\partial}} X \in T^{1,0}$ also, so this implies that $\Box_{\overline{\partial}} X = 0$. By the first part $\overline{\partial} X = 0$, which implies that $\mathcal{L}_{X'} J = 0$ by Proposition 5.2. □

11.5 The musical isomorphisms

We recall the following from Riemannian geometry. The metric gives an isomorphism between TM and T^*M,

$$\flat : TM \to T^*M \tag{11.41}$$

defined by

$$\flat(X)(Y) = g(X, Y). \tag{11.42}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp \omega_1, \sharp \omega_2). \tag{11.43}$$

Note that if g has components g_{ij}, then $\langle \cdot, \cdot \rangle$ has components g^{ij}, the inverse matrix of g_{ij}.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{11.44}$$

where

$$X_i = g_{ij} X^j, \tag{11.45}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{11.46}$$

where

$$\omega^i = g^{ij} \omega_j, \tag{11.47}$$

thus the sharp operator "raises" an index.

The $\flat$ operator extends to a complex linear mapping

$$\flat : TM \otimes \mathbb{C} \to T^*M \otimes \mathbb{C}. \tag{11.48}$$

We have the following

Proposition 11.6. *The operator $\flat$ is a complex anti-linear isomorphism*

$$\flat : T^{(1,0)} \to \Lambda^{0,1} \tag{11.49}$$

$$\flat : T^{(0,1)} \to \Lambda^{1,0}. \tag{11.50}$$

Proof. These mapping properties follow from the Hermitian property of g. Next, for any two vectors X and Y

$$\flat(JX)Y = g(JX, Y), \tag{11.51}$$

while

$$J(\flat X)(Y) = (\flat X)(JY) = g(X, JY) = -g(JX, Y). \tag{11.52}$$

□

In components, we have the following. The metric on $T^*Z \otimes \mathbb{C}$ has components

$$g(dz^\alpha, d\overline{z}^\beta) = g^{\alpha\overline{\beta}} \tag{11.53}$$

where these are the components of the inverse matrix of $g_{\alpha\overline{\beta}}$. We have the identities

$$g^{\alpha\overline{\beta}} g_{\overline{\beta}\gamma} = \delta^\alpha_\gamma, \tag{11.54}$$

$$g^{\overline{\alpha}\beta} g_{\beta\overline{\gamma}} = \delta^{\overline{\alpha}}_{\overline{\gamma}}, \tag{11.55}$$

If $X = X^\alpha Z_\alpha$ is in $T^{(1,0)}$, then $\flat X$ has components

$$(\flat X)_{\overline{\alpha}} = g_{\overline{\alpha}\beta} X^\beta, \tag{11.56}$$

and if $X = X^{\overline{\alpha}} Z_{\overline{\alpha}}$ is in $T^{(0,1)}$, then $\flat X$ has components

$$(\flat X)_\alpha = g_{\alpha\overline{\beta}} X^{\overline{\beta}}, \tag{11.57}$$

Similarly, if $\omega = \omega_\alpha Z^\alpha$ is in $\Lambda^{1,0}$, then $\sharp\omega$ has components

$$(\sharp\omega)^{\overline{\alpha}} = g^{\overline{\alpha}\beta}\omega_\beta, \tag{11.58}$$

and if $\omega = \omega_{\overline{\alpha}} Z^{\overline{\alpha}}$ is in $\Lambda^{0,1}$, then $\sharp X$ has components

$$(\sharp\omega)^\alpha = g^{\alpha\overline{\beta}}\omega_{\overline{\beta}}. \tag{11.59}$$

If (M, J, g) is almost Hermitian, then the $\flat$ operator gives an identification

$$End_{\mathbb{R}}(TM) \cong T^*M \otimes TM \cong T^*M \otimes T^*M, \tag{11.60}$$

This yields a trace map defined on $T^*M \otimes T^*M$ defined as follows. If

$$h = h_{\alpha\beta} dz^\alpha dz^\beta + h_{\overline{\alpha}\overline{\beta}} d\overline{z}^\alpha d\overline{z}^\beta + h_{\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta + h_{\overline{\alpha}\beta} d\overline{z}^\alpha dz^\beta, \tag{11.61}$$

then

$$tr(h) = g^{\alpha\overline{\beta}} h_{\alpha\overline{\beta}} + g^{\overline{\alpha}\beta} h_{\overline{\alpha}\beta}. \tag{11.62}$$

Note that the components $h_{\alpha\beta}$ and $h_{\overline{\alpha}\overline{\beta}}$ do not contribute to the trace.

Remark 11.7. *In Kähler geometry one sometimes sees the trace of some* 2*-tensor defined as just the first term in* (11.62)*. If h is the complexification of a real tensor, then this is* (1/2) *of the Riemannian trace.*

12 Lecture 12

12.1 Serre duality

For a real oriented Riemannian manifold of dimension n, the Hodge star operator is a mapping

$$* : \Lambda^p \to \Lambda^{n-p} \tag{12.1}$$

defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV_g, \tag{12.2}$$

for $\alpha, \beta \in \Lambda^p$, where dV_g is the oriented Riemannian volume element.

If M is a complex manifold of complex dimension $m = n/2$, and g is a Hermitian metric, then the Hodge star extends to the complexification

$$* : \Lambda^p \otimes \mathbb{C} \to \Lambda^{2m-p} \otimes \mathbb{C}, \tag{12.3}$$

and it is not hard to see that

$$* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}. \tag{12.4}$$

Therefore the operator

$$\overline{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \tag{12.5}$$

is a $\mathbb{C}$-antilinear mapping and satisfies

$$\alpha \wedge \overline{*}\beta = \langle \alpha, \overline{\beta} \rangle dV_g. \tag{12.6}$$

for $\alpha, \beta \in \Lambda^p \otimes \mathbb{C}$.

The L^2-adjoint of $\overline{\partial}$ is given by

$$\overline{\partial}^* = - * \overline{\partial} *, \tag{12.7}$$

and the $\overline{\partial}$-Laplacian is defined by

$$\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*. \tag{12.8}$$

Letting

$$\mathbb{H}^{p,q}(M,g) = \{\alpha \in \Lambda^{p,q} | \Delta_{\overline{\partial}} \alpha = 0\}, \tag{12.9}$$

Hodge theory tells us that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathbb{H}^{p,q}(M,g), \tag{12.10}$$

is finite-dimensional, and that

$$\Lambda^{p,q} = \mathbb{H}^{p,q}(M,g) \oplus Im(\Delta_{\overline{\partial}}) \tag{12.11}$$

$$= \mathbb{H}^{p,q}(M,g) \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^*), \tag{12.12}$$

with this being an orthogonal direct sum in L^2.

