Math 222C, Complex Variables and Geometry

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Contents

Introduction

This quarter will be about complex manifolds and Kähler geometry. One goal is to give a fairly self-contained proof of the Newlander-Nirenberg Theorem regarding existence of local holomorphic coordinates for any integrable almost complex structure. If time permits, we will then discuss various results regarding which manifolds can admit an almost complex structure (a topological condition) or moreover a complex structure (an analytic condition).

1 Lecture 1

1.1 Example of $\mathbb{R}^{2n} = \mathbb{C}^n$

Remark 1.1. For now we will denote $\sqrt{-1}$ by i. However, later we will not do this, because the letter i is sometimes used as an index.

We consider $\mathbb{R}^{2n}$ and denote the coordinates as $x^1, y^1, \ldots, x^n, y^n$. Letting $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, define complex one-forms

$$dz^j = dx^j + idy^j,$$
$$d\bar{z}^j = dx^j - idy^j,$$

and complex tangent vectors

$$\partial/\partial z^j = (1/2)\left(\partial/\partial x^j - i\partial/\partial y^j\right),$$
$$\partial/\partial \bar{z}^j = (1/2)\left(\partial/\partial x^j + i\partial/\partial y^j\right).$$

Note that

$$dz^j(\partial/\partial z^k) = d\bar{z}^j(\partial/\partial \bar{z}^k) = \delta^{jk},$$
$$dz^j(\partial/\partial \bar{z}^k) = d\bar{z}^j(\partial/\partial z^k) = 0.$$

The standard complex structure $J_0 : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ on $\mathbb{R}^{2n}$ is given by

$$J_0(\partial/\partial x^j) = \partial/\partial y^j, \quad J_0(\partial/\partial y^j) = -\partial/\partial x^j,$$

which in matrix form is written

$$J_0 = \operatorname{diag}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}. \tag{1.1}$$

Next, we complexify the tangent space $T \otimes \mathbb{C}$, and let

$$T^{(1,0)}(J_0) = \operatorname{span}\{\partial/\partial z^j, j = 1 \ldots n\} = \{X - iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{1.2}$$

be the i-eigenspace and

$$T^{(0,1)}(J_0) = \operatorname{span}\{\partial/\partial \bar{z}^j, j = 1 \ldots n\} = \{X + iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{1.3}$$

be the $-i$-eigenspace of J_0, so that

$$T \otimes \mathbb{C} = T^{(1,0)}(J_0) \oplus T^{(0,1)}(J_0). \tag{1.4}$$

The map J_0 also induces an endomorphism of 1-forms by

$$J_0(\omega)(v_1) = \omega(J_0 v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0(dx_j) = -dy_j, \quad J_0(dy_j) = +dx_j.$$

Then complexifying the cotangent space $T^* \otimes \mathbb{C}$, we have

$$\Lambda^{1,0}(J_0) = \text{span}\{dz^j, j = 1 \dots n\} = \{\alpha - iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.5}$$

is the i-eigenspace, and

$$\Lambda^{0,1}(J_0) = \text{span}\{d\bar{z}^j, j = 1 \dots n\} = \{\alpha + iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.6}$$

is the $-i$-eigenspace of J_0, and

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J_0) \oplus \Lambda^{0,1}(J_0). \tag{1.7}$$

We note that

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{1.8}$$

and similarly

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{1.9}$$

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{1.10}$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{1.11}$$

Note that we can characterize $\Lambda^{p,q}$ as those forms satisfying

$$\alpha(v_1, \dots, v_{p+q}) = 0, \tag{1.12}$$

if more than p if the v_j-s are in $T^{(1,0)}$ or if more than q of the v_j-s are in $T^{(0,1)}$.

Finally, we can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{1.13}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J is not a complex structure on the space $\Lambda^k_{\mathbb{C}}$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J-invariant.

1.2 Cauchy-Riemann equations

Let $f : \mathbb{C}^n \to \mathbb{C}^m$. Let the coordinates on $\mathbb{C}^n$ be given by

$$\{z^1, \dots z^n\} = \{x^1 + iy^1, \dots, x^n + iy^n\}, \tag{1.14}$$

and coordinates on $\mathbb{C}^m$ given by

$$\{w^1, \dots w^m\} = \{u^1 + iv^1, \dots, u^m + iv^m\} \tag{1.15}$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \text{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{1.16}$$
$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial u^1, \dots \partial/\partial u^m, \partial/\partial v^1, \dots \partial/\partial v^m\}. \tag{1.17}$$

Then the real Jacobian of

$$f = (f^1, \dots f^{2m}) = (u^1 \circ f, u^2 \circ f, \dots, v^{2m} \circ f). \tag{1.18}$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \dots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \dots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \dots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \tag{1.19}$$

Definition 1.2. A differentiable mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is pseudo-holomorphic if

$$f_* \circ J_{0,\mathbb{C}^n} = J_{0,\mathbb{C}^m} \circ f_*. \tag{1.20}$$

That is, the differential of f commutes with J_0.

We have the following characterization of pseudo-holomorphic maps.

Proposition 1.3. *A mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ is pseudo-holomorphic if and only if the Cauchy-Riemann equations are satisfied, that is, writing*

$$f(z^1, \dots z^m) = (f_1, \dots, f_n) = (u_1 + iv_1, \dots u_n + iv_n), \tag{1.21}$$

and $z^j = x^j + iy^j$, for each $j = 1 \dots n$, we have

$$\frac{\partial u_j}{\partial x^k} = \frac{\partial v_j}{\partial y^k} \qquad \frac{\partial u_j}{\partial y^k} = -\frac{\partial v_j}{\partial x^k}, \tag{1.22}$$

for each $k = 1 \dots m$, and these equations are equivalent to

$$\frac{\partial}{\partial \bar{z}^k} f_j = 0, \tag{1.23}$$

for each $j = 1 \dots n$ and each $k = 1 \dots m$

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{1.24}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{1.25}$$

which is exactly the Cauchy-Riemann equations. In the general case, rearrange the coordinates so that $(x^1, \dots, x^m, y^1, \dots, y^m)$ are the real coordinates on $\mathbb{R}^{2m}$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$, such that the complex structure J_0 is given by

$$J_0(\mathbb{R}^{2m}) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{1.26}$$

and similarly for $J_0(\mathbb{R}^{2n})$. Then the computation in matrix form is entirely analogous to the case of $m = n = 1$.

Finally, we compute

$$\frac{\partial}{\partial \bar{z}^k} f_j = \frac{1}{2}\Big(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\Big)(u_j + iv_j) \tag{1.27}$$

$$= \frac{1}{2}\Big\{\frac{\partial}{\partial x^k}u_j - \frac{\partial}{\partial y^k}v_j + i\Big(\frac{\partial}{\partial x^k}v_j + \frac{\partial}{\partial y^k}u_j\Big)\Big\}, \tag{1.28}$$

the vanishing of which again yields the Cauchy-Riemann equations. □

From now on, if f is a mapping satisfying the Cauchy-Riemann equations, we will just say that f is *holomorphic*.

For any differentiable f, the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^m). \tag{1.29}$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \mathrm{span}\{\partial/\partial z^1, \dots \partial/\partial z^n, \partial/\partial \overline{z}^1, \dots \partial/\partial \overline{z}^n\}, \tag{1.30}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \mathrm{span}\{\partial/\partial w^1, \dots \partial/\partial w^m, \partial/\partial \overline{w}^1, \dots \partial/\partial \overline{w}^m\}. \tag{1.31}$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \dots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \overline{z}^1} & \dots & \frac{\partial f^1}{\partial \overline{z}^n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \dots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \overline{z}^1} & \dots & \frac{\partial f^m}{\partial \overline{z}^n} \\ \frac{\partial \overline{f}^1}{\partial z^1} & \dots & \frac{\partial \overline{f}^1}{\partial z^n} & \frac{\partial \overline{f}^1}{\partial \overline{z}^1} & \dots & \frac{\partial \overline{f}^1}{\partial \overline{z}^n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial \overline{f}^m}{\partial z^1} & \dots & \frac{\partial \overline{f}^m}{\partial z^n} & \frac{\partial \overline{f}^m}{\partial \overline{z}^1} & \dots & \frac{\partial \overline{f}^m}{\partial \overline{z}^n}, \end{pmatrix} \tag{1.32}$$

where $(f^1, \ldots, f^m) = f$ now denotes the complex components of f. This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{1.33}$$

Notice that (1.32) is of the form

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{1.34}$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.35}$$

where $\mathbb{C}$-linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \tag{1.36}$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{1.37}$$

This corresponds to the embeddings

$$Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.38}$$

where the left $\mathbb{C}$-linear is with respect to J_0, via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{1.39}$$

Note that since the latter embedding is just a change of basis, if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB) \det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{1.40}$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(1,0)}. \tag{1.41}$$

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0, then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{1.42}$$

This corresponds to the embeddings

$$Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \tag{1.43}$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \tag{1.44}$$

We see that f is anti-holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(0,1)}. \tag{1.45}$$

Note that if f is antiholomorphic, then is it holomorphic with respect to the complex structure $-J_0$ on the domain (but still J_0 on the range).

Note that we can decompose $f_* = f_*^C + f_*^A$, where

$$f_*^C = \frac{1}{2}(f_* - Jf_*J) \tag{1.46}$$

$$f_*^A = \frac{1}{2}(f_* + Jf_*J), \tag{1.47}$$

and f_*^C is holomorphic, while f_*^A is anti-holomorphic. In block matrix form, this just says that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \tag{1.48}$$

2 Lecture 2

2.1 Almost complex manifolds

Definition 2.1. An *almost complex manifold* is a real manifold with an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$.

The following lemma shows that we can always take J to be standard at any *point*.

Lemma 2.2. *Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear mapping satisfying $J^2 = -Id$. Then there exists an invertible matrix A such that $A^{-1}JA = J_{Euc}$.*

Proof. For $X \in \mathbb{R}^{2n}$, define

$$(a + ib)X = aX + bJX. \tag{2.1}$$

Then $\mathbb{R}^{2n}$ becomes an n-dimensional complex vector space. Let $X_1, \ldots, X_n$ be a complex basis. Then $X_1, JX_1, \ldots, X_n, JX_n$ is a basis of $\mathbb{R}^{2n}$ as a real vector space, and J is obviously standard in this basis. □

Remark 2.3. The Newlander-Nirenberg Theorem deals with the following question: when can we make J standard in a *neighborhood* of a point? As we will see shortly, this cannot possibly be true for an arbitrary almost complex structure; there is an *integrability condition* which must be satisfied.

All of the linear algebra we discussed above in $\mathbb{C}^n$ can be done on an almost complex manifold (M, J). We can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \tag{2.2}$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \tag{2.3}$$

is the i-eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \tag{2.4}$$

is the $-i$-eigenspace of J.

The map J also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{2.5}$$

where

$$\Lambda^{1,0} = \{\alpha - iJ\alpha, \alpha \in T_p^*M\} \tag{2.6}$$

is the i-eigenspace of J, and

$$\Lambda^{0,1} = \{\alpha + iJ\alpha, \alpha \in T_p^*M\} \tag{2.7}$$

is the $-i$-eigenspace of J.

Next, we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} \tag{2.8}$$

decomposes as a direct sum.

Remark 2.4. This gives a necessary topological obstruction for existence of an almost complex structure: the bundle of complex k-forms must decompose into to a direct sum of subbundles as in (2.8).

We can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{2.9}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$. Note we can also extend J to k-forms by

$$J\alpha(X_1, \dots, X_k) = \alpha(JX_1, \dots, JX_k). \tag{2.10}$$

Exercise 2.5. Check that these two definitions of J on k-forms agree.

Definition 2.6. A triple (M, J, g) where J is an almost complex structure, and g is a Riemannian metric is *almost Hermitian* if

$$g(X, Y) = g(JX, JY) \tag{2.11}$$

for all $X, Y \in TM$. We also say that g is *compatible* with J.

Proposition 2.7. *Given a linear J with $J^2 = -Id$ on $\mathbb{R}^{2n}$, and a positive definite inner product g on $\mathbb{R}^{2n}$ which is compatible with J, there exist elements $\{X_1, \dots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{2.12}$$

is an ONB for $\mathbb{R}^{2n}$ with respect to g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{2.13}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of $\mathrm{span}\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. □

Definition 2.8. To an almost Hermitian structure (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y) \tag{2.14}$$

called the *Kähler form* or *fundamental 2-form*.

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{2.15}$$

Furthermore, since

$$\omega(JX, JY) = \omega(X, Y), \tag{2.16}$$

this form is a real form of type $(1, 1)$. That is, $\omega \in \Gamma(\Lambda^{1,1}_{\mathbb{R}})$, where $\Lambda^{1,1}_{\mathbb{R}} \subset \Lambda^{1,1}$ is the real subspace of elements satisfying $\overline{\omega} = \omega$.

Remark 2.9. Note that it wouldn't make sense to say a form is in $\Lambda^{2,0}_{\mathbb{R}}$ since conjugation maps $\Lambda^{2,0}$ to $\Lambda^{0,2}$. However, the real subspace $\{\Lambda^{2,0} \oplus \Lambda^{0,2}\}_{\mathbb{R}}$ makes sense. So for example we have

$$\Lambda^2 = \Lambda^{1,1}_{\mathbb{R}} \oplus \{\Lambda^{2,0} \oplus \Lambda^{0,2}\}_{\mathbb{R}}. \tag{2.17}$$

For example, this gives some topological obstruction for existence of an almost complex structure: the real bundle $\Lambda^2(M)$ must decompose into real sub-bundles of dimension n^2 and $n(n-1)$. Similarly, the decomposition (2.8) of $\Lambda^k \otimes \mathbb{C}$ implies a corresponding decomposition of the bundle

$$\Lambda^k = \begin{cases} \bigoplus_{p+q=k, p<q} \{\Lambda^{p,q} \oplus \Lambda^{q,p}\}_{\mathbb{R}} & k \text{ odd} \\ \Lambda_{\mathbb{R}}^{k/2,k/2} \oplus \left\{ \bigoplus_{p+q=k, p<q} \{\Lambda^{p,q} \oplus \Lambda^{q,p}\}_{\mathbb{R}} \right\} & k \text{ even} \end{cases}. \tag{2.18}$$

In Euclidean space $(\mathbb{R}^{2n}, J_0, g_{Euc})$, the fundamental 2-form is

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{2.19}$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{2.20}$$

Note that this defines an orientation on $\mathbb{C}^n$, which we will refer to as the natural orientation. Note also that

$$\omega_{Euc}^n = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{2.21}$$

Proposition 2.10. *If (M, J) is almost complex, then* $\dim(M)$ *is even and M is orientable.*

Proof. If M is of real dimension m, and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^m, \tag{2.22}$$

which implies that m is even. We will henceforth write $m = 2n$. Next, let g be any Riemannian metric on M. Then define

$$h(X, Y) = g(X, Y) + g(JX, JY). \tag{2.23}$$

Then $h(JX, JY) = h(X, Y)$ is J-invariant, so (M, J, h) is almost Hermitian. We then consider the fundamental 2-form

$$\omega(X, Y) = h(JX, Y). \tag{2.24}$$

This is a form of type $(1, 1)$, so $\omega^n \in \Lambda_{\mathbb{R}}^{n,n} \cong \Lambda_{\mathbb{R}}^{2n}$ is a top degree $2n$-form. It is nowhere-vanishing since at any point $x \in M$ by Proposition 2.7 we can assume that both $J_x = J_{Euc}$ and $g_x = g_{Euc}$, so $\omega^n(x) \neq 0$ by (2.21). Therefore, ω gives a globally defined orientation on M. $\square$

Example 2.11. For example, $\mathbb{RP}^n$ does not admit any almost complex structure, since it is non-orientable for n even.

Remark 2.12. This is a continuation of Remark 2.9. From the proof of Proposition 2.10, we see that any almost complex manifold admits a compatible Riemannian metric h, and therefore carries fundamental 2-form ω which is non-zero at every point. Thus (2.17) must always decompose further:

$$\Lambda^2 = (\mathbb{R}\omega) \oplus (\Lambda_0^{1,1})_{\mathbb{R}} \oplus \{\Lambda^{2,0} \oplus \Lambda^{0,2}\}_{\mathbb{R}}, \tag{2.25}$$

where $(\Lambda_0^{1,1})_{\mathbb{R}}$ is the space of *primitive* real $(1,1)$-forms which are the $(1,1)$- forms orthogonal to ω.

For $n = 2$ (real dimension 4), since M is orientable, the space of 2-forms decomposes into self-dual and anti-self-dual forms

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2 \tag{2.26}$$

where $\Lambda_\pm^2$ is the subspace of forms satisfying $*\omega = \pm\omega$, the ± 1 eigenspaces of the Hodge star operator with respect to the complex orientation. Since $\omega \in \Lambda_+^2$, comparing with (2.25), we must have that

$$\Lambda_+^2 = (\mathbb{R}\omega) \oplus \{\Lambda^{2,0} \oplus \Lambda^{0,2}\}_{\mathbb{R}} \tag{2.27}$$

$$\Lambda_-^2 = (\Lambda_0^{1,1})_{\mathbb{R}}. \tag{2.28}$$

These are a special case of the Hodge-Riemann bilinear relations.

Definition 2.13. A smooth mapping between $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is *pseudo-holomorphic* if

$$f_* \circ J_M = J_N \circ f_* \tag{2.29}$$

We have a useful characterization of pseudo-holomorphic mappings.

Proposition 2.14. *A mapping $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is pseudo-holomorphic if and only if*

$$f_*(T^{1,0}(M)) \subset T^{1,0}(N), \tag{2.30}$$

if and only if

$$f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M). \tag{2.31}$$

2.2 Complex manifolds

We next define a complex manifold.

Definition 2.15. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

Example 2.16. Since holomorphic mappings are orientation-preserving by (1.40), any complex manifold is necessarily orientable. For example, $\mathbb{RP}^n$ does not admit any complex structure. Note that we knew from Example 2.11 above that there is no almost complex structure.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 2.17. *In any coordinate chart, define* $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ *by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{2.32}$$

Then $J_\alpha = J_\beta$ *on* $U_\alpha \cap U_\beta$ *and therefore gives a globally defined almost complex structure* $J : TM \to TM$ *satisfying* $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{2.33}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{2.34}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{2.35}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

The next proposition follows from the above discussion on Cauchy-Riemann equations.

Proposition 2.18. *If* (M, J_M) *and* (N, J_N) *are complex manifolds, then* $f : M \to N$ *is pseudo-holomorphic if and only if is a holomorphic mapping in local holomorphic coordinate systems.*

Definition 2.19. An almost complex structure J is said to be a *complex structure* if J is induced from a collection of holomorphic coordinates on M.

Proposition 2.20. *An almost complex structure* J *is a complex structure if and only if for any* $x \in M$*, there is a neighborhood* U *of* x *and a pseudo-holomorphic mapping* $\phi : (U, J) \to (\mathbb{C}^n, J_0)$ *which has non-vanishing Jacobian at* x*. Equivalently, there exist* n *pseudo-holomorphic functions* $f^j : U \to \mathbb{C}, j = 1 \ldots n$*, with linearly independent differentials at* x*.*

Proof. By the inverse function theorem, ϕ gives a coordinate system in a possible smaller neighborhood of of x. The overlap mappings are pseudo-holomorphic mappings with respect to J_0, so they satisfy the Cauchy-Riemann equations, and are therefore holomorphic. The components of ϕ are functions $f^j, j = 1 \dots n$ with linearly independent differentials, and conversely, $\phi = (f^1, \dots, f^n)$ is a local coordinate system. □

Proposition 2.21. *A real* 2*-dimensional manifold admits an almost complex structure if and only if it is oriented.*

Proof. We have already proved the forward direction. Let M^2 be any oriented surface, and choose any Riemannian metric g on M. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric to identify $\Lambda^1 \cong TM$, we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure. □

Remark 2.22. In this case, any such J is necessarily a complex structure. This is equivalent to the problem of existence of isothermal coordinates, we will prove this soon.

3 Lecture 3

3.1 The Nijenhuis tensor

When does an almost complex structure arise from a true complex structure? To answer this question, we define the following tensor associated to an almost complex structure.

