

Math 222B, Complex Variables and Geometry

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1 Lecture 1

1.1 Example of $\mathbb{R}^{2n} = \mathbb{C}^n$

Remark 1.1. For now we will denote $\sqrt{-1}$ by i . However, later we will not do this, because the letter i is sometimes used as an index.

We consider \mathbb{R}^{2n} and denote the coordinates as $x^1, y^1, \dots, x^n, y^n$. Letting $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, define complex one-forms

$$\begin{aligned} dz^j &= dx^j + idy^j, \\ d\bar{z}^j &= dx^j - idy^j, \end{aligned}$$

and complex tangent vectors

$$\begin{aligned} \partial/\partial z^j &= (1/2) (\partial/\partial x^j - i\partial/\partial y^j), \\ \partial/\partial \bar{z}^j &= (1/2) (\partial/\partial x^j + i\partial/\partial y^j). \end{aligned}$$

Note that

$$\begin{aligned} dz^j(\partial/\partial z^k) &= d\bar{z}^j(\partial/\partial \bar{z}^k) = \delta^{jk}, \\ dz^j(\partial/\partial \bar{z}^k) &= d\bar{z}^j(\partial/\partial z^k) = 0. \end{aligned}$$

The standard complex structure $J_0 : T\mathbb{R}^{2n} \rightarrow T\mathbb{R}^{2n}$ on \mathbb{R}^{2n} is given by

$$J_0(\partial/\partial x^j) = \partial/\partial y^j, \quad J_0(\partial/\partial y^j) = -\partial/\partial x^j,$$

which in matrix form is written

$$J_0 = \text{diag} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \quad (1.1)$$

Next, we complexify the tangent space $T \otimes \mathbb{C}$, and let

$$T^{(1,0)}(J_0) = \text{span}\{\partial/\partial z^j, j = 1 \dots n\} = \{X - iJ_0X, X \in T_p\mathbb{R}^{2n}\} \quad (1.2)$$

be the i -eigenspace and

$$T^{(0,1)}(J_0) = \text{span}\{\partial/\partial \bar{z}^j, j = 1 \dots n\} = \{X + iJ_0X, X \in T_p\mathbb{R}^{2n}\} \quad (1.3)$$

be the $-i$ -eigenspace of J_0 , so that

$$T \otimes \mathbb{C} = T^{(1,0)}(J_0) \oplus T^{(0,1)}(J_0). \quad (1.4)$$

The map J_0 also induces an endomorphism of 1-forms by

$$J_0(\omega)(v_1) = \omega(J_0v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0(dx_j) = -dy_j, \quad J_0(dy_j) = +dx_j.$$

Then complexifying the cotangent space $T^* \otimes \mathbb{C}$, we have

$$\Lambda^{1,0}(J_0) = \text{span}\{dz^j, j = 1 \dots n\} = \{\alpha - iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \quad (1.5)$$

is the i -eigenspace, and

$$\Lambda^{0,1}(J_0) = \text{span}\{d\bar{z}^j, j = 1 \dots n\} = \{\alpha + iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \quad (1.6)$$

is the $-i$ -eigenspace of J_0 , and

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J_0) \oplus \Lambda^{0,1}(J_0). \quad (1.7)$$

We note that

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \quad (1.8)$$

and similarly

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \quad (1.9)$$

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \quad (1.10)$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \quad (1.11)$$

Note that we can characterize $\Lambda^{p,q}$ as those forms satisfying

$$\alpha(v_1, \dots, v_{p+q}) = 0, \quad (1.12)$$

if more than p of the v_j -s are in $T^{(1,0)}$ or if more than q of the v_j -s are in $T^{(0,1)}$.

Finally, we can extend $J : \Lambda^k \otimes \mathbb{C} \rightarrow \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \quad (1.13)$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J is not a complex structure on the space $\Lambda_{\mathbb{C}}^k$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J -invariant.

1.2 Cauchy-Riemann equations

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$. Let the coordinates on \mathbb{C}^n be given by

$$\{z^1, \dots, z^n\} = \{x^1 + iy^1, \dots, x^n + iy^n\}, \quad (1.14)$$

and coordinates on \mathbb{C}^m given by

$$\{w^1, \dots, w^m\} = \{u^1 + iv^1, \dots, u^m + iv^m\} \quad (1.15)$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \text{span}\{\partial/\partial x^1, \dots, \partial/\partial x^n, \partial/\partial y^1, \dots, \partial/\partial y^n\}, \quad (1.16)$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial u^1, \dots, \partial/\partial u^m, \partial/\partial v^1, \dots, \partial/\partial v^m\}. \quad (1.17)$$

Then the real Jacobian of

$$f = (f^1, \dots, f^{2m}) = (u^1 \circ f, u^2 \circ f, \dots, v^{2m} \circ f). \quad (1.18)$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}}f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \cdots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \quad (1.19)$$

Definition 1.2. A differentiable mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is pseudo-holomorphic if

$$f_* \circ J_{0, \mathbb{C}^n} = J_{0, \mathbb{C}^m} \circ f_*. \quad (1.20)$$

That is, the differential of f commutes with J_0 .

We have the following characterization of pseudo-holomorphic maps.

Proposition 1.3. *A mapping $f : \mathbb{C}^m \rightarrow \mathbb{C}^n$ is pseudo-holomorphic if and only if the Cauchy-Riemann equations are satisfied, that is, writing*

$$f(z^1, \dots, z^m) = (f_1, \dots, f_n) = (u_1 + iv_1, \dots, u_n + iv_n), \quad (1.21)$$

and $z^j = x^j + iy^j$, for each $j = 1 \dots n$, we have

$$\frac{\partial u_j}{\partial x^k} = \frac{\partial v_j}{\partial y^k} \quad \frac{\partial u_j}{\partial y^k} = -\frac{\partial v_j}{\partial x^k}, \quad (1.22)$$

for each $k = 1 \dots m$, and these equations are equivalent to

$$\frac{\partial}{\partial \bar{z}^k} f_j = 0, \quad (1.23)$$

for each $j = 1 \dots n$ and each $k = 1 \dots m$

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \quad (1.24)$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \quad (1.25)$$

which is exactly the Cauchy-Riemann equations. In the general case, rearrange the coordinates so that $(x^1, \dots, x^m, y^1, \dots, y^m)$ are the real coordinates on \mathbb{R}^{2m} and $(u^1, \dots, u^n, v^1, \dots, v^n)$, such that the complex structure J_0 is given by

$$J_0(\mathbb{R}^{2m}) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \quad (1.26)$$

and similarly for $J_0(\mathbb{R}^{2n})$. Then the computation in matrix form is entirely analogous to the case of $m = n = 1$.

Finally, we compute

$$\frac{\partial}{\partial \bar{z}^k} f_j = \frac{1}{2} \left(\frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right) (u_j + iv_j) \quad (1.27)$$

$$= \frac{1}{2} \left\{ \frac{\partial}{\partial x^k} u_j - \frac{\partial}{\partial y^k} v_j + i \left(\frac{\partial}{\partial x^k} v_j + \frac{\partial}{\partial y^k} u_j \right) \right\}, \quad (1.28)$$

the vanishing of which again yields the Cauchy-Riemann equations. \square

From now on, if f is a mapping satisfying the Cauchy-Riemann equations, we will just say that f is *holomorphic*.

For any differentiable f , the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \rightarrow T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \rightarrow T_{\mathbb{C}}(\mathbb{C}^m). \quad (1.29)$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \text{span}\{\partial/\partial z^1, \dots, \partial/\partial z^n, \partial/\partial \bar{z}^1, \dots, \partial/\partial \bar{z}^n\}, \quad (1.30)$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial w^1, \dots, \partial/\partial w^m, \partial/\partial \bar{w}^1, \dots, \partial/\partial \bar{w}^m\}. \quad (1.31)$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}}f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \dots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \bar{z}^1} & \dots & \frac{\partial f^1}{\partial \bar{z}^n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \dots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \bar{z}^1} & \dots & \frac{\partial f^m}{\partial \bar{z}^n} \\ \frac{\partial \bar{f}^1}{\partial z^1} & \dots & \frac{\partial \bar{f}^1}{\partial z^n} & \frac{\partial \bar{f}^1}{\partial \bar{z}^1} & \dots & \frac{\partial \bar{f}^1}{\partial \bar{z}^n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial \bar{f}^m}{\partial z^1} & \dots & \frac{\partial \bar{f}^m}{\partial z^n} & \frac{\partial \bar{f}^m}{\partial \bar{z}^1} & \dots & \frac{\partial \bar{f}^m}{\partial \bar{z}^n} \end{pmatrix} \quad (1.32)$$

where $(f^1, \dots, f^m) = f$ now denotes the complex components of f . This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^j}{\partial \bar{z}^k} d\bar{z}^k. \quad (1.33)$$

Notice that (1.32) is of the form

$$\mathcal{J}_{\mathbb{C}}f = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad (1.34)$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$\text{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \quad (1.35)$$

where \mathbb{C} -linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \quad (1.36)$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \quad (1.37)$$

This corresponds to the embeddings

$$\text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset \text{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \quad (1.38)$$

where the left \mathbb{C} -linear is with respect to J_0 , via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \quad (1.39)$$

Note that since the latter embedding is just a change of basis, if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB) \det(A - iB) = |\det(A + iB)|^2 \geq 0, \quad (1.40)$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(1,0)}. \quad (1.41)$$

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0 , then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \quad (1.42)$$

This corresponds to the embeddings

$$\text{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset \text{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset \text{Hom}_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \quad (1.43)$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \quad (1.44)$$

We see that f is anti-holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(0,1)}. \quad (1.45)$$

Note that if f is antiholomorphic, then is it holomorphic with respect to the complex structure $-J_0$ on the domain (but still J_0 on the range).

Note that we can decompose $f_* = f_*^C + f_*^A$, where

$$f_*^C = \frac{1}{2} (f_* - Jf_*J) \quad (1.46)$$

$$f_*^A = \frac{1}{2} (f_* + Jf_*J), \quad (1.47)$$

and f_*^C is holomorphic, while f_*^A is anti-holomorphic. In block matrix form, this just says that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \quad (1.48)$$

2 Lecture 2

2.1 Cauchy's formula in one complex variable

For now, just consider $f : U \rightarrow \mathbb{C}$, where $U \subset \mathbb{C}$ is an open set. Assume that f , as a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, is differentiable. This means that, for each $z \in U$, there exists a linear mapping $L_z : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(z+h) - f(z) - L_z h\|}{\|h\|} = 0. \quad (2.1)$$

This implies that the partial derivatives of f exist. Conversely, if the partial derivatives exist and are continuous at z , then the mapping L_z exists.

We say that f is holomorphic in U if it is C^1 and satisfies the Cauchy-Riemann equations. Writing $f = u + iv$, then Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \iff \frac{\partial}{\partial \bar{z}} f = 0, \quad (2.2)$$

Note that the linear mapping L is given by

$$L = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}. \quad (2.3)$$

If we consider h as a complex number, then f being holomorphic is equivalent to

$$\lim_{h \rightarrow 0} \frac{\|f(z+h) - f(z) - \frac{\partial f}{\partial z} h\|}{\|h\|} = 0. \quad (2.4)$$

Definition 2.1. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad (2.5)$$

which converges absolutely and uniformly in a disc $\Delta(z_0, \epsilon)$ around z_0 , for some $\epsilon > 0$.

Proposition 2.2 (Cauchy-Pompiou Formula). *Let $\Omega \subset \mathbb{C}$ be a bounded domain in \mathbb{C} with C^1 boundary. For $z \in \Omega$ and $f \in C^1(\bar{\Omega})$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)dw}{w-z} + \frac{1}{2\pi i} \int_{\Omega} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}, \quad (2.6)$$

where the boundary has the counterclockwise orientation.

Proof. The 1-form

$$\eta = \frac{1}{2\pi i} \frac{f(w)dw}{w-z}, \quad (2.7)$$

satisfies

$$d\eta = -\frac{1}{2\pi i} \frac{\partial f}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w - z}. \quad (2.8)$$

Apply Stokes' Theorem to the annular domain $\Omega \setminus \Delta(z, \epsilon)$, to get

$$\int_{\partial(\Omega \setminus \Delta(z, \epsilon))} \eta = \int_{\Omega \setminus \Delta(z, \epsilon)} d\eta. \quad (2.9)$$

The left hand side of (2.9) is

$$\int_{\partial\Omega} \eta - \int_{\partial\Delta(z, \epsilon)} \eta, \quad (2.10)$$

and the right hand side of (2.9) is

$$\int_{\Omega} d\eta - \int_{\Delta(z, \epsilon)} d\eta, \quad (2.11)$$

since $d\eta$ is obviously in $L^1(\Omega)$. A calculation shows that the inner boundary integral limits to $f(z)$, and the error term in the solid integral goes to 0 as $\epsilon \rightarrow 0$. For details, see [GH78, page 3]. \square

Proposition 2.3. *Let U be an open set in \mathbb{C} . Then f is holomorphic in U if and only if f is complex analytic in U .*

Proof. If f is holomorphic in U the Cauchy-Pompiou formula in a small disc $\Delta = \Delta(z_0, \epsilon)$ yields for $z \in \Delta$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)dw}{w - z}. \quad (2.12)$$

Then expand

$$\frac{1}{z - w} = \frac{1}{z - z_0 + z_0 - w} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \quad (2.13)$$

$$= \frac{1}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k, \quad (2.14)$$

with the sum converging absolutely and uniformly in any smaller disc. So the above yield the power series expansion

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)dw}{(w - z_0)^{k+1}} \right) (z - z_0)^k, \quad (2.15)$$

which also converges absolutely and uniformly in any smaller disc.

For the converse, if f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula without solid integral. So then f does also by uniform convergence. So we have

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} \left(\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)dw}{w - z} \right) = \frac{1}{2\pi i} \int_{\partial\Delta} f(w) \left(\frac{\partial}{\partial \bar{z}} \frac{1}{w - z} \right) dw = 0. \quad (2.16)$$

For more details, see [GH78, page 4]. \square

Definition 2.4. We will let $\Omega \subset \mathbb{C}$ be a bounded domain with C^1 boundary. If u is holomorphic in an open set Ω , then we write $u \in \mathcal{O}(\Omega)$.

First, let's recall the basic result about differentiating under an integral.

Proposition 2.5. *Let*

$$f(z) = \int_{\Omega} a(z, w) dw \wedge d\bar{w}. \quad (2.17)$$

(Note this notation does not mean that f is holomorphic in z or that a is holomorphic as a function of 2 variables!). Assume that

1. $a(z, w) \in L^1(\Omega)$, in the w variable.
2. $\frac{\partial a}{\partial z}$ and $\frac{\partial a}{\partial \bar{z}}$ exist for all z , for almost every $w \in \Omega$.
3. $|\frac{\partial a}{\partial z}| + |\frac{\partial a}{\partial \bar{z}}| \leq h(w)$, where $h \in L^1(\Omega)$.

Then

$$\frac{\partial f}{\partial z} = \int_{\Omega} \frac{\partial}{\partial z} (a(z, w)) dw \wedge d\bar{w} \quad (2.18)$$

$$\frac{\partial f}{\partial \bar{z}} = \int_{\Omega} \frac{\partial}{\partial \bar{z}} (a(z, w)) dw \wedge d\bar{w}. \quad (2.19)$$

Proof. Recall that

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right). \quad (2.20)$$

The real part of the left hand side of (2.18) is

$$\operatorname{Re} \left(\frac{\partial f}{\partial z} \right) = \frac{1}{2} \left(\frac{\partial \operatorname{Re} f}{\partial x} + \frac{\partial \operatorname{Im} f}{\partial y} \right) \quad (2.21)$$

The real part of the right hand side of (2.18) is

$$\int_{\Omega} \frac{1}{2} \left(\frac{\partial \operatorname{Re}(a(x + iy, w))}{\partial x} + \frac{\partial \operatorname{Im}(a(x + iy, w))}{\partial y} \right) dw \wedge d\bar{w}. \quad (2.22)$$

Therefore we can consider real-valued functions, and prove for partials with respect to the real variables x and y . We have that

$$\frac{\partial f}{\partial x}(x, y) = \lim_{\delta \rightarrow 0} \frac{f(x + \delta, y) - f(x, y)}{\delta} \quad (2.23)$$

For $\delta \neq 0$, consider

$$\frac{f(x + \delta, y) - f(x, y)}{\delta} = \int_{\Omega} \frac{a(x + \delta + iy, w) - a(x + iy, w)}{\delta} dw \wedge d\bar{w}. \quad (2.24)$$

By the mean value theorem, given $\delta > 0$, there exists x' on the line segment from (x, y) to $(x + \delta, y)$ such that

$$a(x + \delta + iy, w) - a(x + iy, w) = \frac{\partial a}{\partial x}(x' + iy, w)\delta, \quad (2.25)$$

so

$$\begin{aligned} \left| \frac{a(x + \delta + iy, w) - a(x + iy, w)}{\delta} \right| &\leq \left| \frac{\partial a}{\partial x}(x' + iy, w) \right| \\ &\leq \left| \frac{\partial a}{\partial z}(x' + iy, w) \right| + \left| \frac{\partial a}{\partial \bar{z}}(x' + iy, w) \right| \leq |h(w)|. \end{aligned} \quad (2.26)$$

We can do this for any sequence $\delta_n \rightarrow 0$, so the result follows from Lebesgue's dominated convergence theorem. The proof for the other derivative (2.19) is similar. \square

We next go through several corollaries of the Cauchy-Pompiou formula; see [Hör90, Chapter 1] for more details.

Corollary 2.6. *Let $K \subset \Omega$ be a compact subset. Then there exist constant C_k , depending only upon K and Ω such that*

$$\sup_{z \in K} \left| \left(\frac{\partial}{\partial z} \right)^k u(z) \right| \leq C_k \|u\|_{L^1(\Omega)}, \quad (2.27)$$

for all $u \in \mathcal{O}(\Omega)$.

Proof. Choose a $\psi \in C_0^\infty(\Omega)$ (compact support) such that $\psi \equiv 1$ in a neighborhood of K . If $u \in \mathcal{O}(\Omega)$, then

$$\frac{\partial}{\partial \bar{z}}(\psi u) = u \frac{\partial}{\partial \bar{z}}\psi. \quad (2.28)$$

Now we apply (2.6) to ψu in Ω to get

$$\psi(z)u(z) = \frac{1}{2\pi i} \int_{\Omega} u(w) \frac{\partial \psi(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}, \quad (2.29)$$

Now consider

$$a(z, w) = u(w) \frac{\partial \psi(w)}{\partial \bar{w}} \frac{1}{w - z}, \quad (2.30)$$

If $z \in K$, then $|w - z| > \delta > 0$, since the support of $\partial \psi / \partial \bar{w}$ is at a positive distance from K . So using Proposition 2.5, we can differentiate under the integral as many times as we like, and obtain

$$\left(\frac{\partial}{\partial z} \right)^k (\psi u(z)) = \frac{1}{2\pi i} \int_{\Omega} u(w) \frac{\partial \psi(w)}{\partial \bar{w}} \left(\frac{\partial}{\partial z} \right)^k \left(\frac{1}{w - z} \right) dw \wedge d\bar{w} \quad (2.31)$$

If $z \in K$, then ψu is equal to u in a neighborhood of z , so (2.27) follows. \square

Corollary 2.7. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \rightarrow u$ converges uniformly to u in the C^0 norm as $n \rightarrow \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Proof. Let $K \subset \Omega$, be a compact subset. Then given $\epsilon > 0$, there exist N such that

$$\sup_{z \in K} |u_m(z) - u_n(z)| < \epsilon, \quad (2.32)$$

for $m, n \geq N$. The difference $u_m - u_n \in \mathcal{O}(\Omega)$. Corollary 5.11 implies that

$$\sup_{z \in K} \left| \frac{\partial}{\partial z} (u_m - u_n)(z) \right| \leq C\epsilon. \quad (2.33)$$

This says that $\partial u_n / \partial z$ converges uniformly on K . But $\partial u_n / \partial \bar{z} = 0$, so the real partial derivatives $\partial u_n / \partial x$ and $\partial u_n / \partial y$ converge uniformly. It is an elementary result that if a sequence of functions converges uniformly, and the derivatives converge uniformly, then the limit of the derivatives is the derivative of the limit. This implies that $u \in C^1$ and $\partial u / \partial \bar{z} = \lim_{n \rightarrow \infty} \partial u_n / \partial \bar{z} = 0$. \square

Corollary 2.8. *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

Proof. Corollary 5.11 yield a uniform bound on derivatives of u_n on any compact subset. By Arzela-Ascoli Theorem, some subsequence converges to a limit u uniformly on compact subsets. Then the previous corollary yields that $u \in \mathcal{O}(\Omega)$. \square

3 Lecture 3

3.1 The $\bar{\partial}$ -equation in domains in \mathbb{C}

Theorem 3.1. *If $\Omega \subset \mathbb{C}$ is any bounded domain, then $H_{\bar{\partial}}^{0,1}(\Omega) = 0$.*

Recall that

$$H_{\bar{\partial}}^{0,1}(\Omega) = \frac{\text{Ker}\{\bar{\partial} : \Lambda^{0,1} \rightarrow \Lambda^{0,2}\}}{\text{Im}\{\bar{\partial} : \Lambda^{0,0} \rightarrow \Lambda^{0,1}\}}, \quad (3.1)$$

where these are spaces of C^∞ forms. Since $n = 1$, $\Lambda^{0,2} = \{0\}$. So $\omega \in \text{Ker}\{\bar{\partial} : \Lambda^{0,1} \rightarrow \Lambda^{0,2}\}$ just means that $\omega = g d\bar{z}$ for $g \in C^\infty(\Omega)$. Also, $f \in \Lambda^{0,0}$ is just a function, and $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$. So the theorem is equivalently stated as the following.