Corollary 12.1. *Let (M, J) be a compact complex manifold of real dimension $n = 2m$. Then*

$$H^{p,q}_{\overline{\partial}}(M) \cong (H^{n-p,n-q}_{\overline{\partial}}(M))^*, \tag{12.13}$$

and therefore

$$b^{p,q}(M) = b^{n-p,n-q}(M) \tag{12.14}$$

Proof. One verifies that

$$\overline{*}\Delta_{\overline{\partial}} = \Delta_{\overline{\partial}}\overline{*}, \tag{12.15}$$

so the mapping $\overline{*}$ preserves the space of harmonic forms, and is invertible. The result then follows from Hodge theory. □

This same argument works with form taking values in a holomorphic bundle, and the conclusion of Serre duality is that

$$H^p(M,E) \cong (H^{n-p}(M, K \otimes E^*), \tag{12.16}$$

where $K = \Lambda^{n,0}$ is the canonical bundle. Note that

$$H^p(M, \Omega^q(E)) \cong \mathbb{H}^{q,p}(M,E). \tag{12.17}$$

Proposition 12.2. *If (M,J) is a compact complex manifold then*

$$b^k(M) \leq \sum_{p+q=k} b^{p,q}(M), \tag{12.18}$$

and

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b^k(M) = \sum_{p,q=0}^{m} (-1)^{p+q} b^{p,q}(M). \tag{12.19}$$

Proof. This requires some machinery; it follows from the Frölicher spectral sequence [?]. □

12.2 Hodge numbers of a Kähler manifold

Now let us assume that (M,J,g) is Kähler. That is, the fundamental 2-form ω is closed. Consider the 3 Laplacians

$$\Delta_H = d^*d + dd^*, \tag{12.20}$$
$$\Delta_\partial = \partial^*\partial + \partial\partial^* \tag{12.21}$$
$$\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*, \tag{12.22}$$

where $\cdot^*$ denotes the L^2-adjoint. The key is the following

Proposition 12.3. *If (M,J,g) is Kähler, then*

$$\Delta_H = 2\Delta_\partial = 2\Delta_{\overline{\partial}}. \tag{12.23}$$

Proof. Let L denote the mapping

$$L : \Lambda^p \to \Lambda^{p+2} \tag{12.24}$$

given by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form. Then we have the identities

$$[\overline{\partial}^*, L] = i\partial \tag{12.25}$$
$$[\partial^*, L] = -i\overline{\partial}. \tag{12.26}$$

These are proved proved first in $\mathbb{C}^n$ and then on a Kähler manifold using Kähler normal coordinates. The proposition then follows from these identities (proof omitted). □

Proposition 12.4. *If* (M, J, g) *is a compact Kähler manifold, then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M), \tag{12.27}$$

and

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{q,p}_{\overline{\partial}}(M)^*. \tag{12.28}$$

Consequently,

$$b^k(M) = \sum_{p+q=k} b^{p,q}(M) \tag{12.29}$$

$$b^{p,q}(M) = b^{q,p}(M). \tag{12.30}$$

Proof. This follows because if a harmonic k-form is decomposed as

$$\phi = \phi^{p,0} + \phi^{p-1,1} + \cdots + \phi^{1,p-1} + \phi^{0,p}, \tag{12.31}$$

then

$$0 = \Delta_H \phi = 2\Delta_{\overline{\partial}}\phi^{p,0} + 2\Delta_{\overline{\partial}}\phi^{p-1,1} + \cdots + 2\Delta_{\overline{\partial}}\phi^{1,p-1} + 2\Delta_{\overline{\partial}}\phi^{0,p}, \tag{12.32}$$

therefore

$$\Delta_{\overline{\partial}}\phi^{p-k,k} = 0, \tag{12.33}$$

for $k = 0 \ldots p$.

Next,

$$\overline{\Delta_{\overline{\partial}}\phi} = \Delta_{\partial}\overline{\phi}, \tag{12.34}$$

so conjugation sends harmonic forms to harmonic forms. □

This yields a topologicial obstruction for a complex manifold to admit a Kähler metric:

Corollary 12.5. *If* (M, J, g) *is a compact Kähler manifold, then the odd Betti numbers of* M *are even.*

Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$

$$(z_1, z_2) \to 2^k(z_1, z_2). \tag{12.35}$$

This is a free and properly discontinuous action, so the quotient $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ is a manifold, which is called a primary Hopf surface. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$, which has $b^1 = 1$, therefore it does not admit any Kähler metric.

12.3 The Hodge diamond

The following picture is called the Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & & \vdots & & & \\
h^{n,0} & \dots & & \vdots & & \dots & h^{0,n}. \\
 & & & \vdots & & & \\
 & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & &
\end{array}
\tag{12.36}
$$

Reflection about the center vertical is conjugation. Reflection about the center horizontal is Hodge star. The composition of these two operations, or rotation by π, is Serre duality.

For a surface, the Hodge diamond is

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
\quad .
\tag{12.37}
$$

13 Lecture 13

13.1 Complex projective space

Complex projective spaces is defined to be the space of lines through the origin in $\mathbb{C}^{n+1}$. This is equivalent to $\mathbb{C}^{n+1}/\sim$, where $\sim$ is the equivalence relation

$$(z^0, \dots, z^n) \sim (w^0, \dots, w^n) \tag{13.1}$$

if there exists $\lambda \in \mathbb{C}^*$ so that $z^j = \lambda w^j$ for $j = 1 \dots n$. The equivalence class of $(z^0, \dots, z^n)$ will be denoted by $[z^0 : \dots : z^n]$. Letting $U_j = \{[z^0 : \dots : z^n] | z^j \neq 0\}$, $\mathbb{CP}^n$ is covered by $(n+1)$ coordinate charts $\phi_j : U_j \to \mathbb{C}^n$ defined by

$$\phi_j : [z^0 : \dots : z^n] \mapsto \Big(\frac{z^0}{z_j}, \dots, \frac{z^{j-1}}{z_j}, \frac{z^{j+1}}{z_j}, \dots, \frac{z^n}{z_j} \Big), \tag{13.2}$$

with inverse given by

$$\phi_j^{-1} : (w^1, \dots, w^n) \mapsto [w^1 : \dots w^{j-1} : 1 : w^j : \dots : w^n]. \tag{13.3}$$

The overlap maps are holomorphic, which gives $\mathbb{CP}^n$ the structure of a complex manifold.

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\}$ denote the projection map, and consider the form

$$\tilde{\Phi} = -4i\partial\overline{\partial} \log \Big(\sum_{j=0}^{n} |z^j|^2 \Big) \tag{13.4}$$

It is easy to see that this form is the pull-back of a $(1,1)$-form on $\mathbb{CP}^n$, that is $\tilde{\Phi} = \pi^* \Phi$. Furthermore, Φ is positive-definite, which implies that Φ is the fundamental 2-form of a Hermitian metric g_{FS}. Furthermore, since $d = \partial + \overline{\partial}$, it follows that $d\Phi = 0$, so g_{FS} is a Kähler metric. Note that $\tilde{\Phi}$ is invariant under the action of $\mathrm{U}(n+1)$, which implies that the isometry group of g_{FS} contains $\mathrm{PU}(n+1)$, the projective Unitary group. Moreover, these isometries are holomorphic. The full isometry group has 2 components; the non-identity component consists of anti-holomoprhic isometries (the $\mathrm{U}(n+1)$-action composed with conjugation of the coordinates).