Proposition 3.1. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \tag{3.1}$$

is in $\Gamma(T^*M \otimes T^*M \otimes TM)$ *and satisfies*

$$N(Y, X) = -N(X, Y), \tag{3.2}$$
$$N(JX, JY) = -N(X, Y), \tag{3.3}$$
$$N(X, JY) = N(JX, Y) = -J(N(X, Y)). \tag{3.4}$$

Proof. Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned} N(fX, Y) &= 2\{[J(fX), JY] - [fX, Y] - J[fX, JY] - J[J(fX), Y]\} \\ &= 2\{[fJX, JY] - [fX, Y] - J[fX, JY] - J[fJX, Y]\} \\ &= 2\{f[JX, JY] - (JY(f))JX - f[X, Y] + (Yf)X \\ &\quad - J(f[X, JY] - (JY(f))X) - J(f[JX, Y] - (Yf)JX)\} \\ &= fN(X, Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X\}. \end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X, fY) = fN(X, Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y (3.2) is obvious, and (3.3) follows easily using $J^2 = -Id$. For (3.4)

$$N(X, JY) = -N(JX, J^2Y) = N(JX, Y), \tag{3.5}$$

and

$$\begin{aligned} N(X, JY) &= 2\{[JX, J^2Y] - [X, JY] - J[X, J^2Y] - J[JX, JY]\} \\ &= 2\{-[JX, Y] - [X, JY] + J[X, Y] - J[JX, JY]\} \\ &= 2J\{J[JX, Y] + J[X, JY] + [X, Y] - [JX, JY]\} \\ &= -2J\{N(X, Y)\}. \end{aligned} \tag{3.6}$$

□

Proposition 3.2. *For a C^1 almost complex structure J,*

$$N_J \in \Gamma\Big(\big\{\big(\Lambda^{2,0} \otimes T^{0,1}\big) \oplus \big(\Lambda^{0,2} \otimes T^{1,0}\big)\big\}_{\mathbb{R}}\Big). \tag{3.7}$$

Consequently, if $\dim(M) = 2n$, then the Nijenhuis tensor has $n^2(n-1)$ independent real components. In particular, if $n = 1$, then $N_J \equiv 0$.

Proof. If we complexify, just using (3.2), we have

$$\begin{aligned} N_J &\in \Gamma((\Lambda^2 \otimes TM) \otimes \mathbb{C})) \\ &= \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}\big) \otimes \big(T^{1,0} \oplus T^{0,1}\big)\Big). \end{aligned} \tag{3.8}$$

But (3.3) says that the $\Lambda^{1,1}$ component vanishes. So we have

$$N_J \in \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2}\big) \otimes \big(T^{1,0} \oplus T^{0,1}\big)\Big). \tag{3.9}$$

Using (3.4), for $X', Y' \in \Gamma(TM)$, we have

$$\begin{aligned} &N_J(X' - iJX', Y' - iJY') \\ &\quad = N_J(X', Y') - N_J(JX', JY') - iN_J(JX', Y') - iN_J(X', JY') \\ &\quad = N_J(X', Y') + N_J(X', Y') + iJN_J(X', Y') + iJN_J(X', Y') \\ &\quad = 2N_J(X', Y') + 2iJN_J(X', Y'), \end{aligned} \tag{3.10}$$

which lies in $T^{0,1}$. This shows that the $\Lambda^{2,0} \otimes T^{1,0}$ component vanishes, so the $\Lambda^{0,2} \otimes T^{0,1}$ component also vanishes, and (3.7) follows since N_J is a real tensor. □

We have the following local formula for the Nijenhuis tensor.

Proposition 3.3. *In local coordinates, the Nijenhuis tensor is given by*

$$N^i_{jk} = 2\sum_{h=1}^{2n}(J^h_j \partial_h J^i_k - J^h_k \partial_h J^i_j - J^i_h \partial_j J^h_k + J^i_h \partial_k J^h_j) \tag{3.11}$$

Proof. We compute

$$\begin{aligned} \frac{1}{2}N(\partial_j, \partial_k) &= [J\partial_j, J\partial_k] - [\partial_j, \partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j, \partial_k] \\ &= [J^l_j \partial_l, J^m_k \partial_m] - [\partial_j, \partial_k] - J[\partial_j, J^l_k \partial_l] - J[J^l_j \partial_l, \partial_k] \\ &= I + II + III + IV. \end{aligned}$$

The first term is

$$\begin{aligned} I &= J_j^l \partial_l (J_k^m \partial_m) - J_k^m \partial_m (J_j^l \partial_l) \\ &= J_j^l (\partial_l J_k^m) \partial_m + J_j^l J_k^m \partial_l \partial_m - J_k^m (\partial_m J_j^l) \partial_l - J_k^m J_j^l \partial_m \partial_l \\ &= J_j^l (\partial_l J_k^m) \partial_m - J_k^m (\partial_m J_j^l) \partial_l. \end{aligned}$$

The second term is obviously zero. The third term is

$$III = -J(\partial_j (J_k^l) \partial_l) = -\partial_j (J_k^l) J_l^m \partial_m. \tag{3.12}$$

Finally, the fourth term is

$$III = \partial_k (J_j^l) J_l^m \partial_m. \tag{3.13}$$

Combining these, we are done. □

Corollary 3.4. *If (M, J) is an integrable complex structure then $N(J) \equiv 0$.*

Proof. In local holomorphic coordinates $J = J_0$ is a constant tensor, and $N(J) = 0$ follows from Proposition 3.3. □

Next, we have an alternative characterization of the vanishing of the Nijenhuis tensor.

Proposition 3.5. *For an almost complex structure J the Nijenhius tensor $N(J) = 0$ if and only if for any 2 vector fields $X, Y \in \Gamma(T^{1,0})$, their Lie bracket $[X, Y] \in \Gamma(T^{1,0})$.*

Proof. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \tag{3.14}$$

But this is also a $(1, 0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \tag{3.15}$$

applying J, and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \tag{3.16}$$

which is exactly the vanishing of the Nijenhuis tensor. □

4 Lecture 4

4.1 The operators ∂ and $\overline{\partial}$

Recall that on any almost complex manifold (M, J), we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{4.1}$$

We define $\Omega^k, \Omega^k_{\mathbb{C}}, \Omega^{p,q}$ to be the space of sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. The real operator $d : \Omega^k_{\mathbb{R}} \to \Omega^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{4.2}$$

by complexification.

Proposition 4.1. *For a C^1 almost complex structure J*

$$d(\Omega^{p,q}) \subset \Omega^{p+2,q-1} \oplus \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p-1,q+2}, \tag{4.3}$$

and $N_J = 0$ if and only if

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}. \tag{4.4}$$

if and only if

$$d(\Omega^{1,0}) \subset \Omega^{2,0} \oplus \Omega^{1,1} \tag{4.5}$$

if and only if

$$d(\Omega^{0,1}) \subset \Omega^{1,1} \oplus \Omega^{0,2} \tag{4.6}$$

Proof. Let $\alpha \in \Omega^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) = &\sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{4.7}$$

This is easily seen to vanish if more than $p+2$ of the X_j are of type $(1,0)$ or if more than $q+2$ are of type $(0,1)$, and (4.3) follows.

Next, assume that (4.6) is satisfied. Let $\alpha \in \Omega^{0,1}$, then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{4.8}$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X, Y]$. Proposition 3.5 implies that $N(J) \equiv 0$. Conversely, if $N(J) \equiv 0$, then we can reverse the steps in this argument to obtain (4.6). Equation (4.5) is just the conjugate of (4.6).

Recall that if $\alpha \in \Omega^k$ and $\beta \in \Omega^l$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{4.9}$$

The formula (4.4) then follows from this. □

If $N_J = 0$, we can therefore define operators

$$\partial : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{4.10}$$

$$\overline{\partial} : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{4.11}$$

using (4.1) and

$$\partial|_{\Omega^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{4.12}$$

$$\overline{\partial}|_{\Omega^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{4.13}$$

Corollary 4.2. *For a C^1 almost complex structure J with $N_J = 0$, $d = \partial + \overline{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{4.14}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{4.15}$$

If we plug in a form of type (p,q) the first term is of type $(p+2,q)$, the middle terms are of type $(p+1,q+1)$, and the last term is of type $(p,q+2)$. Since (4.1) is a direct sum, the claim follows. □

This says that if $N(J) = 0$, for each $0 \leq p \leq n = \dim_{\mathbb{C}}(M)$, then we have a co-chain complex

$$\cdots \xrightarrow{\overline{\partial}} \Omega^{p,q-1}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M) \xrightarrow{\overline{\partial}} \cdots. \tag{4.16}$$

which terminates at $\Omega^{p,n}(M)$. Clearly we have that $Im(\overline{\partial}) \subset Ker(\overline{\partial})$, so we can define the following vector spaces.

Definition 4.3. For $0 \leq p,q \leq \dim_{\mathbb{C}}(M)$, the (p,q) Dolbeault cohomology group is

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\{\alpha \in \Lambda^{p,q}(M) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M))}. \tag{4.17}$$

The Dolbeault cohomology groups enjoy the following functorality properties.

Proposition 4.4. *Let $(X, J_X), (Y, J_Y), (Z, J_Z)$ be almost complex manifolds satisfying $N_{J_X} = 0, N_{J_Y} = 0$, and $N_{J_Z} = 0$. Let $f : (X, J_X) \to (Y, J_Y)$ be a C^1 mapping which is pseudo-holomorphic, that is*

$$f_* \circ J_X = J_Y \circ f_*. \tag{4.18}$$

Then there are induced mappings

$$f^* : H^{p,q}(Y) \to H^{p,q}(X). \tag{4.19}$$

If $g : (Y, J_Y) \to (Z, J_Z)$ is C^1 pseudoholmorphic, then so is $g \circ f : (X, J_X) \to (Z, J_Z)$ and

$$(g \circ f)^* = f^* \circ g^* : H^{p,q}(Z) \to H^{p,q}(X). \tag{4.20}$$

In particular, if f is a pseudo-biholomorphism (one-to-one, onto, with pseudo-holomorphic inverse), then the Dolbeault cohomologies of X and Y are isomorphic.

Proof. The equation (4.18) implies that

$$f^* : \Omega^{p,q}(Y) \to \Omega^{p,q}(X). \tag{4.21}$$

To see this, let $\alpha^{p,q} \in \Omega^{p,q}(Y)$, then for vectors $V_1, \ldots, V_{p+q}$ we have

$$f^*\alpha^{p,q}(V_1, \ldots, V_{p+q}) = \alpha^{p,q}(f_*V_1, \ldots, f_*V_{p+q}) \tag{4.22}$$

Note that if $V \in T^{1,0}(X)$, then $J_X V = iV$, so then

$$J_Y f_* V = f_* J_X V = f_* iV = i f_* V, \tag{4.23}$$

therefore $f_*V \in T^{1,0}(Y)$. Similarly, if $V \in T^{0,1}(X)$ then $f_*V \in T^{0,1}(Y)$. If more that p of the V_j are of type $(1,0)$ or more than q of the V_j are of type $(1,0)$, then the same is true for the f_*V_j, and the claim follows.

We also know that the exterior derivative commutes with pullback,

$$d_X \circ f^* = f^* \circ d_Y. \tag{4.24}$$

Since the Nijenhuis tensors vanish, this is

$$(\partial_X + \overline{\partial}_X) \circ f^* = f^* \circ (\partial_Y + \overline{\partial}_Y) \tag{4.25}$$

If we plug in $\alpha^{p,q} \in \Omega^{p,q}(Y)$, we have 2 equations

$$\partial_X \circ f^*\alpha^{p,q} = f^* \circ \partial_Y \alpha^{p,q} \tag{4.26}$$

$$\overline{\partial}_X \circ f^*\alpha^{p,q} = f^* \circ \overline{\partial}_Y \alpha^{p,q} \tag{4.27}$$

The second equation implies that f^* induces a well-defined mapping on cohomology $f^* : H^{p,q}(Y) \to H^{p,q}(X)$ by the following. If $[\alpha^{p,q}] \in H^{p,q}(Y)$ is represented by a form $\alpha^{p,q}$, such that $\overline{\partial}_Y \alpha^{p,q} = 0$, then we have

$$\overline{\partial}_X f^*\alpha^{p,q} = f^*\overline{\partial}_Y \alpha^{p,q} = f^*0 = 0, \tag{4.28}$$

so we can define $f^*[\alpha^{p,q}] = [f^*\alpha^{p,q}]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^{p,q} + \overline{\partial}_Y \beta^{p,q-1}) = f^*\alpha^{p,q} + f^*\overline{\partial}_Y \beta^{p,q-1} = f^*\alpha^{p,q} + \overline{\partial}_X f^*\beta^{p,q-1}, \tag{4.29}$$

so we have

$$[f^*(\alpha^{p,q} + \overline{\partial}_Y \beta^{p,q-1})] = [f^*\alpha^{p,q} + \overline{\partial}_X f^*\beta^{p,q-1}] = [f^*\alpha^{p,q}]. \tag{4.30}$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \tag{4.31}$$

holds on the level of forms. Finally, if f is a pseudo-biholomorphism, then f^{-1} exists and is pseudo-holomorphic, so we have

$$f \circ f^{-1} = id_Y, \quad f^{-1} \circ f = id_X, \tag{4.32}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^{p,q}(X)}, \quad (f^{-1})^* \circ f^* = id_{H^{p,q}(Y)}, \tag{4.33}$$

□

Definition 4.5. A form $\alpha \in \Omega^{p,0}$ is pseudo-holomorphic if $\overline{\partial}\alpha = 0$.

Remark 4.6. We only talk about forms of type $(p,0)$ being pseudo-holomorphic, we never call a (p,q)-form pseudo-holomorphic if $q > 0$. Also, we have (trivially)

$$H^{p,0}(M) = \{\alpha \in \Omega^{p,0} \mid \alpha \text{ is pseudo-holomorphic}\}. \tag{4.34}$$

4.2 On a complex manifold

If (M, J) is a complex manifold, then there exist coordinate systems around any point

$$(z^1, \dots, z^n) = (x^1 + iy^1, \dots, x^n + iy^n) \tag{4.35}$$

such that $T^{1,0}$ is spanned by

$$\frac{\partial}{\partial z^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big), \tag{4.36}$$

$T^{0,1}$ is spanned by

$$\frac{\partial}{\partial \bar{z}^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big), \tag{4.37}$$

$\Lambda^{1,0}$ is spanned by

$$dz^j \equiv dx^j + idy^j, \tag{4.38}$$

and $\Lambda^{0,1}$ is spanned by

$$d\bar{z}^j \equiv dx^j - idy^j, \tag{4.39}$$

for $j = 1 \dots n$. If α is a (p,q)-form, then locally we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{4.40}$$

where I and J are multi-indices of length p and q, respectively, and $\alpha_{I,J}$ are complex-valued functions. Using (1.33), we have the formula

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \Big) \wedge dz^I \wedge d\overline{z}^J. \tag{4.41}$$

It follows that

$$\partial \alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J. \tag{4.42}$$

$$\overline{\partial} \alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J. \tag{4.43}$$

The following proposition is immediate.

Proposition 4.7. *If (M, J) is a complex manifold, then $\alpha \in \Omega^{p,0}$ is pseudo-holomorphic if and only if it can locally be written as*

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{4.44}$$

where the α_I are holomorphic functions.

In this case, we say that α is *holomorphic*.

Remark 4.8. Note, that if we had only assumed that $N_J = 0$, this proposition wouldn't make sense because without the Newlander-Nirenberg Theorem, we would not yet know that $\Lambda^{p,0}$ has a local basis of holomorphic sections.

4.3 The operator d^c

For an almost complex structure J with $N_J = 0$, we know that

$$d = \partial + \overline{\partial}, \tag{4.45}$$

and

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{4.46}$$

We can write these complex operators in the form

$$\overline{\partial} = \frac{1}{2}(d - id^c), \ \partial = \frac{1}{2}(d + id^c). \tag{4.47}$$

for a *real* operator $d^c : \Omega^p \to \Omega^{p+1}$ given by

$$d^c = i(\overline{\partial} - \partial), \tag{4.48}$$

which satisfies

$$d^2 = 0, \ dd^c + d^c d = 0, \ (d^c)^2 = 0. \tag{4.49}$$

We next have an alternative formula for d^c. Recall that $J : TM \to TM$ induces a dual mapping $J : T^*M \to T^*M$, and we extended to $J : \Lambda^r_{\mathbb{C}} \to \Lambda^r_{\mathbb{C}}$ by

$$J\alpha^{p,q} = i^{p-q}\alpha^{p,q}, \tag{4.50}$$

for a α a form of type (p, q). Notice that if $\alpha^r \in \Lambda^r_{\mathbb{C}}$, then

$$J^2\alpha^r = w \cdot \alpha^r, \text{ where } w \cdot \alpha^r = (-1)^r\alpha^r, \tag{4.51}$$

since

$$J^2\alpha^{p,q} = i^{2(p-q)}\alpha^{p,q} = (-1)^{p-q}\alpha^{p,q} = (1)^{p-q+2q}\alpha^{p,q} = (-1)^{p+q}\alpha^{p,q}. \tag{4.52}$$

Proposition 4.9. *For $\alpha \in \Lambda^r$, we have*

$$d^c\alpha = (-1)^{r+1}JdJ\alpha. \tag{4.53}$$

We also have

$$dd^c = 2i\partial\overline{\partial} = (-1)^{r+1}dJdJ\alpha. \tag{4.54}$$

Proof. For $\alpha \in \Lambda^{p,q}$, $p + q = r$, we compute

$$\begin{aligned} JdJ\alpha = i^{p-q}Jd\alpha &= i^{p-q}J(\partial\alpha + \overline{\partial}\alpha) && (4.55)\\ &= i^{p-q}(i^{p+1-q}\partial\alpha + i^{p-q-1}\overline{\partial}\alpha) && (4.56)\\ &= i^{2(p-q)+1}\partial\alpha + i^{2(p-q)-1}\overline{\partial}\alpha && (4.57)\\ &= (-1)^{p+q}(i\partial\alpha - i\overline{\partial}\alpha) = (-1)^{r+1}d^c\alpha. && (4.58) \end{aligned}$$

For (4.54), using (4.46) we have

$$dd^c = (\partial + \overline{\partial})i(\overline{\partial} - \partial) = i(\partial\overline{\partial} + \overline{\partial}^2 - \partial^2 - \overline{\partial}\partial) = 2i\partial\overline{\partial}. \tag{4.59}$$

□

5 Lecture 5

Recall that for $n = 1$, any almost complex structure J satisfies $N_J = 0$, so there is no integrability condition. Let's look at various forms of the equations.

5.1 Real form of the equations

We just look in an open set in real coordinates (x, y), and then we have

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}. \tag{5.1}$$

The only condition is

$$-I = J^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} \tag{5.2}$$

If we assume that J is not too far from J_0, then $b \sim -1$ and $c \sim 1$, so we must have

$$a + d = 0, \quad a^2 + bc = -1. \tag{5.3}$$

Note that since $b \sim -1$, we can solve $c = -(1 + a^2)/b$, but we won't need to do this now. So we just consider

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix}. \tag{5.4}$$

We are desirous to find a pseudo-holomorphic mapping

$$\phi : (U, J) \to (\mathbb{C}, J_0) \tag{5.5}$$

which has non-vanishing Jacobian at 0. So we want to solve

$$\phi_* \circ J = J_0 \circ \phi_* \tag{5.6}$$

If we write

$$\phi(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}, \tag{5.7}$$

then the pseudoholomorphic condition is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{5.8}$$

which yields the 4 equations

$$\begin{matrix} au_x + cu_y = -v_x & bu_x - au_y = -v_y \\ av_x + cv_y = u_x & bv_x - av_y = u_y \end{matrix}. \tag{5.9}$$

This looks like 4 first-order equations for 2 unknown functions, so one wouldn't expect a solution. However, the first two equations imply the second two:

$$av_x + cv_y = a(-au_x - cu_y) + c(-bu_x + au_y) = (-a^2 - bc)u_x = u_x, \tag{5.10}$$

and

$$bv_x - av_y = b(-au_x - cu_y) + a(bu_x - au_y) = (-bc - a^2)u_y = u_y, \tag{5.11}$$

using the condition that $a^2 + bc = -1$.

Example 5.1. Let's now do an example (as was so passionately requested by one class participant). Consider

$$J = \begin{pmatrix} -2x & 1 \\ -1 - 4x^2 & 2x \end{pmatrix}. \tag{5.12}$$

We have

$$J^2 = \begin{pmatrix} -2x & 1 \\ -1 - 4x^2 & 2x \end{pmatrix} \begin{pmatrix} -2x & 1 \\ -1 - 4x^2 & 2x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.13}$$

so this is indeed an almost complex structure.