Theorem 3.2. *If $\Omega \subset \mathbb{C}$ is any bounded domain, and $g \in C^\infty(\Omega)$, then there exists $f \in C^\infty(\Omega)$ with $\frac{\partial f}{\partial \bar{z}} = g$.*

Our goal of this lecture is to prove this result. If we want to solve $\frac{\partial f}{\partial \bar{z}} = g$, it is natural to guess that

$$f(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{g(w) dw \wedge d\bar{w}}{w - z}, \quad (3.2)$$

is a solution. However, letting $a(z, w) = g(w)/(w - z)$, we have

$$\frac{\partial a(z, w)}{\partial z} = g(w) \frac{-1}{(w - z)^2}, \quad (3.3)$$

so the assumptions of Proposition 2.5 are NOT satisfied, so we cannot directly differentiate under the integral sign! Another problem is that g is only assumed to be in $C^\infty(\Omega)$, so it is not in $L^1(\Omega)$ and (3.2) is not necessarily defined. We first give a preliminary result, with a stronger assumption on g .

Proposition 3.3. *If $g \in C^1(\overline{\Omega})$ then the function*

$$f(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{g(w)dw \wedge d\bar{w}}{w - z}, \quad (3.4)$$

satisfies $f \in C^1(\overline{\Omega})$ and $\partial f/\partial \bar{z} = g$ in Ω .

Proof. We fix a point $z_0 \in \Omega$. Choose a C^∞ cutoff function $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on a neighborhood $V \subset \Omega$ containing z_0 . We then write $f = f_1 + f_2$, where

$$f_1(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w - z} \quad (3.5)$$

$$f_2(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{(1 - \psi(w))g(w)dw \wedge d\bar{w}}{w - z}. \quad (3.6)$$

For z in a small neighborhood of z_0 , the integrand in f_2 does not have a singularity. We can therefore differentiate under the integral sign to see that $\partial f_2/\partial \bar{z} = 0$. So we just need to prove that $\partial f_1/\partial \bar{z} = g$ in V . Since ψ has compact support, we can extend ψg to all of \mathbb{C} , and write

$$f_1(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w - z} \quad (3.7)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\psi(\xi + z)g(\xi + z)d\xi \wedge d\bar{\xi}}{\xi}, \quad (3.8)$$

where we used the change of variables $w = \xi + z$. Note that

$$\frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial z} = \frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \xi} \quad (3.9)$$

$$\frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \bar{z}} = \frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \bar{\xi}}. \quad (3.10)$$

This shows that the z and \bar{z} partials of the integrand are uniformly in L^1 , so we can differentiate under the integral sign, to obtain

$$\frac{\partial f_1(z)}{\partial \bar{z}} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \bar{\xi}} \frac{d\xi \wedge d\bar{\xi}}{\xi} \quad (3.11)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial(\psi(w)g(w))}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}. \quad (3.12)$$

Now apply the Cuchy-Pompieu formula in a very large ball in \mathbb{C} , to conclude the right hand side is equal to $\psi(z)g(z)$ which is $g(z)$ if $z \in V$. \square

This result does not directly help us in proving Theorem 4.2. But notice that in the proof, we also proved the following result.

Proposition 3.4. *If $g \in C_c^\infty(\Omega)$, then there exists $f \in C^\infty(\mathbb{C})$ such that $\partial f/\partial\bar{z} = g$.*

Proof. Above, we proved that there is a solution $f \in C^1(\mathbb{C})$, but the same argument allows us to differentiate f_1 infinitely many times, provided g is infinitely differentiable. \square

Now we can prove a special case of Theorem 4.2.

Proposition 3.5. *If $\Omega = \Delta(z, r)$ is a disc in \mathbb{C} , then $H_{\bar{\partial}}^{0,1}(\Omega) = 0$.*

Proof. Take a sequence $0 < r_1 < r_2 < \dots < r$ such that $\lim_{j \rightarrow \infty} r_j = r$. Let $0 \leq \psi_k \in C_0^\infty(\Delta(z, r_{k+1}))$ and $\psi_k \equiv 1$ on $\Delta(z, r_k)$. Then $g_k = \psi_k g \in C_0^\infty(\Delta(z, r_{k+1}))$, and by Proposition 3.4, we can find $f_k \in C^\infty(\mathbb{C})$ such that $\partial f_k = g_k$, which is equal to g in $\Delta(z, r_k)$.

Now there is no reason that the sequence f_k will converge to a limit, so we need to modify as follows. We claim that we can choose f_k so that

$$\sup_{z \in \Delta(z, r_{k-1})} |f_{k+1}(z) - f_k(z)| \leq 2^{-k}. \quad (3.13)$$

Given f_2 , the difference $f_3 - f_2$ is holomorphic in $\overline{\Delta(z, r_1)}$. So there exists a polynomial P_2 such that

$$\sup_{z \in \Delta(z, r_1)} |f_3(z) - f_2(z) - P_2(z)| \leq 2^{-2}. \quad (3.14)$$

So we redefined f_3 to be $f_3 - P_2$. We then proceed by induction. Given f_k , the difference $f_{k+1} - f_k$ is holomorphic in $\overline{\Delta(z, r_{k-1})}$, so we can find a polynomial P_{k+1} such that

$$\sup_{z \in \Delta(z, r_{k-1})} |f_{k+1}(z) - f_k(z) - P_{k+1}(z)| \leq 2^{-k}, \quad (3.15)$$

and we redefine f_{k+1} to be $f_{k+1} - P_{k+1}$.

The sequence of functions f_k will be a Cauchy sequence in any disc $\Delta(z, r')$, when $r' < r$. So there exists a uniform limit f . Fixing any m , then $f - f_m$ is then a uniform limit of holomorphic functions in $\Delta(z, r_{m-1})$, so is holomorphic by Corollary 2.7, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m / \partial \bar{z} \rightarrow \partial f / \partial \bar{z} = g, \quad (3.16)$$

and the proof is finished. \square

To prove for a general domain, we require the following result.

Theorem 3.6 (Runge's approximation Theorem, first version). *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z \in K} |f(z) - f_\epsilon(z)| < \epsilon, \quad (3.17)$$

and such that poles of f_ϵ are contained in $\mathbb{C} \setminus K$.

Proof. The proof is from [Sar07, Theorem IX.15], we just give an outline. From elementary arguments, there exists a contour $\gamma : S^1 \rightarrow U \setminus K$ such that $K \subset \text{Int}\Gamma \subset U$, which has winding number 1 around any point $z_0 \in K$. Note that K might have several components, so γ will also. Since the winding number is 1, by Cauchy's Integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw \quad (3.18)$$

for any $z \in K$. By dividing the plane into a sufficiently fine grid, we can assume that γ is piecewise smooth and $\gamma = \gamma_1 + \cdots + \gamma_n$, with each γ_j a line segment parallel to one of the coordinate axes. Consider each term

$$f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw \quad (3.19)$$

We can approximate this arbitrarily closely with a Riemann sum R_k , which will be of the form

$$\frac{c_1}{w_1 - z} + \cdots + \frac{c_l}{w_l - z}, \quad (3.20)$$

where the w_j are points on γ_k . Doing this for every γ_j , the proof is complete. \square

4 Lecture 4

4.1 Runge's Theorem

Theorem 4.1 (Runge's approximation Theorem, second version). *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Let $S \subset \mathbb{C} \setminus K$ which contains at least one point from each connected component of $\mathbb{C} \setminus K$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z \in K} |f(z) - f_\epsilon(z)| < \epsilon, \quad (4.1)$$

and such that poles of f_ϵ are contained in S . If $\mathbb{C} \setminus K$ is connected, the rational functions can be taken to be polynomials.

Proof. The proof is from [Sar07, Theorem IX.17]. In the proof of Theorem 3.6, each term in the approximation was of the form $c/(w - z)$, where $w \in \gamma$. Now choose a piecewise linear path α from w to any point w_0 in the same connected component of $\mathbb{C} \setminus K$. Choose points w_i on α so that

$$|w_{i-1} - w_i| < \text{dist}(\gamma, K). \quad (4.2)$$

We show that any rational R_{j-1} function with a pole only at w_{j-1} may be uniformly approximated on K by a rational function R_j with a pole only at w_j . But this follows from considering the Laurent series expansion of R_{j-1} centered at w_j : R_{j-1} is holomorphic in the region $U = \mathbb{C} \setminus \Delta(w_j, |w_j - w_{j-1}|)$, so the Laurent series of R_{j-1} centered at w_j converges

uniformly on compact subsets of U . Then we can approximate R_{j-1} by a rational function with a pole only at w_j , uniformly on K , since K is a compact subset of U , which follows from (4.2).

If $\mathbb{C} \setminus K$ is connected, by the above argument, we can move the pole of the rational function R_j to a single point z_0 so that $K \subset \Delta(0, |z_0|)$. The Taylor series of R_j converges uniformly on K , so we can approximate by the partial sums of the Taylor series. \square

Theorem 4.2. *If $\Omega \subset \mathbb{C}$ is any domain, and $g \in C^\infty(\Omega)$, then there exists $f \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \bar{z}} f = g$.*

Proof. We choose a sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \dots$, so that $\overline{K_j} \subset \text{Int}K_{j+1}$ and $\cup K_j = \Omega$. Note that $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, and we can assume that for large j , each component of $\mathbb{C} \setminus K_j$ contains a component of $\mathbb{C} \setminus \Omega$. Let $0 \leq \psi_j \in C_0^\infty(K_{j+1})$ and $\psi_k \equiv 1$ on K_j . Then $g_j = \psi_j g \in C_0^\infty(K_{j+1})$, and by Proposition 3.4, we can find $f_j \in C^\infty(\mathbb{C})$ such that $\partial f_j = g_j$.

We claim that we can choose $f_j \in C^\infty(\Omega)$ so that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z)| \leq 2^{-j}. \quad (4.3)$$

We proceed by induction. Given f_j , the difference $f_{j+1} - f_j$ is holomorphic in

$$U \equiv \text{Int}K_j \supset K = \overline{K_{j-1}}. \quad (4.4)$$

We have $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, so by Theorem 4.1, there exists a rational function R_{j+1} such that it poles are in $\mathbb{C} \setminus \Omega$ and such that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z) - R_{j+1}(z)| \leq 2^{-j}, \quad (4.5)$$

and we redefine f_{j+1} to be $f_{j+1} - R_{j+1}$.

The sequence of functions f_j will be a Cauchy sequence in any subset K_m for fixed m . So there exists a limit f , with uniform convergence on compact subsets. Fixing any m , then $f - f_m$ is then a uniform limit of holomorphic functions in K_{m-1} , so is holomorphic by Corollary 2.7, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m / \partial \bar{z} \rightarrow \partial f / \partial \bar{z} = g, \quad (4.6)$$

and the proof is finished. \square

4.2 Meromorphic Functions

The above solution of the inhomogeneous Cauchy-Riemann equations has many corollaries. We give some applications to the theory of meromorphic functions. References for this section are [Hör90, Chapter 1] and [Eps91, Section 1.6].

Definition 4.3. For a domain $\Omega \subset \mathbb{C}$, we say that $f \in \mathcal{M}(\Omega)$, or f is meromorphic in Ω , if there is an open covering U_j of Ω such that $f|_{U_j} = \frac{g_j}{h_j}$, where g_j and h_j are in $\mathcal{O}(U_j)$.

Note this is equivalent to saying that f has a Laurent series expansion near any $z_0 \in \Omega$ with only finitely many negative terms. That is we have

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k. \quad (4.7)$$

The finite sum of the negative terms is called the principal part of f at z_0 . Note the set of poles will be some discrete subset $\{w_j\}$ of Ω .

Theorem 4.4 (Mittag-Leffler). *Let Ω be a domain in \mathbb{C} and $\{w_j\}$ a discrete subset of Ω . Let P_j be any principal sum at w_j . Then there exists a meromorphic function $h \in \mathcal{M}(\Omega)$ such that the principal part of h at w_j is P_j .*

Proof. Let ψ_j be a cutoff function supported in a small neighborhood of w_j which doesn't contain any other points in the discrete subset. Consider $g = \sum_j \psi_j P_j$. Then $\partial g / \partial \bar{z} \in C^\infty(\Omega)$. By Theorem 4.2, there exists a solution $f \in C^\infty(\Omega)$ of $\partial f / \partial \bar{z} = \partial g / \partial \bar{z}$. Then $h = g - f$ satisfies $\partial h / \partial \bar{z} = 0$, and the principal part of h at w_j is P_j . \square

Definition 4.5. The order of $f \in \mathcal{M}(\Omega)$ at $z_0 \in \Omega$ is the least integer n such that the coefficient $a_n \neq 0$ in (4.7)

Theorem 4.6 (Weierstrass). *Let Ω be a domain in \mathbb{C} , $\{w_j\}$ a discrete subset of Ω , and $n_j \in \mathbb{Z}$. Then there exists a meromorphic function $f \in \mathcal{M}(\Omega)$ with the order of f at w_j equal to n_j .*

Proof. We cover Ω by discs $U_i = \Delta(z_i, r_i)$ such that each w_i is contained in exactly one of these discs. Define the function $f_i = (z - w_j)^{n_j}$ if $w_j \in U_i$, and let $f_i = 1$ if U_i doesn't contain any of the discrete points. On $U_i \cap U_j$, let $f_{ij} = f_i / f_j$. Then $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ is a non-vanishing holomorphic function. Since $U_i \cap U_j$ is simply-connected, we can define $g_{ij} = \log f_{ij}$. Note that $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ is only defined up to adding an integer multiple of $2\pi i$. Since

$$f_{ik} = \frac{f_i}{f_k} = \frac{f_i}{f_j} \frac{f_j}{f_k} = f_{ij} f_{jk}, \quad (4.8)$$

the g_{ij} satisfy on triple intersections $U_i \cap U_j \cap U_k$

$$g_{ij} - g_{ik} + g_{jk} = 2\pi i n_{ijk}, \quad (4.9)$$

where $n_{ijk} \in \mathbb{Z}$. The n_{ijk} satisfy the condition on intersections $U_i \cap U_j \cap U_k \cap U_l$,

$$n_{jkl} - n_{ikl} + n_{ijl} - n_{ijk} = 0. \quad (4.10)$$

So $n_{ijk} \in H^2(\mathfrak{U}, \mathbb{Z})$ defines a Čech 2-cocycle. Since \mathfrak{U} is a good cover of Ω , $H^2(\mathfrak{U}, \mathbb{Z}) = H^2(\Omega, \mathbb{Z})$. However, since Ω is a domain in \mathbb{C} , it is in particular a non-compact 2-manifold, so this latter group vanishes. Therefore, there exists integers n_{ij} such that

$$n_{ijk} = n_{jk} - n_{ik} + n_{ij}. \quad (4.11)$$

Now we define $g'_{ij} = g_{ij} - 2\pi i n_{ij}$, which now satisfy

$$g'_{ij} - g'_{ik} + g'_{jk} = 0 \quad (4.12)$$

Now choose a partition of unity ψ_i subordinate to U_i , and define

$$h_i = \sum_j g'_{ij} \psi_j, \quad (4.13)$$

which satisfies $h_i \in C^\infty(U_i)$. On $U_i \cap U_j$, we have

$$h_i - h_j = \sum_k (g'_{ik} - g'_{jk}) \psi_k = \sum_k g'_{ij} \psi_k = g'_{ij}. \quad (4.14)$$

We then have that

$$\frac{\partial}{\partial \bar{z}}(h_i - h_j) = \frac{\partial}{\partial \bar{z}} g'_{ij} = 0, \quad (4.15)$$

So we can define $h \in C^\infty(\Omega)$ by letting

$$h|_{U_i} = \frac{\partial h_i}{\partial \bar{z}} \quad (4.16)$$

By Theorem 4.2, we can solve the equation $\partial f / \partial \bar{z} = h$ for $f \in C^\infty(\Omega)$. Then we redefine $h'_i = h_i - f$. These now satisfy $h'_i \in \mathcal{O}(U_i)$, and $h'_i - h'_j = g'_{ij}$.

So going back to the above, we define $f'_i = e^{-h'_i} f_i$. On overlaps, we now have

$$\frac{f'_i}{f'_j} = \frac{e^{-h'_i} f_i}{e^{-h'_j} f_j} = e^{-h'_i + h'_j} \frac{f_i}{f_j} = e^{-g_{ij}'} \frac{f_i}{f_j} = e^{-g_{ij}} \frac{f_i}{f_j} = 1, \quad (4.17)$$

so the f'_i patch together to define $f \in \mathcal{M}(\Omega)$. Since we only multiplied the f_i by a non-zero holomorphic function, the order of f at w_j is equal to n_j . \square

Corollary 4.7. *If $f \in \mathcal{M}(\Omega)$, then there exists $g, h \in \mathcal{O}(\Omega)$ such that $f = g/h$ in all of Ω .*

Proof. If f has poles of order n_j at w_j , then by the Weierstrass Theorem, there exists a holomorphic function $h \in \mathcal{O}(\Omega)$ which has a zero of order n_j at w_j . Then $g = hf$ has no poles so $g \in \mathcal{O}(\Omega)$. \square

5 Lecture 5

5.1 Power series in several variables

We review some basic facts about power series in several variables. Some good references for this material are [FG02, Chapter 1], [JP08, Chapter 1], or [KP02, Chapter 2.1]. We write a point $z = (z_1, \dots, z_n)$. The open polydisc with polyradius $r = (r_1, \dots, r_n)$ about a point $z_0 = (z_1^0, \dots, z_n^0)$ is the set

$$\Delta(z_0, r) = \{z \mid |z_j - z_j^0| < r_j, j = 1 \dots n\}. \quad (5.1)$$

We will let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ denote a multi-index, where \mathbb{Z}_+ denotes the non-negative integers. Define

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \quad (5.2)$$

$$|z|^\alpha = |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \quad (5.3)$$

$$\alpha! = \alpha_1! \cdots \alpha_n! \quad (5.4)$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \quad (5.5)$$

Definition 5.1. The series $\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha$ converges at z if some rearrangement converges, that is, give some bijection $\phi: \mathbb{Z}_+ \rightarrow (\mathbb{Z}_+)^n$, the series

$$\sum_{j=0}^{\infty} a_{\phi(j)} (z - z_0)^{\phi(j)} \quad (5.6)$$

converges. The *domain of convergence* of the power series is the interior of the set of points of convergence.

In 1 variable we know that domains of convergence are discs. Regions of convergence in several variable can be more complicated.

Example 5.2. The domain of convergence of the series $\sum_{k=0}^{\infty} z^k w^k$ is $\{(z, w) \mid |zw| < 1\}$.

Example 5.3 (Boas). The series $\sum_{n=1}^{\infty} z^n w^{n!}$ converges in the 3 sets

$$U_1 = \{(z, w) \mid |w| < 1\}, \quad U_2 = \{(0, w)\}, \quad U_3 = \{(z, w) \mid |z| < 1 \text{ and } |w| = 1\}. \quad (5.7)$$

Only U_1 is an open set; the sets U_2 and U_3 are 1 dimensional, and are not domains. The domain of convergence is U_1 .