Remark 13.1. *This metric seems to just come from nowhere, but we will see in a bit that is a very natural definition (but we need to discuss line bundles first to understand this). Also, the normalization in* (13.4) *is to arrange that the holomorphic sectional curvature of g_{FS} is equal to* 1*, we will discuss this later.*

The only non-trivial integral cohomology of $\mathbb{CP}^n$ is in even degrees

$$H^{2j}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z} \tag{13.5}$$

for $j = 1 \dots n$. Using Proposition 12.4, it follows that the Hodge numbers are given by

$$b^{p,q}(\mathbb{CP}^n) = \begin{cases} 1 & p = q \\ 0 & p \neq q. \end{cases} \tag{13.6}$$

For example, the Hodge diamond of $\mathbb{CP}^1$ is given by

$$\begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix} \quad , \tag{13.7}$$

and the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{13.8}$$

13.2 Line bundles and divisors

A line bundle over a complex manifold M is a rank 1 complex vector bundle $\pi : E \to M$. The transition functions are defined as follows. A trivialization is a mapping

$$\Phi_\alpha : U_\alpha \times \mathbb{C} \to E \tag{13.9}$$

which maps $x \times \mathbb{C}$ linearly onto a fiber. The transition functions are

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*, \tag{13.10}$$

defined by

$$\varphi_{\alpha\beta}(x) = \frac{1}{v}\pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{13.11}$$

for $v \neq 0$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{13.12}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satifsying (13.12), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$, and this bundle is uniquely defined up to bundle equivalence, which we will define below. If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle, while if they are holomorphic, we say that E is a holomorphic vector bundle. Note that total space of a holomorphic vector bundle over a complex manifold is a complex manifold.

A vector bundle mapping is a mapping $f : E_1 \to E_2$ which is linear on fibers, and covers the identity map. Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions $f_\alpha : U_\alpha \to \mathbb{C}$ defined by

$$f_\alpha(x) = \frac{1}{v}\pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)) \tag{13.13}$$

for $v \neq 0$. It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi^{E_2}_{\alpha\beta} f_\beta \varphi^{E_1}_{\beta\alpha}, \tag{13.14}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} f_\alpha = f_\beta \varphi^{E_1}_{\beta\alpha}. \tag{13.15}$$

We say that two bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $f : E_1 \to E_2$. This is equivalent to non-vanishing of the local representatives, that is, $f_\alpha : U_\alpha \to \mathbb{C}^*$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$\phi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{13.16}$$

The tensor product $E_1 \otimes E_2$ of two line bundles E_1 and E_2 is again a line bundle, and has transition functions

$$\varphi^{E_1 \otimes E_2}_{\alpha\beta} = \varphi^{E_1}_{\alpha\beta} \varphi^{E_2}_{\alpha\beta}. \tag{13.17}$$

The dual E^* of a line bundle E, is again a line bundle, and has transition functions

$$\varphi^{E^*}_{\alpha\beta} = (\varphi^{E}_{\beta\alpha})^{-1}. \tag{13.18}$$

Note that for any line bundle,

$$E \otimes E^* \cong \mathbb{C}, \tag{13.19}$$

is the trivial line bundle.

For our purpose, a divisor is defined to be the zero set of a holomorphic section of a nontrivial line bundle. Conversely, an irreducible holomorphic subvariety of codimension 1 defines a line bundle by taking local defining functions to be the transition functions, that is,

$$\varphi_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \tag{13.20}$$

13.3 Line bundles on complex projective space

If M is any smooth manifold, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^* \to 1. \tag{13.21}$$

where $\mathcal{E}$ is the sheaf of germs of C^∞ functions, and $\mathcal{E}^*$ is the sheaf of germs of non-vanishing C^∞ functions. The associated long exact sequence in cohomology is

$$\begin{aligned} \ldots \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{E}) \to H^1(M,\mathcal{E}^*) \\ \to H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{E}) \to H^2(M,\mathcal{E}^*) \to \ldots. \end{aligned} \tag{13.22}$$

But $\mathcal{E}$ is a flabby sheaf due to existence of partitions of unity in the smooth category, so $H^k(M,\mathcal{E}) = \{0\}$ for $k \geq 1$. This implies that

$$H^1(M,\mathcal{E}^*) \cong H^2(M,\mathbb{Z}). \tag{13.23}$$

Using Čech cohomology, the left hand side is easily seen to be the set of smooth line bundles on M up to equivalence.

Next, if M is a complex manifold, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1. \tag{13.24}$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions, and $\mathcal{O}^*$ is the sheaf of germs of non-vanishing holomorphic functions. The associated long exact sequence in cohomology is

$$\begin{aligned} \ldots \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \\ \overset{c_1}{\to} H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{O}) \to H^2(M,\mathcal{O}^*) \to \ldots. \end{aligned} \tag{13.25}$$

Now $\mathcal{O}$ is not flabby (there are no nontrivial holomorphic partitions of unity!). However

$$\dim(H^k(M, \mathcal{O})) = b^{0,k}. \tag{13.26}$$

Since $b^{0,1} = b^{0,2} = 0$ for $\mathbb{CP}^n$, we have

$$H^1(\mathbb{CP}^n, \mathcal{O}^*) \cong H^2(M, \mathbb{Z}) \cong \mathbb{Z}. \tag{13.27}$$

Again, using Čech cohomology, the left hand side is easily seen to be the set of holomorphic line bundles on M up to equivalence. Consequently, on $\mathbb{CP}^n$ the smooth line bundles are the same as holomorphic line bundles up to equivalence:

Corollary 13.2. *The set of holomorphic line bundles on $\mathbb{CP}^n$ up to equivalence is isomorphic to $\mathbb{Z}$, with the tensor product corresponding to addition.*

The line bundles on $\mathbb{CP}^n$ are denoted by $\mathcal{O}(k)$, where k is the integer obtained under the above isomorphism, which is the first Chern class. Of course, every line bundle must be a tensor power of a generator. If $H \subset \mathbb{CP}^n$ is a hyperplane, then the line bundle corresponding to H, denoted by $[H]$ is $\mathcal{O}(1)$. The dual of this bundle, $\mathcal{O}(-1)$ has a nice description, it is called the tautological bundle. This is

$$\mathcal{O}(-1) = \{([x], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v \in [x]\}. \tag{13.28}$$

To see that $[H]$ corresponds to $\mathcal{O}(1)$, use the following:

Proposition 13.3 ([GH78, page 141])**.** *The first Chern class of a complex line bundle L is equal to the Euler class of the underlying oriented real rank 2 bundle, and is the Poincaré dual to the zero locus of a transverse section. Furthermore, if g is a Hermitian metric on L, then the curvature form of the Chern connection on L is given by*

$$\Theta = 2\pi i \partial\overline{\partial}|\sigma|^2, \tag{13.29}$$

where σ is is any locally defined holomorphic section. Finally,

$$c_1(L) = \Big[\frac{i}{2\pi}\Theta\Big]. \tag{13.30}$$

Returning to the Fubini-Study metric: note the $\mathcal{O}(-1)$ admits a Hermitian metric h by restricting the inner product in $\mathbb{C}^{n+1}$ to a fiber. Thus we see that

Proposition 13.4. *The Kähler form of the Fubini-Study metric is $(-i/2\pi)$ times the curvature form of h.*

13.4 Adjunction formula

Let $V \subset M^n$ be a smooth complex hypersurface. The exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}M\big|_V \to N_V \to 0, \tag{13.31}$$

defines the holomorphic normal bundle. The adjunction formula says that

$$N_V = [V]|_V. \tag{13.32}$$

To see this, let f_α be local defining functions for V, so that the transition functions of $[V]$ are $g_{\alpha\beta} = f_\alpha f_\beta^{-1}$. Apply d to the equation

$$f_\alpha = g_{\alpha\beta} f_\beta \tag{13.33}$$

to get

$$df_\alpha = d(g_{\alpha\beta}) f_\beta + g_{\alpha\beta} df_\beta. \tag{13.34}$$

Restricting to V, since $f_\beta = 0$ defines V, we have

$$df_\alpha = g_{\alpha\beta} df_\beta. \tag{13.35}$$

Note that df_α is a section of N_V^*. For a smooth hypersurface, the differential of a local defining function is nonzero on normal vectors. Consequently, $N_V^* \otimes [V]$ is the trivial bundle when restricted to V since it has a non-vanishing section.