From (5.9), the pseudoholomorphic equations are

$$-2xu_x - (1 + 4x^2)u_y = -v_x \tag{5.14}$$

$$u_x + 2xu_y = -v_y. \tag{5.15}$$

If a sufficiently smooth solution exists, then we have $v_{xy} = v_{yx}$, which yields

$$(-2xu_x - (1 + 4x^2)u_y)_y = (u_x + 2xu_y)_x \tag{5.16}$$

This expands out to

$$u_{xx} + 4xu_{xy} + (1 + 4x^2)u_{yy} + 2u_y = 0. \tag{5.17}$$

By inspection, we find that $u = x$ is obviously a solution. We then return to the pseudo-holomorphic equations, and find that

$$v_x = 2x, \quad v_y = -1, \tag{5.18}$$

so we can choose $v = x^2 - y$. So our solution is $\phi = (u, v) = (x, -y + x^2)$. The Jacobian at the origin is clearly non-degenerate, so we have found a holomorphic coordinate system. Note that the mapping $\phi : \mathbb{R}^2 \to \mathbb{C}$ is defined everywhere. It is injective: if we have $(x_1, -y_1 + x_1^2) = (x_2, -y_2 + x_2^2)$ then the first component says that $x_1 = x_2$ and the second component then implies that $y_1 = y_2$. It is also surjective: given any $(u, v) \in \mathbb{C}$, we let $x_2 = u$, and then we need to solve $-y + u^2 = v$, which obviously has a solution $y = u^2 - v$. Thus we have found that

$$\phi : (\mathbb{R}^2, J) \to (\mathbb{C}, J_0) \tag{5.19}$$

is a global biholomorphism! Note that any function of the form $f(x, y) = h(x + i(-y + x^2))$, where h is a holomorphic function with respect to J_0, is then holomorphic for J_0, for example

$$f(x, y) = e^x(\cos(-y + x^2) + i\sin(-y + x^2)). \tag{5.20}$$

5.2 Another real method

Given J of the form

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix} \tag{5.21}$$

satisfying $a^2 + bc = -1$, let's first just find a matrix $M(x,y)$ such that

$$MJ = J_0 M \tag{5.22}$$

Given Lemma 2.2, this is easy to find: consider the basis

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = JX_1 = \begin{pmatrix} a \\ c \end{pmatrix} \tag{5.23}$$

shows that we could choose M to be

$$M = \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix} \tag{5.24}$$

To check, we have

$$MJ = \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & bc + a^2 = -1 \\ c & -a \end{pmatrix}. \tag{5.25}$$

The other side is

$$J_0 M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ c & -a \end{pmatrix}. \tag{5.26}$$

In order to find a coordinate system, we need to find $\phi = (u, v)$ such that

$$\phi_* J = J_0 \phi_* \tag{5.27}$$

We would need that

$$M = \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \tag{5.28}$$

But this cannot be satisfied in general because

$$u_x = c, \quad u_y = -a \tag{5.29}$$

would imply that

$$u_{xy} = c_y = -a_x = u_{yx}, \tag{5.30}$$

but a and c are arbitrary functions.

We need to examine the freedom in our construction. Our choice of matrix M is clearly not unique: we can multiply M on the left by any matrix of the form

$$A = \begin{pmatrix} f & -g \\ g & f \end{pmatrix}, \tag{5.31}$$

because we will then have

$$(AM)J = A(MJ) = AJ_0M = J_0(AM), \tag{5.32}$$

since the matrix A commutes with J_0.

Now we have a chance to make AM a Jacobian matrix, we need to solve

$$AM = \begin{pmatrix} f & -g \\ g & f \end{pmatrix} \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} cf & -af - g \\ cg & -ag + f \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \tag{5.33}$$

This results in the 4 equations

$$\begin{aligned} u_x = cf \quad u_y = -af - g \\ v_x = cg \quad v_y = -ag + f \end{aligned} \tag{5.34}$$

There is an obvious integrability condition:

$$(cf)_y = -(af + g)_x \tag{5.35}$$

$$(cg)_y = (-ag + f)_x. \tag{5.36}$$

But this is now just a first order system for the functions f and g.

Note that if we find a solution of this system, then

$$au_x + cu_y = acf + c(-af - g) = -c_g = -v_x, \tag{5.37}$$

and

$$bu_x - au_y = bcf + a(af + g) = (a^2 + bc)f + ag = -f + ag = -v_y, \tag{5.38}$$

So the pseudoholomorphic equations (5.9) are satisfied.

Exercise 5.2. For the J in Example 5.1, find holomorphic coordinates using this method instead of the previous.

5.3 Complex form of the equations

In the basis $\{\partial/\partial x, \partial/\partial y\}$ we have J of the form

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix} \tag{5.39}$$

satisfying $a^2 + bc = -1$. Using (1.36) to change to the complex basis $\{\partial/\partial z, \partial/\partial \overline{z}\}$, then we have

$$J = \frac{1}{2} \begin{pmatrix} i(c - b) & 2a + i(b + c) \\ 2a - i(b + c) & -i(c - b) \end{pmatrix} \tag{5.40}$$

For a complex valued function w, the equation $\overline{\partial}_J w = 0$ is $\Pi_{\Lambda^{0,1}} dw = 0$, which is

$$\begin{aligned} 0 &= dw + iJdw = w_z dz + w_{\overline{z}} d\overline{z} + iJ(w_z dz + w_{\overline{z}} d\overline{z}) \\ &= w_z dz + w_{\overline{z}} d\overline{z} + i w_z J dz + i w_{\overline{z}} J d\overline{z}. \end{aligned} \tag{5.41}$$

Note that we need to use $J : \Lambda^1 \to \Lambda^1$ here, which is the transpose matrix of the above J. So we have

$$\begin{aligned} 0 &= w_z dz + w_{\overline{z}} d\overline{z} + \frac{i}{2} w_z (i(c-b)dz + (2a + i(b+c))d\overline{z}) + \frac{i}{2} w_{\overline{z}} ((2a - i(b+c))dz - i(c-b)d\overline{z}) \\ &= \Big(w_z + \frac{1}{2}(b-c) w_z + \frac{1}{2}(2ai + b + c) w_{\overline{z}} \Big) dz + \Big(w_{\overline{z}} + \frac{1}{2}(2ai - b - c) w_z + \frac{1}{2}(c-b) w_{\overline{z}} \Big) d\overline{z}. \end{aligned} \tag{5.42}$$

Let's look only at the second equation which is

$$\Big(1 + \frac{1}{2}(c-b) \Big) w_{\overline{z}} = -\frac{1}{2}(2ai - b - c) w_z. \tag{5.43}$$

If $b - c \neq 2$, which is certainly the case if J is close to J_0, then the leading coefficient is non-zero, and we can divide to get

$$w_{\overline{z}} = -\frac{2ai - b - c}{2 + c - b} w_z \tag{5.44}$$

Note that the first equation is

$$\Big(1 + \frac{1}{2}(b-c) \Big) w_z = -\frac{1}{2}(2ai + b + c) w_{\overline{z}}. \tag{5.45}$$

If $2ai - b - c \neq 0$, then we can divide to get

$$w_{\overline{z}} = -\frac{2 + b - c}{2ai + b + c} w_z. \tag{5.46}$$

I claim these are the same equation. For this, we would need

$$\frac{2ai - b - c}{2 + c - b} = \frac{2 + b - c}{2ai + b + c}, \tag{5.47}$$

which yields

$$(2ai - b - c)(2ai + b + c) = (2 + c - b)(2 + b - c), \tag{5.48}$$

which is

$$-4a^2 - (b+c)^2 = 4 - (c-b)^2. \tag{5.49}$$

Expanding this out

$$-4a^2 - b^2 - 2bc - c^2 = 4 - c^2 + 2bc - b^2, \tag{5.50}$$

which is true since $a^2 + bc = -1$!

Definition 5.3. The equation

$$w_{\overline{z}} + \mu(z, \overline{z}) w_z = 0 \tag{5.51}$$

is called the *Beltrami equation*.

6 Lecture 6

6.1 Complex form of the equations

Today, we give an alternative derivation of the Beltrami equation. We begin with the following characterization of pseudo-holomorphic functions (which holds in any dimension).

Proposition 6.1. *Let (M, J) be almost complex. Then the following are equivalent.*

(i) $f : (M, J) \to (\mathbb{C}, J_0)$ *is pseudo-holomorphic.*

(ii) $\overline{\partial}_J f = 0$.

(iii) $Xf = 0$ *for all vector fields* $X \in \Gamma(T_J^{0,1})$.

Proof. Note that if we take any $X \in \Gamma(TM \otimes \mathbb{C})$

$$(df + iJdf)(X) = Xf + idf(JX) = Xf + iJXf = (X + iJX)f \tag{6.1}$$

so we always have

$$\overline{\partial}_J f(X) = (\Pi_{\Lambda_J^{0,1}} df)(X) = (\Pi_{T_J^{0,1}} X)f. \tag{6.2}$$

If condition (ii) is satisfied, then (iii) follows immediately from (6.2) Conversely, if condition (iii) is satisfied then taking $X \in \Gamma(T_J^{0,1})$ and using (6.2), then

$$(\Pi_{\Lambda_J^{0,1}} df)(X) = 0 \tag{6.3}$$

for all such X, which implies that condition (ii) is satisfied.

We next show that (ii) is equivalent to (i). If (ii) is satisfied, then

$$Jdf = idf. \tag{6.4}$$

Recall that if $u : M \to \mathbb{R}$ is a *real-valued* function, and $X \in TM$ is a real tangent vector, then there is a canonical identification

$$du(X) = u_*(X), \tag{6.5}$$

where the right hand side is interpreted as a real number. So then if $f = u + iv$ is a *complex-valued* function, then we have for $X \in TM$,

$$df(X) = u_*(X) + iv_*(X), \tag{6.6}$$

and then we extend this to complex vectors by complex linearity. We plug in a complex tangent vector to (6.4) to get

$$Jdf(X) = idf(X), \tag{6.7}$$

which is (using the definition of J on 1-forms as the transpose)

$$df(JX) = idf(X). \tag{6.8}$$

Using the above, we then write this as

$$(du + idv)(JX) = i(du + idv)(X) \tag{6.9}$$

which is

$$(u_* + iv_*)(JX) = i(u_* + iv_*)(X). \tag{6.10}$$

This yields the equations

$$u_*(JX) = -v_*(X), \quad v_*(JX) = u_*(X). \tag{6.11}$$

But as a real-valued function, we have

$$f = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{6.12}$$

so we can write f_* in the form

$$f_* = \begin{pmatrix} u_* \\ v_* \end{pmatrix}. \tag{6.13}$$

The equation $f_*J = J_0 f_*$ is then

$$\begin{pmatrix} u_* \\ v_* \end{pmatrix} J = \begin{pmatrix} u_*J \\ v_*J \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_* \\ v_* \end{pmatrix} = \begin{pmatrix} -v_* \\ u_* \end{pmatrix}. \tag{6.14}$$

Therefore (ii) implies (i). Reversing the above argument, we see that (i) implies (ii), and we are done.

□

We will next discuss a "better" way to think about almost complex structures (for $n = 1$; we will consider $n > 1$ shortly).

Proposition 6.2. *If J is defined in an open set U which induces the standard orientation on U, then there exists a unique complex valued function $\mu : U \to B(0,1) \subset \mathbb{C}$ so that*

$$T^{0,1}_J = \{v + \mu\overline{v} \mid v \in T^{0,1}_{J_0}\} \subset T_{\mathbb{C}}U. \tag{6.15}$$

Explicitly, if

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{6.16}$$

with $a^2 + bc = -1$, then

$$\mu = \frac{2ai - b - c}{2 + c - b}. \tag{6.17}$$

Conversely, given a function $\mu : U \to B(0,1) \subset \mathbb{C}$, writing $\mu = f + ig$, there is a uniquely determined almost complex structure J given by

$$J = \frac{1}{1 - f^2 - g^2} \begin{pmatrix} 2g & -(1+f)^2 - g^2 \\ g^2 + (1-f)^2 & -2g \end{pmatrix} \tag{6.18}$$

which has $T^{0,1}_J$ given by the above.

Proof. Given any such J, then we have previously defined

$$T_J^{0,1} = \{X \in T_{\mathbb{C}}U \mid JX = -iX\} = \{X' + iJX' \mid X \in T_{\mathbb{R}}U\}. \tag{6.19}$$

We next claim that the projection $\pi : T_J^{0,1} \to T_{J_0}^{0,1}$ is a complex linear isomorphism. These are two 1-dimensional complex subspaces of the 2-dimensional space $TU \otimes \mathbb{C}$, so there is a complex linear projection mapping, which is given by

$$X' + iJX' \mapsto X' + iJX' + iJ_0(X' + iJX') = (X' - J_0JX') + i(J + J_0)X'. \tag{6.20}$$

Since both spaces are 1-dimensional, and π is complex linear, it is an isomorphism provided it is not the zero map. Obviously, from (6.20), if $J \neq -J_0$ then it is not the zero mapping. We may therefore write $T_J^{0,1}$ as a graph over $T_{J_0}^{0,1}$. To do this, we compute like last time: using (1.36) to change to the complex basis $\{\partial/\partial z, \partial/\partial \overline{z}\}$, then we have

$$J = \frac{1}{2}\begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix}. \tag{6.21}$$

Then a basis for the 1-dimensional space $T_J^{0,1}$ is given by

$$\frac{\partial}{\partial \overline{z}} + iJ\Big(\frac{\partial}{\partial \overline{z}}\Big) = \frac{\partial}{\partial \overline{z}} + \frac{i}{2}\Big(i(b-c)\frac{\partial}{\partial \overline{z}} + (2a + i(b+c))\frac{\partial}{\partial z}\Big) \tag{6.22}$$

$$= \Big(1 + \frac{c-b}{2}\Big)\frac{\partial}{\partial \overline{z}} + \frac{1}{2}(2ai - b - c)\frac{\partial}{\partial z}. \tag{6.23}$$

From this, we find that

$$\mu = \frac{2ai - b - c}{2 + c - b}, \tag{6.24}$$

as claimed. Using $a^2 + bc = -1$, we compute

$$|\mu|^2 = \frac{4(-1 - bc) + (b+c)^2}{(2 + c - b)^2} = \frac{2 + b - c}{-2 + b - c}. \tag{6.25}$$

To show that $|\mu| < 1$, we use the orientation condition. Notice that the condition $bc = -1 - a^2$ says that $bc < 0$, so there are 2 components to the set of almost complex structures, determined by the sign of b: if $b < 0$, then this is the component inducing the standard orientation. In this case, we have

$$\frac{2 + b - c}{-2 + b - c} < 1 \tag{6.26}$$

is equivalent to

$$2 + b - c > -2 + b - c, \tag{6.27}$$

which is obviously true.

Next, given any such function μ, we define

$$T^{0,1}_{\mu} = \text{span}\Big\{\frac{\partial}{\partial \overline{z}} + \mu\frac{\partial}{\partial z}\Big\}. \tag{6.28}$$

Define

$$T^{1,0}_{\mu} = \text{span}\Big\{\frac{\partial}{\partial z} + \overline{\mu}\frac{\partial}{\partial \overline{z}}\Big\}. \tag{6.29}$$

We claim that $T^{1,0}_{\mu} \cap T^{0,1}_{\mu} = \{0\}$. To see this, if the intersection was non-zero, then there would exist $\alpha \in \mathbb{C}$ so that

$$\frac{\partial}{\partial \overline{z}} + \mu\frac{\partial}{\partial z} = \alpha\Big(\frac{\partial}{\partial z} + \overline{\mu}\frac{\partial}{\partial \overline{z}}\Big). \tag{6.30}$$

This clearly implies that $\alpha = \mu$ and then $|\mu|^2 = 1$. But we have assumed that $|\mu| < 1$, so the claim follows. To find the corresponding almost complex structure J, we must have

$$\frac{\partial}{\partial \overline{z}} + \mu\frac{\partial}{\partial z} = X' + iJX', \tag{6.31}$$

for some real tangent vector X'. We then write the real and imaginary parts of the left hand side:

$$\begin{aligned} \frac{\partial}{\partial \overline{z}} + \mu\frac{\partial}{\partial z} &= \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) + (f+ig)\frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \\ &= \frac{1}{2}\Big((1+f)\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\Big) + \frac{i}{2}\Big(g\frac{\partial}{\partial x} + (1-f)\frac{\partial}{\partial y}\Big). \end{aligned} \tag{6.32}$$

So we must have

$$J\Big((1+f)\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\Big) = g\frac{\partial}{\partial x} + (1-f)\frac{\partial}{\partial y}, \tag{6.33}$$

and since $J^2 = -Id$,

$$J\Big(g\frac{\partial}{\partial x} + (1-f)\frac{\partial}{\partial y}\Big) = -\Big((1+f)\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\Big). \tag{6.34}$$

A simple change of basis computation shows that

$$J = \frac{1}{1-f^2-g^2}\begin{pmatrix} 2g & -(1+f)^2 - g^2 \\ g^2 + (1-f)^2 & -2g \end{pmatrix} \tag{6.35}$$

□

Propositions 6.1 and 6.2 enable us to more easily understand the Beltrami equation. Given $\mu : U \to \mathbb{C}$ with $|\mu| < 1$, then since

$$T^{0,1}_{\mu} = \text{span}\Big\{\frac{\partial}{\partial \overline{z}} + \mu\frac{\partial}{\partial z}\Big\}, \tag{6.36}$$

Proposition 6.1 tells us that a function $w : U \to \mathbb{C}$ is holomorphic if and only if

$$\Big(\frac{\partial}{\partial \overline{z}} + \mu \frac{\partial}{\partial z}\Big) w = 0, \tag{6.37}$$

or

$$w_{\overline{z}} + \mu w_z = 0, \tag{6.38}$$

which is exactly the Beltrami equation. For many purposes, one can just completely forget about the matrix version of J, and parametrize almost complex structures by a single function $\mu : U \to \mathbb{C}$!

Remark 6.3. We see that given any M, an almost complex structure is a section of a smooth fiber bundle, with fiber F the union of 2 open discs. We have already seen that the component inducing the complex orientation corresponds to the unit ball $|\mu| < 1$. The component with the reversed orientation is $|\mu| > 1$ together with the point at infinity, which corresponds to the complex structure $-J_0$. Note also that fiber can be described as the homogeneous space

$$F = GL(2, \mathbb{R})/GL(1, \mathbb{C}) \tag{6.39}$$

7 Lecture 7

7.1 Power series in several variables

We review some basic facts about power series in several variables. We write a point $z = (z_1, \ldots, z_n)$, where these case be either real or complex coordinates. The open polydisc with polyradius $\vec{r} = (r_1, \ldots, r_n)$ about a point $a = (a_1, \ldots, a_n)$ is the set

$$P(a, \vec{r}) = \{z \mid |z_j - a_j| < r_j,\ j = 1 \ldots n\}. \tag{7.1}$$

We will let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ denote a multi-index, where $\mathbb{Z}_+$ denotes the non-negative integers. Define

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \tag{7.2}$$
$$|z|^\alpha = |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \tag{7.3}$$
$$\alpha! = \alpha_1! \cdots \alpha_n! \tag{7.4}$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_n \tag{7.5}$$
$$\tag{7.6}$$

Definition 7.1. The series $\sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha z^\alpha$ converges if some rearrangement converges, that is, give some bijection $\phi : \mathbb{Z}_+ \to (\mathbb{Z}_+)^n$, the series

$$\sum_{j=0}^{\infty} a_{\phi(j)} (z - a)^{\phi(j)} \tag{7.7}$$

converges.

Proposition 7.2. *If $\sum_\alpha a_\alpha z^\alpha$ (centered at $z = 0$) converges at the point z' then it converges uniformly and absolutely for point z of the form $z_j = \rho_j z'_j$ where $-1 < \rho_j < 1$.*

Proof. Since the series converges at the point z', the terms must be bounded, so there exists a constant C so that $|a_\alpha||z'|^\alpha \leq C$. We choose $0 < \rho < 1$ and consider any point $z = (z_1, \ldots, z_n)$ so that $|z_j| < \rho|z'_j|$. We then have

$$|a_\alpha||z|^\alpha \leq |a_\alpha|\rho^{|\alpha|}|z'|^\alpha \leq C\rho^{|\alpha|}. \tag{7.8}$$

So given an integer $N > 0$, we have

$$\begin{aligned}\sum_{|\alpha|\leq N} |a_\alpha||z|^\alpha &= \sum_{j=0}^{N} \sum_{|\alpha|=j} |a_\alpha||z|^\alpha \\ &\leq \sum_{j=0}^{N} \sum_{|\alpha|=j} C\rho^j\end{aligned} \tag{7.9}$$

How many multi-indices of length j are there? This is counting the number of non-negative integer solutions of

$$\alpha_1 + \cdots + \alpha_n = j. \tag{7.10}$$

To see this, let $\alpha' = \alpha_1 + 1$, then we are interested in the number of positive integer solutions to

$$\alpha'_1 + \cdots + \alpha'_n = j + n. \tag{7.11}$$

So we have a total of $j + n$ integers, dividing this up into n integers is the same as putting $n - 1$ partitions somewhere in the spaces between them, so the number is

$$\binom{j+n-1}{n-1}. \tag{7.12}$$

Continuing with the above calculation,

$$\begin{aligned}\sum_{|\alpha|\leq N} |a_\alpha||z|^\alpha &\leq C\sum_{j=0}^{N} \binom{j+n-1}{n-1}\rho^j \\ &= C\sum_{j=0}^{N} \frac{(j+n-1)!}{j!(n-1)!}\rho^j \\ &= \frac{C}{(n-1)!}\sum_{j=0}^{N}(j+n-1)(j+n-2)\cdots(j+1)\rho^j \\ &\leq C_n \sum_{j=0}^{N} j^n \rho^j.\end{aligned} \tag{7.13}$$

Applying the ratio test, we have

$$\lim_{j\to\infty} \frac{(j+1)^n \rho^{j+1}}{j^n \rho^j} = \lim_{j\to\infty} \left(\frac{j+1}{j}\right)^n \rho = \rho, \tag{7.14}$$

so the series converges provided $\rho < 1$.