Lemma 5.4 (Abel). *If $\sum_{\alpha} a_{\alpha} z^{\alpha}$ (centered at $z = 0$) converges at the point z' then it converges uniformly and absolutely for any point z of the form $z_j = \rho_j z'_j$ where $|\rho_j| < 1$. Furthermore, a point p belongs to the domain of convergence of the power series $\sum_{\alpha} a_{\alpha} z^{\alpha}$ if and only if there exists a neighborhood U of p , a constant C , and $r < 1$ such that $|a_{\alpha} z^{\alpha}| \leq C r^{|\alpha|}$ for all $z \in U$.*

Proof. Since the series converges at the point z' , the terms must be bounded, so there exists a constant C so that $|a_{\alpha}| |z'|^{\alpha} \leq C$. Let $\rho = \max\{|\rho_1|, \dots, |\rho_n|\} < 1$, and consider any point $z = (z_1, \dots, z_n)$ so that $|z_j| < \rho |z'_j|$. We then have

$$|a_{\alpha}| |z|^{\alpha} \leq |a_{\alpha}| \rho^{|\alpha|} |z'|^{\alpha} \leq C \rho^{|\alpha|}. \quad (5.8)$$

So given an integer $N > 0$, we have

$$\begin{aligned} \sum_{|\alpha| \leq N} |a_{\alpha}| |z|^{\alpha} &= \sum_{j=0}^N \sum_{|\alpha|=j} |a_{\alpha}| |z|^{\alpha} \\ &\leq \sum_{j=0}^N \sum_{|\alpha|=j} C \rho^j \end{aligned} \quad (5.9)$$

How many multi-indices of length j are there? This is counting the number of non-negative integer solutions of

$$\alpha_1 + \cdots + \alpha_n = j. \quad (5.10)$$

To see this, let $\alpha'_1 = \alpha_1 + 1$, then we are interested in the number of positive integer solutions to

$$\alpha'_1 + \cdots + \alpha'_n = j + n. \quad (5.11)$$

So we have a total of $j + n$ integers, dividing this up into n integers is the same as putting $n - 1$ partitions somewhere in the spaces between them, so the number is

$$\binom{j + n - 1}{n - 1}. \quad (5.12)$$

Continuing with the above calculation,

$$\begin{aligned} \sum_{|\alpha| \leq N} |a_\alpha| |z|^\alpha &\leq C \sum_{j=0}^N \binom{j + n - 1}{n - 1} \rho^j \\ &= C \sum_{j=0}^N \frac{(j + n - 1)!}{j!(n - 1)!} \rho^j \\ &= \frac{C}{(n - 1)!} \sum_{j=0}^N (j + n - 1)(j + n - 2) \cdots (j + 1) \rho^j \\ &\leq C_n \sum_{j=0}^N j^n \rho^j. \end{aligned} \quad (5.13)$$

Applying the ratio test, we have

$$\lim_{j \rightarrow \infty} \frac{(j + 1)^n \rho^{j+1}}{j^n \rho^j} = \lim_{j \rightarrow \infty} \left(\frac{j + 1}{j} \right)^n \rho = \rho, \quad (5.14)$$

so the series converges provided $\rho < 1$.

If p belongs to the domain of convergence, then by definition the series converges in a neighborhood of p . Then by the first part it converges in some polydisc around the origin containing z , and we follow the first part of the proof. □

Definition 5.5. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha \quad (5.15)$$

which converges absolutely and uniformly in a polydisc $\Delta(z_0, \hat{\epsilon})$ around z_0 , for some positive polyradius $\hat{\epsilon}$.

5.2 Cauchy's formula in several complex variables

Basic reference are [GH78, Hör90, Nog16].

Definition 5.6. We say that f is holomorphic in U if it is $C^1(U)$ and satisfies the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial \bar{z}^j} = 0, \quad j = 1 \cdots n. \quad (5.16)$$

Proposition 5.7. *Let U be an open set in \mathbb{C} . Then f is holomorphic in U if and only if f is complex analytic in U .*

Proof. Consider $n = 2$, the higher-dimensional case is similar. We assume that $U = \Delta(0, r_1) \times \Delta(0, r_2)$ is a polydisc, and $f \in C^1(\bar{U})$. If f is holomorphic in U , then for fixed z_1 , the slice $f(z_1, z_2)$ is a 1-variable holomorphic function for $z_2 \in \Delta(0, r_2)$. This holds similarly for the other variable, so the Cauchy-Pompiou formula applied twice yields

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi i} \int_{|w_2|=r_2} \frac{f(z_1, w_2)dw}{w_2 - z_2} \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{|w_2|=r_2} \int_{|w_1|=r_1} \frac{f(w_1, w_2)dw}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned} \quad (5.17)$$

For any $(z_1^0, z_2^0) \in U$, we expand

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0 + z_1^0 - z_1} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \frac{1}{1 - \frac{z_1 - z_1^0}{w_1 - z_1^0}} \quad (5.18)$$

$$= \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0}\right)^k \quad (5.19)$$

$$= \frac{1}{(w_1 - z_1^0)(w_2 - z_2^0)} \sum_{l=0}^{\infty} \left(\frac{z_2 - z_2^0}{w_2 - z_2^0}\right)^l \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0}\right)^k \quad (5.20)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \quad (5.21)$$

We next show that we are justified in the last step. Let $(z_1^0, z_2^0) \in \Delta(0, r_1') \times \Delta(0, r_2')$ with $r_1' < r_1$ and $r_2' < r_2$. Then we have $|w_1 - z_1^0| > r_1 - r_1'$, and $|w_2 - z_2^0| > r_2 - r_2'$. For $|z_1 - z_1^0| < (r_1 - r_1')/2$ and $|z_2 - z_2^0| < (r_2 - r_2')/2$, we then have

$$|a_{kl}| = \left| \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}} \right| \leq \frac{1}{(r_1 - r_1')(r_2 - r_2')} 2^{-k} 2^{-l}, \quad (5.22)$$

so the sum converges absolutely and uniformly in any smaller polydisc by Lemma 5.4. Interchanging the integration and summation in (5.17) then yields a power series expansion for f .

The converse is similar to the 1-variable case. If f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula (5.17). So then f does also by uniform convergence. Then we can differentiate under the integral to see that f is holomorphic. For more details, see [GH78, page 6]. \square

Remark 5.8. Note that the integral in (5.17) is just over a 2-dimensional torus contained in the boundary of the polydisc. The topological boundary of the polydisc is 3-dimensional, but it is not a manifold, it is $\partial(\Delta \times \Delta) = S^1 \times \Delta \cup \Delta \times S^1$, and these 2 sets intersect along the torus.

Similar to the 1 variable case, we have the following corollaries. In the proofs, we just consider the case of 2 dimensions, the higher dimensional cases are similar.

Corollary 5.9. *If f is analytic at z_0 then f is infinitely differentiable at z_0 and*

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(z_0)}{\partial z^\alpha} = \frac{1}{\alpha_1! \alpha_2!} \frac{\partial^{\alpha_1 + \alpha_2} f(z_0)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2}} \quad (5.23)$$

Corollary 5.10 (The maximum principle). *Let $\Omega \subset \mathbb{C}^n$, and $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$. Then $|f|$ does not assume its maximum at an interior point unless f is constant.*

Proof. Assume that $|f|$ attains local maximum at some interior point $z_0 \in \Omega$. Since f is holomorphic, it admits a power series expansion $f(z) = \sum_\alpha a_\alpha (z - z_0)^\alpha$ which converges uniformly in the closure of a polydisc $\Delta(z_0, r) = \Delta(z_0^1, r_1) \times \Delta(z_0^2, r_2)$. Write $S^1 \times S^1 = \partial\Delta(z_0^1, r_1) \times \partial\Delta(z_0^2, r_2)$. Then $|f(z)|^2 \leq |f(z_0)|^2$ for $z \in \Delta(z_0, r)$, so

$$|f(z_0)|^2 \geq \left(\frac{1}{2\pi}\right)^2 \int_{S^1 \times S^1} |f(z_1, z_2)|^2 dV_{S^1 \times S^1} = \sum_{k,l=0}^{\infty} |a_{kl}|^2 r_1^{2k} r_2^{2l} \geq |a_{00}|^2 = |f(z_0)|^2. \quad (5.24)$$

This implies that a_{kl} is zero except for a_{00} and therefore f is constant. \square

Corollary 5.11. *Let $K \subset \Omega$ be a compact subset. Then there exist constants $C_{|\alpha|}$, depending only upon K and Ω such that*

$$\sup_{z \in K} \left| \frac{\partial^\alpha f(z)}{\partial z^\alpha} \right| \leq C_{|\alpha|} \sup_{z \in \Omega} |f(z)|. \quad (5.25)$$

for all $u \in \mathcal{O}(\Omega)$.

Proof. Again, we just consider the case of 2 dimensions, the higher dimensional case is similar. Fix $(z_1^0, z_2^0) \in \Omega$, and let $\Delta(z^0, (r_1, r_2)) \subset \Omega$ be a polydisc. Then for $(z_1, z_2) \in \Delta(z^0, (r_1', r_2'))$ with $r_1' < r_1$ and $r_2' < r_2$, we have

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl} (z_1 - z_1^0)^k (z_2 - z_2^0)^l, \quad (5.26)$$

where

$$a_{kl} = \left(\frac{1}{2\pi i}\right)^2 \int_{|w_2|=r_2} \int_{|w_1|=r_1} \frac{f(w_1, w_2) dw}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \quad (5.27)$$

We then get *Cauchy's inequalities*

$$\left| \frac{\partial^{k+l} f}{\partial z_1^k \partial z_2^l} (z_1^0, z_2^0) \right| = k!l! |a_{kl}| \leq \frac{k!l! r_1 r_2}{(r_1 - r_1')^k (r_2 - r_2')^l} \sup_{w=(w_1, w_2), |w_1|=r_1, |w_2|=r_2} |f(w)|. \quad (5.28)$$

The claim follows by covering K with finitely many polydiscs contained in Ω . \square

The following corollaries are proved exactly as before.

Corollary 5.12. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \rightarrow u$ converges uniformly to u in the C^0 norm as $n \rightarrow \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Corollary 5.13. *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

6 Lecture 6

6.1 The operators ∂ and $\bar{\partial}$ in \mathbb{C}^n

Using the coordinates

$$(z^1, \dots, z^n) = (x^1 + iy^1, \dots, x^n + iy^n), \quad (6.1)$$

recall that that $T^{1,0}$ is spanned by

$$\frac{\partial}{\partial z^j} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j} \right), \quad (6.2)$$

$T^{0,1}$ is spanned by

$$\frac{\partial}{\partial \bar{z}^j} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \right), \quad (6.3)$$

$\Lambda^{1,0}$ is spanned by

$$dz^j \equiv dx^j + idy^j, \quad (6.4)$$

and $\Lambda^{0,1}$ is spanned by

$$d\bar{z}^j \equiv dx^j - idy^j, \quad (6.5)$$

for $j = 1 \dots n$.

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \quad (6.6)$$

We define $\Omega^k, \Omega_{\mathbb{C}}^k, \Omega^{p,q}$ to be the space of sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. So we have that

$$\Omega^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}. \quad (6.7)$$

If $\alpha \in \Omega^{p,q}(U)$, then we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\bar{z}^J, \quad (6.8)$$

where I and J are multi-indices of length p and q , respectively, and $\alpha_{I,J} : U \rightarrow \mathbb{C}$ are complex-valued functions.

The real operator $d : \Omega_{\mathbb{R}}^k \rightarrow \Omega_{\mathbb{R}}^{k+1}$, extends to an operator

$$d : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1} \quad (6.9)$$

by complexification.

Proposition 6.1. *We have*

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}. \quad (6.10)$$

Proof. Recall that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^j}{\partial \bar{z}^k} d\bar{z}^k. \quad (6.11)$$

Applying d to (6.8), we obtain

$$d\alpha = \sum_{I,J} \left(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \right) \wedge dz^I \wedge d\bar{z}^J, \quad (6.12)$$

and we are done. \square

We can therefore define operators

$$\partial : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1} \quad (6.13)$$

$$\bar{\partial} : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1} \quad (6.14)$$

by

$$\partial\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\bar{z}^J \quad (6.15)$$

$$\bar{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J. \quad (6.16)$$

using (9.17) and we have

$$\partial|_{\Omega^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \quad (6.17)$$

$$\bar{\partial}|_{\Omega^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \quad (6.18)$$

Corollary 6.2. *We have $d = \partial + \bar{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0. \quad (6.19)$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2. \quad (6.20)$$

If we plug in a form of type (p, q) the first term is of type $(p+2, q)$, the middle terms are of type $(p+1, q+1)$, and the last term is of type $(p, q+2)$. Since (9.17) is a direct sum, the claim follows. \square

6.2 Dolbeault cohomology

Definition 6.3. Let $U \subset \mathbb{C}^n$ be a domain. For $0 \leq p, q \leq n$, the (p, q) Dolbeault cohomology group is

$$H_{\bar{\partial}}^{p,q}(U) = \frac{\{\alpha \in \Omega^{p,q}(U) \mid \bar{\partial}\alpha = 0\}}{\bar{\partial}(\Omega^{p,q-1}(U))}, \quad (6.21)$$

where the forms have C^∞ regularity.

The Dolbeault cohomology groups enjoy the following functoriality properties.

Proposition 6.4. Let $U \subset \mathbb{C}^m, V \subset \mathbb{C}^n, W \subset \mathbb{C}^l$ be domains. Let $f : U \rightarrow V$ be a C^1 mapping which is holomorphic, that is

$$f_* \circ J_0 = J_0 \circ f_*. \quad (6.22)$$

Then there are induced mappings

$$f^* : H^{p,q}(V) \rightarrow H^{p,q}(U). \quad (6.23)$$

If $g : V \rightarrow W$ is C^1 holomorphic, then so is $g \circ f : U \rightarrow W$ and

$$(g \circ f)^* = f^* \circ g^* : H^{p,q}(W) \rightarrow H^{p,q}(U). \quad (6.24)$$

In particular, if f is a biholomorphism (one-to-one, onto, with holomorphic inverse), then the Dolbeault cohomologies of U and V are isomorphic.

Proof. The equation (6.22) implies that

$$f^* : \Omega^{p,q}(V) \rightarrow \Omega^{p,q}(U). \quad (6.25)$$

To see this, let $\alpha^{p,q} \in \Omega^{p,q}(V)$, then for vectors X_1, \dots, X_{p+q} we have

$$f^* \alpha^{p,q}(X_1, \dots, X_{p+q}) = \alpha^{p,q}(f_* X_1, \dots, f_* X_{p+q}) \quad (6.26)$$

Note that if $X \in T^{1,0}(U)$, then $J_U X = iX$, so then

$$J_V f_* X = f_* J_U X = f_* iX = i f_* X, \quad (6.27)$$

therefore $f_* X \in T^{1,0}(V)$. Similarly, if $X \in T^{0,1}(U)$ then $f_* X \in T^{0,1}(V)$. If more than p of the X_j are of type $(1,0)$ or more than q of the X_j are of type $(0,1)$, then the same is true for the $f_* X_j$, and the claim follows.

We also know that the exterior derivative commutes with pullback,

$$d_U \circ f^* = f^* \circ d_V. \quad (6.28)$$

This is equivalent to

$$(\partial_U + \bar{\partial}_U) \circ f^* = f^* \circ (\partial_V + \bar{\partial}_V) \quad (6.29)$$

If we plug in $\alpha^{p,q} \in \Omega^{p,q}(V)$, we have 2 equations

$$\partial_U \circ f^* \alpha^{p,q} = f^* \circ \partial_V \alpha^{p,q} \quad (6.30)$$

$$\bar{\partial}_U \circ f^* \alpha^{p,q} = f^* \circ \bar{\partial}_V \alpha^{p,q} \quad (6.31)$$

The second equation implies that f^* induces a well-defined mapping on cohomology $f^* : H^{p,q}(V) \rightarrow H^{p,q}(U)$ by the following. If $[\alpha^{p,q}] \in H^{p,q}(V)$ is represented by a form $\alpha^{p,q}$, such that $\bar{\partial}_V \alpha^{p,q} = 0$, then we have

$$\bar{\partial}_U f^* \alpha^{p,q} = f^* \bar{\partial}_V \alpha^{p,q} = f^* 0 = 0, \quad (6.32)$$

so we can define $f^*[\alpha^{p,q}] = [f^* \alpha^{p,q}]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^{p,q} + \bar{\partial}_V \beta^{p,q-1}) = f^* \alpha^{p,q} + f^* \bar{\partial}_V \beta^{p,q-1} = f^* \alpha^{p,q} + \bar{\partial}_U f^* \beta^{p,q-1}, \quad (6.33)$$

so we have

$$[f^*(\alpha^{p,q} + \bar{\partial}_V \beta^{p,q-1})] = [f^* \alpha^{p,q} + \bar{\partial}_U f^* \beta^{p,q-1}] = [f^* \alpha^{p,q}]. \quad (6.34)$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \quad (6.35)$$

holds on the level of forms. Finally, if f is a pseudo-biholomorphism, then f^{-1} exists and is pseudo-holomorphic, so we have

$$f \circ f^{-1} = id_V, \quad f^{-1} \circ f = id_U, \quad (6.36)$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^{p,q}(U)}, \quad (f^{-1})^* \circ f^* = id_{H^{p,q}(V)}, \quad (6.37)$$

□

Definition 6.5. A form $\alpha \in \Omega^{p,0}(U)$ is *holomorphic* if $\bar{\partial}\alpha = 0$.

Remark 6.6. We only talk about forms of type $(p,0)$ being holomorphic, we never call a (p,q) -form holomorphic if $q > 0$. Also, we have (trivially)

$$H^{p,0}(U) = \{\alpha \in \Omega^{p,0}(U) \mid \alpha \text{ is holomorphic}\}. \quad (6.38)$$

Proposition 6.7. A p -form $\alpha \in \Omega^{p,0}(U)$ is holomorphic if and only if it can be written as

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \quad (6.39)$$

where the $\alpha_I : U \rightarrow \mathbb{C}$ are holomorphic functions.

Proof. We have

$$\bar{\partial}\alpha = \sum_{|I|=p,k} \frac{\partial\alpha_I}{\partial\bar{z}^k} dz^k \wedge dz^I. \quad (6.40)$$

So $\bar{\partial}\alpha = 0$ if and only if the α_I are holomorphic. \square

Remark 6.8. So for $U \subset \mathbb{C}^n$ a domain, $\dim_{\mathbb{C}} H^{p,0}(U) = \infty$ is always infinite-dimensional for $0 \leq p \leq n$, in particular because any polynomial function in the z -variables is holomorphic.

Example 6.9. Let's review the case of a domain $U \subset \mathbb{C}$. First, $H_{\bar{\partial}}^{0,0}(U) = \mathcal{O}(U)$. Theorem 4.2 shows that $H_{\bar{\partial}}^{0,1}(U) = \{0\}$. The space $H_{\bar{\partial}}^{0,1}(U)$ consists of holomorphic 1-forms, but since $n = 1$, any holomorphic 1-form is of the form $f(z)dz$, where $f \in \mathcal{O}(U)$. So $H_{\bar{\partial}}^{0,1}(U) \cong \mathcal{O}(U)$. Finally,

$$H_{\bar{\partial}}^{1,1}(U) = \frac{\text{Ker } \bar{\partial} : \Omega^{1,1} \rightarrow \Omega^{1,2}}{\text{Image } \bar{\partial} : \Omega^{1,0} \rightarrow \Omega^{1,1}} = \frac{gdz \wedge d\bar{z}}{(\partial f / \partial \bar{z})d\bar{z} \wedge dz} = \{0\}, \quad (6.41)$$

which also follows from Theorem 4.2.

7 Lecture 7

7.1 The $\bar{\partial}$ -equation for $(0, 1)$ -forms and Hartogs' Theorem

A reference for this section is [HL84, Section 1.2]. For $n \geq 2$, and $g \in \Omega^{0,1}(U)$, the equation $\bar{\partial}f = g$ is not always solvable. This follows from (9.30): applying $\bar{\partial}$ yields a compatibility condition $\bar{\partial}g = 0$. The following is in sharp contrast to the case $n = 1$.