For any short exact sequence

$$0 \to A \to B \to C \to 0, \tag{13.36}$$

it holds that

$$\Lambda^{\dim(B)}(B) \cong \Lambda^{\dim(A)}(A) \otimes \Lambda^{\dim(C)}(C), \tag{13.37}$$

so the adjunction formula can be rephrased as

$$K_V = (K_M \otimes [V])|_V. \tag{13.38}$$

14 Lecture 14

14.1 Characteristic numbers of hypersurfaces

Let $V \subset \mathbb{P}^n$ be a smooth complex hypersurface. We know that the line bundle $[V] = \mathcal{O}(d)$ for some $d \geq 1$. We have the exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}\mathbb{P}^n\big|_V \to N_V \to 0. \tag{14.1}$$

The adjunction formula says that

$$N_V = \mathcal{O}(d)\big|_V. \tag{14.2}$$

We have the smooth splitting of (14.1),

$$T^{(1,0)}\mathbb{P}^n\big|_V = T^{(1,0)}(V) \oplus \mathcal{O}(d)\big|_V. \tag{14.3}$$

Taking Chern classes,

$$c(T^{(1,0)}\mathbb{P}^n\big|_V) = c(T^{(1,0)}(V)) \cdot c(\mathcal{O}(d)\big|_V). \tag{14.4}$$

From the Euler sequence [GH78, page 409],

$$0 \to \mathbb{C} \to \mathcal{O}(1)^{\oplus(n+1)} \to T^{(1,0)}\mathbb{P}^n \to 0, \tag{14.5}$$

it follows that

$$c(T^{(1,0)}\mathbb{P}^n) = (1 + c_1(\mathcal{O}(1)))^{n+1}. \tag{14.6}$$

Note that for any divisor D,

$$c_1([D]) = \eta_D, \tag{14.7}$$

where η_D is the Poincaré dual to D. That is

$$\int_D \xi = \int_{\mathbb{P}^n} \xi \wedge \eta_D, \tag{14.8}$$

for all $\xi \in H^{2n-2}(\mathbb{P})$, see [GH78, page 141]. So in particular $c_1(\mathcal{O}(1)) = \omega$, where ω is the Poincaré dual of a hyperplane in $\mathbb{P}^n$ (note that ω is integral, and is some multiple of the Fubini-Study metric). Therefore

$$c(T^{(1,0)}\mathbb{P}^n) = (1 + \omega)^{n+1}. \tag{14.9}$$

Also $c_1(\mathcal{O}(d)) = d \cdot \omega$, since $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. The formula (14.4) is then

$$(1 + \omega)^{n+1}\big|_V = (1 + c_1 + c_2 + \dots)(1 + d \cdot \omega\big|_V). \tag{14.10}$$

A crucial tool in the following is the Lefschetz hyperplane theorem:

Theorem 14.1 ([GH78, page 156])**.** *Let $M \subset \mathbb{P}^n$ be a hypersurface of dimension $n-1$, and H be a hyperplane, and let $V = M \cap H$. Then the inclusion $\iota : V \to M$ induces a mapping*

$$\iota^* : H^q(M, \mathbb{Q}) \to H^q(V, \mathbb{Q}) \tag{14.11}$$

which is an isomorphism for $q \leq n-3$ and injective for $q = n-2$.

Furthermore, the mapping

$$\iota_* : \pi_q(V) \to \pi_q(M) \tag{14.12}$$

is an isomorphism for $q \leq n-3$ and surjective for $q = n-2$.

14.2 Dimension $n = 2$

Consider a curve in $\mathbb{P}^2$. The formula (14.10) is

$$(1 + 3\omega)\big|_V = (1 + c_1)(1 + d \cdot \omega\big|_V), \tag{14.13}$$

which yields

$$c_1 = (3 - d)\omega\big|_V. \tag{14.14}$$

The top Chern class is the Euler class, so we have

$$\chi(V) = \int_V (3 - d)\omega \tag{14.15}$$
$$= (3 - d) \int_{\mathbb{P}^2} \omega \wedge (d \cdot \omega) \tag{14.16}$$
$$= d(3 - d) \int_{\mathbb{P}^2} \omega^2 = d(3 - d). \tag{14.17}$$

Here we used the fact that $d \cdot \omega$ is Poincaré dual to V, and ω^2 is a positive generator of $H^4(\mathbb{P}^2, \mathbb{Z})$. Equivalently, we can write

$$\int_V \omega = \int_{\mathbb{P}^2} \omega \wedge \eta_V = \int_{\mathbb{P}^2} \eta_H \wedge \eta_V. \tag{14.18}$$

Since cup product is dual to intersection under Poincaré duality, the integral simply counts the number of intersection points of V with a generic hyperplane.

In term of the genus g,

$$g = \frac{(d-1)(d-2)}{2}. \tag{14.19}$$

14.3 Dimension $n = 3$

We consider a hypersurface in $\mathbb{P}^3$, which is topologically a 4-manifold. The formula (14.10) is

$$(1 + 4\omega + 6\omega^2)\big|_V = (1 + c_1 + c_2) \cdot (1 + d \cdot \omega\big|_V), \tag{14.20}$$

so that

$$c_1 = (4 - d)\omega\big|_V, \tag{14.21}$$

and then

$$c_2 = (6 - d(4 - d))\omega^2\big|_V. \tag{14.22}$$

Since the top Chern class is the Euler class,

$$\chi(V) = \int_V (6 - d(4-d))\omega^2 \tag{14.23}$$

$$= (6 - d(4-d)) \int_{\mathbb{P}^3} \omega^2 \wedge (d \cdot \omega) \tag{14.24}$$

$$= d(6 - d(4-d)), \tag{14.25}$$

again using the fact that $d \cdot \omega$ is Poincaré dual to V, and that ω^3 is a positive generator of $H^6(\mathbb{P}^3, \mathbb{Z})$.

It follows from the Lefschetz hyperplane theorem that M has $b_1 = 0$, therefore $b^{1,0} = b^{0,1} = 0$.