Note that we could have also estimated this as follows. Let $\rho = (\rho_1, \dots, \rho_n)$ with $0 < \rho_j < 1$, and let $z_j = \rho_j z'_i$. Then

$$\begin{aligned} \sum_\alpha |a_\alpha||z|^\alpha &= \sum_\alpha |a_\alpha| \rho^\alpha |z'|^\alpha \\ &\leq C \sum_\alpha \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n} \\ &\leq C \frac{1}{(1-\rho_1)\cdots(1-\rho_n)}. \end{aligned} \tag{7.15}$$

so converges if $\rho_i < 1$. □

We can also state the convergence criterion as follows.

Proposition 7.3. *A point p belongs to the domain of convergence of the power series $\sum_\alpha c_\alpha z^\alpha$ if and only if there exists a neighborhood U of p, a constant C, and $r < 1$ such that $|c_\alpha z^\alpha| \leq C r^{|\alpha|}$ for all $z \in U$.*

Example 7.4. In 1 variable we know that domains of converge are intervals (in the case of a real variable) or a disc (in the case of a complex variables). Domains of convergence in several variable can be more complicated. For example, the series $\sum_{k=0}^\infty x^k y^k$ converges in the domain $|xy| < 1$.

Now we consider the Beltrami equation

$$w_{\overline{z}} = \mu(z, \overline{z}) w_z \tag{7.16}$$

Assuming μ is analytic, we have a convergent power series expansion

$$\mu(z, \overline{z}) = \sum_{j,k} \mu_{jk} z^j \overline{z}^k = \sum_{l=0}^\infty \sum_{j+k=l} \mu_{jk} z^j \overline{z}^k = \sum_{l=0}^\infty \mu_l. \tag{7.17}$$

Using Lemma 2.2, we can make the ACS standard at the origin, which implies that $\mu_0 = 0$, that is, μ has no constant term.

We also write

$$w = \sum_{j,k} w_{jk} z^j \overline{z}^k = \sum_{l=0}^\infty \sum_{j+k=l} w_{jk} z^j \overline{z}^k = \sum_{l=0}^\infty w_l. \tag{7.18}$$

We want to find a holomorphic coordinate system, so we make the assumption that $w_0 = 0$ and $w_1 = z$.

We then have

$$w_{\overline{z}} = \sum_{l=2}^{\infty} \partial_{\overline{z}} w_l \tag{7.19}$$

$$w_z = 1 + \sum_{l=2}^{\infty} \partial_z w_l. \tag{7.20}$$

We then want to solve

$$w_{\overline{z}} = \sum_{l=2}^{\infty} \partial_{\overline{z}} w_l = \mu w_z = \Big(\sum_{l=1}^{\infty} \mu_l \Big) \Big(1 + \sum_{k=2}^{\infty} \partial_z w_k \Big) = \Big(\sum_{l=1}^{\infty} \mu_l \Big) + \sum_{l=2}^{\infty} \sum_{j+k=l, j \geq 1, k \geq 2} \mu_j \partial_z w_k. \tag{7.21}$$

We then find the recursion relation

$$\partial_{\overline{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j\geq 1, k \geq 2} \mu_j \partial_z w_k. \tag{7.22}$$

Note that in the sum on the right hand side, we must have $k \leq l$, so this in indeed a recursion relation, provided that we can solve for w_{l+1}.

Fixing l, the right hand side is just a homogeneous polynomial of degree l in the variables z and $\overline{z}$. In general, if $f_l = \sum_{j+k=l, j\geq 0, k \geq 0} h_{jk} z^j \overline{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j \geq 0, k \geq 0} \frac{1}{k+1} h_{jk} z^j \overline{z}^{k+1} \tag{7.23}$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\overline{z}} F = f$.

Remark 7.5. Notice that our "inverse" of the $\overline{\partial}$-operator on polynomials of degree l does not contain any terms proportional to z^{l+1}. Our inverse operator is unique with this condition. If we had not imposed this condition, one could have chosen $w_l = l! z^l + O(\overline{z})$, in which case our series would definitely not converge! Also, if we view our series as a power series in 2 complex variables, then formally $w(z, 0) = z$ exactly because of this choice of inverse to $\overline{\partial}$.

We need to show if the estimate

$$|\mu_j|(z) \leq C \rho^j \tag{7.24}$$

holds for z such that $|z_j| < \rho$, then there exists a constant C', and $0 < \rho' < \rho$ such that for z' with $|z'_j| < \rho'$, then

$$|w_l|(z') \leq C' (\rho')^j \tag{7.25}$$

This is not so easy to show directly, we will discuss how to show this in the next lecture.

To illustrate, let's do an example.

Example 7.6. Let $\mu(z,\overline{z}) = z$. Then $\mu_1 = z$ and $\mu_j = 0$ for $j \neq 1$. We have $w_0 = 0, w_1 = z$. The next term is

$$\partial_{\overline{z}} w_2 = \mu_1 = z \Longrightarrow w_2 = z\overline{z}. \tag{7.26}$$

We then have

$$\partial_{\overline{z}} w_3 = \mu_1 \partial_z w_2 = z\overline{z} \Longrightarrow w_3 = \frac{1}{2} z\overline{z}^2. \tag{7.27}$$

Then

$$\partial_{\overline{z}} w_4 = \mu_1 \partial_z w_3 = \frac{1}{2} z\overline{z}^3 \Longrightarrow w_4 = \frac{1}{6} z\overline{z}^3. \tag{7.28}$$

In general we get that

$$w_j = \frac{1}{(j-1)!} z\overline{z}^j, \tag{7.29}$$

so we have found the solution

$$w = ze^{\overline{z}}. \tag{7.30}$$

8 Lecture 8

8.1 More examples

For fun, let's do a few more examples.

Example 8.1. Let $\mu(z,\overline{z}) = \overline{z}$. Then $\mu_1 = \overline{z}$ and $\mu_j = 0$ for $j \neq 1$. We have $w_0 = 0, w_1 = z$. The next term is

$$\partial_{\overline{z}} w_2 = \mu_1 = \overline{z} \Longrightarrow w_2 = \frac{1}{2}\overline{z}^2. \tag{8.1}$$

We then have

$$\partial_{\overline{z}} w_3 = \mu_1 \partial_z w_2 = 0 \Longrightarrow w_3 = 0. \tag{8.2}$$

Then

$$\partial_{\overline{z}} w_4 = \mu_1 \partial_z w_3 = 0 \Longrightarrow w_4 = 0. \tag{8.3}$$

It is clear that the series terminates, so we have found the solution

$$w = z + \frac{1}{2}\overline{z}^2 \tag{8.4}$$

Example 8.2. Let $\mu(z,\overline{z}) = z + \overline{z}$. Then $\mu_1 = z + \overline{z}$ and $\mu_j = 0$ for $j \neq 1$. We have $w_0 = 0, w_1 = z$. The next term is

$$\partial_{\overline{z}} w_2 = \mu_1 = z + \overline{z} \Longrightarrow w_2 = z\overline{z} + \frac{1}{2}\overline{z}^2. \tag{8.5}$$

We then have

$$\partial_{\overline{z}} w_3 = \mu_1 \partial_z w_2 = (z + \overline{z})\overline{z} \Longrightarrow w_3 = \frac{1}{2} z\overline{z}^2 + \frac{1}{3}\overline{z}^3. \tag{8.6}$$

Then

$$\partial_{\overline{z}} w_4 = \mu_1 \partial_z w_3 = \frac{1}{2}(z + \overline{z})\overline{z}^2 \Longrightarrow w_4 = \frac{1}{6} z\overline{z}^3 + \frac{1}{6}\overline{z}^3. \tag{8.7}$$

Apparently, the solution is of the form

$$w = ze^{\overline{z}} + f(\overline{z}). \tag{8.8}$$

Plugging this into the Beltrami equation yields

$$ze^{\overline{z}} + \partial_{\overline{z}} f(\overline{z}) = (z + \overline{z})e^{\overline{z}}, \tag{8.9}$$

or

$$\partial_{\overline{z}} f(\overline{z}) = \overline{z}e^{\overline{z}} \tag{8.10}$$

This has the solution

$$f(\overline{z}) = \overline{z}e^{\overline{z}} - e^{\overline{z}} + 1. \tag{8.11}$$

So we have found the solution

$$w = (z + \overline{z} - 1)e^{\overline{z}} + 1. \tag{8.12}$$

Example 8.3. Let $\mu(z,\overline{z}) = z\overline{z}$. Then $\mu_2 = z\overline{z}$ and $\mu_j = 0$ for $j \neq 2$. We have $w_0 = 0, w_1 = z$. The next term is

$$\partial_{\overline{z}} w_2 = \mu_1 = 0 \Longrightarrow w_2 = 0. \tag{8.13}$$

We then have

$$\partial_{\overline{z}} w_3 = \mu_2 = z\overline{z} \Longrightarrow w_3 = \frac{1}{2} z\overline{z}^2. \tag{8.14}$$

Then

$$\partial_{\overline{z}} w_4 = \mu_2 \partial_z w_2 = 0 \Longrightarrow w_4 = 0. \tag{8.15}$$

One more term:

$$\partial_{\overline{z}} w_5 = \mu_2 \partial_z w_3 = \frac{1}{2} z\overline{z}^3 \Longrightarrow w_5 = \frac{1}{8} z\overline{z}^4. \tag{8.16}$$

It is not hard to see that we have found the solution

$$w = ze^{\overline{z}^2/2}. \tag{8.17}$$

Example 8.4. Let $\mu(z,\overline{z}) = z^2$. Then $\mu_2 = z^2$ and $\mu_j = 0$ for $j \neq 2$. We have $w_0 = 0, w_1 = z$. The next term is

$$\partial_{\overline{z}} w_2 = \mu_1 = 0 \Longrightarrow w_2 = 0. \tag{8.18}$$

We then have

$$\partial_{\overline{z}} w_3 = \mu_2 = z^2 \Longrightarrow w_3 = z^2\overline{z}. \tag{8.19}$$

Then

$$\partial_{\overline{z}} w_4 = \mu_2 \partial_z w_2 = 0 \Longrightarrow w_4 = 0. \tag{8.20}$$

Next,

$$\partial_{\overline{z}} w_5 = \mu_2 \partial_z w_3 = 2z^3\overline{z} \Longrightarrow w_5 = z^3\overline{z}^2. \tag{8.21}$$

It is not hard to see that we get

$$\begin{aligned} w &= z(1 + z\overline{z} + z^2\overline{z}^2 + z^3\overline{z}^3 + \cdots) && (8.22)\\ &= z(1 + |z|^2 + |z|^4 + |z|^6 + \cdots) && (8.23)\\ &= \frac{z}{1 - |z|^2} && (8.24) \end{aligned}$$

for $|z| < 1$.

Example 8.5. Let $\mu(z,\overline{z}) = \overline{z}^2$. Then $\mu_2 = \overline{z}^2$ and $\mu_j = 0$ for $j \neq 2$. We have $w_0 = 0, w_1 = z$. The next term is

$$\partial_{\overline{z}} w_2 = \mu_1 = 0 \Longrightarrow w_2 = 0. \tag{8.25}$$

We then have

$$\partial_{\overline{z}} w_3 = \mu_2 = \overline{z}^2 \Longrightarrow w_3 = \frac{1}{3}\overline{z}^3. \tag{8.26}$$

Then

$$\partial_{\overline{z}} w_4 = \mu_2 \partial_z w_2 = 0 \Longrightarrow w_4 = 0. \tag{8.27}$$

Next,

$$\partial_{\overline{z}} w_5 = \mu_2 \partial_z w_3 = 0 \Longrightarrow w_5 = 0, \tag{8.28}$$

and it is easy to see that the process terminates. So we have found the solution

$$w = z + \frac{1}{3}\overline{z}^3. \tag{8.29}$$

From examining the above, looks like we have found the following general cases.

Proposition 8.6 (Separation of variables). *If $\mu(z,\overline{z}) = \mu_1(z)\mu_2(\overline{z})$, then*

$$w(z,\overline{z}) = C' e^{-C\left(M_1(z)+M_2(\overline{z})\right)} \tag{8.30}$$

solves the Beltrami equation

$$w_{\overline{z}} = \mu_1(z)\mu_2(\overline{z}) w_z, \tag{8.31}$$

where $M_1(z)$ is an antiderivative of μ_1^{-1}, that is $\partial_z M_1(z) = \mu_1^{-1}(z)$, $M_2(\overline{z})$ be an antiderivative of $\mu_2(\overline{z})$, and C, C' are constants.

Proof. We assume that there is a solution of the form $w(z,\overline{z}) = f(z)g(\overline{z})$. Then the Beltrami equation is

$$w_{\overline{z}} = f(z)\partial_{\overline{z}} g(\overline{z}) = \mu_1(z)\mu_2(\overline{z})\partial_z f(z) g(\overline{z}). \tag{8.32}$$

Equivalently,

$$\frac{\partial_{\overline{z}} g(\overline{z})}{\mu_2(\overline{z}) g(\overline{z})} = \frac{\partial_z f(z)\mu_1(z)}{f(z)}. \tag{8.33}$$

The left hand side is a function of $\overline{z}$ and the right hand side is a function of z, so this is only possible if both sides are constant C. So we have the following equation

$$\partial_z f(z) - C\mu_1^{-1}(z) f(z) = 0. \tag{8.34}$$

This is a first order linear equation, and can be solved by integrating factor. Let $M_1(z)$ be an antiderivative of $\mu_1^{-1}(z)$, that is $\partial_z M_1(z) = \mu_1^{-1}(z)$. Then

$$f(z) = C_1 e^{-CM_1(z)} \tag{8.35}$$

Similarly, letting $M_2(\overline{z})$ be an antiderivative of $\mu_2(\overline{z})$, we have

$$g(\overline{z}) = C_2 e^{-CM_2(\overline{z})} \tag{8.36}$$

So we have found the solution

$$w(z,\overline{z}) = C' e^{-C\left(M_1(z)+M_2(\overline{z})\right)}. \tag{8.37}$$

□

Let's analyze this formula in a few cases. First, for Example 7.6, we have $\mu(z,\overline{z}) = z$, so we have $\mu_1(z) = z$ and $\mu_2(\overline{z}) = 1$. Then

$$M_1 = \log(z), \quad M_2(\overline{z}) = \overline{z}, \tag{8.38}$$

so we have

$$w = e^{\log(z)+\overline{z}} = z e^{\overline{z}}, \tag{8.39}$$

which agrees with before.

Next, for Example 8.3, we have $\mu(z,\overline{z}) = z\overline{z}$. Then $\mu_1(z) = z$ and $\mu_2(\overline{z}) = \overline{z}$. Then

$$M_1 = \log(z), \quad M_2(\overline{z}) = \frac{1}{2}\overline{z}^2, \tag{8.40}$$

so we have

$$w = e^{\log(z)+\overline{z}^2/2} = ze^{\overline{z}^2/2}, \tag{8.41}$$

which agrees with before.

Let's next look at Example 8.4, where $\mu(z,\overline{z}) = z^2$, so $\mu_1(z) = z^2$ and $\mu_2(\overline{z}) = 1$. Then

$$M_1 = -\frac{1}{z}, \quad M_2(\overline{z}) = \overline{z}, \tag{8.42}$$

so we have

$$w = e^{-\frac{1}{z}+\overline{z}}, \tag{8.43}$$

which is definitely NOT the previous solution

$$w = \frac{z}{1-|z|^2}. \tag{8.44}$$

What is going on here? The fact is that our problem has an infinite dimensional gauge group!

Proposition 8.7. *Let $w(z,\overline{z})$ solve the Beltrami equation $w_{\overline{z}} = \mu(z,\overline{z})w_z$. Then if $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is any holomorphic function, then $f \circ w$ is also a solution of the Beltrami equation.*

Proof. We have two ways to prove this. First, the direct way. Using the chain rule

$$\begin{aligned}\partial_{\overline{z}}(f\circ w)(z,\overline{z}) &= (\partial_z f)(w(z,\overline{z}))\partial_{\overline{z}}w = (\partial_z f)(w(z,\overline{z}))\mu(z,\overline{z})\partial_z w \\ &= \mu(z,\overline{z})(\partial_z f)(w(z,\overline{z}))\partial_z w = \mu(z,\overline{z})\partial_{\overline{z}}(f\circ w)(z,\overline{z}).\end{aligned} \tag{8.45}$$

The easier way is to recall that the solution of the Beltrami equation is a holomorphic function from the almost complex structure determined by μ to the standard complex structure. So then obviously the composition with any holomorphic function on $\mathbb{C}$ is still holomorphic. □

Now then we see how (8.43) and (8.44) are related. Taking $f(w) = -1/\log(w)$, we have

$$f \circ (e^{-\frac{1}{z}+\overline{z}}) = -\frac{1}{-\frac{1}{z}+\overline{z}} = \frac{z}{1-|z|^2}, \tag{8.46}$$

which is the first solution!

Remark 8.8. The solution from our power series method satisfies the Cauchy data $w(z,0) = z$, so once we prove the series converges, we will have uniqueness of this solution. Then $f \circ w$ will solve the Beltrami equation with Cauchy data $f \circ w(z,0) = f(z)$, so we actually will have uniqueness for any holomorphic Cauchy data.

Note that all of our examples are in separated form, except for (8.2). We will come back to this later, but for now, let's solve for a particular $\mu(z, \overline{z})$. It might seem random, but this will be crucial for our proof of convergence of our above series for analytic μ.

Proposition 8.9. *Let*

$$\mu^* = \frac{C}{\left(1 - \frac{z}{r}\right)\left(1 - \frac{\overline{z}}{r}\right)}, \tag{8.47}$$

which is analytic in the polydisc $P(r) = \{(z, \overline{z}) \mid |z| < r, |\overline{z}| < r\}$. Then a solution of the Beltrami equation satisfying $w(z, 0) = z$ is given by

$$w^*(z, \overline{z}) = r - \sqrt{(r - z)^2 + Cr \log\left(1 - \frac{\overline{z}}{r}\right)}, \tag{8.48}$$

and is also analytic in $P(r)$.

Proof. We let

$$\mu_1 = \frac{1}{1 - \frac{z}{r}}, \quad \mu_2 = \frac{C}{1 - \frac{\overline{z}}{r}}. \tag{8.49}$$

Then

$$M_1(z) = \int \left(1 - \frac{z}{r}\right) dz = z - \frac{z^2}{2r}, \tag{8.50}$$

and

$$M_2(\overline{z}) = \int \frac{C}{1 - \frac{\overline{z}}{r}} d\overline{z} = -Cr \log\left(1 - \frac{\overline{z}}{r}\right). \tag{8.51}$$

From Proposition 8.6, we obtain the solution

$$w = Ae^{B(M_1(z) + M_2(\overline{z})}. \tag{8.52}$$

We choose $A = 1, B = -1$, and then compose with $f(w) = \log(w)$ to get the solution

$$w = M_1 + M_2 = z - \frac{z^2}{2r} - Cr \log\left(1 - \frac{\overline{z}}{r}\right). \tag{8.53}$$

We want to obtain a solution which satisfies $\mu(z, 0) = z$. For this, we solve

$$w - \frac{w^2}{2r} = z - \frac{z^2}{2r} - Cr \log\left(1 - \frac{\overline{z}}{r}\right), \tag{8.54}$$

for w, and (8.47) follows from the quadratic formula. □

9 Lecture 9

9.1 Proof of convergence

We assume that $\mu(z,\overline{z})$ is analytic, so there is a power series

$$\mu(z,\overline{z}) = \sum_{j,k} \mu_{j\bar{k}} z^j \overline{z}^k, \tag{9.1}$$

which converges in a neighborhood of the origin. The above method produces a formal power series solution

$$w(z,\overline{z}) = \sum_{j,k} w_{j\bar{k}} z^j \overline{z}^k, \tag{9.2}$$

but we do not yet know if this series converges in a neighborhood of the origin. The next is a key step in Cauchy's method of majorants.

Proposition 9.1. *The coefficients $w_{j\bar{k}}$ for $j+k=l$ are a polynomial of degree $l-1$ in the $\mu_{p\bar{q}}$ for $p+q<l$ with all coefficients non-negative rational numbers.*

Proof. Let us examine the first few steps of the iteration. We have $w_{00}=1$, $w_{10}=1$, and $w_{0\bar{1}}=0$. The term w_2 is determined by

$$\partial_{\overline{z}} w_2 = \mu_1 = \mu_{1\bar{0}} z + \mu_{0\bar{1}} \overline{z}, \tag{9.3}$$

so

$$w_2 = \mu_{1\bar{0}} z\overline{z} + \frac{1}{2}\mu_{0\bar{1}} \overline{z}^2, \tag{9.4}$$

so

$$w_{2\bar{0}} = 0, \quad w_{1\bar{1}} = \mu_{1\bar{0}}, \quad w_{0\bar{2}} = \frac{1}{2}\mu_{0\bar{1}}. \tag{9.5}$$

To illustrate, let's do one more step. The term w_3 is determined by

$$\begin{aligned} \partial_{\overline{z}} w_3 &= \mu_2 + \mu_1 \partial_z w_2 = \mu_{2\bar{0}} z^2 + \mu_{1\bar{1}} z\overline{z} + \mu_{0\bar{2}} \overline{z}^2 + (\mu_{1\bar{0}} z + \mu_{0\bar{1}} \overline{z})(\mu_{1\bar{0}} \overline{z}) \\ &= \mu_{2\bar{0}} z^2 + (\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z\overline{z} + (\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}})\overline{z}^2. \end{aligned} \tag{9.6}$$

so

$$w_3 = \mu_{2\bar{0}} z^2 \overline{z} + \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z\overline{z}^2 + \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}})\overline{z}^3. \tag{9.7}$$

so

$$w_{3\bar{0}} = 0, \quad w_{2\bar{0}} = \mu_{2\bar{0}}, \quad w_{1\bar{2}} = \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2), \quad w_{0\bar{3}} = \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}}), \tag{9.8}$$

and the claim is evidently true.