Proposition 7.1. *Let $g \in \Omega_0^{0,1}(\mathbb{C}^n)$ (compact support) have C^∞ regularity and satisfy $\bar{\partial}g = 0$. Then there exists a smooth $f \in \Omega_0^0(\mathbb{C}^n)$ (also having compact support) with $\bar{\partial}f = g$. Furthermore, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus \text{supp}(g)$.*

Proof. We write $g = \sum_{j=1}^n g_j d\bar{z}^j$. Define

$$f(z_1, \dots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(w, z_2, \dots, z_n)}{w - z_1} dw \wedge d\bar{w}. \quad (7.1)$$

The integral is defined since g_1 has compact support. Make the change of variable $\xi = w - z_1$, and we can write f as

$$f(z_1, \dots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\xi + z_1, z_2, \dots, z_n)}{\xi} d\xi \wedge d\bar{\xi}. \quad (7.2)$$

This shows that we can differentiate under the integral sign to conclude that f has C^∞ regularity. Furthermore,

$$\begin{aligned} \frac{\partial f(z_1, \dots, z_n)}{\partial\bar{z}^1} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(\xi + z_1, z_2, \dots, z_n)}{\partial\bar{z}^1} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(\xi + z_1, z_2, \dots, z_n)}{\partial\bar{\xi}} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(w, z_2, \dots, z_n)}{\partial\bar{w}} \frac{1}{w - z_1} dw \wedge d\bar{w} = g_1(z_1, \dots, z_n), \end{aligned} \quad (7.3)$$

by the Cauchy-Pompiou formula applied to a large ball containing the support of g . The condition that $\bar{\partial}g = 0$ means that

$$0 = \sum_{j=1}^n \sum_{k=1}^n \frac{\partial g_j}{\partial \bar{z}_k} dz^k \wedge d\bar{z}^j, \quad (7.4)$$

so

$$\frac{\partial g_j}{\partial \bar{z}_k} = \frac{\partial g_k}{\partial \bar{z}_j} \quad (7.5)$$

for all $1 \leq j, k \leq n$. Then differentiating (7.6) for $j \geq 2$, we obtain

$$\begin{aligned} \frac{\partial f(z_1, \dots, z_n)}{\partial \bar{z}_j} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(\xi + z_1, z_2, \dots, z_n)}{\partial \bar{z}^j} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_j(\xi + z_1, z_2, \dots, z_n)}{\partial \bar{z}^1} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_j(w, z_2, \dots, z_n)}{\partial \bar{w}} \frac{1}{w - z_1} dw \wedge d\bar{w} = g_j(z_1, \dots, z_n). \end{aligned} \quad (7.6)$$

So the equation $\bar{\partial}f = g$ is satisfied everywhere. Finally, since g has compact support, it follows that f is holomorphic on the complement of a large ball $B_r(0)$ containing the support of g . But (7.1) shows that f vanishes when $\max\{|z_2|, \dots, |z_n|\} > r$. Therefore f is a holomorphic function on $\mathbb{C}^n \setminus B_r(0)$ which vanishes on the open subset $V = \{\max\{|z_2|, \dots, |z_n|\} > r\}$. By unique continuation, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus \text{supp}(g)$. \square

Theorem 7.2 (Hartogs). *Let $n \geq 2$, U a domain, and $K \subset U$ a compact subset of U such that $U \setminus K$ is connected. Then if $u \in \mathcal{O}(U \setminus K)$, there exists $\tilde{u} \in \mathcal{O}(U)$ with $\tilde{u}|_{U \setminus K} = u$.*

Proof. Let $0 \leq \chi \in C_0^\infty(U)$ and $\chi \equiv 1$ on K . Define $g = \bar{\partial}(\chi \cdot u)$. Since

$$\bar{\partial}(\chi u) = u \bar{\partial}(\chi) + \chi \bar{\partial}(u) = u \bar{\partial}(\chi) + 0, \quad (7.7)$$

we see that g extends smoothly to U , that is, $g \in \Omega_0^{0,1}(U)$, and $\bar{\partial}g = 0$. By Proposition 7.1, there exists $f \in C_0^\infty(\mathbb{C}^n)$ with $\bar{\partial}f = g$. So then we let $\tilde{u} = (1 - \chi)u + f$. This satisfies

$$\bar{\partial}\tilde{u} = -g + \bar{\partial}(f) = 0, \quad (7.8)$$

so $\tilde{u} \in \mathcal{O}(U)$. Let V denote the unbounded component of the complement of the support of χ . Since $\text{supp}(g) \subset \text{supp}(\chi)$, from Proposition 7.1, we have that $f \equiv 0$ in V , so $\tilde{u} = u$ in $U \cap V$. But since $U \setminus K$ is connected and $V \cap (U \setminus K) \neq \emptyset$, we have $\tilde{u} = u$ in $U \setminus K$ from unique continuation. \square

Example 7.3. For example, point singularities are removable for $n \geq 2$. Even polydiscs are removable: if u is holomorphic on $\Delta \setminus \Delta'$, where $\Delta' \subset \Delta$ are polydiscs with $\bar{\Delta}' \subset \Delta$, then u extends to a holomorphic function on Δ . Same for $B_{r_1}(0) \subset B_{r_2}(0)$ with $r_1 < r_2$.

7.2 Dolbeault cohomology of a polydisc

Some references for this section are [GH78, Section 0.2] or [Nog16, Section 3.6].

Proposition 7.4. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), and $\omega \in \Omega^{p,q}(U)$ satisfies $\bar{\partial}\omega = 0$ for $q \geq 1$, then given any polyradius $s < r$, there exists $\eta \in \Omega^{p,q-1}(\Delta(r))$ with $\bar{\partial}\eta = \omega$ satisfied in $\Delta(s)$.*

Proof. Step 1: reduce to case of $\Omega^{0,q}$. If $\omega \in \Omega^{p,q}(U)$,

$$\omega = \sum_{|I|=p, |J|=q} \omega_{IJ} dz^I \wedge d\bar{z}^J. \quad (7.9)$$

Define

$$\omega_I = \sum_{|J|=q} \omega_{IJ} d\bar{z}^J. \quad (7.10)$$

Then $\omega_I \in \Omega^{0,q}$, and $\bar{\partial}\omega_I = 0$. If $\omega_I = \bar{\partial}\eta_I$, then

$$\bar{\partial}(dz^I \wedge \eta_I) = (-1)^p dz^I \wedge \bar{\partial}\eta_I = (-1)^p dz^I \wedge \omega_I, \quad (7.11)$$

and we are done with Step 1.

Step 2. Given $s < r$, if $\omega \in \Omega^{0,q}(\Delta(r))$ and $\bar{\partial}\omega = 0$ in $\Delta(r)$, then there exists $\eta \in \Omega^{0,q-1}(\Delta(r))$ with $\bar{\partial}\eta = \omega$ satisfied in $\Delta(s)$. Choose cutoff functions $0 \leq \chi_j(t) \leq 1$ so that

$$\chi_i(t) = \begin{cases} 1 & t \leq s_j \\ 0 & t \geq r_j \end{cases}. \quad (7.12)$$

We begin with $q = 1$. Note that $\omega \in \Omega^{0,1}(\Delta(r))$, but it does not have compact support, so we proceed differently than in the proof of Proposition 7.1. Write

$$\omega = \sum_k \omega_k d\bar{z}^k, \quad (7.13)$$

and define

$$\eta_1(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|w^j| \leq r_j} \frac{\chi_1(w_1) \omega_j(w^1, z^2, \dots, z^n)}{w^1 - z^1} dw^1 \wedge d\bar{w}^1. \quad (7.14)$$

Then $\partial\eta_1/\partial\bar{z}^1 = \chi_1\omega_1$, and we have

$$\bar{\partial}\eta_1 = \sum_l \frac{\partial\eta_1}{\partial\bar{z}^l} d\bar{z}^l = \chi_1\omega_1 d\bar{z}^1 + \sum_{j>1} \frac{\partial\eta_1}{\partial\bar{z}^j} d\bar{z}^j. \quad (7.15)$$

That is, we have solved the $d\bar{z}^1$ -term, modulo terms involving $d\bar{z}^j$ for $j > 1$ (we have not even used the fact that $\bar{\partial}\omega = 0$ yet!) Next, we consider the case

$$\omega = \sum_{k>1} \omega_k d\bar{z}^k, \quad (7.16)$$

Since $\bar{\partial}\omega = 0$, this tells us that $\partial\omega_2/\partial\bar{z}^1 = 0$. Next, we define

$$\eta_2(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|w^2| \leq r_2} \frac{\chi_2(w_2)\omega_2(z^1, w^2, z^3, \dots, z^n)}{w^2 - z^2} dw^2 \wedge d\bar{w}^2. \quad (7.17)$$

Then $\partial\eta_2/\partial\bar{z}^2 = \chi_2\omega_2$ and $\partial\eta_2/\partial\bar{z}^1 = 0$, so we have

$$\bar{\partial}\eta_2 = \sum_l \frac{\partial\eta_2}{\partial\bar{z}^j} d\bar{z}^j = \chi_2\omega_2 d\bar{z}^2 + \sum_{j>2} \frac{\partial\eta_2}{\partial\bar{z}^j} d\bar{z}^j. \quad (7.18)$$

Assume that we can solve all the terms involving $d\bar{z}^k$ for $k \leq l$, and

$$\omega = \sum_{k>l} \omega_k d\bar{z}^k. \quad (7.19)$$

Since $\bar{\partial}\omega = 0$, this tells us that $\partial\omega_{l+1}/\partial\bar{z}^j = 0$ for $j \leq l$. Then we define

$$\eta_{l+1}(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|w^{l+1}| \leq r_{l+1}} \frac{\chi_{l+1}(w_{l+1})\omega_{l+1}(z^1, \dots, w^{l+1}, \dots, z^n)}{w^{l+1} - z^{l+1}} dw^{l+1} \wedge d\bar{w}^{l+1}. \quad (7.20)$$

Then $\partial\eta_{l+1}/\partial\bar{z}^{l+1} = \chi_{l+1}\omega_{l+1}$ and $\partial\eta_{l+1}/\partial\bar{z}^j = 0$ for $j \leq l$, so we have

$$\bar{\partial}\eta_{l+1} = \sum_{j>l} \frac{\partial\eta_{l+1}}{\partial\bar{z}^j} d\bar{z}^j = \chi_{l+1}\omega_{l+1} d\bar{z}^{l+1} + \sum_{j>l+1} \frac{\partial\eta_{l+1}}{\partial\bar{z}^j} d\bar{z}^j. \quad (7.21)$$

By induction, we are done with the case of $q = 1$.

Next, consider the case of $q = 2$. Then

$$\omega = \sum_{1 \leq k < l} \omega_{kl} d\bar{z}^k \wedge d\bar{z}^l = \sum_{1 < l} \omega_{1l} d\bar{z}^1 \wedge d\bar{z}^l + \sum_{1 < k < l} \omega_{kl} d\bar{z}^k \wedge d\bar{z}^l. \quad (7.22)$$

Define $\eta = \sum_{1 < k} \eta_{1k} d\bar{z}^k$, where

$$\eta_{1k}(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|w^1| \leq r_1} \frac{\chi_1(w^1)\omega_{1k}(w^1, z^2, \dots, z^n)}{w^1 - z^1} dw^1 \wedge d\bar{w}^1. \quad (7.23)$$

Then η_{1k} solves $\partial\eta_{1k}/\partial\bar{z}^1 = \chi_1\omega_{1k}$. So then

$$\bar{\partial}\eta = \sum_{1 < k} \frac{\partial\eta_{1k}}{\partial\bar{z}^l} d\bar{z}^l \wedge d\bar{z}^k = \sum_{1 < k} \chi_1\omega_{1k} d\bar{z}^1 \wedge d\bar{z}^k + R \quad (7.24)$$

where R doesn't include any $d\bar{z}^1$ -s. So we have solved the terms in ω involving $d\bar{z}^1$ -s. We next assume that ω is of the form

$$\omega = \sum_{1 < k < l} \omega_{kl} d\bar{z}^k \wedge d\bar{z}^l = \sum_{2 < l} \omega_{2l} d\bar{z}^2 \wedge d\bar{z}^l + \sum_{2 < k < l} \omega_{kl} d\bar{z}^k \wedge d\bar{z}^l. \quad (7.25)$$

Let $\eta = \sum_{2 < k} \eta_{2k} d\bar{z}^k$ where

$$\eta_{2k} = \frac{1}{2\pi i} \int_{|w^2| \leq r_2} \frac{\chi_2(w^2) \omega_{2k}(z^1, w^2, z^3, \dots, z^n)}{w^2 - z^2} dw^2 \wedge d\bar{w}^2. \quad (7.26)$$

Then $\partial\eta_{2k}/\partial\bar{z}^2 = \chi_2\omega_{2k}$. Furthermore, since $\bar{\partial}\omega = 0$, $\partial\eta_{2k}/\partial\bar{z}^1 = 0$. So then

$$\bar{\partial}\eta = \sum_{2 < k} \bar{\partial}(\eta_{2k} d\bar{z}^k) = \sum_{2 < k} \sum_{2 \leq l} \frac{\partial\eta_{2k}}{\partial\bar{z}^l} d\bar{z}^l \wedge d\bar{z}^k = \sum_{2 < k} \chi_2\omega_{2k} d\bar{z}^2 \wedge d\bar{z}^k + R, \quad (7.27)$$

where R only has terms $d\bar{z}^k \wedge d\bar{z}^l$ for $k, l \geq 3$. So we have solved as the term in ω having $d\bar{z}^1$ -s or $d\bar{z}^2$ -s. By a similar induction argument as in the $q = 1$ case, we can solve all terms in this manner. The case of $q > 2$ is similar, and details left as an exercise. \square

Theorem 7.5. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), then $H_{\bar{\partial}}^{p,q}(U) = \{0\}$ for $q \geq 1$.*

Proof. Choose a monotone increasing sequence of polyradii $r_1 < r_2 < \dots$ with $\lim_{j \rightarrow \infty} r_j = r$. Given $\omega \in \Omega^{0,q}(\Delta(r))$, by Step 2, we can find $\eta_j \in \Omega^{0,q-1}(\Delta(r))$ with $\bar{\partial}\eta_j = \omega$ on $\Delta(r_j)$. We do not know that the sequence η_j will converge. However, $\bar{\partial}(\eta_{j+1} - \eta_j) = 0$ in $\Delta(r_j)$. If $q \geq 2$, then by Step 2, we can find $\beta_{j+1} \in \Omega^{0,q-2}(\Delta(r_j))$ solving $\bar{\partial}(\beta_{j+1}) = \eta_{j+1} - \eta_j$ in $\Delta(r_{j-1})$. We then consider the sequence $\eta'_{j+1} = \eta_{j+1} - \bar{\partial}(\beta_{j+1})$. Then $\eta'_{j+1} \in \Omega^{0,q-2}(\Delta(r_j))$ and

$$\bar{\partial}(\eta'_{j+1}) = \bar{\partial}\eta_{j+1} - \bar{\partial}^2(\beta_{j+1}) = \omega \quad (7.28)$$

in $\Delta(r_{j-1})$, and this new sequence now obviously converges to a solution $\eta \in \Omega^{0,q}(\Delta(r))$ with $\bar{\partial}\eta = \omega$ in $\Delta(r)$.

If $q = 1$, then we prove exactly like we did in the case of $n = 1$, by approximating the difference $\eta_{j+1} - \eta_j$ by a polynomial P_{j+1} to obtain a sequence so that

$$\sup_{z \in K} |\eta_{j+1}(z) - \eta_j(z)| < 2^{-j}, \quad (7.29)$$

and we obtain a sequence converging on compact subsets to a solution. \square

Remark 7.6. Using Laurent series instead of polynomials, a similar proof works to prove that Theorem 7.5 also holds for products $\Delta^*(r_1) \times \dots \times \Delta^*(r_k) \times \Delta(r_{k+1}) \times \dots \times \Delta(r_{k+l})$, that is, we can allow punctured 1-dimensional disks. With a lot more work, one can also show that Theorem 7.5 holds for $\Omega_1 \times \dots \times \Omega_n$ with $\Omega_j \subset \mathbb{C}$ are domains. Note the result is NOT true for a punctured polydisc $\Delta(0, r) \setminus \{0\}$ for $n \geq 2$, but we cannot prove that yet.

Remark 7.7. Theorem 7.5 also holds for a ball $B(0, r) \subset \mathbb{C}^n$. However, this is difficult to prove directly. One could use the Bochner-Martinelli kernel instead of the Cauchy kernel to prove Proposition 7.4. Then one would also need to prove that the $B(0, r)$ is a Runge domain, that is, $\mathcal{O}(B(0, r))$ can be approximated by holomorphic polynomials uniformly on compact subsets. However, it seems actually easier to prove this more generally for any pseudoconvex domain (using Hörmander's L^2 methods), and then show that $B(0, r)$ is pseudoconvex.

8 Lecture 8

8.1 Almost complex manifolds

Definition 8.1. An *almost complex manifold* is a real manifold with an endomorphism $J : TM \rightarrow TM$ satisfying $J^2 = -Id$.

The following lemma shows that we can always take J to be standard at any *point*.

Lemma 8.2. Let $J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a linear mapping satisfying $J^2 = -Id$. Then there exists an invertible matrix A such that $A^{-1}JA = J_{Euc}$.

Proof. For $X \in \mathbb{R}^{2n}$, define

$$(a + ib)X = aX + bJX. \quad (8.1)$$

Then \mathbb{R}^{2n} becomes an n -dimensional complex vector space. Let X_1, \dots, X_n be a complex basis. Then $X_1, JX_1, \dots, X_n, JX_n$ is a basis of \mathbb{R}^{2n} as a real vector space, and J is obviously standard in this basis. \square

Remark 8.3. The Newlander-Nirenberg Theorem deals with the following question: when can we make J standard in a *neighborhood* of a point? As we will see shortly, this cannot possibly be true for an arbitrary almost complex structure; there is an *integrability condition* which must be satisfied.

All of the linear algebra we discussed above in \mathbb{C}^n can be done on an almost complex manifold (M, J) . We can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \quad (8.2)$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \quad (8.3)$$

is the i -eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \quad (8.4)$$

is the $-i$ -eigenspace of J .

The map J also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \quad (8.5)$$

where

$$\Lambda^{1,0} = \{\alpha - iJ\alpha, \alpha \in T_p^*M\} \quad (8.6)$$

is the i -eigenspace of J , and

$$\Lambda^{0,1} = \{\alpha + iJ\alpha, \alpha \in T_p^*M\} \quad (8.7)$$

is the $-i$ -eigenspace of J .

Next, we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} \quad (8.8)$$

decomposes as a direct sum.

Remark 8.4. This gives a necessary topological obstruction for existence of an almost complex structure: the bundle of complex k -forms must decompose into a direct sum of subbundles as in (8.8).

We can extend $J : \Lambda^k \otimes \mathbb{C} \rightarrow \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \quad (8.9)$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$. Note we can also extend J to k -forms by

$$J\alpha(X_1, \dots, X_k) = \alpha(JX_1, \dots, JX_k). \quad (8.10)$$

Exercise 8.5. Check that these two definitions of J on k -forms agree.

Definition 8.6. A triple (M, J, g) where J is an almost complex structure, and g is a Riemannian metric is *almost Hermitian* if

$$g(X, Y) = g(JX, JY) \quad (8.11)$$

for all $X, Y \in TM$. We also say that g is *compatible* with J .

Proposition 8.7. *Given a linear J with $J^2 = -Id$ on \mathbb{R}^{2n} , and a positive definite inner product g on \mathbb{R}^{2n} which is compatible with J , there exist elements $\{X_1, \dots, X_n\}$ in \mathbb{R}^{2n} so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \quad (8.12)$$

is an ONB for \mathbb{R}^{2n} with respect to g .

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \quad (8.13)$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1 , and let W be the orthogonal complement of $\text{span}\{X_1, JX_1\}$. We claim that $J : W \rightarrow W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J -invariance of g , we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. \square

Definition 8.8. To an almost Hermitian structure (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y) \quad (8.14)$$

called the *Kähler form* or *fundamental 2-form*.

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \quad (8.15)$$

Furthermore, since

$$\omega(JX, JY) = \omega(X, Y), \quad (8.16)$$

this form is a real form of type $(1, 1)$. That is, $\omega \in \Gamma(\Lambda_{\mathbb{R}}^{1,1})$, where $\Lambda_{\mathbb{R}}^{1,1} \subset \Lambda^{1,1}$ is the real subspace of elements satisfying $\bar{\omega} = \omega$.