14.4 Hirzebruch Signature Theorem

We think of V as a real 4-manifold, with complex structure given by J. Then the kth Pontrjagin Class is defined to be

$$p_k(V) = (-1)^k c_{2k}(TV \otimes \mathbb{C}) \tag{14.26}$$

Since (V, J) is complex, we have that

$$TV \otimes \mathbb{C} = TV \oplus \overline{TV}, \tag{14.27}$$

so

$$c(TV \otimes \mathbb{C}) = c(TV) \cdot c(\overline{TV}) \tag{14.28}$$

$$= (1 + c_1 + c_2) \cdot (1 - c_1 + c_2) \tag{14.29}$$

$$= 1 + 2c_2 - c_1^2, \tag{14.30}$$

which yields

$$p_1(V) = c_1^2 - 2c_2. \tag{14.31}$$

Consider next the intersection pairing $H^2(V) \times H^2(V) \to \mathbb{R}$, given by

$$(\alpha, \beta) \to \int \alpha \wedge \beta \in \mathbb{R}. \tag{14.32}$$

Let b_2^+ denote the number of positive eigenvalues, and b_2^- denote the number of negative eigenvalues. By Poincaré duality the intersection pairing is non-degenerate, so

$$b_2 = b_2^+ + b_2^-. \tag{14.33}$$

The *signature* of V is defined to be

$$\tau = b_2^+ - b_2^-. \tag{14.34}$$

The Hirzebruch Signature Theorem [MS74, page 224] states that

$$\tau = \frac{1}{3}\int_V p_1(V) \tag{14.35}$$

$$= \frac{1}{3}\int_V (c_1^2 - 2c_2). \tag{14.36}$$

Rewriting this,

$$2\chi + 3\tau = \int_V c_1^2. \tag{14.37}$$

Remark 14.2. *This implies that S^4 does not admit any almost complex structure, since the left hand side is 4, but the right hand side trivially vanishes.*

14.5 Representations of $U(2)$

As discussed above, some representations which are irreducible for $SO(4)$ become reducible when restricted to $U(2)$. Under $SO(4)$, we have

$$\Lambda^2 T^* = \Lambda^2_+ \oplus \Lambda^2_-, \tag{14.38}$$

where

$$\Lambda_2^+ = \{\alpha \in \Lambda^2(M,\mathbb{R}) : *\alpha = \alpha\} \tag{14.39}$$

$$\Lambda_2^- = \{\alpha \in \Lambda^2(M,\mathbb{R}) : *\alpha = -\alpha\}. \tag{14.40}$$

But under U(2), we have the decomposition

$$\Lambda^2 T^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}. \tag{14.41}$$

Notice that these are the complexifications of real vector spaces. The first is of dimension 2, the second is of dimension 4. Let ω denote the 2-form $\omega(X,Y) = g(JX,Y)$. This yields the orthogonal decomposition

$$\Lambda^2 T^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R}\cdot\omega \oplus \Lambda_0^{1,1}, \tag{14.42}$$

where $\Lambda_0^{1,1} \subset \Lambda^{1,1}$ is the orthogonal complement of the span of ω, and is therefore 2-dimensional (the complexification of which is the space of *primitive* $(1,1)$-forms).

Proposition 14.3. *Under* U(2)*, we have the decomposition*

$$\Lambda^2_+ = \mathbb{R}\cdot\omega \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \tag{14.43}$$

$$\Lambda^2_- = \Lambda_0^{1,1}. \tag{14.44}$$

Proof. We can choose an oriented orthonormal basis of the form

$$\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}. \tag{14.45}$$

Let $\{e^1, e^2, e^3, e^4\}$ denote the dual basis. The space of $(1,0)$ forms, $\Lambda^{1,0}$ has generators

$$\theta^1 = e^1 + ie^2, \quad \theta^2 = e^3 + ie^4. \tag{14.46}$$

We have

$$\begin{aligned}\omega &= \frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 + \theta^2 \wedge \overline{\theta}^2) \\ &= \frac{i}{2}\Big((e^1 + ie^2) \wedge (e^1 - ie^2) + (e^3 + ie^4) \wedge (e^3 - ie^4)\Big) \\ &= e^1 \wedge e^2 + e^3 \wedge e^4 = \omega_+^1.\end{aligned} \tag{14.47}$$

Similarly, we have

$$\frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 - \theta^2 \wedge \overline{\theta}^2) = e^1 \wedge e^2 - e^3 \wedge e^4 = \omega_-^1, \tag{14.48}$$

so ω_-^1 is of type $(1,1)$, so lies in $\Lambda_0^{1,1}$. Next,

$$\begin{aligned}\theta^1 \wedge \theta^2 &= (e^1 + ie^2) \wedge (e^3 + ie^4) \\ &= (e^1 \wedge e^3 - e^2 \wedge e^4) + i(e^1 \wedge e^4 + e^2 \wedge e^3) \\ &= \omega_+^2 + i\omega_+^3.\end{aligned} \tag{14.49}$$

Solving, we obtain

$$\omega_+^2 = \frac{1}{2}(\theta^1 \wedge \theta^2 + \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{14.50}$$

$$\omega_+^3 = \frac{1}{2i}(\theta^1 \wedge \theta^2 - \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{14.51}$$

which shows that ω_+^2 and ω_+^3 are in the space $\Lambda^{2,0} \oplus \Lambda^{0,2}$. Finally,

$$\begin{aligned}\theta^1 \wedge \overline{\theta}^2 &= (e^1 + ie^2) \wedge (e^3 - ie^4) \\ &= (e^1 \wedge e^3 + e^2 \wedge e^4) + i(-e^1 \wedge e^4 + e^2 \wedge e^3) \\ &= \omega_-^2 - i\omega_-^3,\end{aligned} \tag{14.52}$$

which shows that ω_-^2 and ω_-^3 are in the space $\Lambda_0^{1,1}$. □

This decomposition also follows from the proof of the Hodge-Riemann bilinear relations [GH78, page 123].

Corollary 14.4. *If (M^4, g) is Kähler, then*

$$b_2^+ = 1 + 2b^{2,0}, \tag{14.53}$$

$$b_2^- = b^{1,1} - 1, \tag{14.54}$$

$$\tau = b_2^+ - b_2^- = 2 + 2b^{2,0} - b^{1,1}. \tag{14.55}$$

Proof. This follows from Proposition 14.3, and Hodge theory on Kähler manifolds, see [GH78]. □

So we have that

$$\chi = 2 + b_2 = 2 + b^{1,1} + 2b^{2,0} \tag{14.56}$$
$$\tau = 2 + 2b^{2,0} - b^{1,1}. \tag{14.57}$$

Remark 14.5. *Notice that*

$$\chi + \tau = 4(1 + b^{2,0}), \tag{14.58}$$

so in particular, the integer $\chi + \tau$ is divisible by 4 on a Kähler manifold with $b_1 = 0$. This is in fact true for any *almost complex manifold of real dimension 4, this follows from a version of Riemann-Roch Theorem which holds for almost complex manifolds, see [Gil95, Lemma 3.5.3]. This implies that there is no almost complex structure on $\overline{\mathbb{P}}^2$, that is, there is no almost complex structure on $\mathbb{P}^2$ which induces the reversed orientation to that induced by the usual complex structure on $\mathbb{P}^2$.*

Applying these formulas to our example, we find that

$$2\chi + 3\tau = (4-d)^2 \int_V \omega^2 = d(4-d)^2. \tag{14.59}$$

Using the formula for the Euler characteristic from above,

$$\chi = d(6 - d(4-d)), \tag{14.60}$$

we find that

$$\tau = -\frac{1}{3}d(d+2)(d-2). \tag{14.61}$$

Some arithmetic shows that

$$b_2 = d^3 - 4d^2 + 6d - 2 \tag{14.62}$$
$$b_2^+ = \frac{1}{3}(d^3 - 6d^2 + 11d - 3) \tag{14.63}$$
$$b_2^- = \frac{1}{3}(d-1)(2d^2 - 4d + 3) \tag{14.64}$$
$$b^{2,0} = b^{0,2} = \frac{1}{6}(d-3)(d-2)(d-1) \tag{14.65}$$
$$b^{1,1} = \frac{1}{3}d(2d^2 - 6d + 7) \tag{14.66}$$
$$b^{1,0} = b^{0,1} = 0. \tag{14.67}$$