To do the general case, we prove by induction: assume the claim is true up to for $0, \dots, l$, and we prove for $l+1$. Recall that

$$\partial_{\overline{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j\geq 1, k\geq 2} \mu_j \partial_z w_k. \tag{9.9}$$

By induction, the coefficients of w_k for $k \leq l$ are polynomials with non-negative coefficients in the $\mu_{p\bar{q}}$ with $p + \bar{q} < k < l$, so that $\partial_z w_k$ is also of this form. Then since

$$\mu_j = \sum_{k+l=j} \mu_{k\bar{l}} z^k \overline{z}^l, \tag{9.10}$$

any term $\mu_j \partial_z w_k$ is also a polynomial in the $\mu_{k\bar{l}}$ with non-negative coefficients.

To get w_{l+1}, recall that if $f_l = \sum_{j+k=l, j\geq 0, k\geq 0} h_{jk} z^j \overline{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j\geq 0, k\geq 0} \frac{1}{k+1} h_{jk} z^j \overline{z}^{k+1} \tag{9.11}$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\overline{z}} F = f$. Clearly, this preserves non-negativity of the coefficients, and we are done. ☐

Theorem 9.2. *If $\mu(z, \overline{z})$ is analytic in the closed polydisc $|z| \leq \rho, |\overline{z}| \leq \rho$, there there exists a unique solution of the Beltrami equation*

$$w_{\overline{z}} = \mu(z, \overline{z}) w_z \tag{9.12}$$

which is analytic in the polydisc $|z| < \rho, |\overline{z}| < \rho$ and satisfies the Cauchy data

$$w(z, 0) = z. \tag{9.13}$$

Proof. By assumption, the series

$$\mu = \sum_{j,k} \mu_{j\bar{k}} z^j \overline{z}^k \tag{9.14}$$

converges for any point $(z, \overline{z})$ with $|z| \leq \rho, |\overline{z}| \leq \rho$. So for any such point, there exists a constant C so that

$$|\mu_{j\bar{k}} z^j \overline{z}^k| < C. \tag{9.15}$$

If we let $z = (0, \dots, 1_j, \dots, 0)$, and $\overline{z} = (0, \dots, 1_k, \dots, 0)$, then $(\rho z, \rho \overline{z})$ is in the above polydisc. So

$$|\mu_{j\bar{k}} (\rho z)^j (\rho \overline{z})^k| < C, \tag{9.16}$$

which obviously implies that

$$|\mu_{j\bar{k}}| < C \rho^{-j-k}. \tag{9.17}$$

Then we define

$$\mu^*(z, \overline{z}) = \frac{C}{\left(1 - \frac{z}{\rho}\right)\left(1 - \frac{\overline{z}}{\rho}\right)}, \tag{9.18}$$

which is analytic in the polydisc $P(\rho) = \{(z, \overline{z}) \mid |z| < \rho, |\overline{z}| < \rho\}$. The power series of μ^* is given by

$$\mu^*(z, \overline{z}) = \sum_{j,k} C\rho^{-j-k} z^j \overline{z}^k, \tag{9.19}$$

so

$$\mu^*_{j\bar{k}} = C\rho^{-j-k}. \tag{9.20}$$

We therefore have

$$|\mu_{j\bar{k}}| \leq C\rho^{-j-k} = \mu^*_{j\bar{k}}. \tag{9.21}$$

Recall that the solution of the Beltrami equation for μ^* satisfying $w(z, 0) = z$ is given by

$$w^*(z, \overline{z}) = \rho - \sqrt{(\rho - z)^2 + C\rho \log\left(1 - \frac{\overline{z}}{\rho}\right)}, \tag{9.22}$$

and is also analytic in $P(\rho)$, and has a power series expansion

$$w^*(z, \overline{z}) = \sum_{j,k} w^*_{j\bar{k}} z^j \overline{z}^k. \tag{9.23}$$

Recall that our formal power series solves

$$w_{j\bar{k}} = P_{j\bar{k}}(\mu_{*\bar{*}}), \tag{9.24}$$

where $P_{j\bar{k}}$ is a polynomial with positive coefficients depending only upon $\mu_{p\bar{q}}$ for $p+q < j+k$. Since w^* is an analytic solution of the Beltrami equation with μ^*, we must also have

$$w^*_{j\bar{k}} = P_{j\bar{k}}(\mu^*_{*\bar{*}}). \tag{9.25}$$

We then estimate

$$|w_{j\bar{k}}| = |P_{j\bar{k}}(\mu_{*\bar{*}})| \leq P_{j\bar{k}}(|\mu_{*\bar{*}}|) \leq P_{j\bar{k}}(\mu^*_{*\bar{*}}) = w^*_{j\bar{k}}. \tag{9.26}$$

The inequalities hold since $P_{j\bar{k}}$ is a polynomial with real non-negative coefficients, and using (9.21). This shows that our power series is majorized by the power series of w^*, which implies that the power series for w converges in the open polydisc $P(\rho)$. $\square$

9.2 Method of Characteristics

We next discuss an equivalent way to approach the above problem, but which also allows us to solve the Beltrami equation explicitly in some cases where separation of variables doesn't work. This is called the method of characteristics, and allows us to reduce to solving an ODE in the analytic case. The only "drawback" of this method is that the ODE is nonlinear.

Let consider a first order PDE for a single function of two variables. We will call the independent variables x, y, and independent variable w. We will assume that all variables are real for the time being. The Beltrami equation has the form

$$w_y = \mu(x, y) w_x, \tag{9.27}$$

for $w : U \to \mathbb{R}$. We consider the graph of the solution as a hypersurface in $\mathbb{R}^3$:

$$G = \{(x, y, w(x, y)) \,|(x, y) \in U\}. \tag{9.28}$$

The normal to the graph is

$$\vec{N} = (w_x, w_y, -1). \tag{9.29}$$

Define the vector field

$$\vec{F} = (-\mu(x, y), 1, 0). \tag{9.30}$$

Then

$$\vec{F} \cdot \vec{N} = -\mu(x, y) w_x + w_y = 0. \tag{9.31}$$

So the vector field $\vec{F}$ is everywhere tangent to the graph G. Consequently, G is stratified by the integral curves of $\vec{F}$.

We then solve the ODE system:

$$\frac{dx}{ds} = -\mu(x, y), \quad \frac{dy}{ds} = 1, \quad \frac{dw}{ds} = 0, \tag{9.32}$$

with initial conditions

$$x(0) = z, \quad y(0) = 0, \quad w(0) = z. \tag{9.33}$$

The solution is of the form

$$x = X(z, s), \quad y = s, \quad w = z. \tag{9.34}$$

So we have

$$x = X(w, y), \tag{9.35}$$

which determines w implicitly as a function of x and y. By the implicit function theorem, we can solve for w in terms of x, y provided that

$$\frac{\partial X}{\partial w}|_{(0,0)} \neq 0. \tag{9.36}$$

But along the curve $(x, 0)$ we have $X(x, 0) = z = w$, so this is obviously true.

We assumed the variables were real, but now we consider $x = z$, and $y = \overline{z}$ as two independent complex variables. The same method as above gives us an implicit solution

$$z = X(w, \overline{z}) \tag{9.37}$$

with initial conditions $w(z, 0) = z$, which we can then solve for w.

If $\mu(z, \overline{z})$ is analytic, then the solution to the ODE

$$\frac{dz}{ds} = -\mu(z, s), \quad z(0) = w, \tag{9.38}$$

will also be analytic in both variables (proved using the majorant method, but slightly easier since it's just an ODE).

Example 9.3. Let $\mu(z, \overline{z}) = z + \overline{z}$. We solve the ODE system

$$\frac{dz}{ds} = -\mu(z, s) = -z - s, \quad z(0) = w. \tag{9.39}$$

The ODE is

$$\frac{dz}{ds} + z = -s, \tag{9.40}$$

which can be solved using the integrating factor e^s:

$$\frac{d}{ds}(e^s z) = -se^s. \tag{9.41}$$

Integrating gives

$$e^s z = -se^s + e^s + C. \tag{9.42}$$

Plugging in the intial conditions gives

$$z(0) = w = 1 + C, \tag{9.43}$$

or $C = w - 1$. We then have

$$w = e^{\overline{z}}(z + \overline{z} - 1) + 1, \tag{9.44}$$

which agrees with our previous solution which was found by ad hoc methods.

Example 9.4. We can also solve our majorant problem with this method, in that case, we had

$$\mu^* = \frac{C}{\left(1 - \frac{z}{r}\right)\left(1 - \frac{\overline{z}}{r}\right)}. \tag{9.45}$$

We solve the ODE

$$\frac{dz}{ds} = \frac{C}{\Big(1-\frac{z}{r}\Big)\Big(1-\frac{s}{r}\Big)}. \tag{9.46}$$

with initial condition $z(0) = w$. This is separable, so we have

$$\Big(1-\frac{z}{r}\Big)dz = \frac{C}{\Big(1-\frac{s}{r}\Big)}ds, \tag{9.47}$$

which we can integrate, and then let $s = \overline{z}$, to get

$$z - \frac{z^2}{2r} = -Cr\log\Big(1-\frac{\overline{z}}{r}\Big) + C'. \tag{9.48}$$

The initial conditions give

$$z - \frac{z^2}{2r} = -Cr\log\Big(1-\frac{\overline{z}}{r}\Big) + w - \frac{w^2}{2r}, \tag{9.49}$$

and we can use the quadratic formula to solve for w, to get the same answer as before.

10 Lecture 10

10.1 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the smooth case into the analytic case [?, ?]. In the z coodinates, our Beltrami equation is $w_{\overline{z}} = \mu(z,\overline{z})w_z$. We want to change coordinates $\xi = \xi(z,\overline{z})$ so that

$$\Big|\frac{\partial\xi}{\partial z}(0,0)\Big|^2 - \Big|\frac{\partial\xi}{\partial\overline{z}}(0,0)\Big|^2 \neq 0, \tag{10.1}$$

such that our Beltrami equation transform into another Beltrami equation, but with $\mu(z,\bar{z})$ analytic. Write

$$w(z,\overline{z}) = W(\xi(z,\overline{z}),\bar{\xi}(z,\overline{z})) \tag{10.2}$$
$$\mu(z,\overline{z}) = U(\xi(z,\overline{z}),\bar{\xi}(z,\overline{z})). \tag{10.3}$$

Then

$$\frac{\partial w}{\partial\overline{z}} = \frac{\partial W}{\partial\xi}\frac{\partial\xi}{\partial\overline{z}} + \frac{\partial W}{\partial\bar{\xi}}\frac{\partial\bar{\xi}}{\partial\overline{z}} \tag{10.4}$$
$$\frac{\partial w}{\partial z} = \frac{\partial W}{\partial\xi}\frac{\partial\xi}{\partial z} + \frac{\partial W}{\partial\bar{\xi}}\frac{\partial\bar{\xi}}{\partial z}. \tag{10.5}$$

So the Beltrami equation becomes

$$\frac{\partial W}{\partial\xi}\frac{\partial\xi}{\partial\overline{z}} + \frac{\partial W}{\partial\bar{\xi}}\frac{\partial\bar{\xi}}{\partial\overline{z}} = U(\xi,\bar{\xi})\Big(\frac{\partial W}{\partial\xi}\frac{\partial\xi}{\partial z} + \frac{\partial W}{\partial\bar{\xi}}\frac{\partial\bar{\xi}}{\partial z}\Big), \tag{10.6}$$

which we can write as

$$\frac{\partial W}{\partial \bar{\xi}} = \left(\frac{\frac{\partial \xi}{\partial \bar{z}} - U(\xi, \bar{\xi}) \frac{\partial \xi}{\partial z}}{\frac{\partial \bar{\xi}}{\partial \bar{z}} - U(\xi, \bar{\xi}) \frac{\partial \bar{\xi}}{\partial z}} \right) \frac{\partial W}{\partial \xi}, \tag{10.7}$$

which is another Beltrami equation with a new right hand side

$$\tilde{U}(\xi, \bar{\xi}) = \frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z}. \tag{10.8}$$

Let us try to find the coordinates so that

$$\frac{\partial}{\partial \xi} \tilde{U}(\xi, \bar{\xi}) = 0. \tag{10.9}$$

From the chain rule, we have

$$\frac{\partial}{\partial \xi} = \frac{\partial z}{\partial \xi} \frac{\partial}{\partial z} + \frac{\partial \overline{z}}{\partial \xi} \frac{\partial}{\partial \overline{z}}, \tag{10.10}$$

and we have

$$\begin{aligned} \frac{\partial}{\partial \xi} \tilde{U}(\xi, \bar{\xi}) &= \frac{\partial}{\partial \xi} \Big(\frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z} \Big) \\ &= \Big(z_\xi \partial_z + \overline{z}_\xi \partial_{\overline{z}} \Big) \Big(\frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z} \Big) \\ &= z_\xi \partial_z \Big(\frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z} \Big) + \overline{z}_\xi \partial_{\overline{z}} \Big(\frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z} \Big). \end{aligned} \tag{10.11}$$

By the inverse function theorem, we have

$$\begin{pmatrix} z_\xi & z_{\bar{\xi}} \\ \overline{z}_\xi & \overline{z}_{\bar{\xi}} \end{pmatrix} = \begin{pmatrix} \xi_z & \xi_{\bar{z}} \\ \bar{\xi}_z & \bar{\xi}_{\bar{z}} \end{pmatrix}^{-1} = \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \begin{pmatrix} \bar{\xi}_{\bar{z}} & -\xi_{\bar{z}} \\ -\bar{\xi}_z & \xi_z \end{pmatrix}, \tag{10.12}$$

so

$$z_\xi = \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \bar{\xi}_{\bar{z}} \tag{10.13}$$

$$\overline{z}_\xi = \frac{-1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \bar{\xi}_z. \tag{10.14}$$

We therefore have

$$\frac{\partial}{\partial \xi} \tilde{U}(\xi, \bar{\xi}) = \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \bar{\xi}_{\bar{z}} \partial_z \Big(\frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z} \Big) - \frac{1}{|\xi_z|^2 - |\xi_{\bar{z}}|^2} \bar{\xi}_z \partial_{\overline{z}} \Big(\frac{\xi_{\bar{z}} - U(\xi, \bar{\xi}) \xi_z}{\bar{\xi}_{\bar{z}} - U(\xi, \bar{\xi}) \bar{\xi}_z} \Big). \tag{10.15}$$

The equation (10.9) is *quasilinear* of the form

$$F(D^2 \xi, D\xi, \xi) = 0. \tag{10.16}$$

The linearization of F at a function ξ is given by

$$F'_\xi(h) = \frac{d}{dt} F(D^2(\xi + th), D(\xi + th), \xi + th)\Big|_{t=0}. \tag{10.17}$$

The linearization of F at the function $\xi = z$ is a multiple of the Laplacian (plus lower order terms), so we can write

$$F(z + h) = F(z) + \frac{1}{4}\Delta h + Q(h), \tag{10.18}$$

where the $Q(h)$ terms are higher order. We define a mapping $T : C^{k,\alpha} \to C^{k+2,\alpha}_0$ by letting Tf be the unique solution to the Dirichlet problem

$$\frac{1}{4}\Delta(Tf) = f \text{ in } B(0,1), \quad Tf = 0 \text{ on } \partial B(0,1). \tag{10.19}$$

From the basic theory of the Laplace operator, there exists a constant C so that

$$\|Tf\|_{C^{k+2,\alpha}(B(0,1))} \leq C\|f\|_{C^{k,\alpha}(B(0,1))}. \tag{10.20}$$

We would like to solve the equation

$$F(z) + \frac{1}{4}\Delta h + Q(h) = 0, \tag{10.21}$$

with h very small. Writing $h = Tf$, we can write this as

$$F(z) + f + Q(Tf) = 0, \tag{10.22}$$

that is

$$f = -Q(Tf) - F(z). \tag{10.23}$$

So we would like to find a fixed point of the operator $S : C^{k,\alpha}(B(0,1)) \to C^{k,\alpha}(B(0,1))$ defined by

$$Sf = -Q(Tf) - F(z). \tag{10.24}$$

This will be a contraction mapping provided certain norms are sufficiently small. The precise statement is the following version of the implicit function theorem in Banach spaces, and we will leave the proof as an exercise.

Lemma 10.1. *Let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be a C^1-map between two Banach spaces such that $F(x) = F(0) + L(x) + Q(x)$, where the operator $L : \mathcal{B}_1 \to \mathcal{B}_2$ is linear and $Q(0) = 0$. Assume that*

1. *L is an isomorphism with inverse T satisfying $\|T\| \leq C_1$,*
2. *there are constants $r > 0$ and $C_2 > 0$ with $r < \frac{1}{3C_1C_2}$ such that*
   (a) *$\|Q(x) - Q(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x - y\|_{\mathcal{B}_1}$ for all $x, y \in B_r(0) \subset \mathcal{B}_1$,*

(b) $\|F(0)\|_{\mathcal{B}_2} \le \frac{r}{2C_1}$.

Then there exists a unique solution to $F(x) = 0$ in $\mathcal{B}_1$ such that

$$\|x\|_{\mathcal{B}_1} \le 2C_1 \cdot \|F(0)\|_{\mathcal{B}_2}.$$

Returning to our problem, using Taylor's Theorem, we estimate

$$\begin{aligned}\|Q(Tf_1) - Q(Tf_2)\|_{C^{k,\alpha}} &\le C(\|Tf_1\|_{C^{k+2,\alpha}} + \|Tf_2\|_{C^{k+2,\alpha}})(\|Tf_1 - Tf_2\|_{C^{k+2,\alpha}} \\ &\le C'(\|f_1\|_{C^{k,\alpha}} + \|f_2\|_{C^{k,\alpha}})(\|f_1 - f_2\|_{C^{k,\alpha}}.\end{aligned} \tag{10.25}$$

By scaling the coordinates $z = \epsilon z'$, and letting $w'(z', \overline{z}') = w(\epsilon z', \epsilon \overline{z}')$, then we have

$$w'_{\overline{z}'} = \mu(\epsilon z', \epsilon \overline{z}') w'_{z'}. \tag{10.26}$$

So by choosing ϵ small, we can assume that

$$[\mu]_{C^{k,\alpha}(B(0,1))} < C\epsilon^{k+\alpha}. \tag{10.27}$$

Notice that we possibly included some linear terms in our Q, but the coefficients of those terms will depend on U, so these will also satisfy the estimate in (a) for ϵ sufficiently small. Also, by taking ϵ sufficiently small, we can always arrange so that condition (b) is satisfied. The implicit function theorem yields a solution h with

$$\|h\|_{C^{k,\alpha}(B(0,1))} = o(\epsilon), \tag{10.28}$$

as $\epsilon \to 0$. As long as $k \ge 1$, we will have

$$|h(0)| = o(\epsilon), \quad |\nabla h|(0) = o(\epsilon), \tag{10.29}$$

as $\epsilon \to 0$. Then if ϵ is sufficiently small, then condition (10.1) will also be satisfied.

11 Lecture 11

11.1 Relation with isothermal coordinates

Proposition 11.1. *For $n = 1$, an oriented conformal structure $(M, [g])$ is equivalent to an almost complex structure $J : TM \to TM$.*

Proof. Choose any representative g of the conformal class $[g]$. The Hodge star operator on 1-forms is uniquely defined by

$$\alpha \wedge *\beta = g(\alpha, \beta) dV_g, \tag{11.1}$$

where dV_g is the oriented volume element. Then $* : T^*M \to T^*M$ satisfies $*^2 = -Id$, so is an almost complex structure, which is clearly conformally invariant, since if $\tilde{g} = fg$, then $dV_{\tilde{g}} = f dV_g$, and $\tilde{g}(\alpha, \beta) = f^{-1} g(\alpha, \beta)$.