In Euclidean space $(\mathbb{R}^{2n}, J_0, g_{Euc})$, the fundamental 2-form is

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^n dz^j \wedge d\bar{z}^j. \quad (8.17)$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \quad (8.18)$$

Note that this defines an orientation on \mathbb{C}^n , which we will refer to as the natural orientation. Note also that

$$\omega_{Euc}^n = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \quad (8.19)$$

Proposition 8.9. *If (M, J) is almost complex, then $\dim(M)$ is even and M is orientable.*

Proof. If M is of real dimension m , and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^m, \quad (8.20)$$

which implies that m is even. We will henceforth write $m = 2n$. Next, let g be any Riemannian metric on M . Then define

$$h(X, Y) = g(X, Y) + g(JX, JY). \quad (8.21)$$

Then $h(JX, JY) = h(X, Y)$ is J -invariant, so (M, J, h) is almost Hermitian. We then consider the fundamental 2-form

$$\omega(X, Y) = h(JX, Y). \quad (8.22)$$

This is a form of type $(1, 1)$, so $\omega^n \in \Lambda_{\mathbb{R}}^{n,n} \cong \Lambda_{\mathbb{R}}^{2n}$ is a top degree $2n$ -form. It is nowhere-vanishing since at any point $x \in M$ by Proposition 8.7 we can assume that both $J_x = J_{Euc}$ and $g_x = g_{Euc}$, so $\omega^n(x) \neq 0$ by (8.19). Therefore, ω gives a globally defined orientation on M . \square

Example 8.10. For example, $\mathbb{R}P^n$ does not admit any almost complex structure, since it is non-orientable for n even.

Definition 8.11. A smooth mapping between $f : M \rightarrow N$ between almost complex manifolds (M, J_M) and (N, J_N) is *pseudo-holomorphic* if

$$f_* \circ J_M = J_N \circ f_* \quad (8.23)$$

We have a useful characterization of pseudo-holomorphic mappings.

Proposition 8.12. A mapping $f : M \rightarrow N$ between almost complex manifolds (M, J_M) and (N, J_N) is pseudo-holomorphic if and only if

$$f_*(T^{1,0}(M)) \subset T^{1,0}(N), \quad (8.24)$$

if and only if

$$f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M). \quad (8.25)$$

8.2 Complex manifolds

We next define a complex manifold.

Definition 8.13. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M , such that $\phi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

Example 8.14. Since holomorphic mappings are orientation-preserving by (1.40), any complex manifold is necessarily orientable. For example, $\mathbb{R}P^n$ does not admit any complex structure. Note that we knew from Example 8.10 above that there is no almost complex structure.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 8.15. In any coordinate chart, define $J_\alpha : TM_{U_\alpha} \rightarrow TM_{U_\alpha}$ by

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \quad (8.26)$$

Then $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$ and therefore gives a globally defined almost complex structure $J : TM \rightarrow TM$ satisfying $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \quad (8.27)$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \quad (8.28)$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \quad (8.29)$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations. Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

The next proposition follows from the above discussion on Cauchy-Riemann equations.

Proposition 8.16. *If (M, J_M) and (N, J_N) are complex manifolds, then $f : M \rightarrow N$ is pseudo-holomorphic if and only if it is a holomorphic mapping in local holomorphic coordinate systems.*

Definition 8.17. An almost complex structure J is said to be a *complex structure* if J is induced from a collection of holomorphic coordinates on M .

Proposition 8.18. *An almost complex structure J is a complex structure if and only if for any $x \in M$, there is a neighborhood U of x and a pseudo-holomorphic mapping $\phi : (U, J) \rightarrow (\mathbb{C}^n, J_0)$ which has non-vanishing Jacobian at x . Equivalently, there exist n pseudo-holomorphic functions $f^j : U \rightarrow \mathbb{C}, j = 1 \dots n$, with linearly independent differentials at x .*

Proof. By the inverse function theorem, ϕ gives a coordinate system in a possible smaller neighborhood of x . The overlap mappings are pseudo-holomorphic mappings with respect to J_0 , so they satisfy the Cauchy-Riemann equations, and are therefore holomorphic. The components of ϕ are functions $f^j, j = 1 \dots n$ with linearly independent differentials, and conversely, $\phi = (f^1, \dots, f^n)$ is a local coordinate system. □

Proposition 8.19. *A real 2-dimensional manifold admits an almost complex structure if and only if it is oriented.*

Proof. We have already proved the forward direction. Let M^2 be any oriented surface, and choose any Riemannian metric g on M . Then $*$: $\Lambda^1 \rightarrow \Lambda^1$ satisfies $*^2 = -Id$, and using the metric to identify $\Lambda^1 \cong TM$, we obtain an endomorphism $J : TM \rightarrow TM$ satisfying $J^2 = -Id$, which is an almost complex structure. □

Remark 8.20. In this case, any such J is necessarily a complex structure. This is equivalent to the problem of existence of isothermal coordinates, we will prove this soon.

9 Lecture 9

9.1 The Nijenhuis tensor

When does an almost complex structure arise from a true complex structure? To answer this question, we define the following tensor associated to an almost complex structure.

Proposition 9.1. *The Nijenhuis tensor of an almost complex structure defined by*

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \quad (9.1)$$

is in $\Gamma(T^*M \otimes T^*M \otimes TM)$ and satisfies

$$N(Y, X) = -N(X, Y), \quad (9.2)$$

$$N(JX, JY) = -N(X, Y), \quad (9.3)$$

$$N(X, JY) = N(JX, Y) = -J(N(X, Y)). \quad (9.4)$$

Proof. Given a function $f : M \rightarrow \mathbb{R}$, we compute

$$\begin{aligned} N(fX, Y) &= 2\{[J(fX), JY] - [fX, Y] - J[fX, JY] - J[J(fX), Y]\} \\ &= 2\{[fJX, JY] - [fX, Y] - J[fX, JY] - J[fJX, Y]\} \\ &= 2\{f[JX, JY] - (JY(f))JX - f[X, Y] + (Yf)X \\ &\quad - J(f[X, JY] - (JY(f))X) - J(f[JX, Y] - (Yf)JX)\} \\ &= fN(X, Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X\}. \end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X, fY) = fN(X, Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y (9.2) is obvious, and (9.3) follows easily using $J^2 = -Id$. For (9.4)

$$N(X, JY) = -N(JX, J^2Y) = N(JX, Y), \quad (9.5)$$

and

$$\begin{aligned} N(X, JY) &= 2\{[JX, J^2Y] - [X, JY] - J[X, J^2Y] - J[JX, JY]\} \\ &= 2\{-[JX, Y] - [X, JY] + J[X, Y] - J[JX, JY]\} \\ &= 2J\{J[JX, Y] + J[X, JY] + [X, Y] - [JX, JY]\} \\ &= -2J\{N(X, Y)\}. \end{aligned} \quad (9.6)$$

□

Proposition 9.2. *For a C^1 almost complex structure J ,*

$$N_J \in \Gamma\left(\{(\Lambda^{2,0} \otimes T^{0,1}) \oplus (\Lambda^{0,2} \otimes T^{1,0})\}_{\mathbb{R}}\right). \quad (9.7)$$

Consequently, if $\dim(M) = 2n$, then the Nijenhuis tensor has $n^2(n-1)$ independent real components. In particular, if $n = 1$, then $N_J \equiv 0$.

Proof. If we complexify, just using (9.2), we have

$$\begin{aligned} N_J &\in \Gamma((\Lambda^2 \otimes TM) \otimes \mathbb{C}) \\ &= \Gamma\left((\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}) \otimes (T^{1,0} \oplus T^{0,1})\right). \end{aligned} \quad (9.8)$$

But (9.3) says that the $\Lambda^{1,1}$ component vanishes. So we have

$$N_J \in \Gamma\left((\Lambda^{2,0} \oplus \Lambda^{0,2}) \otimes (T^{1,0} \oplus T^{0,1})\right). \quad (9.9)$$

Using (9.4), for $X', Y' \in \Gamma(TM)$, we have

$$\begin{aligned} N_J(X' - iJX', Y' - iJY') & \\ &= N_J(X', Y') - N_J(JX', JY') - iN_J(JX', Y') - iN_J(X', JY') \\ &= N_J(X', Y') + N_J(X', Y') + iJN_J(X', Y') + iJN_J(X', Y') \\ &= 2N_J(X', Y') + 2iJN_J(X', Y'), \end{aligned} \quad (9.10)$$

which lies in $T^{0,1}$. This shows that the $\Lambda^{2,0} \otimes T^{1,0}$ component vanishes, so the $\Lambda^{0,2} \otimes T^{0,1}$ component also vanishes, and (9.7) follows since N_J is a real tensor. \square

We have the following local formula for the Nijenhuis tensor.

Proposition 9.3. *In local coordinates, the Nijenhuis tensor is given by*

$$N_{jk}^i = 2 \sum_{h=1}^{2n} (J_j^h \partial_h J_k^i - J_k^h \partial_h J_j^i - J_h^i \partial_j J_k^h + J_h^i \partial_k J_j^h) \quad (9.11)$$

Proof. We compute

$$\begin{aligned} \frac{1}{2}N(\partial_j, \partial_k) &= [J\partial_j, J\partial_k] - [\partial_j, \partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j, \partial_k] \\ &= [J_j^l \partial_l, J_k^m \partial_m] - [\partial_j, \partial_k] - J[\partial_j, J_k^l \partial_l] - J[J_j^l \partial_l, \partial_k] \\ &= I + II + III + IV. \end{aligned}$$

The first term is

$$\begin{aligned} I &= J_j^l \partial_l (J_k^m \partial_m) - J_k^m \partial_m (J_j^l \partial_l) \\ &= J_j^l (\partial_l J_k^m) \partial_m + J_j^l J_k^m \partial_l \partial_m - J_k^m (\partial_m J_j^l) \partial_l - J_k^m J_j^l \partial_m \partial_l \\ &= J_j^l (\partial_l J_k^m) \partial_m - J_k^m (\partial_m J_j^l) \partial_l. \end{aligned}$$

The second term is obviously zero. The third term is

$$III = -J(\partial_j(J_k^l) \partial_l) = -\partial_j(J_k^l) J_l^m \partial_m. \quad (9.12)$$

Finally, the fourth term is

$$IV = \partial_k(J_j^l) J_l^m \partial_m. \quad (9.13)$$

Combining these, we are done. \square

Definition 9.4. If J is an almost complex structure of class C^1 satisfying $N_J \equiv 0$, then we say that J is *integrable*.

Corollary 9.5. If (M, J) arises from a complex structure, then J is integrable.

Proof. In local holomorphic coordinates $J = J_0$ is a constant tensor, and $N(J) = 0$ follows from Proposition 9.3. \square

Next, we have an alternative characterization of the vanishing of the Nijenhuis tensor.

Proposition 9.6. For an almost complex structure J the Nijenhuis tensor $N(J) = 0$ if and only if for any 2 vector fields $X, Y \in \Gamma(T^{1,0})$, their Lie bracket $[X, Y] \in \Gamma(T^{1,0})$.

Proof. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y' . The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \quad (9.14)$$

But this is also a $(1, 0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \quad (9.15)$$

applying J , and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \quad (9.16)$$

which is exactly the vanishing of the Nijenhuis tensor. \square

9.2 The operators ∂ and $\bar{\partial}$

Recall that on any almost complex manifold (M, J) , we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \quad (9.17)$$

We define $\Omega^k, \Omega_{\mathbb{C}}^k, \Omega^{p,q}$ to be the space of sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. The real operator $d : \Omega_{\mathbb{R}}^k \rightarrow \Omega_{\mathbb{R}}^{k+1}$, extends to an operator

$$d : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1} \quad (9.18)$$

by complexification.

Proposition 9.7. For a C^1 almost complex structure J

$$d(\Omega^{p,q}) \subset \Omega^{p+2,q-1} \oplus \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p-1,q+2}, \quad (9.19)$$

and $N_J = 0$ if and only if

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}. \quad (9.20)$$

if and only if

$$d(\Omega^{1,0}) \subset \Omega^{2,0} \oplus \Omega^{1,1} \quad (9.21)$$

if and only if

$$d(\Omega^{0,1}) \subset \Omega^{1,1} \oplus \Omega^{0,2} \quad (9.22)$$

Proof. Let $\alpha \in \Omega^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) &= \sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &+ \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \quad (9.23)$$

This is easily seen to vanish if more than $p + 2$ of the X_j are of type $(1, 0)$ or if more than $q + 2$ are of type $(0, 1)$, and (9.19) follows.

Next, assume that (9.22) is satisfied. Let $\alpha \in \Omega^{0,1}$, then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \quad (9.24)$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X, Y]$. Proposition 9.6 implies that $N(J) \equiv 0$. Conversely, if $N(J) \equiv 0$, then we can reverse the steps in this argument to obtain (9.22). Equation (9.21) is just the conjugate of (9.22).

Recall that if $\alpha \in \Omega^k$ and $\beta \in \Omega^l$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \quad (9.25)$$

The formula (9.20) then follows from this. \square

If $N_J = 0$, we can therefore define operators

$$\partial : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1} \quad (9.26)$$

$$\bar{\partial} : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1} \quad (9.27)$$

using (9.17) and

$$\partial|_{\Omega^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \quad (9.28)$$

$$\bar{\partial}|_{\Omega^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \quad (9.29)$$

Corollary 9.8. *For a C^1 almost complex structure J with $N_J = 0$, $d = \partial + \bar{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0. \quad (9.30)$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2. \quad (9.31)$$

If we plug in a form of type (p, q) the first term is of type $(p + 2, q)$, the middle terms are of type $(p + 1, q + 1)$, and the last term is of type $(p, q + 2)$. Since (9.17) is a direct sum, the claim follows. \square

10 Lecture 10

Recall that for $n = 1$, any almost complex structure J satisfies $N_J = 0$, so there is no integrability condition. Let's look at various forms of the equations.

10.1 Real form of the equations

We just look in an open set in real coordinates (x, y) , and then we have

$$J = \begin{pmatrix} a(x, y) & b(x, y) \\ c(x, y) & d(x, y) \end{pmatrix}. \quad (10.1)$$

The only condition is

$$-I = J^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{pmatrix} \quad (10.2)$$

If we assume that J is not too far from J_0 , then $b \sim -1$ and $c \sim 1$, so we must have

$$a + d = 0, \quad a^2 + bc = -1. \quad (10.3)$$

Note that since $b \sim -1$, we can solve $c = -(1 + a^2)/b$, but we won't need to do this now. So we just consider

$$J = \begin{pmatrix} a(x, y) & b(x, y) \\ c(x, y) & -a(x, y) \end{pmatrix}. \quad (10.4)$$

We want to find a pseudo-holomorphic mapping

$$\phi : (U, J) \rightarrow (\mathbb{C}, J_0) \quad (10.5)$$

which has non-vanishing Jacobian at 0. So we want to solve

$$\phi_* \circ J = J_0 \circ \phi_* \quad (10.6)$$

If we write

$$\phi(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}, \quad (10.7)$$

then the pseudoholomorphic condition is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \quad (10.8)$$

which yields the 4 equations

$$\begin{aligned} au_x + cu_y &= -v_x & bu_x - au_y &= -v_y \\ av_x + cv_y &= u_x & bv_x - av_y &= u_y \end{aligned} \quad (10.9)$$

This looks like 4 first-order equations for 2 unknown functions, so one wouldn't expect a solution. However, the first two equations imply the second two:

$$av_x + cv_y = a(-au_x - cu_y) + c(-bu_x + au_y) = (-a^2 - bc)u_x = u_x, \quad (10.10)$$

and

$$bv_x - av_y = b(-au_x - cu_y) + a(bu_x - au_y) = (-bc - a^2)u_y = u_y, \quad (10.11)$$

using the condition that $a^2 + bc = -1$.

Example 10.1. Let's now do an example. Consider

$$J = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix}. \quad (10.12)$$

We have

$$J^2 = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (10.13)$$

so this is indeed an almost complex structure.

From (10.9), the pseudoholomorphic equations are

$$2xu_x + (1 + 4x^2)u_y = -v_x \quad (10.14)$$

$$-u_x - 2xu_y = -v_y. \quad (10.15)$$

If a sufficiently smooth solution exists, then we have $v_{xy} = v_{yx}$, which yields

$$(2xu_x + (1 + 4x^2)u_y)_y = -(u_x + 2xu_y)_x \quad (10.16)$$

This can be rewritten as

$$u_{xx} + 4xu_{xy} + (1 + 4x^2)u_{yy} + 2u_y = 0. \quad (10.17)$$

By inspection, we find that $u = x$ is obviously a solution. We then return to the pseudoholomorphic equations, and find that

$$v_x = -2x, \quad v_y = 1, \quad (10.18)$$

so we can choose $v = -x^2 + y$. So our solution is $\phi = (u, v) = (x, y - x^2)$. The Jacobian at the origin is clearly non-degenerate, so we have found a holomorphic coordinate system. Note that the mapping $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$ is defined everywhere. It is injective: if we have $(x_1, y_1 - x_1^2) = (x_2, y_2 - x_2^2)$ then the first component says that $x_1 = x_2$ and the second component then implies that $y_1 = y_2$. It is also surjective: given any $(u, v) \in \mathbb{C}$, we let $x_2 = u$, and then we need to solve $y - u^2 = v$, which obviously has a solution $y = -u^2 + v$. Thus we have found that

$$\phi : (\mathbb{R}^2, J) \rightarrow (\mathbb{C}, J_0) \quad (10.19)$$

is a global biholomorphism! Note that any function of the form $f(x, y) = h(x + i(y - x^2))$, where h is a holomorphic function with respect to J_0 , is then holomorphic for J_0 , for example

$$f(x, y) = e^x(\cos(y - x^2) + i \sin(y - x^2)). \quad (10.20)$$

10.2 Complex form of the equations

In the basis $\{\partial/\partial x, \partial/\partial y\}$ we have J of the form

$$J = \begin{pmatrix} a(x, y) & b(x, y) \\ c(x, y) & -a(x, y) \end{pmatrix} \quad (10.21)$$

satisfying $a^2 + bc = -1$. Using (1.36) to change to the complex basis $\{\partial/\partial z, \partial/\partial \bar{z}\}$, then we have

$$J = \frac{1}{2} \begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix} \quad (10.22)$$

For a complex valued function w , the equation $\bar{\partial}_J w = 0$ is $\Pi_{\Lambda^{0,1}} dw = 0$, which is

$$\begin{aligned} 0 &= dw + iJdw = w_z dz + w_{\bar{z}} d\bar{z} + iJ(w_z dz + w_{\bar{z}} d\bar{z}) \\ &= w_z dz + w_{\bar{z}} d\bar{z} + iw_z Jdz + iw_{\bar{z}} Jd\bar{z}. \end{aligned} \quad (10.23)$$

Note that we need to use $J : \Lambda^1 \rightarrow \Lambda^1$ here, which is the transpose matrix of the above J . So we have

$$\begin{aligned} 0 &= w_z dz + w_{\bar{z}} d\bar{z} + \frac{i}{2} w_z (i(c-b)dz + (2a + i(b+c))d\bar{z}) + \frac{i}{2} w_{\bar{z}} ((2a - i(b+c))dz - i(c-b)d\bar{z}) \\ &= \left(w_z + \frac{1}{2}(b-c)w_z + \frac{1}{2}(2ai + b+c)w_{\bar{z}} \right) dz + \left(w_{\bar{z}} + \frac{1}{2}(2ai - b - c)w_z + \frac{1}{2}(c-b)w_{\bar{z}} \right) d\bar{z}. \end{aligned} \quad (10.24)$$

Let's look only at the second equation which is

$$\left(1 + \frac{1}{2}(c-b) \right) w_{\bar{z}} = -\frac{1}{2}(2ai - b - c)w_z. \quad (10.25)$$

If $b - c \neq 2$, which is certainly the case if J is close to J_0 , then the leading coefficient is non-zero, and we can divide to get

$$w_{\bar{z}} = -\frac{2ai - b - c}{2 + c - b} w_z \quad (10.26)$$

Note that the first equation is

$$\left(1 + \frac{1}{2}(b-c) \right) w_z = -\frac{1}{2}(2ai + b + c)w_{\bar{z}}. \quad (10.27)$$

If $2ai - b - c \neq 0$, then we can divide to get

$$w_{\bar{z}} = -\frac{2 + b - c}{2ai + b + c} w_z. \quad (10.28)$$

I claim these are the same equation. For this, we would need

$$\frac{2ai - b - c}{2 + c - b} = \frac{2 + b - c}{2ai + b + c}, \quad (10.29)$$

which yields

$$(2ai - b - c)(2ai + b + c) = (2 + c - b)(2 + b - c), \quad (10.30)$$

which is

$$-4a^2 - (b + c)^2 = 4 - (c - b)^2. \quad (10.31)$$

Expanding this out

$$-4a^2 - b^2 - 2bc - c^2 = 4 - c^2 + 2bc - b^2, \quad (10.32)$$

which is true since $a^2 + bc = -1$!