For $d = 2$, we find that $b_2^+ = 1$, $b_2^- = 1$. This is not surprising, as any non-degenerate quadric is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ [GH78, page 478]. The Hodge numbers are $b^{1,1} = 2$, $b^{0,2} = b^{2,0} = 0$, so the Hodge diamond is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{14.68}$$

For $d = 3$, we find $b_2^+ = 1$, $b_2^- = 6$. This is expected, since any non-degenerate cubic is biholomorphic to $\mathbb{P}^2$ blown up at 6 points, and is therefore diffeomorphic to $\mathbb{P}^2 \# 6\overline{\mathbb{P}^2}$ [GH78, page 489]. The Hodge numbers in this case are $b^{1,1} = 7$, $b^{0,2} = b^{2,0} = 0$, so the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{14.69}$$

For $d = 4$, this is a $K3$ surface [GH78, page 590]. We find $b_2^+ = 3$, $b_2^- = 19$, so $\chi = 24$, and $\tau = -16$. The intersection form is given by

$$2E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{14.70}$$

Since $c_1 = 0$, the canonical bundle is trivial. The Hodge numbers in this case are $b^{1,1} = 20$, $b^{0,2} = b^{2,0} = 1$, so the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{14.71}$$

For $d = 5$, we find $b_2^+ = 9$, $b_2^- = 44$, so $\chi = 55$, and $\tau = -35$. From Freedman's topological classification of simply-connected 4-manifolds, V must be homeomorphic to $9\mathbb{P}^2 \# 44\overline{\mathbb{P}^2}$, see [FQ90]. By the work of Gromov-Lawson [GL80], this latter smooth manifold admits a metric of positive scalar curvature, and therefore all of its Seiberg-Witten invariants vanish [Wit94]. But V is Kähler, so it has some non-zero Seiberg-Witten invariant [Mor96, Theorem 7.4.4]. We conclude that V is homeomorphic to $9\mathbb{P}^2 \# 44\overline{\mathbb{P}^2}$, but not diffeomorphic.

15 Lecture 15

15.1 Complete Intersections

Let $V^k \subset \mathbb{P}^n$ be a smooth complete intersection of $n - k$ homogeneous polynomials of degree $d_1, \ldots, d_{n-k}$. Consider again the exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}\mathbb{P}^n\big|_V \to N_V \to 0, \tag{15.1}$$

where N_V is now a bundle of rank $n - k$ bundle. The adjunction formula says that

$$N_V = \mathcal{O}(d_1)\big|_V \oplus \cdots \oplus \mathcal{O}(d_{n-k})\big|_V. \tag{15.2}$$

We have the smooth splitting of (14.1),

$$T^{(1,0)}\mathbb{P}^n\big|_V = T^{(1,0)}(V) \oplus N_V. \tag{15.3}$$

Taking Chern classes,

$$c(T^{(1,0)}\mathbb{P}^n\big|_V) = c(T^{(1,0)}(V)) \cdot c\left(\mathcal{O}(d_1)\big|_V \oplus \cdots \oplus \mathcal{O}(d_{n-k})\big|_V\right), \tag{15.4}$$

which is

$$(1+\omega)^{n+1}\big|_V = (1 + c_1 + \cdots + c_k)(1 + d_1 \cdot \omega\big|_V) \cdots (1 + d_{n-k} \cdot \omega\big|_V). \tag{15.5}$$

15.2 Calabi-Yau complete intersections

Note that if

$$n + 1 = d_1 + \cdots + d_{n-k} \tag{15.6}$$

then V has vanishing first Chern class, and therefore carries a Ricci-flat metric by Yau's theorem. If any of the degrees d_j is equal to 1, then this reduces to a complete intersection in a lower dimensional projective space. So without loss of generality, assume that $d_j \geq 2$. Then (15.6) implies the inequality

$$n \leq 2k + 1. \tag{15.7}$$

A Calabi-Yau manifold admits a non-zero holomorphic $(n, 0)$-form, which is denoted by Ω. This form yield an isomorphism of bundles

$$\Theta \cong \Lambda^{n-1,0} \tag{15.8}$$

by the mapping $X \mapsto \iota_X \Omega$, where ι is interior multiplication. Consequently, the lowest cohomologies of the holomorphic tangent sheaf are given by

$$H^0(V, \Theta) = H^0(V, \Lambda^{n-1}) \tag{15.9}$$
$$H^1(V, \Theta) = H^1(V, \Lambda^{n-1}) \tag{15.10}$$
$$H^2(V, \Theta) = H^2(V, \Lambda^{n-1}), \tag{15.11}$$

so that

$$\dim(H^0(V, \Theta)) = h^{n-1,0} \tag{15.12}$$
$$\dim(H^1(V, \Theta)) = h^{n-1,1} \tag{15.13}$$
$$\dim(H^2(V, \Theta)) = h^{n-2,2}. \tag{15.14}$$

Consider the case of Calabi-Yau surfaces, $k = 2$, so that (15.7) implies that $n \leq 5$. The possibilites are in Table 15.1. The computation of the Euler characteristic is straightforward from the above formulas, using that the integral of the top Chern class is the Euler class. Notice that all of these have the same Euler characteristic. This is not an accident, it turns out that all of these are in fact diffeomorphic [**?**]. By

Degrees	$\subset \mathbb{P}^n$	$\chi(V)$	$\dim(H^1(V,\Theta)) = h^{1,1}$
(4)	$\mathbb{P}^3$	24	20
$(2,3)$	$\mathbb{P}^4$	24	20
$(2,2,2)$	$\mathbb{P}^5$	24	20

Table 15.1: Complete intersection Calabi-Yau surfaces.

the Tian-Todorov theorem, the moduli space of complex structures is of dimension 20. A computation shows that the dimension of the space of quartics in $\mathbb{P}^3$ modulo the action of the automorphism group of $\mathbb{P}^3$, which is $\mathrm{PGL}(4,\mathbb{C})$, is equal to 19. Therefore "most" $K3$ surfaces are not algebraic.

Consider the case of Calabi-Yau threefolds, $k = 3$, so that (15.7) implies that the number $n \leq 7$. The possibilites are in Table 15.2. This shows that, in contrast to surfaces, Calabi-Yau threefolds are not necesssarily diffeomorphic, and their Hodge numbers are not always the same. In fact, this leads to the big subject of mirror symmetry, which we will not discuss.

Degrees	$\subset \mathbb{P}^n$	$\chi(V)$	$\dim(H^1(V,\Theta)) = h^{2,1}$	$h^{1,1}$
(5)	$\mathbb{P}^4$	-200	101	1
$(4,2)$	$\mathbb{P}^5$	-176	89	1
$(3,3)$	$\mathbb{P}^5$	-144	73	1
$(3,2,2)$	$\mathbb{P}^6$	-144	73	1
$(2,2,2,2)$	$\mathbb{P}^7$	-128	65	1

Table 15.2: Complete intersection Calabi-Yau threefolds.