Conversely, given an almost complex structure J, we know this determines an orientation. Choose any non-zero 2-form compatible with this orientation, and call it dV_g. Then (11.1) defines an inner product on 1-forms. It is positive definite because by Lemma 2.2, at any point, we can assume that J is standard. So there is a basis dx, dy of T_p^*M such that $Jdx = -dy$. The complex orientation is $dy \wedge dx$. Writing any form $\alpha = \alpha_1 dx + \alpha_2 dy$, we then have

$$\alpha \wedge *\alpha = (\alpha_1 dx + \alpha_2 dy) \wedge (-\alpha_1 dy + \alpha_2 dx) = (|\alpha_1|^2 + |\alpha_2|^2) dy \wedge dx. \tag{11.2}$$

Clearly, different choices of volume elements lead to conformally equivalent metrics. □

We can write down explicity the above in coordinates.

Proposition 11.2. *Given $g = g_{ij} dx^i \otimes dx^j$, then*

$$*_g = \frac{\pm 1}{\sqrt{\det(g)}} \begin{pmatrix} g_{12} & -g_{11} \\ g_{22} & -g_{12} \end{pmatrix}, \tag{11.3}$$

depending upon choice of orientation. Consequently, if $\tilde{g} = fg$, then $*_{\tilde{g}} = *_g$. Conversely, given any

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{11.4}$$

with $a^2 + bc = -1$, and a choice of volume form $f dx^1 \wedge dx^2$, we define a Riemannian metric up to scaling by

$$g = \pm f \begin{pmatrix} -b & a \\ a & c \end{pmatrix}, \tag{11.5}$$

for the sign choice which makes this positive definite.

Proof. We choose a coordinate system $\{x^1, x^2\}$, and write

$$g(\partial_i, \partial_j) = g_{ij}. \tag{11.6}$$

We then have

$$g(dx^i, dx^j) = g^{ij}, \tag{11.7}$$

where g^{ij} are the components of the inverse matrix of g_{ij}. Also, we have

$$dV_g = \sqrt{\det(g)} dx^1 \wedge dx^2. \tag{11.8}$$

In matrix form, we write

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}. \tag{11.9}$$

We write

$$*dx^1 = a_{11}dx^1 + a_{21}dx^2, \quad *dx^2 = a_{12}dx^1 + a_{22}dx^2, \tag{11.10}$$

which in matrix form is just

$$* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{11.11}$$

We then have

$$dx^1 \wedge *dx^1 = dx^1 \wedge (a_{11}dx^1 + a_{21}dx^2) = a_{21}dx^1 \wedge dx^2 \tag{11.12}$$
$$dx^1 \wedge *dx^2 = dx^1 \wedge (a_{12}dx^1 + a_{22}dx^2) = a_{22}dx^1 \wedge dx^2 \tag{11.13}$$
$$dx^2 \wedge *dx^1 = dx^2 \wedge (a_{11}dx^1 + a_{21}dx^2) = -a_{11}dx^1 \wedge dx^2 \tag{11.14}$$
$$dx^2 \wedge *dx^2 = dx^2 \wedge (a_{12}dx^1 + a_{22}dx^2) = -a_{12}dx^1 \wedge dx^2. \tag{11.15}$$

On the other hand, by definition of the Hodge star operator, these must be equal to

$$g(dx^1, dx^1)dV_g = g^{11}\sqrt{\det(g)}dx^1 \wedge dx^2 = \frac{g_{22}}{\sqrt{\det(g)}} \tag{11.16}$$
$$g(dx^1, dx^2)dV_g = g^{12}\sqrt{\det(g)}dx^1 \wedge dx^2 = -\frac{g_{12}}{\sqrt{\det(g)}} \tag{11.17}$$
$$g(dx^2, dx^1)dV_g = g^{21}\sqrt{\det(g)}dx^1 \wedge dx^2 = -\frac{g_{21}}{\sqrt{\det(g)}} \tag{11.18}$$
$$g(dx^2, dx^2)dV_g = g^{22}\sqrt{\det(g)}dx^1 \wedge dx^2 = \frac{g_{11}}{\sqrt{\det(g)}}. \tag{11.19}$$

Comparing these equations, we obtain

$$*_g = \frac{\pm 1}{\sqrt{\det(g)}} \begin{pmatrix} g_{21} & -g_{11} \\ g_{22} & -g_{12} \end{pmatrix}. \tag{11.20}$$

This expression is obviously conformally invariant.

Conversely, given J and a volume element $dV = f dx^1 \wedge dx^2$, we define an inner product by

$$g(\alpha, \beta)dV_g = \alpha \wedge J\beta. \tag{11.21}$$

We then have

$$g(dx^i, dx^j)dV_g = g^{ij} f dx^1 \wedge dx^2, \tag{11.22}$$

and by the above, we see that

$$g^{-1} = \frac{1}{f} \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}, \tag{11.23}$$

so then

$$g = f \begin{pmatrix} -b & a \\ a & c \end{pmatrix}. \tag{11.24}$$

□

Corollary 11.3. *The problem of isothermal coordinates for a Riemannian metric g is equivalent to solving the Beltrami equation for the almost complex structure determined by $*_g$. That is, solving the Beltrami equation in a neighborhood of a point is equivalent to finding a coordinate system $\phi : U \to \mathbb{R}^2$ so that*

$$\phi_* g = e^{\lambda(x,y)}(dx^2 + dy^2), \tag{11.25}$$

for some function $\lambda : \phi(U) \to \mathbb{R}$.

Proof. A solution of the Beltrami equation $w_{\overline{z}} = \mu(z, \overline{z}) w_z$ is a holomorphic function. As long as $\partial_z w(p) \neq 0$, then we know that

$$w : (U, J) \to (\mathbb{C}, J_0) \tag{11.26}$$

is pseudoholomorphic. By the arguments above, w must be conformal and orientation preserving. But the conformal class determined by J_0 is the conformal class of the Euclidean metric, so we have isothermal coordinates.

Conversely, given an isothermal coordinate system $\phi : (U, [g]) \to (\mathbb{R}^2, [g_{Euc}])$. Then $[g]$ induces a unique J, and by the above, ϕ must be pseudoholomorphic with respect to J, so yields a solution of the Beltrami equation. □

11.2 Reduction to harmonic functions

Proposition 11.4. *If (M^2, J) is a real 2-dimensional almost complex manifold with J of class C^2, then J is a complex 1-manifold.*

Proof. As before, choose a compatible Riemannian metric g, and let $*$ be the Hodge star operator with respect to the almost complex orientation. Then on 1-forms, $J = *$. Given any point x in M, by Proposition 6.1, we need to find a function $f : U \to \mathbb{C}$ where U is a neighborhood of x satisfying $\overline{\partial}_J f = 0$ in U, and $\partial_J f(x) \neq 0$. This equation is

$$0 = \overline{\partial}_J f = df + iJdf = df + i * df. \tag{11.27}$$

Let us write $f = u + iv$, where u and v are real-valued. Then we need

$$0 = du + idv + iJ(du + idv) = (du - *dv) + i(dv + *du). \tag{11.28}$$

Note that applying the Hodge star to $du = *dv$, results in $dv = - * du$, so if we solve the single equation

$$du = *dv, \tag{11.29}$$

then $f = u + iv$ will be pseudo-holomorphic. Note that

$$\partial f = (du + *dv) + i(dv - *du), \tag{11.30}$$

so if $du(x) \neq 0$, then $\partial f(x) \neq 0$. To solve (11.29), we apply $*d$ to get

$$*d * dv = \delta dv = \Delta_g v = 0. \tag{11.31}$$

So if v is harmonic, then $*dv$ is closed, and by the Poincaré Lemma, we can solve $*dv = du$ in any simply-connected neighborhood U of x. To summarize, we have reduced the problem to finding a simply-connected neighborhood U of x, and a harmonic function $v : U \to \mathbb{R}$ with $dv(x) \neq 0$. □

We will finish the proof in the next lecture.

12 Lecture 12

The remaining ingredient we need is the following.

Theorem 12.1 (Harmonic coordinates of Sabitov-Shefel (1976) DeTurck-Kazdan (1981))**.** *If (M^n, g) is any Riemannian manifold with g of class $C^{k,\alpha}$ for $k \geq 1$, and $p \in M$, then there exists a coordinate system $(x_1, \ldots, x_n)$ defined in some neighborhood of p such that $\Delta_g(x_j) = 0$ for $j = 1, \ldots, n$, with x_i of class $C^{k+1,\alpha}$. If g is C^∞ then so are x_i. If g is real analytic, then so are x_i.*

Proof. We will prove the 2-dimensional case. The higher-dimensional case is identical. From above, we have that in local coordinates $\{x, y\}$,

$$* = \sqrt{\det(g)} \begin{pmatrix} -g^{21} & -g^{22} \\ g^{11} & g^{12} \end{pmatrix}. \tag{12.1}$$

Let us write $dv = v_1 dx + v_2 dy$, then

$$*dv = (-g^{21}v_1 - g^{22}v_2)\sqrt{\det(g)}dx + (g^{11}v_1 + g^{12}v_2)\sqrt{\det(g)}dy. \tag{12.2}$$

Then we have

$$*d*dv = \frac{1}{\sqrt{\det(g)}}\Big(\partial_2\big((g^{21}v_1 + g^{22}v_2)\sqrt{\det(g)}\big) + \partial_1\big((g^{11}v_1 + g^{12}v_2)\sqrt{\det(g)}\big)\Big) \tag{12.3}$$

In local coordinates, the Laplacian therefore has the form

$$\Delta v = \frac{1}{\sqrt{\det(g)}}\partial_i\big(g^{ij}u_j\sqrt{\det(g)}\big). \tag{12.4}$$

(This formula holds in any dimension). Expanding this out yields

$$\Delta v = g^{ij}\partial_i\partial_j u + (\partial_i g^{ij})u_j + \partial_i(\log(\sqrt{\det(g)}))g^{ij}u_j. \tag{12.5}$$

Jacobi's formula for the determinant is

$$\frac{1}{2}g^{pq}\partial_i g_{pq} = \partial_i(\log(\sqrt{\det(g)})), \tag{12.6}$$

so we have

$$\Delta v = g^{ij}\partial_i\partial_j u + (\partial_i g^{ij})u_j + \frac{1}{2}g^{pq}\partial_i g_{pq} g^{ij}u_j. \tag{12.7}$$

So we can expand

$$\Delta v = \Delta_0 u + Q(u), \tag{12.8}$$

where

$$Q(u) = a^{ij}\partial_i\partial_j u + b^j u_j \tag{12.9}$$

$$a^{ij} = g^{ij} - \delta^{ij} \tag{12.10}$$

$$b^j = \partial_i g^{ij} + \frac{1}{2}g^{pq}\partial_i g_{pq} g^{ij}. \tag{12.11}$$

Let us assume that $g \in C^{1,\alpha}(B(0,1))$. Using normal coordinates (which are OK under this regularity assumption: the geodesic equation has C^α coefficients), we have that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$. It follows that there exists a constant C so that

$$|g^{ij}(x) - \delta^{ij}| \leq C|x|^{1+\alpha} \tag{12.12}$$
$$|\partial_k g^{ij}(x)| \leq C|x|^{\alpha}. \tag{12.13}$$

Consider the mapping $\phi_\epsilon : B(0,1) \to B(0,\epsilon)$ defined by $\phi_\epsilon(x') = \epsilon x'$. Then

$$\phi_\epsilon^* g(x') = g_{ij}(\epsilon x')\epsilon^2 dx'_i \otimes dx'_j. \tag{12.14}$$

So the metrics $g_\epsilon = \epsilon^{-2}\phi_\epsilon^* g$ has components $(g_\epsilon)_{ij} = g_{ij}(\epsilon x')$ in the x' coordinates. We then have

$$|g_\epsilon^{ij}(x') - \delta^{ij}| \leq C\epsilon^{1+\alpha}|x'|^{1+\alpha} \tag{12.15}$$
$$|\partial_k g_\epsilon^{ij}(x')| \leq C\epsilon^{1+\alpha}|x'|^{\alpha}. \tag{12.16}$$

We can assume that there exists a constant C so that

$$|g_{ij}(x) - g_{ij}(y)| \leq C|x-y|^{\alpha}, \tag{12.17}$$

which implies that

$$|g'_{ij}(x') - g'_{ij}(y')| \leq C\epsilon^{\alpha}|x'-y'|^{\alpha}. \tag{12.18}$$

Also, there exists a constant C so that

$$|\partial_k g_{ij}(x) - \partial_k g_{ij}(y) \leq C|x-y|^{\alpha}, \tag{12.19}$$

which implies that

$$|\partial_k g'_{ij}(x') - \partial_k g'_{ij}(y')| \leq \epsilon^{1+\alpha}|x'-y'|^{\alpha}, \tag{12.20}$$

Consequently, we have that

$$\|a_\epsilon^{ij}\|_{C^{1,\alpha}(B(0,1))} \leq C\epsilon^{1+\alpha} \tag{12.21}$$
$$\|b_\epsilon^{ij}\|_{C^{0,\alpha}(B(0,1))} \leq C\epsilon^{\alpha}. \tag{12.22}$$

We then have that the exists a constant C so that.

$$\begin{aligned}\|Q(f)\|_{C^{0,\alpha}(B(0,1))} &= \|a^{ij}\partial_i\partial_j u + b^j u_j\|_{C^{0,\alpha}(B(0,1))} \\ &\leq \|a^{ij}\|_{C^{0,\alpha}(B(0,1))} \cdot \|\partial_i\partial_j u\|_{C^{0,\alpha}(B(0,1))} + \|b^j\|_{C^{0,\alpha}(B(0,1))} \cdot \|u_j\|_{C^{0,\alpha}(B(0,1))} \\ &\leq C\epsilon^{\alpha}\|f\|_{C^{2,\alpha}(B(0,1))}.\end{aligned} \tag{12.23}$$

We define $F : C^{2,\alpha}(B(0,1)) \to C^{0,\alpha}(B(0,1))$ by

$$F(h) = \Delta_{g_\epsilon}(x+h). \tag{12.24}$$

We define Q by

$$F(h) = F(0) + \Delta_0(h) + Q(h). \tag{12.25}$$

We define a mapping $T : C^{0,\alpha} \to C^{2,\alpha}_0$ by letting Tf be the unique solution to the Dirichlet problem

$$\Delta_0(Tf) = f \text{ in } B(0,1), \quad Tf = 0 \text{ on } \partial B(0,1). \tag{12.26}$$

From the basic theory of the Laplace operator, there exists a constant C so that

$$\|Tf\|_{C^{2,\alpha}(B(0,1))} \leq C\|f\|_{C^{0,\alpha}(B(0,1))}. \tag{12.27}$$

We would like to solve the equation

$$F(0) + \Delta_0 h + Q(h) = 0, \tag{12.28}$$

with h very small. Writing $h = Tf$, we can write this as

$$F(0) + f + Q(Tf) = 0, \tag{12.29}$$

that is

$$f = -Q(Tf) - F(0). \tag{12.30}$$

So we would like to find a fixed point of the operator $S : C^{0,\alpha}(B(0,1)) \to C^{0,\alpha}(B(0,1))$ defined by

$$Sf = -Q(Tf) - F(0). \tag{12.31}$$

We next claim that for ϵ sufficiently small, S is a contraction mapping. To see this, we compute

$$\begin{aligned} \|Sf_1 - Sf_2\|_{C^{0,\alpha}(B(0,1))} &= \|Q(Tf_1) - Q(Tf_2)\|_{C^{0,\alpha}(B(0,1))} \\ &\leq C\epsilon^\alpha(\|Tf_1 - Tf_2\|_{C^{2,\alpha}(B(0,1))} \\ &\leq C\epsilon^\alpha(\|f_1 - f_2\|_{C^{0,\alpha}(B(0,1))}. \end{aligned} \tag{12.32}$$

We then let $f_0 = 0$, and define $f_{j+1} = Sf_j$. If $n \geq m$, we have

$$\begin{aligned} \|f_n - f_m\|_{C^{0,\alpha}(B(0,1))} &\leq \sum_{j=m+1}^{n} \|f_j - f_{j-1}\|_{C^{0,\alpha}(B(0,1))} \\ &= \sum_{j=m+1}^{n} \|S^{j-1}f_1 - S^{j-1}f_0\|_{C^{0,\alpha}(B(0,1))} \\ &\leq \sum_{j=m+1}^{n} (C\epsilon^\alpha)^{j-1}\|f_1 - f_0\|_{C^{0,\alpha}(B(0,1))} \\ &\leq \frac{(C\epsilon^\alpha)^m}{1 - C\epsilon^\alpha}\|f_1 - f_0\|_{C^{0,\alpha}(B(0,1))}. \end{aligned} \tag{12.33}$$

For ϵ sufficiently small, we see that the right hand side limits to 0 as $m \to \infty$. This proves that the sequence f_j is a Cauchy sequence in the Banach space $C^{0,\alpha}(B(0,1))$, which therefore converges to a limit f_∞. Since S is continuous, we therefore have

$$Tf_\infty = T \lim_{j\to\infty} f_j = \lim_{j\to\infty} Tf_j = \lim_{j\to\infty} f_{j+1} = f_\infty. \tag{12.34}$$

Take $m = 1$ in (12.33) to get

$$\|f_n - f_1\|_{C^{0,\alpha}(B(0,1))} \leq \frac{C\epsilon^\alpha}{1 - C\epsilon^\alpha}\|f_1 - f_0\|_{C^{0,\alpha}(B(0,1))}. \tag{12.35}$$

Letting $n \to \infty$ yields

$$\|f_\infty - f_1\|_{C^{0,\alpha}(B(0,1))} \leq \frac{C\epsilon^\alpha}{1 - C\epsilon^\alpha}\|f_1 - f_0\|_{C^{0,\alpha}(B(0,1))}. \tag{12.36}$$

Since $f_0 = 0$, using the triangle inequality, this implies

$$\|f_\infty\|_{C^{0,\alpha}(B(0,1))} \leq C\|f_1\|_{C^{0,\alpha}(B(0,1))}. \tag{12.37}$$

We have found a harmonic function $x + h_\infty$, where $h_\infty = Tf_\infty$, but we also need to verify that it has nonvanishing differential at the origin. For this, we estimate

$$\|h_\infty\|_{C^{2,\alpha}(B(0,1))} = \|Tf_\infty\|_{C^{2,\alpha}(B(0,1))} \leq C\|f_\infty\|_{C^{0,\alpha}(B(0,1))} \leq C\|F(0)\|_{C^{0,\alpha}(B(0,1))}, \tag{12.38}$$

since $f_1 = S(0) = -F(0)$. We then estimate

$$\|F(0)\|_{C^{0,\alpha}(B(0,1))} = \|\Delta_{g_\epsilon} x\|_{C^{0,\alpha}(B(0,1))} \leq \|b^1\|_{C^{0,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \tag{12.39}$$

At the origin,

$$\partial_1(x + h_\epsilon)(0) = 1 + \partial_1 h_\epsilon(0), \tag{12.40}$$

and

$$|\partial_1 h_\epsilon(0)| \leq \|h_\epsilon\|_{C^1(B(0,1))} \leq \|h_\epsilon\|_{C^{2,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \tag{12.41}$$

So if ϵ is sufficiently small, $\partial_1(x + h_\epsilon)(0) \neq 0$, and we are done.

For higher regularity, we argue as follows. If $g \in C^{k,\alpha}$ then in particular $g \in C^{1,\alpha}$. By the above, we can find $C^{2,\alpha}$ harmonic coordinates $\{x_1, x_2\}$. We then write

$$0 = \Delta x_k = g^{ij}\partial_i\partial_j x_k + b^j. \tag{12.42}$$

That is

$$g^{ij}\partial_i\partial_j x_k = -b^j \in C^{k-1,\alpha}. \tag{12.43}$$

The left hand side is an elliptic operator with $C^{k,\alpha}$ coefficients, so by elliptic regularity arguments, $x_k \in C^{k+2,\alpha}$. If $g \in C^\infty$, the right hand side is also in C^∞, so again by elliptic regularity we see that $x_k \in C^{k,\alpha}$ for any $k \geq 0$, so $x_k \in C^\infty$. For the real analytic case, there is a general result that solutions of elliptic equations with real analytic coefficients are real analytic. (This is proved by Cauchy's majorant method, similar to how we did before.)

□

Unfortunately, the trick in this subsection does not help us to solve the Newlander-Nirenberg problem in higher dimensions. However, the method in the previous section *can* be extended to the higher dimensional case, which we will discuss soon.

Remark 12.2. The above methods require that $\mu \in C^{1,\alpha}$. The Beltrami equation can be solved locally for $\mu \in C^{0,\alpha}$ by inverting the $\partial_{\bar{z}}$ operator using the Cauchy-Pompeiu formula. However, this is a bit technical so we will omit. There are many great references for this method, see for example [?], [?], [?], [?], [?].

13 Lecture 13

13.1 Endomorphisms

Let $End_{\mathbb{R}}(TM)$ denotes the real endomorphisms of the tangent bundle.