Definition 10.2. The equation

$$w_{\bar{z}} + \mu(z, \bar{z})w_z = 0 \quad (10.33)$$

is called the *Beltrami equation*.

10.3 Method of characteristics

This is a general method for solving linear PDE by solving nonlinear ODEs, we just explain for the Beltrami equation. Let's solve the nonlinear ODE

$$\frac{\partial z}{\partial s} = \mu(z, s), \quad z(0) = w. \quad (10.34)$$

The solution will depend on the independent variable s and the initial conditions w , call the solution $\Phi(s, w)$, and we write

$$z = \Phi(s, w). \quad (10.35)$$

By the implicit function theorem, we can write $w = w(z, s)$ in a neighborhood of $(s, w) = (0, 0)$, provided that $\frac{\partial \Phi}{\partial w}|_{0,0} \neq 0$. But this is

$$\frac{\partial \Phi}{\partial w} \Big|_{0,0} = \lim_{h \rightarrow 0} \frac{\Phi(0, h) - \Phi(0, 0)}{h} = 1. \quad (10.36)$$

So we have

$$z = \Phi(s, w(z, s)). \quad (10.37)$$

Taking the partial derivative of (10.37) with respect to z yields

$$1 = \frac{\partial \Phi}{\partial w} \frac{\partial w}{\partial z}. \quad (10.38)$$

Taking the partial derivative of (10.37) with respect to s yields

$$0 = \frac{\partial \Phi}{\partial s} + \frac{\partial \Phi}{\partial w} \frac{\partial w}{\partial s}, \quad (10.39)$$

which is

$$0 = \mu(z, s) + \left(\frac{\partial w}{\partial z}\right)^{-1} \frac{\partial w}{\partial s}, \quad (10.40)$$

which is the Beltrami equation upon letting $s = \bar{z}$.

Let's return to Example (10.1), and solve using this method. Recall

$$J = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \quad (10.41)$$

So we need to solve the Beltrami equation with

$$\mu = \frac{2ai - b - c}{2 + c - b} = \frac{4xi + 1 - 1 - 4x^2}{2 + 1 + 4x^2 + 1} = \frac{xi - x^2}{1 + x^2} = -\frac{x}{i + x}. \quad (10.42)$$

Since $x = (z + \bar{z})/2$, we have

$$\mu(z, \bar{z}) = -\frac{z + \bar{z}}{2i + z + \bar{z}}. \quad (10.43)$$

Let's solve the ODE

$$\frac{dz}{ds} = \mu(z, s), \quad z(0) = w. \quad (10.44)$$

For our example, this is

$$\frac{dz}{ds} = -\frac{z + s}{2i + z + s}. \quad (10.45)$$

To solve this, let's make a change of variables $p = z + s$. Then

$$\frac{dp}{ds} - 1 = -\frac{p}{2i + p}, \quad (10.46)$$

which gives

$$\frac{dp}{ds} = 1 - \frac{p}{2i + p} = \frac{2i}{2i + p}, \quad (10.47)$$

or

$$(2i + p)dp = 2ids, \quad (10.48)$$

which integrates to

$$2ip + \frac{1}{2}p^2 = 2is + C, \quad (10.49)$$

which is

$$2i(z + s) + \frac{1}{2}(z + s)^2 = 2is + C. \quad (10.50)$$

Our initial conditions are $z(0) = w$, so we get

$$2i(z + s) + \frac{1}{2}(z + s)^2 = 2is + 2iw + \frac{1}{2}w^2. \quad (10.51)$$

This is

$$w^2 + 4iw - 4iz - (z + s)^2 = 0 \quad (10.52)$$

Using the quadratic formula and letting $s = \bar{z}$ yields

$$w = -2i + \sqrt{-4 + 4iz + (z + \bar{z})^2}, \quad (10.53)$$

and we take the branch of the square root satisfying $\sqrt{-4} = 2i$. Note that this does not agree with the above method, but this is because the initial conditions are different. The above solution satisfies $w(z, 0) = z$, but the solution found in the previous section was

$$w = z - ix^2 = z - \frac{i}{4}(z + \bar{z})^2, \quad (10.54)$$

which satisfies $w(z, 0) = z - \frac{i}{4}z^2$.

11 Lecture 11

11.1 Another example

We solve another example using the method of characteristics. This example will be crucial in proving convergence in the analytic case, and is called a *Cauchy majorant*.

Proposition 11.1. *For $\rho > 0$, let*

$$\mu^* = \frac{C}{\left(1 - \frac{z}{\rho}\right)\left(1 - \frac{\bar{z}}{\rho}\right)}, \quad (11.1)$$

which is analytic in the polydisc $P(\rho) = \{(z, \bar{z}) \mid |z| < \rho, |\bar{z}| < \rho\}$. Then a solution of the Beltrami equation $w_{\bar{z}} + \mu(z, \bar{z})w_z = 0$ satisfying $w(z, 0) = z$ is given by

$$w^*(z, \bar{z}) = \rho - \sqrt{(\rho - z)^2 - 2C\rho^2 \log\left(1 - \frac{\bar{z}}{\rho}\right)}, \quad (11.2)$$

and is also analytic in $P(\rho)$, where we take the branch of the logarithm satisfying $\sqrt{(\rho - z)^2} = \rho - z$.

Proof. We solve the ODE

$$\frac{dz}{ds} = \frac{C}{\left(1 - \frac{z}{\rho}\right)\left(1 - \frac{s}{\rho}\right)}. \quad (11.3)$$

with initial condition $z(0) = w$. This is separable, so we have

$$\left(1 - \frac{z}{\rho}\right) dz = \frac{C}{\left(1 - \frac{s}{\rho}\right)} ds, \quad (11.4)$$

which we can integrate, and then let $s = \bar{z}$, to get

$$z - \frac{z^2}{2\rho} = -C\rho \log\left(1 - \frac{\bar{z}}{\rho}\right) + C'. \quad (11.5)$$

The initial conditions give

$$z - \frac{z^2}{2\rho} = -C\rho \log\left(1 - \frac{\bar{z}}{\rho}\right) + w - \frac{w^2}{2\rho}, \quad (11.6)$$

which is

$$w^2 - 2\rho w - z^2 + 2\rho z + 2C\rho^2 \log\left(1 - \frac{\bar{z}}{\rho}\right) = 0 \quad (11.7)$$

and we can use the quadratic formula to solve for w . □

11.2 Equivalence of J and μ

The following proposition gives another way to think about almost complex structures for $n = 1$.

Proposition 11.2. *If J is defined in an open set U which induces the standard orientation on U , then there exists a unique complex valued function $\mu : U \rightarrow B(0, 1) \subset \mathbb{C}$ so that*

$$T_J^{0,1} = \{v + \mu\bar{v} \mid v \in T_{J_0}^{0,1}\} \subset T_{\mathbb{C}}U. \quad (11.8)$$

Explicitly, if

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad (11.9)$$

with $a^2 + bc = -1$, then

$$\mu = \frac{2ai - b - c}{2 + c - b}. \quad (11.10)$$

Conversely, given a function $\mu : U \rightarrow B(0, 1) \subset \mathbb{C}$, writing $\mu = f + ig$, there is a uniquely determined almost complex structure J given by

$$J = \frac{1}{1 - f^2 - g^2} \begin{pmatrix} 2g & -(1 + f)^2 - g^2 \\ g^2 + (1 - f)^2 & -2g \end{pmatrix} \quad (11.11)$$

which has $T_J^{0,1}$ given by the above.

Proof. Given any such J , then we have previously defined

$$T_J^{0,1} = \{X \in T_{\mathbb{C}}U \mid JX = -iX\} = \{X' + iJX' \mid X \in T_{\mathbb{R}}U\}. \quad (11.12)$$

We next claim that the projection $\pi : T_J^{0,1} \rightarrow T_{J_0}^{0,1}$ is a complex linear isomorphism. These are two 1-dimensional complex subspaces of the 2-dimensional space $TU \otimes \mathbb{C}$, so there is a complex linear projection mapping, which is given by

$$X' + iJX' \mapsto X' + iJX' + iJ_0(X' + iJX') = (X' - J_0JX') + i(J + J_0)X'. \quad (11.13)$$

Since both spaces are 1-dimensional, and π is complex linear, it is an isomorphism provided it is not the zero map. Obviously, from (11.13), if $J \neq -J_0$ then it is not the zero mapping. We may therefore write $T_J^{0,1}$ as a graph over $T_{J_0}^{0,1}$. To do this, we compute like last time: using (1.36) to change to the complex basis $\{\partial/\partial z, \partial/\partial \bar{z}\}$, then we have

$$J = \frac{1}{2} \begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix}. \quad (11.14)$$

Then a basis for the 1-dimensional space $T_J^{0,1}$ is given by

$$\frac{\partial}{\partial \bar{z}} + iJ\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{\partial}{\partial \bar{z}} + \frac{i}{2} \left(i(b-c) \frac{\partial}{\partial \bar{z}} + (2a + i(b+c)) \frac{\partial}{\partial z} \right) \quad (11.15)$$

$$= \left(1 + \frac{c-b}{2} \right) \frac{\partial}{\partial \bar{z}} + \frac{1}{2} (2ai - b - c) \frac{\partial}{\partial z}. \quad (11.16)$$

From this, we find that

$$\mu = \frac{2ai - b - c}{2 + c - b}, \quad (11.17)$$

as claimed. Using $a^2 + bc = -1$, we compute

$$|\mu|^2 = \frac{4(-1 - bc) + (b+c)^2}{(2+c-b)^2} = \frac{2+b-c}{-2+b-c}. \quad (11.18)$$

To show that $|\mu| < 1$, we use the orientation condition. Notice that the condition $bc = -1 - a^2$ says that $bc < 0$, so there are 2 components to the set of almost complex structures, determined by the sign of b : if $b < 0$, then this is the component inducing the standard orientation. In this case, we have

$$\frac{2+b-c}{-2+b-c} < 1 \quad (11.19)$$

is equivalent to

$$2+b-c > -2+b-c, \quad (11.20)$$

which is obviously true.

Next, given any such function μ , we define

$$T_\mu^{0,1} = \text{span}\left\{\frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z}\right\}. \quad (11.21)$$

Define

$$T_\mu^{1,0} = \text{span}\left\{\frac{\partial}{\partial z} + \bar{\mu} \frac{\partial}{\partial \bar{z}}\right\}. \quad (11.22)$$

We claim that $T_\mu^{1,0} \cap T_\mu^{0,1} = \{0\}$. To see this, if the intersection was non-zero, then there would exist $\alpha \in \mathbb{C}$ so that

$$\frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z} = \alpha \left(\frac{\partial}{\partial z} + \bar{\mu} \frac{\partial}{\partial \bar{z}} \right). \quad (11.23)$$

This clearly implies that $\alpha = \mu$ and then $|\mu|^2 = 1$. But we have assumed that $|\mu| < 1$, so the claim follows. To find the corresponding almost complex structure J , we must have

$$\frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z} = X' + iJX', \quad (11.24)$$

for some real tangent vector X' . We then write the real and imaginary parts of the left hand side:

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + (f + ig) \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\ &= \frac{1}{2} \left((1+f) \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right) + \frac{i}{2} \left(g \frac{\partial}{\partial x} + (1-f) \frac{\partial}{\partial y} \right). \end{aligned} \quad (11.25)$$

So we must have

$$J \left((1+f) \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right) = g \frac{\partial}{\partial x} + (1-f) \frac{\partial}{\partial y}, \quad (11.26)$$

and since $J^2 = -Id$,

$$J \left(g \frac{\partial}{\partial x} + (1-f) \frac{\partial}{\partial y} \right) = - \left((1+f) \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right). \quad (11.27)$$

A simple change of basis computation shows that

$$J = \frac{1}{1-f^2-g^2} \begin{pmatrix} 2g & -(1+f)^2 - g^2 \\ g^2 + (1-f)^2 & -2g \end{pmatrix}. \quad (11.28)$$

□

This gives another way to understand the Beltrami equation. Given $\mu : U \rightarrow \mathbb{C}$ with $|\mu| < 1$, then since

$$T_\mu^{0,1} = \text{span}\left\{\frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z}\right\}, \quad (11.29)$$

a function $w : U \rightarrow \mathbb{C}$ is holomorphic if and only if

$$\left(\frac{\partial}{\partial \bar{z}} + \mu \frac{\partial}{\partial z}\right)w = 0, \quad (11.30)$$

or

$$w_{\bar{z}} + \mu w_z = 0, \quad (11.31)$$

which is exactly the Beltrami equation. We can just completely forget about the matrix version of J , and parametrize almost complex structures by a single function $\mu : U \rightarrow B(0, 1)$.

Remark 11.3. This proposition also shows us that the regularity of $J : U \rightarrow \text{GL}(2, \mathbb{R})$ is the same as the regularity of $\mu : U \rightarrow B(0, 1)$. That is, J is $C^{k, \alpha}, C^\infty, C^\omega$ if and only if μ is also.

Remark 11.4. The complex structures inducing the reversed orientation correspond $|\mu| > 1$ together with the point at infinity, which corresponds to the complex structure $-J_0$.

11.3 The Beltrami equation: analytic case

Now we consider the Beltrami equation

$$w_{\bar{z}} = \mu(z, \bar{z})w_z \quad (11.32)$$

Assuming μ is analytic, we have a convergent power series expansion

$$\mu(z, \bar{z}) = \sum_{j, k} \mu_{jk} z^j \bar{z}^k = \sum_{l=0}^{\infty} \sum_{j+k=l} \mu_{jk} z^j \bar{z}^k = \sum_{l=0}^{\infty} \mu_l. \quad (11.33)$$

Using Lemma 8.2, we can make the ACS standard at the origin, which implies that $\mu_0 = 0$, that is, μ has no constant term.

We also write

$$w = \sum_{j, k} w_{jk} z^j \bar{z}^k = \sum_{l=0}^{\infty} \sum_{j+k=l} w_{jk} z^j \bar{z}^k = \sum_{l=0}^{\infty} w_l. \quad (11.34)$$

We want to find a holomorphic coordinate system, so we make the assumption that $w_0 = 0$ and $w_1 = z$.

We then have

$$w_{\bar{z}} = \sum_{l=2}^{\infty} \partial_{\bar{z}} w_l \quad (11.35)$$

$$w_z = 1 + \sum_{l=2}^{\infty} \partial_z w_l. \quad (11.36)$$

We then want to solve

$$w_{\bar{z}} = \sum_{l=2}^{\infty} \partial_{\bar{z}} w_l = \mu w_z = \left(\sum_{l=1}^{\infty} \mu_l \right) \left(1 + \sum_{k=2}^{\infty} \partial_z w_k \right) = \left(\sum_{l=1}^{\infty} \mu_l \right) + \sum_{l=2}^{\infty} \sum_{j+k=l, j \geq 1, k \geq 2} \mu_j \partial_z w_k. \quad (11.37)$$

We then find the recursion relation

$$\partial_{\bar{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j \geq 1, k \geq 2} \mu_j \partial_z w_k. \quad (11.38)$$

Note that in the sum on the right hand side, we must have $k \leq l$, so this is indeed a recursion relation, provided that we can solve for w_{l+1} .

Fixing l , the right hand side is just a homogeneous polynomial of degree l in the variables z and \bar{z} . In general, if $f_l = \sum_{j+k=l, j \geq 0, k \geq 0} h_{jk} z^j \bar{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j \geq 0, k \geq 0} \frac{1}{k+1} h_{jk} z^j \bar{z}^{k+1} \quad (11.39)$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\bar{z}} F = f$.

Remark 11.5. Notice that our “inverse” of the $\bar{\partial}$ -operator on polynomials of degree l does not contain any terms proportional to z^{l+1} . Our inverse operator is unique with this condition. If we had not imposed this condition, one could have chosen $w_l = l!z^l + O(\bar{z})$, in which case our series would definitely not converge! Also, if we view our series as a power series in 2 complex variables, then formally $w(z, 0) = z$ exactly because of this choice of inverse to $\bar{\partial}$.

Proposition 11.6. *The coefficients $w_{j\bar{k}}$ for $j+k=l$ are a polynomial of degree $l-1$ in the $\mu_{p\bar{q}}$ for $p+q < l$ with all coefficients non-negative rational numbers.*

Proof. Let us examine the first few steps of the iteration. We have $w_{00} = 1$, $w_{10} = 1$, and $w_{0\bar{1}} = 0$. The term w_2 is determined by

$$\partial_{\bar{z}} w_2 = \mu_1 = \mu_{1\bar{0}} z + \mu_{0\bar{1}} \bar{z}, \quad (11.40)$$

so

$$w_2 = \mu_{1\bar{0}} z \bar{z} + \frac{1}{2} \mu_{0\bar{1}} \bar{z}^2, \quad (11.41)$$

so

$$w_{2\bar{0}} = 0, \quad w_{1\bar{1}} = \mu_{1\bar{0}}, \quad w_{0\bar{2}} = \frac{1}{2} \mu_{0\bar{1}}. \quad (11.42)$$

To illustrate, let's do one more step. The term w_3 is determined by

$$\begin{aligned} \partial_{\bar{z}} w_3 &= \mu_2 + \mu_1 \partial_z w_2 = \mu_{2\bar{0}} z^2 + \mu_{1\bar{1}} z \bar{z} + \mu_{0\bar{2}} \bar{z}^2 + (\mu_{1\bar{0}} z + \mu_{0\bar{1}} \bar{z})(\mu_{1\bar{0}} \bar{z}) \\ &= \mu_{2\bar{0}} z^2 + (\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z \bar{z} + (\mu_{0\bar{2}} + \mu_{0\bar{1}} \mu_{1\bar{0}}) \bar{z}^2. \end{aligned} \quad (11.43)$$

so

$$w_3 = \mu_{2\bar{0}}z^2\bar{z} + \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2)z\bar{z}^2 + \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}})\bar{z}^3. \quad (11.44)$$

so

$$w_{3\bar{0}} = 0, \quad w_{2\bar{0}} = \mu_{2\bar{0}}, \quad w_{1\bar{2}} = \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2), \quad w_{0\bar{3}} = \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}}), \quad (11.45)$$

and the claim is evidently true.

To do the general case, we prove by induction: assume the claim is true up to for $0, \dots, l$, and we prove for $l+1$. Recall that

$$\partial_{\bar{z}}w_{l+1} = \mu_l + \sum_{j+k=l+1, j \geq 1, k \geq 2} \mu_j \partial_z w_k. \quad (11.46)$$

By induction, the coefficients of w_k for $k \leq l$ are polynomials with non-negative coefficients in the $\mu_{p\bar{q}}$ with $p + \bar{q} < k < l$, so that $\partial_z w_k$ is also of this form. Then since

$$\mu_j = \sum_{k+l=j} \mu_{k\bar{l}} z^k \bar{z}^l, \quad (11.47)$$

any term $\mu_j \partial_z w_k$ is also a polynomial in the $\mu_{k\bar{l}}$ with non-negative coefficients.