Next, we make some remarks on how to compute the numbers appearing in Table 15.2. Again, the computation of the Euler characteristic is straightforward from the above formulas, using that the integral of the top Chern class is the Euler class.Next, note that

$$H^{2,0}(V) \cong H^{3,1}(V) \cong H^1(V,\Omega^3) \cong H^1(V,\mathcal{O}) \cong H^{0,1}(V). \tag{15.15}$$

Consequently, the Hodge diamond of a simply-connected Calabi-Yau threefold is given by

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}\ , \tag{15.16}$$

and one has

$$\chi(V) = 2(h^{1,1} - h^{2,1}). \tag{15.17}$$

Note that $h^{1,1}(V) = 1$ for the above examples by the Lefschetz hyperplane theorem. However, there are many Calabi-Yau threefolds which have $h^{1,1}(V) > 1$ [**?**].

The space of quintics in $\mathbb{P}^4$ modulo the automorphism group $\mathrm{PGL}(5, \mathbb{C})$ has dimension 101, and therefore all deformations of the quintic are still quintics, in contrast to what happens in the $K3$ case. It turns out that all Calabi-Yau manifolds in dimensions 3 and greater are algebraic [**?**]. To see this, we use Bochner's vanishing theorem:

Theorem 15.1 (Bochner)**.** *If (M, g, J) is Kähler and has non-negative Ricci tensor, and not identically zero, then there are no nontrivial holomorphic $(p, 0)$-forms. Furthermore, if Ric $\equiv 0$, then holomorphic $(p, 0)$-forms are parallel.*

If (M, g, J) is Kähler and has non-positive Ricci tensor, and not identically zero, then there are no non-trivial holomorphic vector fields. Furthermore, if Ric $\equiv 0$, then any holomorphic vector field is parallel.

Proof. One just goes through the usual Weitzenbock argument on p-forms, and show that for $(p, 0)$-forms, the curvature term is given by the Ricci tensor (but need to check the sign of this term). Note that if a $(p, 0)$-form is harmonic for the Hodge Laplacian, then it is harmonic for the $\overline{\partial}$-Laplacian, and thus $\overline{\partial}$-closed and $\overline{\partial}$-co-closed, but a $(p, 0)$-form is automatically $\overline{\partial}$-co-closed, so the harmonic $(p, 0)$-forms are exactly the holomorphic $(p, 0)$-forms. Equivalently,

The statment on holomorphic vector fields is the dual to the statement on holomorphic $(1, 0)$-forms, and the sign of the curvature term in the Weitzenbock formula is opposite. If time, we will go through the details later. □

So assume we have a Calabi-Yau manifold (V^n, J, g) with holonomy exactly $\mathrm{SU}(n)$. This implies that the canonical bundle is flat, and since the curvature form of the canonical bundle is a multiple of the Ricci form, the metric g must be Ricci-flat. Then Bochner's Theorem implies that all harmonic $(p, 0)$-forms are parallel. We already know that the canonical bundle admit a parallel section. For $0 < p < n$, existence of such a parallel form would imply reduction of the holonomy group to a proper subgroup of $\mathrm{SU}(n)$. So if $n \geq 3$, we have that $h^{2,0} = 0$. By the Kähler identities, we also have that $h^{0,2} = 0$, and therefore

$$H^2(V, \mathbb{C}) = H^{1,1}(V). \tag{15.18}$$

The Kähler cone is therefore an open cone in $H^2(V, \mathbb{C})$, so it must contain an integral class in $H^2(V, \mathbb{Z})$. Consequently, by Kodaira's embedding theorem, V is projective, and by Chow's Theorem, it is algebraic.

Note that a flat torus cross a $K3$ surface is not algebraic, but this does not contradict the above because the holonomy in this case is a proper subgroup of $\mathrm{SU}(3)$. For threefolds, above we proved that is V^3 is simply connected with trivial canonical bundle, then $h^{2,0} = 0$. Finally, note that the second part of Bochner's Theorem implies that Calabi-Yau metrics have discrete automorphism group (in fact, it must be finite).

15.3 Riemann surface complete intersections

Let us now just consider the simple case of a complete intersection of $n-1$ hypersurfaces in $\mathbb{P}^n$, of degrees $d_1, \dots, d_{n-1}$. We have

$$1+(n+1)\omega\big|_V = 1 + c_1 + (d_1 + \cdots + d_{n-1})\omega\big|_V, \tag{15.19}$$

which yields

$$c_1 = (n+1-d_1-\cdots-d_{n-1})\omega\big|_V. \tag{15.20}$$

The Euler characteristic is

$$\chi(V) = (n+1-d_1-\cdots-d_{n-1})\int_V \omega. \tag{15.21}$$

By definition of the Poincaré dual,

$$\int_V \omega = \int_{\mathbb{P}^n} \omega \wedge \eta_V = \int_{\mathbb{P}^n} \eta_H \wedge \eta_V. \tag{15.22}$$

We use some intersection theory to understand the integral. Intersecting cycles is Poincaré dual to the cup product, thus the integral counts the number of intersection points of V with a generic hyperplane. Consequently,

$$\chi(V) = (n+1-d_1-\cdots-d_{n-1})d_1 d_2 \cdots d_{n-1}. \tag{15.23}$$

The genus g is given by

$$g = 1 - \frac{1}{2}(n+1-d_1-\cdots-d_{n-1})d_1 d_2 \cdots d_{n-1}. \tag{15.24}$$

Proposition 15.2. *For Riemann surface complete intersections, we have the following:*

- *A curve of genus zero arises as a nontrivial complete intersection only if it is a quadric in $\mathbb{P}^2$.*
- *A curve of genus 1 arises as a complete intersection only if it is a cubic in $\mathbb{P}^2$ or the intersection of two quadrics in $\mathbb{P}^3$.*
- *A curve of genus 2 does not arise as a complete intersection.*

Proof. The first two cases are an easy computation. For the last case, assume by contradiction that it does. If any of the $d_i = 1$, then it is a complete intersection in a lower dimensional projective space. So without loss of generality, assume that $d_i \geq 2$. We would then have

$$2 = -(n+1-d_1-\cdots-d_{n-1})d_1 d_2 \cdots d_{n-1}. \tag{15.25}$$

The right hand side is a product of integers. Since 2 is prime, the only possibility is that $n = 2$, and $d_1 = 2$, in which case the above equation reads

$$2 = -2, \tag{15.26}$$

which is a contradiction. ∎

15.4 The twisted cubic

Here is an example of a surface which is not a complete intersection, called the twisted cubic. Consider

$$\phi : \mathbb{P}^1 \to \mathbb{P}^3, \tag{15.27}$$

by

$$\phi([u,v]) = [u^3, u^2 v, u v^2, v^3]. \tag{15.28}$$

Using coordinates $[z_0, z_1, z_2, z_3]$, the image of ϕ lies on the intersection of 3 quadrics,

$$z_0 z_2 = z_1^2 \tag{15.29}$$

$$z_1 z_3 = z_2^2 \tag{15.30}$$

$$z_0 z_3 = z_1 z_2. \tag{15.31}$$

The intersection of any 2 of these equations vanishes on the twisted cubic, but has another zero component, and the third equation then picks out the correct component.