Proposition 13.1. *On an almost complex manifold (M, J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{+}(TM) \oplus End_{-}(TM) \tag{13.1}$$

where the first factor on the left consists of endomorphisms I commuting with J,

$$IJ = JI \tag{13.2}$$

and the second factor consists of endomorphisms I anti-commuting with J,

$$IJ = -JI \tag{13.3}$$

Proof. Given J, we define

$$I_{+} = \frac{1}{2}(I - JIJ) \tag{13.4}$$

$$I_{-} = \frac{1}{2}(I + JIJ). \tag{13.5}$$

Then

$$I_{+}J = \frac{1}{2}(IJ - JIJ^{2}) = \frac{1}{2}(IJ + JI),$$

and

$$JI_{+} = \frac{1}{2}(JI - J^{2}IJ) = \frac{1}{2}(JI + IJ).$$

Next,

$$I_{-}J = \frac{1}{2}(IJ + JIJ^{2}) = \frac{1}{2}(IJ - JI),$$

and

$$JI_- = \frac{1}{2}(JI + J^2 IJ) = \frac{1}{2}(JI - IJ).$$

Clearly, $I = I_+ + I_-$. To prove it is a direct sum, if $IJ = JI$ and $IJ = -JI$, then $IJ = 0$ which implies that $I = 0$ since J is invertible.

□

We write down the above in a basis. Choose a real basis $\{e_1, \dots e_{2n}\}$ such that the complex structure J_0 is given by

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{13.6}$$

Then in matrix terms, the proposition is equivalent to the following decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A+D & B-C \\ C-B & A+D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A-D & B+C \\ B+C & D-A \end{pmatrix}. \tag{13.7}$$

13.2 The space of almost complex structures

We define

$$\mathcal{J}(\mathbb{R}^{2n}) \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n, \mathbb{R}), J^2 = -I_{2n}\} \tag{13.8}$$

We next give some alternative descriptions of this space.

Proposition 13.2. *The space $\mathcal{J}(\mathbb{R}^{2n})$ is the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, and thus*

$$\dim(\mathcal{J}(\mathbb{R}^{2n})) = 2n^2. \tag{13.9}$$

Proof. We note that $GL(2n, \mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n, \mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$\Phi_A : J \mapsto AJA^{-1}. \tag{13.10}$$

Obviously,

$$(AJA^{-1})^2 = AJA^{-1}AJA^{-1} = AJ^2A^{-1} = -I, \tag{13.11}$$

and

$$\Phi_{AB}(J) = (AB)J(AB)^{-1} = ABJB^{-1}A^{-1} = \Phi_A\Phi_B(J), \tag{13.12}$$

so is indeed a group action (on the left). Given J and J', there exists bases

$$\{e_1, \dots, e_n, Je_1, \dots, Je_n\} \text{ and } \{e'_1, \dots, e'_n, J'e'_1, \dots, J'e'_n\}. \tag{13.13}$$

Define $S \in GL(2n, \mathbb{R})$ by $Se_k = e'_k$ and $S(Je_k) = J'e'_k$. Then $J' = SJS^{-1}$, and the action is therefore transitive. The stabilizer subgroup of J_0 is

$$Stab(J_0) = \{A \in GL(2n, \mathbb{R}) : AJ_0A^{-1} = J_0\}, \tag{13.14}$$

that is, A commutes with J_0. We have seen previously that this can be identified with $GL(n, \mathbb{C})$. □

13.3 The space of orthogonal complex structures

We define

$$\mathcal{O}(\mathbb{R}^{2n}) \equiv \{J \in \mathrm{O}(2n,\mathbb{R}), J^2 = -I_{2n}\} \tag{13.15}$$
$$= \{J \in \mathrm{O}(2n,\mathbb{R}) | J^T = -J\}. \tag{13.16}$$

which we refer to as the set of *orthogonal* complex structures on $\mathbb{R}^{2n}$, since these are by definition compatible with the Euclidean metric.

Proposition 13.3. *The space $\mathcal{O}(\mathbb{R}^{2n})$ is the homogeneous space $\mathrm{O}(2n)/\mathrm{U}(n)$ and thus*

$$\dim(\mathcal{O}(\mathbb{R}^{2n})) = n(n-1). \tag{13.17}$$

Proof. The proof is similar to Proposition 13.2 above. □

Remark 13.4. Actually, it is moreover true that there is a smooth projection $\pi : \mathcal{J}(\mathbb{R}^{2n}) \to \mathcal{O}(\mathbb{R}^{2n})$ which is a fiber bundle with fiber diffeomorphic to a Euclidean ball of dimension $n(n+1)$. We will prove this over the next couple of lectures.

Example 13.5. For $n = 1$, we have a fiber bundle

$$B^2 \to \mathcal{J}(\mathbb{R}^2) \to \mathcal{O}(\mathbb{R}^2). \tag{13.18}$$

But $O(2)/U(1) = \pm 1$, so this agree with what we previously observed: $\mathcal{J}(\mathbb{R}^2)$ is the disjoint union of two 2-balls. Let us illustrate the above, in this case. A skew-symmetric 1 by 1 matrix vanishes, so we have

$$M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \tag{13.19}$$

We then compute that

$$\exp(M) J_0 \exp(-M) = \frac{(e^{4\sqrt{a^2+b^2}} - 1)e^{-2\sqrt{a^2+b^2}}}{2\sqrt{a^2+b^2}} \begin{pmatrix} b & -a - \sqrt{a^2+b^2}\frac{e^{4\sqrt{a^2+b^2}}+1}{e^{4\sqrt{a^2+b^2}}-1} \\ -a + \sqrt{a^2+b^2}\frac{e^{4\sqrt{a^2+b^2}}+1}{e^{4\sqrt{a^2+b^2}}-1} & -b \end{pmatrix} \tag{13.20}$$

which is a pretty elaborate way to parmetrize a hyperbola.

Example 13.6. For $n = 2$, we have a fiber bundle

$$B^6 \to \mathcal{J}(\mathbb{R}^4) \to \mathcal{O}(\mathbb{R}^4). \tag{13.21}$$

The space $\mathcal{O}(\mathbb{R}^4)$ has 2 components which are diffeomorphic, so we only need to consider $\mathrm{SO}(4)/\mathrm{U}(2)$. But $\mathrm{SO}(4)/\mathrm{U}(2) = S^2$, as can be see by the following:

$$\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)/\pm(1,1). \tag{13.22}$$

Equivalently, there is a double covering $\pi : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$. It is not hard to see that $\pi^{-1}(\mathrm{U}(2)) = S^1 \times \mathrm{SU}(2)$, so we see that

$$\mathrm{SO}(4)/\mathrm{U}(2) = \mathrm{SU}(2) \times \mathrm{SU}(2)/S^1 \times \mathrm{SU}(2) = \mathrm{SU}(2)/S^1 = S^3/S^1 = S^2. \tag{13.23}$$

We can see this more directly as follows. Let $\vec{v} = (x, y, z) \in S^2$. Consider the change of basis matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & -y \\ 0 & x & 0 & -z \\ 0 & y & 1 & 0 \\ 0 & z & 0 & x \end{pmatrix}. \tag{13.24}$$

Then a computation yields that

$$J = BJ_0B^{-1} = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & x & 0 \end{pmatrix}. \tag{13.25}$$

The condition that J is compatible with g_0 says that J is a skew-symmetric orthogonal matrix, and the above expression gives all possibilities.

Example 13.7. For $n = 3$, we have

$$B^{12} \to \mathcal{J}(\mathbb{R}^6) \to \mathcal{O}(\mathbb{R}^6). \tag{13.26}$$

The space $\mathrm{SO}(6)/\mathrm{U}(3)$ turns out to be diffeomorphic to $\mathbb{CP}^3$, complex projective 3-space, so $\mathcal{J}(\mathbb{R}^6)$ is the disjoint union of two rank 12 vector bundles over $\mathbb{CP}^3$. To see this, we use the isomorphism

$$\mathrm{Spin}(6) \cong \mathrm{SU}(4). \tag{13.27}$$

The group $\mathrm{SU}(4)$ acts transitively on $\mathbb{CP}^3$ by the following. If $A \in \mathrm{SU}(4)$, and $[L]$ is a line through the origin in $\mathbb{C}^4$, then $A \cdot [L] = [AL]$. The stabilizer of the point $[1, 0, 0, 0]$ is

$$\begin{pmatrix} \alpha & 0 \\ 0 & B \end{pmatrix} \tag{13.28}$$

such that $|\alpha| = 1$ and $B \in \mathrm{U}(3)$ with $\det(B) = \overline{\alpha}$. This is isomorphic to $\mathrm{U}(3)$. Note also that $\pm Id \in \mathrm{SU}(4)$ acts identically, so $\mathrm{SO}(6)$ also acts transitively on $\mathbb{CP}^3$ with the same stabilizer. Consequently, we obtain that $\mathbb{CP}^3 = \mathrm{SO}(6)/\mathrm{U}(3)$. It is possible to write an explicit formula for the J corresponding to a point in $\mathbb{CP}^3$, but we will not do this here.

Example 13.8. Higher dimensions get more complicated, the orthogonal complex strutures are no longer just complex projective spaces. The next case $\mathrm{SO}(8)/\mathrm{U}(4)$ can be identified with a quadric hypersurface in $\mathbb{CP}^7$. In general, $\mathcal{O}(\mathbb{R}^{2n})$ is always a complex subvariety of a complex projective space, in particular, it is a complex manifold [**?**].

14 Lecture 14

14.1 Almost complex structure on the space of almost complex structures

We know that the set of matrices

$$\mathcal{J}_{2n} \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n, \mathbb{R}), J^2 = -I_{2n}\} \tag{14.1}$$

is a manifold of real dimension $2n^2$. If $J(t) : (-\epsilon, \epsilon) \to \mathcal{J}_{2n}$ is a smooth path with $J(0) = J$, then differentiation yields

$$-(I_{2n})' = (J \circ J)' = J' \circ J + J \circ J'. \tag{14.2}$$

So letting $J'(0) = I$, we have that

$$IJ + JI = 0. \tag{14.3}$$

Thus we can identify the tangent space at any J as

$$T_J\mathcal{J}_{2n} = \{I \in End(\mathbb{R}^n)\ |IJ + JI = 0\}, \tag{14.4}$$

the space of endomorphisms which anti-commute with J.

If $I_J \in T_J\mathcal{J}_{2n}$, we compute that

$$(I_J \circ J)J + J(I_J \circ J) = -I_J + JI_JJ = J(JI_J + I_JJ) = 0. \tag{14.5}$$

Therefore, we can define

$$\mathbb{J} : T_J\mathcal{J}_{2n} \to T_J\mathcal{J}_{2n} \tag{14.6}$$

by $\mathbb{J}(I_J) = I_J \circ J$. Clearly

$$\mathbb{J}^2 I_J = \mathbb{J}(I_J \circ J) = I_J \circ J \circ J = -I_J, \tag{14.7}$$

so $\mathbb{J}$ is an almost complex structure.

Question 14.1. Is $\mathbb{J}$ a complex structure?

To answer this question, we need to check check if $N_{\mathbb{J}} = 0$. We need to find coordinates in order to check this.

We also know that

$$\begin{aligned} \mathcal{O}_{2n} &\equiv \{J \in \mathrm{O}(2n, \mathbb{R})\ |\ J^2 = -Id\} \\ &= \{J \in \mathrm{O}(2n, \mathbb{R})\ |\ J^T = -J\} \end{aligned} \tag{14.8}$$

is a submanifold of $\mathcal{J}_{2n}$. The orthogonality condition is $JJ^T = Id$. Differentiating this along a path yields

$$0 = IJ^T + JI^T. \tag{14.9}$$

Since $J \in \mathcal{O}_{2n}$, $J^T = -J$, so this is $IJ = JI^T$. But we also have $IJ + JI = 0$, so

$$J(I + I^T) = 0, \tag{14.10}$$

so $I^T = -I$ since J is invertible. Therefore for $J\mathcal{O}_{2n}$, we have

$$T_J\mathcal{O}_{2n} = \{I \in End(\mathbb{R}^n)\ |\ IJ + JI = 0, I^T = -I\}. \tag{14.11}$$

14.2 The symplectic group

Definition 14.2. The symplectic group is defined as

$$\mathrm{Sp}(2n, \mathbb{R}) = \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^T J_0 A = J_0\}. \tag{14.12}$$

Equivalently, these are the matrices preserving the symplectic form on $\mathbb{R}^{2n}$ defined by J_0, that is,

$$\omega_0(AX, AY) = \omega_0(X, Y), \tag{14.13}$$

where $\omega_0(X, Y) \equiv g_0(J_0 X, Y)$. We note that $\mathrm{Sp}(2n, \mathbb{R})$ is a non-compact Lie group of dimension $n^2 + n(n+1)$. We then define

$$\begin{aligned} \mathcal{S}_{2n} &\equiv \{J \in \mathrm{Sp}(2n, \mathbb{R}) \mid J^2 = -Id\} \\ &= \{J \in \mathrm{Sp}(2n, \mathbb{R}) \mid J^T J_0 + J_0 J = 0\}. \end{aligned} \tag{14.14}$$

Proposition 14.3. *The set $\mathcal{S}_{2n}$ is the homogeneous space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$. Furthermore, $\mathcal{S}_{2n}$ is diffeomorphic to $\mathbb{R}^{n(n+1)}$.*

Proof. This is similar to above, we have an action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathcal{S}_{2n}$ by $A \mapsto \Phi_A$ where

$$\Phi_A : J \mapsto AJA^{-1}. \tag{14.15}$$

This action is transitive, and

$$Stab(J_0) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n, \mathbb{R}) = \mathrm{U}(n). \tag{14.16}$$

Next, we define the Siegel upper half space

$$\mathbb{H}^n = \{Z \in End(\mathbb{C}^n) \mid Z^T = Z, Im(Z) > 0\} \tag{14.17}$$

The group $\mathrm{Sp}(2n, \mathbb{R})$ also acts on $\mathbb{H}^n$ as follows. Writing $M \in \mathrm{Sp}(2n, \mathbb{R})$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{14.18}$$

then we let

$$\Phi_M = (AZ + B)(CZ + D)^{-1}. \tag{14.19}$$

This is a transitive group action, and

$$\mathrm{Stab}(i \cdot Id) = \mathrm{U}(n). \tag{14.20}$$

Thus we see that

$$\mathbb{H}^n = \mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n) = \mathcal{S}_{2n} \tag{14.21}$$

so our original $\mathcal{S}_{2n}$ is diffeomorphic to $\mathbb{R}^{n(n+1)}$. □

Exercise 14.4. (i) Obviously, $\mathbb{H}^n$ is a complex manifold, since it is an open subset of symmetric complex matrices, so is an open subset of $\mathbb{C}^{n(n+1)/2}$. Show that $\mathbb{H}^n$ is biholomorphic to the unit ball in $\mathbb{C}^{n(n+1)/2}$.
(ii) (D. Salamon) Show that there is an $\mathrm{Sp}(2n,\mathbb{R})$-equivariant diffeomorphism $\Psi : \mathbb{H}^n \to \mathcal{S}_{2n}$ given by

$$\Psi(Z) = \begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix}, \tag{14.22}$$

where $Z = X + iY$.

Remark 14.5. For more details of the Siegel upper half space, see [**?**, **?**]. Note that for $n = 1$, the Siegel upper half space is just the upper half space $\mathbb{H} = \{x + iy \mid y > 0\} \subset \mathbb{C}$. The action of $\mathrm{Sp}(2n,\mathbb{R})$ is by fractional linear transformations, so we see that

$$\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) = \mathrm{SO}(2,1) \tag{14.23}$$

acts by hyperbolic isometries, and we have that

$$\mathbb{H} = \mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(1) = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) = \mathrm{SO}(2,1)/\mathrm{SO}(2), \tag{14.24}$$

with the latter isomorphism arising from the hyperboloid model of hyperbolic space. The previous exercise shows that we can parametrize the positively oriented almost complex structures on $\mathbb{R}^2$ by $x + iy \in \mathbb{H}$ as

$$y^{-1}\begin{pmatrix} x & -x^2 - y^2 \\ 1 & -x \end{pmatrix}, \tag{14.25}$$

which is a much nicer parametrization than (13.20).

15 Lecture 15

We see that $\mathcal{S}_{2n}$ is a submanifold of $\mathcal{J}_{2n}$ passing through J_0, and from (15.3), we have that

$$T_{J_0}\mathcal{S}_{2n} = \{I \in End(\mathbb{R}^n) \mid IJ + JI = 0, I^T = I\}. \tag{15.1}$$

We then have the following decomposition

$$T_{J_0}\mathcal{J}_{2n} = T_{J_0}\mathcal{O}_{2n} \oplus T_{J_0}\mathcal{S}_{2n}, \tag{15.2}$$

which is just the decomposition of a matrix which anti-commutes with J_0 into its skew-symmetric and symmetric parts.

Let extend this to any point $J \in \mathcal{O}_{2n}$. For any such J, define

$$\begin{aligned} \mathcal{S}_{2n,J} &\equiv \{A \in \mathrm{GL}(2n,\mathbb{R}) \mid A^T JA = J, A^2 = -Id\} \\ &= \{A \in \mathrm{Sp}(2n,\mathbb{R}) \mid A^T = JAJ\}. \end{aligned} \tag{15.3}$$

Similar to above, we have the $\mathcal{S}_{2n,J}$ is a submanifold of dimension $n(n+1)$ passing through J. Letting $A(t)$ be a path in $\mathcal{S}_{2n,J}$ with $J(0) = J$ and $A'(0) = I$, we have that

$$I^T JJ + J^T JI = 0. \tag{15.4}$$

Since $J \in \mathcal{O}_{2n}$, $J^T = -J$ and therefore $I^T = I$. Consequently, we have shown that

$$T_J\mathcal{S}_{2n,J} = \{I \in End(\mathbb{R}^{2n} \mid IJ + JI = 0, I^T = I\}. \tag{15.5}$$

So we have extended the decomposition (15.2) to any $J \in \mathcal{O}_{2n}$:

$$T_J\mathcal{J}_{2n} = T_J\mathcal{O}_{2n} \oplus T_J\mathcal{S}_{2n,J}, \tag{15.6}$$

which again is just the decomposition of matrices which anti-commute with J into skew-symmetric and symmetric pieces.

Proposition 15.1. *The intersection* $\mathcal{S}_{2n} \cap \mathcal{O}_{2n} = J_0$.

Proof. The Cartan decomposition for $\mathrm{Sp}(2n, \mathbb{R})$ implies that any symplectic matrix can be written as $S = e^s U$ where s is a symmetric matrix in $\mathfrak{sp}(2n, \mathbb{R})$, and $U \in \mathrm{U}(n)$. Note that symmetric elements in the $\mathfrak{sp}(2n, \mathbb{R})$ necessarily anti-commute with J_0, which implies that

$$e^s J_0 = J_0 e^{-s}, \tag{15.7}$$

(by looking at the power series of the matrix exponential). Recall that the action of conjugation of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathcal{S}_{2n}$ is transitive, so any $J \in \mathcal{S}_{2n}$ may be written as

$$J = e^s U J_0 U^{-1} e^{-s} = e^s J_0 e^{-s} = e^{2s} J_0. \tag{15.8}$$

If $J \in \mathcal{O}_{2n}$, then

$$Id = JJ^T = e^{2s} J_0 J_0^T e^{2s} = e^{4s}. \tag{15.9}$$

But the exponential map is a diffeomorphism from the set of symmetric matrices onto the set of positive definite symmetric matrices, which implies that $s = 0$. □

This implies that for $J \in \mathcal{O}_{2n}$, we have

$$\mathcal{S}_{2n,J} \cap \mathcal{O}_{2n} = J. \tag{15.10}$$

We can now prove the following

Proposition 15.2 (Daurtseva-Smolentsev)**.** *There is a smooth projection* $\pi : \mathcal{J}_{2n} \to \mathcal{O}_{2n}$ *with* $\pi^{-1}(J_1) = \mathcal{S}_{2n,J_1}$ *for* $J_1 \in \mathcal{O}_{2n}$.