To get w_{l+1} , recall that if $f_l = \sum_{j+k=l, j \geq 0, k \geq 0} h_{jk} z^j \bar{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j \geq 0, k \geq 0} \frac{1}{k+1} h_{jk} z^j \bar{z}^{k+1} \quad (11.48)$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\bar{z}}F = f$. Clearly, this preserves non-negativity of the coefficients, and we are done. \square

Theorem 11.7. *If $\mu(z, \bar{z})$ is analytic in the closed polydisc $|z| \leq \rho, |\bar{z}| \leq \rho$, there there exists a unique solution of the Beltrami equation*

$$w_{\bar{z}} = \mu(z, \bar{z})w_z \quad (11.49)$$

which is analytic in the polydisc $|z| < \rho, |\bar{z}| < \rho$ and satisfies the Cauchy data

$$w(z, 0) = z. \quad (11.50)$$

Proof. By assumption, the series

$$\mu = \sum_{j,k} \mu_{j\bar{k}} z^j \bar{z}^k \quad (11.51)$$

converges for any point (z, \bar{z}) with $|z| \leq \rho, |\bar{z}| \leq \rho$. So for any such point, there exists a constant C so that

$$|\mu_{j\bar{k}} z^j \bar{z}^k| < C, \quad (11.52)$$

which obviously implies that

$$|\mu_{j\bar{k}}| < C\rho^{-j-k}. \quad (11.53)$$

Then we define

$$\mu^*(z, \bar{z}) = \frac{C}{\left(1 - \frac{z}{\rho}\right)\left(1 - \frac{\bar{z}}{\rho}\right)}, \quad (11.54)$$

which is analytic in the polydisc $P(\rho) = \{(z, \bar{z}) \mid |z| < \rho, |\bar{z}| < \rho\}$. The power series of μ^* is given by

$$\mu^*(z, \bar{z}) = \sum_{j,k} C\rho^{-j-k} z^j \bar{z}^k, \quad (11.55)$$

so

$$\mu_{j\bar{k}}^* = C\rho^{-j-k}. \quad (11.56)$$

We therefore have

$$|\mu_{j\bar{k}}| \leq C\rho^{-j-k} = \mu_{j\bar{k}}^*. \quad (11.57)$$

Recall from Proposition 11.1 that the solution of the Beltrami equation for μ^* satisfying $w(z, 0) = z$ is given by

$$w^*(z, \bar{z}) = \rho - \sqrt{(\rho - z)^2 - 2C\rho^2 \log\left(1 - \frac{\bar{z}}{\rho}\right)}, \quad (11.58)$$

and is also analytic in $P(\rho)$, and has a power series expansion

$$w^*(z, \bar{z}) = \sum_{j,k} w_{j\bar{k}}^* z^j \bar{z}^k. \quad (11.59)$$

Recall that our formal power series solves

$$w_{j\bar{k}} = P_{j\bar{k}}(\mu_{**}), \quad (11.60)$$

where $P_{j\bar{k}}$ is a polynomial with positive coefficients depending only upon $\mu_{p\bar{q}}$ for $p+q < j+k$. Since w^* is an analytic solution of the Beltrami equation with μ^* , we must also have

$$w_{j\bar{k}}^* = P_{j\bar{k}}(\mu_{**}^*). \quad (11.61)$$

We then estimate

$$|w_{j\bar{k}}| = |P_{j\bar{k}}(\mu_{**})| \leq P_{j\bar{k}}(|\mu_{**}|) \leq P_{j\bar{k}}(\mu_{**}^*) = w_{j\bar{k}}^*. \quad (11.62)$$

The inequalities hold since $P_{j\bar{k}}$ is a polynomial with real non-negative coefficients, and using (11.57). This shows that our power series is majorized by the power series of w^* , which implies that the power series for w converges in the open polydisc $P(\rho)$. \square

12 Lecture 12

12.1 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the smooth case into the analytic case [?], [Nir73, Section I.4]. We want to change coordinates $\xi = \xi(z, \bar{z})$ so that such that our solution of the Beltrami equation in the z -coordinates

$$w_{\bar{z}} + \mu(z, \bar{z})w_z \quad (12.1)$$

transforms into another Beltrami equation,

$$W_{\bar{\xi}} + \tilde{U}(\xi, \bar{\xi})W_{\xi} = 0, \quad (12.2)$$

with \tilde{U} analytic. Note that we want a real change of coordinates, so if we write $\xi = \xi_1 + i\xi_2$, we need

$$\det \begin{pmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{pmatrix} (0, 0) \neq 0. \quad (12.3)$$

As we know, after a change of basis, this is

$$\det \begin{pmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial \bar{z}} \\ \frac{\partial \bar{\xi}}{\partial z} & \frac{\partial \bar{\xi}}{\partial \bar{z}} \end{pmatrix} (0, 0) = \left| \frac{\partial \xi}{\partial z}(0, 0) \right|^2 - \left| \frac{\partial \xi}{\partial \bar{z}}(0, 0) \right|^2 \neq 0. \quad (12.4)$$

Write

$$w(z, \bar{z}) = W(\xi(z, \bar{z}), \bar{\xi}(z, \bar{z})) \quad (12.5)$$

$$\mu(z, \bar{z}) = U(\xi(z, \bar{z}), \bar{\xi}(z, \bar{z})). \quad (12.6)$$

Then

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial W}{\partial \xi} \frac{\partial \xi}{\partial \bar{z}} + \frac{\partial W}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{z}} \quad (12.7)$$

$$\frac{\partial w}{\partial z} = \frac{\partial W}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial W}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial z}. \quad (12.8)$$

So the Beltrami equation becomes

$$\frac{\partial W}{\partial \xi} \frac{\partial \xi}{\partial \bar{z}} + \frac{\partial W}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{z}} = -U(\xi, \bar{\xi}) \left(\frac{\partial W}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial W}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial z} \right), \quad (12.9)$$

which we can write as

$$\frac{\partial W}{\partial \bar{\xi}} = - \left(\frac{\frac{\partial \xi}{\partial \bar{z}} + U(\xi, \bar{\xi}) \frac{\partial \xi}{\partial z}}{\frac{\partial \bar{\xi}}{\partial \bar{z}} + U(\xi, \bar{\xi}) \frac{\partial \bar{\xi}}{\partial z}} \right) \frac{\partial W}{\partial \xi}, \quad (12.10)$$

which is another Beltrami equation with a new right hand side

$$\tilde{U}(\xi, \bar{\xi}) = \frac{\xi_{\bar{z}} + U(\xi, \bar{\xi})\xi_z}{\bar{\xi}_{\bar{z}} + U(\xi, \bar{\xi})\bar{\xi}_z}. \quad (12.11)$$

Let us try to find the coordinates so that

$$\frac{\partial}{\partial \xi} \tilde{U}(\xi, \bar{\xi}) = 0. \quad (12.12)$$

Then the new \tilde{U} will be anti-holomorphic and therefore analytic by the Cauchy integral formula. From the chain rule, we have

$$\frac{\partial}{\partial \xi} = \frac{\partial z}{\partial \xi} \frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial \xi} \frac{\partial}{\partial \bar{z}}, \quad (12.13)$$

and we have

$$\begin{aligned} \frac{\partial}{\partial \xi} \tilde{U}(\xi, \bar{\xi}) &= \frac{\partial}{\partial \xi} \left(\frac{\xi_{\bar{z}} + U(\xi, \bar{\xi})\xi_z}{\bar{\xi}_{\bar{z}} + U(\xi, \bar{\xi})\bar{\xi}_z} \right) \\ &= \left(z_\xi \partial_z + \bar{z}_\xi \partial_{\bar{z}} \right) \left(\frac{\xi_{\bar{z}} + U(\xi, \bar{\xi})\xi_z}{\bar{\xi}_{\bar{z}} + U(\xi, \bar{\xi})\bar{\xi}_z} \right). \end{aligned} \quad (12.14)$$

By the inverse function theorem, we have

$$\begin{pmatrix} z_\xi & z_{\bar{\xi}} \\ \bar{z}_\xi & \bar{z}_{\bar{\xi}} \end{pmatrix} = \begin{pmatrix} \xi_z & \xi_{\bar{z}} \\ \bar{\xi}_z & \bar{\xi}_{\bar{z}} \end{pmatrix}^{-1} = \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \begin{pmatrix} \bar{\xi}_{\bar{z}} & -\xi_{\bar{z}} \\ -\bar{\xi}_z & \xi_z \end{pmatrix}, \quad (12.15)$$

so

$$z_\xi = \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \bar{\xi}_{\bar{z}} \quad (12.16)$$

$$\bar{z}_\xi = \frac{-1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \bar{\xi}_z. \quad (12.17)$$

We therefore have

$$\frac{\partial}{\partial \xi} \tilde{U}(\xi, \bar{\xi}) = \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} (\bar{\xi}_{\bar{z}} \partial_z - \bar{\xi}_z \partial_{\bar{z}}) \left(\frac{\xi_{\bar{z}} + U(\xi, \bar{\xi})\xi_z}{\bar{\xi}_{\bar{z}} + U(\xi, \bar{\xi})\bar{\xi}_z} \right). \quad (12.18)$$

If we multiply through by the leading factor, we want to solve

$$0 = (\bar{\xi}_{\bar{z}} \partial_z - \bar{\xi}_z \partial_{\bar{z}}) \left(\frac{\xi_{\bar{z}} + U(\xi, \bar{\xi})\xi_z}{\bar{\xi}_{\bar{z}} + U(\xi, \bar{\xi})\bar{\xi}_z} \right) \quad (12.19)$$

but keep in mind that we need to find a solution with $|\xi_z|^2(0, 0) - |\xi_{\bar{z}}|^2(0, 0) \neq 0$. Converting the $U(\xi, \bar{\xi})$ term back to the (z, \bar{z}) coordinates, we have

$$0 = (\bar{\xi}_{\bar{z}} \partial_z - \bar{\xi}_z \partial_{\bar{z}}) \left(\frac{\xi_{\bar{z}} + \mu(z, \bar{z})\xi_z}{\bar{\xi}_{\bar{z}} + \mu(z, \bar{z})\bar{\xi}_z} \right). \quad (12.20)$$

The equation (12.20) is *quasilinear* of the form

$$F(D^2\xi, D\xi, \xi, z, \bar{z}) = 0. \quad (12.21)$$

Definition 12.1. The linearization of F at a function ξ is given by

$$F'_\xi(h) = \left. \frac{d}{dt} F(D^2(\xi + th), D(\xi + th), \xi + th, z, \bar{z}) \right|_{t=0}. \quad (12.22)$$

The linearization is too complicated to write down in general, but the following is all that we really need.

Proposition 12.2. *Assuming $\mu \in C^1$, then the linearization of F at $\xi = z$ is*

$$F'_z(h) = \partial_z \left(h_{\bar{z}} + \mu(h_z - \bar{h}_{\bar{z}} - \mu \bar{h}_z) \right) + \bar{h}_{\bar{z}} \mu_z - \bar{h}_z \mu_{\bar{z}}. \quad (12.23)$$

If $\mu(0, 0) = 0$, then we have

$$F'_z(h)(0, 0) = \frac{1}{4} \Delta h + c_1 h_z + c_2 \bar{h}_z + c_3 h_{\bar{z}} + c_4 \bar{h}_{\bar{z}}. \quad (12.24)$$

for some constants c_1, c_2, c_3, c_4 . If μ has sufficiently small $C^{1,\alpha}$ norm then F'_z is an elliptic operator with Hölder coefficients bounded in C^α .

Proof. We write out

$$F(D^2(\xi + th), D(\xi + th), \xi + th, z, \bar{z}) \quad (12.25)$$

$$= \left((\overline{\xi + th})_{\bar{z}} \partial_z - (\overline{\xi + th})_z \partial_{\bar{z}} \right) \left(\frac{(\xi + th)_{\bar{z}} + \mu(z, \bar{z})(\xi + th)_z}{(\xi + th)_{\bar{z}} + \mu(z, \bar{z})(\xi + th)_z} \right). \quad (12.26)$$

Letting $\xi = z$, this becomes

$$F(D^2(z + th), D(z + th), z + th, z, \bar{z}) \quad (12.27)$$

$$= \left((1 + t\bar{h}_{\bar{z}}) \partial_z - t\bar{h}_z \partial_{\bar{z}} \right) \left(\frac{th_{\bar{z}} + \mu(z, \bar{z})(1 + th_z)}{(1 + t\bar{h}_{\bar{z}}) + \mu(z, \bar{z})th_z} \right). \quad (12.28)$$

We also see that

$$F'_z(h) = \partial_z \left(h_{\bar{z}} + \mu(h_z - \bar{h}_{\bar{z}} - \mu \bar{h}_z) \right) + \bar{h}_{\bar{z}} \mu_z - \bar{h}_z \mu_{\bar{z}}. \quad (12.29)$$

Noting that

$$\frac{\partial^2 h}{\partial z \partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) h = \frac{1}{4} \Delta h, \quad (12.30)$$

the proposition follows from this. □

We next need the implicit function theorem in Banach spaces.

Lemma 12.3. *Let $\mathcal{F} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a C^1 -map between two Banach spaces such that $\mathcal{F}(x) = \mathcal{F}(0) + \mathcal{L}(x) + \mathcal{Q}(x)$, where the operator $\mathcal{L} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is linear and $\mathcal{Q}(0) = 0$. Assume that*

1. \mathcal{L} is an isomorphism with inverse T satisfying $\|T\| \leq C_1$,

2. there are constants $r > 0$ and $C_2 > 0$ with $r < \frac{1}{3C_1C_2}$ such that

- (a) $\|\mathcal{Q}(x) - \mathcal{Q}(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x - y\|_{\mathcal{B}_1}$ for all $x, y \in B_r(0) \subset \mathcal{B}_1$,
- (b) $\|\mathcal{F}(0)\|_{\mathcal{B}_2} \leq \frac{r}{3C_1}$.

Then there exists a unique solution to $\mathcal{F}(x) = 0$ in \mathcal{B}_1 such that

$$\|x\|_{\mathcal{B}_1} \leq 3C_1 \cdot \|\mathcal{F}(0)\|_{\mathcal{B}_2}. \quad (12.31)$$

Proof. Writing $x = Tf$, we can write the equation $\mathcal{F}(x) = 0$ as

$$\mathcal{F}(0) + f + \mathcal{Q}(Tf) = 0, \quad (12.32)$$

that is

$$f = -\mathcal{Q}(Tf) - \mathcal{F}(0). \quad (12.33)$$

So we would like to find a fixed point of the operator $S : \mathcal{B}_2 \rightarrow \mathcal{B}_2$ defined by

$$Sf = -\mathcal{Q}(Tf) - \mathcal{F}(0). \quad (12.34)$$

We next claim that under the assumptions, S is a contraction mapping from $B_{r/C_1}(0) \subset \mathcal{B}_2$. To see this, we compute

$$\begin{aligned} \|Sf_1 - Sf_2\|_{\mathcal{B}_2} &= \|\mathcal{Q}(Tf_1) - \mathcal{Q}(Tf_2)\|_{\mathcal{B}_2} \\ &\leq C_2(\|Tf_1\|_{\mathcal{B}_1} + \|Tf_2\|_{\mathcal{B}_1})(\|Tf_1 - Tf_2\|_{\mathcal{B}_1}) \\ &\leq C_2(2C_1r/C_1)C_1\|f_1 - f_2\|_{\mathcal{B}_2} \leq \frac{2}{3}(\|f_1 - f_2\|_{\mathcal{B}_2}). \end{aligned} \quad (12.35)$$

We then let $f_0 = 0$, and define $f_{j+1} = Sf_j$. If $n \geq m$, we have

$$\begin{aligned} \|f_n - f_m\|_{\mathcal{B}_2} &\leq \sum_{j=m+1}^n \|f_j - f_{j-1}\|_{\mathcal{B}_2} \\ &= \sum_{j=m+1}^n \|S^{j-1}f_1 - S^{j-1}f_0\|_{\mathcal{B}_2} \\ &\leq \sum_{j=m+1}^n \left(\frac{2}{3}\right)^{j-1} \|f_1 - f_0\|_{\mathcal{B}_2} \\ &\leq \frac{(2/3)^m}{1 - 2/3} \|f_1 - f_0\|_{\mathcal{B}_2}. \end{aligned} \quad (12.36)$$

For ϵ sufficiently small, we see that the right hand side limits to 0 as $m \rightarrow \infty$. This proves that the sequence f_j is a Cauchy sequence in the Banach space \mathcal{B}_2 , which therefore converges to a limit f_∞ . Since S is continuous, we therefore have

$$Sf_\infty = S \lim_{j \rightarrow \infty} f_j = \lim_{j \rightarrow \infty} Sf_j = \lim_{j \rightarrow \infty} f_{j+1} = f_\infty. \quad (12.37)$$

Take $m = 1$ in (12.36) to get

$$\|f_n - f_1\|_{\mathcal{B}_2} \leq 2\|f_1 - f_0\|_{\mathcal{B}_2} = 2\|f_1\|_{\mathcal{B}_2} \quad (12.38)$$

Letting $n \rightarrow \infty$ yields

$$\|f_\infty\|_{\mathcal{B}_2} - \|f_1\|_{\mathcal{B}_2} \leq \|f_\infty - f_1\|_{\mathcal{B}_2} \leq 2\|f_1\|_{\mathcal{B}_2}. \quad (12.39)$$

Then $x_\infty = Tf_\infty$ is a solution to $\mathcal{F}(x_\infty) = 0$ and

$$\|x_\infty\|_{\mathcal{B}_1} \leq C_1\|f_\infty\|_{\mathcal{B}_2} \leq 3C_1\|\mathcal{F}(0)\|_{\mathcal{B}_2} \quad (12.40)$$

which implies (12.31). If x is any solution satisfying (12.31), then letting $f = \mathcal{L}x$, we estimate

$$\|f\|_{\mathcal{B}_2} \leq \frac{1}{C_1}\|x\|_{\mathcal{B}_1} \leq \frac{1}{C_1}3C_1\frac{r}{3C_1} = \frac{r}{C_1} \quad (12.41)$$

and uniqueness then follows from (12.35). \square

Remark 12.4. In the above statment, \mathcal{Q} is a nonlinear operator. But it is easy to see that the result also holds if \mathcal{Q} is linear, but with sufficiently small operator norm satisfying $\|\mathcal{Q}\| < \|T\|$.

Back to our problem, we define

$$\mathcal{B}_1 = C_0^{2,\alpha}(B_1(0)) = \{u \in C^{2,\alpha}(B_1(0)) \mid u = 0 \text{ on } \partial B_1(0)\} \quad (12.42)$$

$$\mathcal{B}_2 = C^{0,\alpha}(B_1(0)) \quad (12.43)$$

$$\mathcal{F}(h) = F(z + h) \quad (12.44)$$

$$\mathcal{L}(h) = F'_z(h) \quad (12.45)$$

$$\mathcal{Q}(h) = F(z + h) - F(z) - F'_z(h) = \mathcal{F}(h) - \mathcal{F}(0) - \mathcal{L}(h). \quad (12.46)$$

By scaling the coordinates $z = \epsilon z'$, and letting $w'(z', \bar{z}') = w(\epsilon z', \epsilon \bar{z}')$, then we have

$$w'_{z'} = \mu(\epsilon z', \epsilon \bar{z}')w'_{z'}. \quad (12.47)$$

So by choosing ϵ small, we can assume that

$$\|\mu\|_{C^{1,\alpha}(B(0,1))} < C\epsilon^{1+\alpha}. \quad (12.48)$$

From Proposition 12.2 we see that \mathcal{L} is an elliptic operator. The mapping $T : C^{0,\alpha} \rightarrow C_0^{2,\alpha}$ is defined to be the unique solution to the Dirichlet problem

$$\mathcal{L}(Tf) = f \text{ in } B(0, 1), \quad Tf = 0 \text{ on } \partial B(0, 1). \quad (12.49)$$

From basic elliptic theory, there exists a constant C so that

$$\|Tf\|_{C^{2,\alpha}(B(0,1))} \leq C\|f\|_{C^{0,\alpha}(B(0,1))}. \quad (12.50)$$

This constant will be uniformly bounded for sufficiently small ϵ since the elliptic estimates only depend upon the $C^{0,\alpha}$ norm of the coefficients. (Note: since \mathcal{L} differs from the Laplacian by lower order terms, it actually suffices to just invert the Laplacian, but we invert \mathcal{L} here for simplicity; more on this later.) We need to estimate

$$\mathcal{Q}(h_2) - \mathcal{Q}(h_1) = F(z + h_2) - F'_z(h_2) - (F(z + h_1) - F'_z(h_1)). \quad (12.51)$$

Consider $f(t) = F(z + th_2 + (1-t)h_1)$. Since $F(z + h) = \tilde{F}(D^2h, Dh, h, z)$, where \tilde{F} is a C^1 function of these variables, then using the fundamental theorem of calculus

$$F(z + h_2) - F(z + h_1) = f(1) - f(0) = \int_0^1 f'(t) dt \quad (12.52)$$

We note that

$$\begin{aligned} f'(t) &= \frac{d}{ds} F(z + sh_2 + (1-s)h_1)|_{s=t} = \frac{d}{ds} F(z + th_1 + (1-t)h_2 + s(h_2 - h_1))|_{s=0} \\ &= F'_{z+th_1+(1-t)h_2}(h_2 - h_1). \end{aligned} \quad (12.53)$$

This gives the expression

$$\begin{aligned} \mathcal{Q}(h_2) - \mathcal{Q}(h_1) &= \left(\int_0^1 F'_{z+th_1+(1-t)h_2}(h_2 - h_1) dt \right) - F'_z(h_2 - h_1) \\ &= \left(\int_0^1 (F'_{z+th_1+(1-t)h_2} - F'_z) dt \right) (h_2 - h_1). \end{aligned} \quad (12.54)$$

We next claim that for any y and h , we have the estimate

$$\|(F'_{z+h} - F'_z)y\|_{C^0} \leq C\|h\|_{C^2}\|y\|_{C^2}. \quad (12.55)$$

To see this, note the linearized operator is of the form

$$\begin{aligned} F'_u(h) &= \frac{d}{dt} F(u + th)|_{t=0} \\ &= a^{ij}(D^2u, Du, u, z)D_{ij}h + b^i(D^2u, Du, u, z)D_ih + c(D^2u, Du, u, z)D_ih. \end{aligned} \quad (12.56)$$

If $\tilde{F}(D^2h, Dh, h, z)$ is C^2 as in the D^2h, Dh, h variables, and continuous in the z variable, then the coefficients a^{ij}, b^i, c are C^1 as functions of D^2u, Du, u , and we have for example

$$c(D^2(z+h), D(z+h), z+h, z) = c(D^2z, Dz, z, z) + O(|D^2h| + |Dh| + |h|), \quad (12.57)$$

so the estimate (12.55) follows. This implies the quadratic estimate for the C^0 -norm.