16 Lecture 16

16.1 Riemann-Roch Theorem

Instead of using the Hirzebruch signature Theorem to compute these characteristic numbers, we can use the Riemann-Roch formula for complex manifolds.

Let $\mathcal{E}$ be a complex vector bundle over V of rank k. Assume that $\mathcal{E}$ splits into a sum of line bundles

$$\mathcal{E} = L_1 \oplus \cdots \oplus L_k. \tag{16.1}$$

Let $a_i = c_1(L_i)$. Then

$$c(\mathcal{E}) = (1 + a_1) \cdots (1 + a_k), \tag{16.2}$$

which shows that $c_j(\mathcal{E})$ is given by the elementary symmetric functions of the a_i, that is

$$c_j(\mathcal{E}) = \sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k}. \tag{16.3}$$

Any other symmetric polynomial can always be expressed as a polynomial in the elementary symmetric functions. We define the *Chern character* as

$$ch(\mathcal{E}) = e^{a_i} + \cdots + e^{a_k}. \tag{16.4}$$

Re-expressing in terms of the Chern classes, we have the first few terms of the Chern character:

$$ch(\mathcal{E}) = rank(E) + c_1(\mathcal{E}) + \frac{1}{2}\big(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\big) + \dots. \tag{16.5}$$

The Todd Class is associated to

$$Td(\mathcal{E}) = \frac{a_1}{1 - e^{-a_1}} \cdots \frac{a_k}{1 - e^{-a_k}} \tag{16.6}$$

Re-expressing in terms of the Chern classes, we have the first few terms of the Todd class:

$$Td(\mathcal{E}) = 1 + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}\left(c_1(\mathcal{E})^2 + c_2(\mathcal{E})\right) + \frac{1}{24}c_1(\mathcal{E})c_2(\mathcal{E}) + \dots. \tag{16.7}$$

For an almost complex manifold V, let $Td(V) = Td(T^{(1,0)}V)$.

Note the following fact: except for ch_0, all of the Chern character and Todd polynomials are independent of the rank of the bundle.

Recall the Dolbeault complex with coefficients in a holomorphic vector bundle,

$$\Omega^p(\mathcal{E}) \xrightarrow{\bar{\partial}} \Omega^{p+1}(\mathcal{E}). \tag{16.8}$$

Let $H^p(V, \mathcal{E})$ denote the pth cohomology group of this complex, and define the *holomorphic Euler characteristic* as

$$\chi(V, \mathcal{E}) = \sum_{p=0}^{k} (-1)^p \dim_{\mathbb{C}}(H^p(V, \mathcal{E})). \tag{16.9}$$

Theorem 16.1. *(Riemann-Roch) Let $\mathcal{E}$ be a holomorphic vector bundle over a complex manifold V. Then*

$$\chi(V, \mathcal{E}) = \int_V ch(\mathcal{E}) \wedge Td(V). \tag{16.10}$$

We look at a few special cases. Let V be a curve, and let $\mathcal{E}$ be a line bundle over V, then we have

$$\dim H^0(V, \mathcal{E}) - \dim H^1(V, \mathcal{E}) = \int_V c_1(\mathcal{E}) + \frac{1}{2}c_1(V). \tag{16.11}$$

Recall that $c_1(V)$ is the Euler class, and $\int_V c_1(\mathcal{E})$ is the degree d of the line bundle. Using Serre duality, this is equivalent to

$$\dim H^0(V, \mathcal{E}) - \dim H^0(V, K \otimes \mathcal{E}^*) = d + 1 - g, \tag{16.12}$$

which is the classical Riemann-Roch Theorem for curves (g is the genus of V).

Next, let V be of dimension 2, and $\mathcal{E}$ be a line bundle, then

$$\begin{aligned} &\dim H^0(V, \mathcal{E}) - \dim H^1(V, \mathcal{E}) + \dim H^2(V, \mathcal{E}) \\ &\qquad = \int_V \frac{1}{2}c_1(\mathcal{E})c_1(V) + \frac{1}{12}\left(c_1(V)^2 + c_2(V)\right). \end{aligned} \tag{16.13}$$

If $\mathcal{E}$ is the trivial line bundle, then this is

$$1 - b^{0,1} + b^{0,2} = \frac{1}{12}\int_V \big(c_1(V)^2 + c_2(V)\big) \tag{16.14}$$

If V is a hypersurface of degree d in $\mathbb{P}^3$, then this gives

$$b^{0,2} = \frac{1}{6}(d-3)(d-2)(d-1), \tag{16.15}$$

which is of course in agreement with (14.65) above. All of the other characteristic numbers follow from this.

If $\mathcal{E}$ is a rank 2 bundle, then

$$\begin{aligned} &\dim H^0(V,\mathcal{E}) - \dim H^1(V,\mathcal{E}) + \dim H^2(V,\mathcal{E}) \\ &= \int_V \frac{1}{2}c_1(\mathcal{E})c_1(V) + \frac{1}{6}\left(c_1(V)^2 + c_2(V)\right) + \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right). \end{aligned} \tag{16.16}$$

For fun, again we let V be a complex hypersurface in $\mathbb{P}^3$, and let $\mathcal{E}$ be $\Omega^1 = \Lambda^{(1,0)} = (T^{(1,0)})^*$, so $c_1(\Omega^1) = -c_1(V)$, and $c_2(\Omega^1) = c_2(V)$. We have $b^{0,1} = b^{1,0} = 0$, and by Serre duality $b^{1,2} = b^{1,0} = 0$. So Riemann-Roch gives

$$\begin{aligned} -b^{1,1} &= \int_V \frac{1}{2}c_1(\Omega^1)c_1(V) + \frac{1}{6}\left(c_1(V)^2 + c_2(V)\right) + \frac{1}{2}\left(c_1(\Omega^1)^2 - 2c_2(\Omega^1)\right) \\ &= \int_V \frac{-1}{2}c_1(V)^2 + \frac{1}{6}\left(c_1(V)^2 + c_2(V)\right) + \frac{1}{2}\left(c_1(V)^2 - 2c_2(V)\right) \\ &= \int_V \left(\frac{1}{6}c_1(V)^2 - \frac{5}{6}c_2(V)\right) \\ &= -\frac{1}{3}d(2d^2 - 6d + 7), \end{aligned}$$

which is of course in agreement with (14.66) from above.

16.2 Hodge numbers of Hopf surface

The Hodge diamond of a Hopf surface is

$$\begin{matrix} & & 1 & & \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{16.17}$$

To see this, obviously $h^{0,0} = 1$ is trivial, and $h^{2,2} = 0$ follows from Serre duality. Next, $h^{1,0} = 0$ and $h^{2,0} = 0$ since there are no holomorphic p-forms on $\mathbb{C}^2$ which are invariant under the group action. By Serre duality, it follows that $h^{1,2} = 0$ and $h^{0,2} = 0$. The Riemann-Roch formula (16.14) yields that

$$h^{0,0} - h^{0,1} + h^{0,2} = 0, \tag{16.18}$$

which implies that $h^{0,1} = 1$. By Serre duality, it follows that $h^{2,1} = 1$. Finally, the Euler characteristic formula (12.19) yields that $h^{1,1} = 0$.

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