For the Hölder norm, similar to the above arguments, we see that any y and h , we have the estimate

$$\|(F'_{z+h} - F'_z)y\|_{C^\alpha} \leq C\|h\|_{C^{2,\alpha}}\|y\|_{C^{2,\alpha}}, \quad (12.58)$$

provided that \tilde{F} is $C^{2,\alpha}$ in the D^2h, Dh, h variables and Hölder continuous in the z variable. This finishes the quadratic estimate.

Also, by taking ϵ sufficiently small, we can always arrange so that condition (b) is satisfied. The implicit function theorem yields a solution h with

$$\|h\|_{C^{2,\alpha}(B(0,1))} = o(\epsilon), \quad (12.59)$$

as $\epsilon \rightarrow 0$. Obviously, we have

$$|h(0)| = o(\epsilon), \quad |\nabla h|(0) = o(\epsilon), \quad (12.60)$$

as $\epsilon \rightarrow 0$. Then if ϵ is sufficiently small, then condition (12.4) will also be satisfied.

Remark 12.5. The minimal regularity required in the above argument is $\mu \in C^{1,\alpha}$. One can actually get away with only assuming $\mu \in C^{0,\alpha}$, but one needs a different method to see this.

13 Lecture 13

13.1 Relation with isothermal coordinates

Proposition 13.1. *For $n = 1$, an oriented conformal structure $(M, [g])$ is equivalent to an almost complex structure $J : TM \rightarrow TM$.*

Proof. Choose any representative g of the conformal class $[g]$. The Hodge star operator on 1-forms is uniquely defined by

$$\alpha \wedge * \beta = g(\alpha, \beta) dV_g, \quad (13.1)$$

where dV_g is the oriented volume element. Then $* : T^*M \rightarrow T^*M$ satisfies $*^2 = -Id$, so is an almost complex structure, which is clearly conformally invariant, since if $\tilde{g} = fg$, then $dV_{\tilde{g}} = f dV_g$, and $\tilde{g}(\alpha, \beta) = f^{-1}g(\alpha, \beta)$.

Conversely, given an almost complex structure J , we know this determines an orientation. Choose any non-zero 2-form compatible with this orientation, and call it dV_g . Then (13.1) defines an inner product on 1-forms. It is positive definite because by Lemma 8.2, at any point, we can assume that J is standard. So there is a basis dx, dy of T_p^*M such that $Jdx = -dy$. The complex orientation is $dy \wedge dx$. Writing any form $\alpha = \alpha_1 dx + \alpha_2 dy$, we then have

$$\alpha \wedge * \alpha = (\alpha_1 dx + \alpha_2 dy) \wedge (-\alpha_1 dy + \alpha_2 dx) = (|\alpha_1|^2 + |\alpha_2|^2) dy \wedge dx. \quad (13.2)$$

Clearly, different choices of volume elements lead to conformally equivalent metrics. \square

We can write down explicitly the above in coordinates.

Proposition 13.2. *Given $g = g_{ij} dx^i \otimes dx^j$, then*

$$*_g = \frac{\pm 1}{\sqrt{\det(g)}} \begin{pmatrix} g_{12} & -g_{11} \\ g_{22} & -g_{12} \end{pmatrix}, \quad (13.3)$$

depending upon choice of orientation. Consequently, if $\tilde{g} = fg$, then $*_{\tilde{g}} = *g$. Conversely, given any

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad (13.4)$$

with $a^2 + bc = -1$, and a choice of volume form $f dx^1 \wedge dx^2$, we define a Riemannian metric up to scaling by

$$g = \pm f \begin{pmatrix} -b & a \\ a & c \end{pmatrix}, \quad (13.5)$$

for the sign choice which makes this positive definite.

Proof. We choose a coordinate system $\{x^1, x^2\}$, and write

$$g(\partial_i, \partial_j) = g_{ij}. \quad (13.6)$$

We then have

$$g(dx^i, dx^j) = g^{ij}, \quad (13.7)$$

where g^{ij} are the components of the inverse matrix of g_{ij} . Also, we have

$$dV_g = \sqrt{\det(g)} dx^1 \wedge dx^2. \quad (13.8)$$

In matrix form, we write

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}. \quad (13.9)$$

We write

$$*dx^1 = a_{11}dx^1 + a_{21}dx^2, \quad *dx^2 = a_{12}dx^1 + a_{22}dx^2, \quad (13.10)$$

which in matrix form is just

$$* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (13.11)$$

We then have

$$dx^1 \wedge *dx^1 = dx^1 \wedge (a_{11}dx^1 + a_{21}dx^2) = a_{21}dx^1 \wedge dx^2 \quad (13.12)$$

$$dx^1 \wedge *dx^2 = dx^1 \wedge (a_{12}dx^1 + a_{22}dx^2) = a_{22}dx^1 \wedge dx^2 \quad (13.13)$$

$$dx^2 \wedge *dx^1 = dx^2 \wedge (a_{11}dx^1 + a_{21}dx^2) = -a_{11}dx^1 \wedge dx^2 \quad (13.14)$$

$$dx^2 \wedge *dx^2 = dx^2 \wedge (a_{12}dx^1 + a_{22}dx^2) = -a_{12}dx^1 \wedge dx^2. \quad (13.15)$$

On the other hand, by definition of the Hodge star operator, these must be equal to

$$g(dx^1, dx^1)dV_g = g^{11}\sqrt{\det(g)}dx^1 \wedge dx^2 = \frac{g_{22}}{\sqrt{\det(g)}} \quad (13.16)$$

$$g(dx^1, dx^2)dV_g = g^{12}\sqrt{\det(g)}dx^1 \wedge dx^2 = -\frac{g_{12}}{\sqrt{\det(g)}} \quad (13.17)$$

$$g(dx^2, dx^1)dV_g = g^{21}\sqrt{\det(g)}dx^1 \wedge dx^2 = -\frac{g_{21}}{\sqrt{\det(g)}} \quad (13.18)$$

$$g(dx^2, dx^2)dV_g = g^{22}\sqrt{\det(g)}dx^1 \wedge dx^2 = \frac{g_{11}}{\sqrt{\det(g)}}. \quad (13.19)$$

Comparing these equations, we obtain

$$*_g = \frac{\pm 1}{\sqrt{\det(g)}} \begin{pmatrix} g_{21} & -g_{11} \\ g_{22} & -g_{12} \end{pmatrix}. \quad (13.20)$$

This expression is obviously conformally invariant.

Conversely, given J and a volume element $dV = f dx^1 \wedge dx^2$, we define an inner product by

$$g(\alpha, \beta)dV_g = \alpha \wedge J\beta. \quad (13.21)$$

We then have

$$g(dx^i, dx^j)dV_g = g^{ij}f dx^1 \wedge dx^2, \quad (13.22)$$

and by the above, we see that

$$g^{-1} = \frac{1}{f} \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}, \quad (13.23)$$

so then

$$g = f \begin{pmatrix} -b & a \\ a & c \end{pmatrix}. \quad (13.24)$$

□

Corollary 13.3. *The problem of isothermal coordinates for a Riemannian metric g is equivalent to solving the Beltrami equation for the almost complex structure determined by $*_g$. That is, solving the Beltrami equation in a neighborhood of a point is equivalent to finding a coordinate system $\phi : U \rightarrow \mathbb{R}^2$ so that*

$$\phi_*g = e^{\lambda(x,y)}(dx^2 + dy^2), \quad (13.25)$$

for some function $\lambda : \phi(U) \rightarrow \mathbb{R}$.

Proof. A solution of the Beltrami equation $w_{\bar{z}} = \mu(z, \bar{z})w_z$ is a holomorphic function. As long as $\partial_z w(p) \neq 0$, then we know that

$$w : (U, J) \rightarrow (\mathbb{C}, J_0) \quad (13.26)$$

is pseudoholomorphic. By the arguments above, w must be conformal and orientation preserving. But the conformal class determined by J_0 is the conformal class of the Euclidean metric, so we have isothermal coordinates.

Conversely, given an isothermal coordinate system $\phi : (U, [g]) \rightarrow (\mathbb{R}^2, [g_{Euc}])$. Then $[g]$ induces a unique J , and by the above, ϕ must be pseudoholomorphic with respect to J , so yields a solution of the Beltrami equation. \square

13.2 Reduction to harmonic functions

Proposition 13.4. *If (M^2, J) is a real 2-dimensional almost complex manifold with J of class C^2 , then J is a complex 1-manifold.*

Proof. As before, choose a compatible Riemannian metric g , and let $*$ be the Hodge star operator with respect to the almost complex orientation. Then on 1-forms, $J = *$. Given any point x in M , by Proposition 8.18, we need to find a function $f : U \rightarrow \mathbb{C}$ where U is a neighborhood of x satisfying $\bar{\partial}_J f = 0$ in U , and $\partial_J f(x) \neq 0$. This equation is

$$0 = \bar{\partial}_J f = df + iJdf = df + i * df. \quad (13.27)$$

Let us write $f = u + iv$, where u and v are real-valued. Then we need

$$0 = du + idv + iJ(du + idv) = (du - *dv) + i(dv + *du). \quad (13.28)$$

Note that applying the Hodge star to $du = *dv$, results in $dv = -*du$, so if we solve the single equation

$$du = *dv, \quad (13.29)$$

then $f = u + iv$ will be pseudo-holomorphic. Note that

$$\partial f = (du + *dv) + i(dv - *du), \quad (13.30)$$

so if $du(x) \neq 0$, then $\partial f(x) \neq 0$. To solve (13.29), we apply $*d$ to get

$$*d * dv = \delta dv = \Delta_g v = 0. \quad (13.31)$$

So if v is harmonic, then $*dv$ is closed, and by the Poincaré Lemma, we can solve $*dv = du$ in any simply-connected neighborhood U of x . To summarize, we have reduced the problem to finding a simply-connected neighborhood U of x , and a harmonic function $v : U \rightarrow \mathbb{R}$ with $dv(x) \neq 0$. \square

We will finish the proof in the next lecture.

14 Lecture 14

The remaining ingredient we need is the following.

Theorem 14.1 (Harmonic coordinates of Sabitov-Shefel (1976) DeTurck-Kazdan (1981)). *If (M^n, g) is any Riemannian manifold with g of class $C^{k,\alpha}$ for $k \geq 1$, and $p \in M$, then there exists a coordinate system (x_1, \dots, x_n) defined in some neighborhood of p such that $\Delta_g(x_j) = 0$ for $j = 1, \dots, n$, with x_i of class $C^{k+1,\alpha}$. If g is C^∞ then so are x_i . If g is real analytic, then so are x_i .*

Proof. We will prove the 2-dimensional case. The higher-dimensional case is identical. From above, we have that in local coordinates $\{x, y\}$,

$$* = \sqrt{\det(g)} \begin{pmatrix} -g^{21} & -g^{22} \\ g^{11} & g^{12} \end{pmatrix}. \quad (14.1)$$

Let us write $dv = v_1 dx + v_2 dy$, then

$$*dv = (-g^{21}v_1 - g^{22}v_2)\sqrt{\det(g)}dx + (g^{11}v_1 + g^{12}v_2)\sqrt{\det(g)}dy. \quad (14.2)$$

Then we have

$$*d* dv = \frac{1}{\sqrt{\det(g)}} \left(\partial_2((g^{21}v_1 + g^{22}v_2)\sqrt{\det(g)}) + \partial_1((g^{11}v_1 + g^{12}v_2)\sqrt{\det(g)}) \right) \quad (14.3)$$

In local coordinates, the Laplacian therefore has the form

$$\Delta v = \frac{1}{\sqrt{\det(g)}} \partial_i (g^{ij} u_j \sqrt{\det(g)}). \quad (14.4)$$

(This formula holds in any dimension). Expanding this out yields

$$\Delta v = g^{ij} \partial_i \partial_j u + (\partial_i g^{ij}) u_j + \partial_i (\log(\sqrt{\det(g)})) g^{ij} u_j. \quad (14.5)$$

Jacobi's formula for the determinant is

$$\frac{1}{2} g^{pq} \partial_i g_{pq} = \partial_i (\log(\sqrt{\det(g)})), \quad (14.6)$$

so we have

$$\Delta v = g^{ij} \partial_i \partial_j u + (\partial_i g^{ij}) u_j + \frac{1}{2} g^{pq} \partial_i g_{pq} g^{ij} u_j. \quad (14.7)$$

So we can expand

$$\Delta v = \Delta_0 u + Q(u), \quad (14.8)$$

where

$$Q(u) = a^{ij} \partial_i \partial_j u + b^j u_j \quad (14.9)$$

$$a^{ij} = g^{ij} - \delta^{ij} \quad (14.10)$$

$$b^j = \partial_i g^{ij} + \frac{1}{2} g^{pq} \partial_i g_{pq} g^{ij}. \quad (14.11)$$

Let us assume that $g \in C^{1,\alpha}(B(0,1))$. Using normal coordinates (which are OK under this regularity assumption: the geodesic equation has C^α coefficients), we have that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$. It follows that there exists a constant C so that

$$|g^{ij}(x) - \delta^{ij}| \leq C|x|^{1+\alpha} \quad (14.12)$$

$$|\partial_k g^{ij}(x)| \leq C|x|^\alpha. \quad (14.13)$$

Consider the mapping $\phi_\epsilon : B(0,1) \rightarrow B(0,\epsilon)$ defined by $\phi_\epsilon(x') = \epsilon x'$. Then

$$\phi_\epsilon^* g(x') = g_{ij}(\epsilon x') \epsilon^2 dx'_i \otimes dx'_j. \quad (14.14)$$

So the metrics $g_\epsilon = \epsilon^{-2} \phi_\epsilon^* g$ has components $(g_\epsilon)_{ij} = g_{ij}(\epsilon x')$ in the x' coordinates. We then have

$$|g_\epsilon^{ij}(x') - \delta^{ij}| \leq C\epsilon^{1+\alpha}|x'|^{1+\alpha} \quad (14.15)$$

$$|\partial_k g_\epsilon^{ij}(x')| \leq C\epsilon^{1+\alpha}|x'|^\alpha. \quad (14.16)$$

By assumption, there exists a constant C so that

$$|g_{ij}(x) - g_{ij}(y)| \leq C|x - y|^\alpha, \quad (14.17)$$

which implies that

$$|g'_{ij}(x') - g'_{ij}(y')| \leq C\epsilon^\alpha|x' - y'|^\alpha. \quad (14.18)$$

Also by assumption, there exists a constant C so that

$$|\partial_k g_{ij}(x) - \partial_k g_{ij}(y)| \leq C|x - y|^\alpha, \quad (14.19)$$

which implies that

$$|\partial_k g'_{ij}(x') - \partial_k g'_{ij}(y')| \leq \epsilon^{1+\alpha}|x' - y'|^\alpha, \quad (14.20)$$

Consequently, we have that

$$\|a_\epsilon^{ij}\|_{C^{1,\alpha}(B(0,1))} \leq C\epsilon^{1+\alpha} \quad (14.21)$$

$$\|b_\epsilon^{ij}\|_{C^{0,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \quad (14.22)$$

We then have that there exists a constant C so that

$$\begin{aligned} \|Q(f)\|_{C^{0,\alpha}(B(0,1))} &= \|a^{ij}\partial_i\partial_j f + b^j f_j\|_{C^{0,\alpha}(B(0,1))} \\ &\leq \|a^{ij}\|_{C^{0,\alpha}(B(0,1))} \cdot \|\partial_i\partial_j f\|_{C^{0,\alpha}(B(0,1))} + \|b^j\|_{C^{0,\alpha}(B(0,1))} \cdot \|f_j\|_{C^{0,\alpha}(B(0,1))} \\ &\leq C\epsilon^\alpha \|f\|_{C^{2,\alpha}(B(0,1))}. \end{aligned} \quad (14.23)$$

We define $F : C^{2,\alpha}(B(0,1)) \rightarrow C^{0,\alpha}(B(0,1))$ by

$$F(h) = \Delta_{g_\epsilon}(x + h), \quad (14.24)$$

which has the expansion

$$F(h) = F(0) + \Delta_0(h) + Q(h). \quad (14.25)$$

By Lemma 12.3 and Remark 12.4, there exists a harmonic function $x + h_\infty$, where $h_\infty \in C^{2,\alpha}(B(0,1))$, but we also need to verify that it has nonvanishing differential at the origin. From the implicit function theorem, we have

$$\|h_\infty\|_{C^{2,\alpha}(B(0,1))} \leq C\|F(0)\|_{C^{0,\alpha}(B(0,1))}. \quad (14.26)$$

We then estimate

$$\|F(0)\|_{C^{0,\alpha}(B(0,1))} = \|\Delta_{g_\epsilon} x\|_{C^{0,\alpha}(B(0,1))} \leq \|b^1\|_{C^{0,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \quad (14.27)$$

At the origin,

$$\partial_1(x + h_\epsilon)(0) = 1 + \partial_1 h_\epsilon(0), \quad (14.28)$$

and

$$|\partial_1 h_\epsilon(0)| \leq \|h_\epsilon\|_{C^1(B(0,1))} \leq \|h_\epsilon\|_{C^{2,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \quad (14.29)$$

So if ϵ is sufficiently small, $\partial_1(x + h_\epsilon)(0) \neq 0$, and we are done.

For higher regularity, we argue as follows. If $g \in C^{k,\alpha}$ then in particular $g \in C^{1,\alpha}$. By the above, we can find $C^{2,\alpha}$ harmonic coordinates $\{x_1, x_2\}$. We then write

$$0 = \Delta x_k = g^{ij} \partial_i \partial_j x_k + b^j. \quad (14.30)$$

That is

$$g^{ij} \partial_i \partial_j x_k = -b^j \in C^{k-1,\alpha}. \quad (14.31)$$

The left hand side is an elliptic operator with $C^{k,\alpha}$ coefficients, so by elliptic regularity arguments, $x_k \in C^{k+2,\alpha}$. If $g \in C^\infty$, the right hand side is also in C^∞ , so again by elliptic regularity we see that $x_k \in C^{k,\alpha}$ for any $k \geq 0$, so $x_k \in C^\infty$. For the real analytic case, there is a general result that solutions of elliptic equations with real analytic coefficients are real analytic. \square

Unfortunately, the trick in this subsection does not help us to solve the Newlander-Nirenberg problem in higher dimensions. However, the method in the previous section *can* be extended to the higher dimensional case, which we will discuss next quarter.

Remark 14.2. The above methods require that $\mu \in C^{1,\alpha}$. The Beltrami equation can be solved locally for $\mu \in C^{0,\alpha}$ by inverting the $\partial_{\bar{z}}$ operator using the Cauchy-Pompeiu formula. However, this is a bit technical so we will omit. There are many great references for this method, see for example [?], [?], [?], [?], [?].

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