Proof. Given any $J \in \mathcal{J}_{2n}$, the metric

$$g_J(X, Y) = \frac{1}{2}\big(g_0(X, Y) + g_0(JX, JY)\big) \tag{15.11}$$

is compatible with J, and the form

$$\omega_J(X,Y) = g(JX,Y) = \frac{1}{2}\big(g_0(JX,Y) - g_0(X,JY)\big) \tag{15.12}$$

is a nondegenerate 2-form since the it the Kähler form with respect to the Riemannian metric g_J. Then we can define an endomorphism B as follows: Fixing X, we have a linear functional.

$$f_X : Y \mapsto \omega_J(X,Y) \in Hom(\mathbb{R}^{2n},\mathbb{R}) \tag{15.13}$$

By the Reisz representation theorem there exists an element B, such that

$$f_X(Y) = \omega_J(X,Y) = g_0(BX,Y). \tag{15.14}$$

Since ω is non-degenerate, so is B. We claim that B is skew-symmetric. To see this,

$$g_0(BX,Y) + g_0(X,BY) = \omega_J(X,Y) + \omega_J(Y,X) = 0. \tag{15.15}$$

Then if $X \neq 0$,

$$g_0(B^2X,X) = -g_0(BX,BX) < 0, \tag{15.16}$$

since B is non-degenerate. So $-B^2$ is symmetric and positive definite. Note the following: from above we have

$$\begin{aligned}\omega_J(X,Y) = g_0(BX,Y) &= \frac{1}{2}\big(g_0(JX,Y) - g_0(X,JY)\big)\\ &= \frac{1}{2}\big(g_0(JX,Y) - g_0(J^TX,Y)\big) = g\Big(\frac{1}{2}(J-J^T)X,Y\Big),\end{aligned} \tag{15.17}$$

so

$$B = \frac{1}{2}(J - J^T) \tag{15.18}$$

Lemma 15.3. *Any symmetric positive definite matrix has a unique positive definite symmetric square root.*

Proof. Any symmetric matrix S is diagonalizeble by an orthogonal matrix, so there exists $O \in \mathrm{O}(2)$ so that $S = O\Lambda O^{-1}$ where Λ is a diagonal matrix with strictly positive entries. This has a unique diagonal square root $\Lambda^{1/2}$ with positive entries. Then we define $S^{1/2} = O\Lambda^{1/2}O^{-1}$. □

Note that any power of S will commute with any other power of S. We then consider the matrix $C = (-B^2)^{-1/2}$. and define $J_1 = CB = BC$. This is an almost complex structure:

$$J_1^2 = (-B^2)^{-1/2}B(-B^2)^{-1/2}B = B(-B^2)^{-1/2}(-B^2)^{-1/2}B = B(-B^{-2})B = -Id. \tag{15.19}$$

Also, $J_1 \in \mathcal{O}_{2n}$:

$$g_0(CBX,CBY) = g_0\big(BX,C^2BY)\big) = g_0\big(X,-B(-B^2)^{-1}BY\big) = g_0(X,Y). \tag{15.20}$$

Next, we claim that $J \in \mathcal{S}_{2n,J_1}$. First, we note that the endomorphism B^2 commutes with J. To see this, from (15.18), we have

$$B^2 = \frac{1}{4}\Big(-2Id - JJ^T - J^T J\Big), \tag{15.21}$$

so

$$JB^2 = \frac{1}{4}\Big(-2J + J^T - JJ^T J\Big), \tag{15.22}$$

and

$$B^2 J = \frac{1}{4}\Big(-2J - JJ^T J + J^T\Big) = JB^2. \tag{15.23}$$

Therefore, and power of B^2 commutes with J. We then define $\omega_{J_1}(X,Y) = g_0(J_1 X, Y)$. Then

$$\begin{aligned}
\omega_{J_1}(JX, JY) &= g_0(BCJX, JY) = \omega_J(CJX, JY)\\
&= \omega_J(JCX, JY) = \omega_J(CX, Y) = \frac{1}{2}\big(g_0(JCX, Y) - g_0(CX, JY)\big)\\
&= \frac{1}{2}\big(g_0(JCX, Y) - g_0(J^T CX, Y)\big)\\
&= g_0\Big(\frac{1}{2}(J - J^T)CX, Y\Big)) = g_0(BCX, Y) = g_0(J_1 X, Y) = \omega_{J_1}(X, Y)
\end{aligned} \tag{15.24}$$

This says that the symplectic form ω_{J_1} in invariant under J, which says that $J \in \mathcal{S}_{2n,J_1}$.

To finish, we need to show that for $J_1 \in \mathcal{O}_{2n}$, $\pi^{-1}(J_1) = \mathcal{S}_{2n,J_1}$. The above argument showed that $\pi^{-1}(J_1) \subset \mathcal{S}_{2n,J_1}$. So we need to show that if $J \in \mathcal{S}_{2n,J_1}$ then $\pi(J) = J_1$. To see this, since $J \in \mathcal{S}_{2n,J_1}$ recall that we can write

$$J = e^I J_1 e^{-I} \tag{15.25}$$

for a matrix I satisfying $I^T = I, IJ_1 + J_1 I = 0$. (This fact follows from the Cartan decomposition...) Since $J_1^T = -J_1$, we have

$$B = \frac{1}{2}(J - J^T) = \frac{1}{2}(e^I J_1 e^{-I} + e^{-I} J_1 e^I). \tag{15.26}$$

Then since $e^I J_1 = J_1 e^{-I}$ (easy to show directly), we have

$$B = \frac{1}{2}(e^{2I} + e^{-2I})J_1. \tag{15.27}$$

Then

$$B^2 = \frac{1}{4}(-2Id + e^{4I} + e^{-4I}) = -\frac{1}{4}(e^{2I} + e^{-2I})^2. \tag{15.28}$$

So we see that

$$C = (-B^2)^{-1/2} = 2(e^{2I} + e^{-2I})^{-1}, \tag{15.29}$$

and we have

$$CB = 2(e^{2I} + e^{-2I})^{-1}\frac{1}{2}(e^{2I} + e^{-2I})J_1 = J_1, \tag{15.30}$$

which proves that $\pi(J) = J_1$. □

15.1 Bundle structure

We begin with a local parametrization, due to Cayley, of $\mathrm{SO}(2n, \mathbb{R})$ by skew-symmetric matrices given by

$$I \mapsto (Id + I)^{-1}(Id - I). \tag{15.31}$$

This makes sense, since $Id + I$ is always invertible if I is skew-symmetric (proof: a kernel element would satisfy $Iv = -v$, and then taking an inner product with v shows that $v = 0$). It is easily seen that the right hand side is orthogonal. Note that that image of this mapping consists of the orthogonal matrices which do not have -1 as an eigenvalue.

However, for the purposes of the following proof, we need a modification of this. As we observed in the above proof if I is skew-symmetric, then $-I^2$ is symmetric and positive definite. Therefore, we can parametrize by

$$I \mapsto (Id - I^2)^{-1/2}(Id - I). \tag{15.32}$$

This motivates the following.

Proposition 15.4 (Daurtseva). *The projection $\pi : \mathcal{J}_{2n} \to \mathcal{O}_{2n}$ has the structure of a fiber bundle with fiber $\mathcal{S}_{2n}$.*

Proof. We need to find local trivializations. So given $J \in \mathcal{O}_{2n}$, define

$$\Phi_J : T_J\mathcal{O}_{2n} \times \mathcal{S}_{2n,J} \to \mathcal{J}_{2n}, \tag{15.33}$$

by

$$\Phi_J(I, J_1) = (Id - I^2)^{-1/2}(Id - I)J_1(Id - I)^{-1}(Id - I^2)^{1/2} \tag{15.34}$$

We claim that for I fixed, $\Phi_J(I, J_1) \in \mathcal{S}_{2n,J'}$ where $J' = (Id - I)J_1(Id + I)$. This means that Φ_J maps (I, J_1) onto a fiber, and is therefore a bundle chart. However, due to lack of time, we will leave the verification of this as an exercise.

□

15.2 Coordinates on $\mathcal{J}_{2n}$

Above, we used the exponential mapping, but this turns out to not be the nicest way to endow $\mathcal{J}_{2n}$ with coordinates, since it is difficult to invert. There is a better parametrization, given by the following.

Lemma 15.5. *Given $J \in \mathcal{J}_{2n}$ The mapping*

$$\Phi_J : T_J\mathcal{J}_{2n} \to \mathcal{J}_{2n} \tag{15.35}$$

given by

$$I \mapsto (Id - I)J(Id - I)^{-1} \tag{15.36}$$

is a diffeomorphism from the subset where $Id - I$ is invertible onto the set of almost complex structures

$$U(J) = \{J' \in \mathcal{J}_{2n} \mid Id - J'J \text{ is invertible}\}. \tag{15.37}$$

Proof. We want to solve

$$J' = (Id - I)J(Id - I)^{-1} = J(Id + I)(Id - I)^{-1} \tag{15.38}$$

multiplying by J, we obtain

$$-JJ' = (Id + I)(Id - I)^{-1} \tag{15.39}$$

The solution of this is given by

$$I = -(Id - JJ')^{-1}(Id + JJ'). \tag{15.40}$$

On the other hand, we write

$$J' = (Id - I)J(Id - I)^{-1} = -(Id - I)J^{-1}(Id - I)^{-1} = -(Id - I)\big((Id - I)J\big)^{-1} \tag{15.41}$$

$$= -(Id - I)\big(J(Id + I)\big)^{-1} = (Id - I)\big(Id + I\big)^{-1}J. \tag{15.42}$$

Multiplying by J on the right, we have

$$-J'J = (Id - I)(Id + I)^{-1}. \tag{15.43}$$

The solution of this is given by

$$I = (Id - J'J)^{-1}(Id + J'J). \tag{15.44}$$

Since both of the solutions must be equal, this implies that $IJ + JI = 0$. So the inverse mapping is given by

$$\Phi_J^{-1}(J') = (Id - J'J)^{-1}(Id + J'J) = -(Id - JJ')^{-1}(Id + JJ'). \tag{15.45}$$

□

Proposition 15.6. *The above charts Φ_J for $J \in \mathcal{O}_{2n}$ give a covering of $\mathcal{J}_{2n}$.*

Proof. We know there exists $M \in \mathrm{GL}(2n, \mathbb{R})$ such that

$$J = MJ_0M^{-1}. \tag{15.46}$$

The Cartan decomposition of $\mathrm{GL}(2n, \mathbb{R})$ enables us to write $M = S \cdot O$, where S is a positive definite symmetric matrix, and O is orthogonal. Then

$$J = SOJ_0(SO)^{-1} = S(OJ_0O^{-1})S^{-1}. \tag{15.47}$$

Then $J_1 = OJ_0O^{-1} \in \mathcal{O}_{2n}$. By Proposition 13.1 we can decompose

$$S = S_+ + S_- \tag{15.48}$$

where S_+ commutes with J_1 and S_- anti-commutes with J_1. Recall that

$$S_+ = \frac{1}{2}(S - J_1SJ_1) = \frac{1}{2}(S + J_1SJ_1^{-1}), \tag{15.49}$$

which shows that S_+ is positive definite, since S is. Consequently, we can write

$$\begin{aligned} J = (S_+ + S_-)J_1(S_+ + S_-)^{-1} &= (Id + S_-S_+^{-1})S_+J_1S_+^{-1}(Id + S_-S_+^{-1})^{-1} \\ &= (Id + S_-S_+^{-1})J_1(Id + S_-S_+^{-1})^{-1}. \end{aligned} \tag{15.50}$$

Finally, since S_+ commutes with J_1 and S_- anti-commutes with J_1, the product $S_-S_+^{-1}$ anti-commutes with J_1, so J lies in the image of Φ_{J_1} by taking $I = -S_-S_+^{-1}$. □

16 Lecture 16

16.1 Integrability

Recall the almost complex structure on the space of almost complex structures

$$\mathbb{J} : T_J\mathcal{J}_{2n} \to T_J\mathcal{J}_{2n} \tag{16.1}$$

defined by $\mathbb{J}(I_J) = I_J \circ J$.

Corollary 16.1. *The almost complex structure on $\mathcal{J}_{2n}$ defined above is integrable.*

Proof. We will show that the Nijenhuis tensor $N_{\mathbb{J}}$ vanishes, that is for any $J \in \mathcal{J}_{2n}$, and any tangent vectors $X, Y \in T_J\mathcal{J}_{2n}$, we have

$$0 = N_{\mathbb{J}}(X,Y) = 2\{[\mathbb{J}X, \mathbb{J}Y] - [X,Y] - \mathbb{J}[X, \mathbb{J}Y] - \mathbb{J}[\mathbb{J}X, Y]\}. \tag{16.2}$$

Recall that $GL(2n,\mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n,\mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$\Phi_A : J \mapsto AJA^{-1}. \tag{16.3}$$

If we let $a \in \mathfrak{gl}(2n,\mathbb{R})$, then let $A(t)$ be any path in $\mathrm{GL}(2n,\mathbb{R})$ with $A(0) = Id$ and $A'(0) = a$. Then for $J \in \mathcal{J}_{2n}$, we see that $A(t)JA(t)^{-1}$ is a path in $\mathcal{J}_{2n}$ through J. We compute

$$\frac{d}{dt}(AJA^{-1}) = A'JA^{-1} + J(-A^{-1}A'A^{-1}), \tag{16.4}$$

so evaluating at $t = 0$, we have that

$$X_a \equiv aJ - Ja \tag{16.5}$$

is a vector field on $\mathcal{J}_{2n}$. We can also see this directly by

$$JX_a + X_aJ = J(aJ - Ja) + (aJ - Ja)J = JaJ + a - a - JaJ = 0. \tag{16.6}$$

To prove (16.2), we just need to prove that for all $a, b \in \mathfrak{gl}(2n,\mathbb{R})$, we have that

$$0 = N_{\mathbb{J}}(X_a, X_b) = 2\{[\mathbb{J}X_a, \mathbb{J}X_b] - [X_a, X_b] - \mathbb{J}[X_a, \mathbb{J}X_b] - \mathbb{J}[\mathbb{J}X_a, X_b]\}. \tag{16.7}$$

This follows since $N_{\mathbb{J}}$ is a tensor, and for any point $J \in \mathcal{J}_{2n}$, the mapping

$$E : \mathfrak{gl}(2n,\mathbb{R}) \to T_J\mathcal{J}_{2n} \tag{16.8}$$

given by $a \mapsto X_a(J)$ is surjective. This follows since on the subspace $\{a \in \mathfrak{gl}(2n,\mathbb{R}) \mid aJ + Ja = 0\}$, the mapping E satisfies $E(a) = aJ - Ja = 2aJ$, which is clearly invertible.

Next, we compute

$$\begin{aligned}
[\mathbb{J}X_a, \mathbb{J}X_b] &= [-a - JaJ, -b - JbJ] \\
&= [a,b] + [a, JbJ] + [JaJ, b] + [JaJ, JbJ] \\
&= a(JbJ) - b(JaJ) + JaJ(JbJ) - JbJ(JaJ) \\
&= abJ + Jba - baJ - Jab + JaJbJ + JbJaJ - JbJaJ - JaJbJ \\
&= [a,b]J - J[a,b] = X_{[a,b]}.
\end{aligned} \tag{16.9}$$

Also,

$$
\begin{aligned}
[\mathbb{J}X_a, X_b] &= [-a - JaJ, bJ - Jb] \\
&= -[a, bJ] + [a, Jb] - [JaJ, bJ] + [JaJ, Jb] \\
&= -a(bJ) + a(Jb) - JaJ(bJ) + bJ(JaJ) + JaJ(Jb) - Jb(JaJ) \\
&= -ba + ab - bJaJ + bJaJ + JabJ + JaJb - JbaJ - JaJb \\
&= -[a, b] + J[a, b]J = -\mathbb{J}X_{[a,b]}.
\end{aligned} \tag{16.10}
$$

This implies that

$$
[X_a, \mathbb{J}X_b] = -[\mathbb{J}X_b, X_a] = \mathbb{J}X_{[b,a]} = -\mathbb{J}X_{[a,b]}. \tag{16.11}
$$

Next, we compute

$$
\begin{aligned}
[X_a, X_b] &= [aJ - Ja, bJ - Jb] \\
&= [aJ, bJ] + [aJ, Jb] - [Ja, bJ] + [Ja, Jb] \\
&= aJ(bJ) - bJ(aJ) + aJ(Jb) - Jb(aJ) - Ja(bJ) + bJ(Ja) + Ja(Jb) - Jb(Ja) \\
&= baJ - abJ + aJb - aJb - bJa + bJa + Jab - Jba \\
&= -[a, b]J + J[a, b] = -X_{[a,b]}.
\end{aligned} \tag{16.12}
$$

Combining the above, we finally have

$$
\begin{aligned}
N_{\mathbb{J}}(X_a, X_b) &= 2\{[\mathbb{J}X_a, \mathbb{J}X_b] - [X_a, X_b] - \mathbb{J}[X_a, \mathbb{J}X_b] - \mathbb{J}[\mathbb{J}X_a, X_b]\} \\
&= 2\{X_{[a,b]} + X_{[a,b]} - \mathbb{J}(-\mathbb{J}X_{[a,b]}) - \mathbb{J}(-\mathbb{J}X_{[a,b]})\} = 0,
\end{aligned} \tag{16.13}
$$

since $\mathbb{J}^2 = -Id$. □

Corollary 16.2. *The group* $\mathrm{GL}(2n, \mathbb{R})$ *acts holomorphically on* $\mathcal{J}_{2n}$.

Proof. Given $A \in \mathrm{GL}(2n, \mathbb{R})$ then $\Phi_A(J) = AJA^{-1}$. The claim is that for any $J \in \mathcal{J}_{2n}$,

$$
(\Phi_A)|_{*,J} \circ \mathbb{J}|_J = \mathbb{J}|_{\Phi_A(J)} \circ (\Phi_A)|_{*,J}. \tag{16.14}
$$

Taking a path $J(t) \in \mathcal{J}_{2n}$ with $J(0) = J$, and $J'(0) = I_J \in T_J\mathcal{J}_{2n}$, we have

$$
(\Phi_A)|_{*,J}(I_J) = AI_JA^{-1} \in T_{\Phi_A(J)}\mathcal{J}_{2n}. \tag{16.15}
$$

The left hand side of (16.14) is then

$$
(\Phi_A)|_{*,J} \circ \mathbb{J}|_J(I_J) = (\Phi_A)|_{*,J}(I_JJ) = AI_JJA^{-1}. \tag{16.16}
$$

The right hand side of (16.14) is

$$
\mathbb{J}|_{\Phi_A(J)} \circ (\Phi_A)|_{*,J}(I_J) = \mathbb{J}|_{\Phi_A(J)}(AI_JA^{-1}) = AI_JA^{-1}\Phi_A(J) = AI_JA^{-1}AJA^{-1} = AI_JJA^{-1}. \tag{16.17}
$$

□

Corollary 16.3. *The submanifolds $\mathcal{O}_{2n}$ and $\mathcal{S}_{2n,J_1}$ for $J_1 \in \mathcal{O}_{2n}$ are complex submanifolds of $\mathcal{J}_{2n}$.*

Proof. For this, we just need to show that $\mathbb{J}$ preserves the tangent spaces to these submanifolds. If $I \in T_{J_1}\mathcal{O}_{2n}$ for $J_1 \in \mathcal{O}_{2n}$, then $IJ_1 + J_1 I = 0$ and $I^T = -I$. By definition,

$$\mathbb{J}I = IJ_1, \tag{16.18}$$

and then

$$(IJ_1)^T = (J_1)^T I^T = (-J_1)I^T = J_1 I = -IJ_1. \tag{16.19}$$

Next, let $I \in T_J\mathcal{S}_{2n,J_1}$, for $J_1 \in \mathcal{O}_{2n}$. Then J satisfies

$$J^T J_1 J = J_1 \tag{16.20}$$

If $J(t)$ is a path in $\mathcal{S}_{2n,J_1}$, with $J(0) = J$ and $J'(0) = I$, then differentiating

$$J(t)^T J_1 J(t) = J_1,\ J(t)^2 = -Id, \tag{16.21}$$

yields

$$I^T J_1 J + J^T J_1 I = 0,\ IJ + JI = 0. \tag{16.22}$$

Since

$$\mathbb{J}I = IJ, \tag{16.23}$$

we compute

$$(IJ)^T J_1 J + J^T J_1(IJ) = J^T I^T J_1 J + J^T J_1(-JI) = J^T(-J^T J_1 I) + (-J_1)I = J_1 I - J_1 I = 0. \tag{16.24}$$

To finish, above we showed that $\mathcal{J}_{2n}$ is a complex manifold, so the Nijenhuis tensor $N(\mathbb{J}_{2n}) = 0$. Since the tangent spaces of these submanifolds are $\mathbb{J}$-invariant, the Nijenhuis tensor of the restricted almost complex structures vanish, and by the Newlander-Nirenberg Theorem, these submanifolds are complex manifolds. □

Exercise 16.4. Show that the projection $\pi : \mathcal{J}_{2n} \to \mathcal{O}_{2n}$ is holomorphic. The fibers are biholomorphic to the unit ball $B \subset \mathbb{C}^{n(n+1)/2}$, so this gives $\mathcal{J}_{2n}$ the structure of a holomorphic disc bundle over $\mathcal{O}_{2n}$.

Remark 16.5. From previous remarks, we knew that $\mathcal{O}_{2n}$ had a complex structure. Similarly, we saw that $\mathcal{S}_{2n}$ has a complex structure (since it is $\mathbb{H}^n$). The induced complex structures induced as submanifolds of $\mathcal{J}_{2n}$ must be the same as these because the above almost complex structure is $\mathrm{GL}(2n,\mathbb{R})$-invariant. Since the $\mathcal{O}_{2n}$ complex structure is $\mathrm{O}(2n,\mathbb{R})$-invariant, and the complex structure on $\mathcal{S}_{2n}$ is $\mathrm{Sp}(2n,\mathbb{R})$-invariant, so they must agree if they agree at a point.

16.2 Holomorphic coordinates

By the Newlander-Nirenberg Theorem, we know that complex coordinates exist. In this subsection, we will write down explicit holomorphic coordinates. Let us recall the coordinates from above. Given $J \in \mathcal{J}_{2n}$ The mapping

$$\Phi_J : T_J\mathcal{J}_{2n} \to \mathcal{J}_{2n} \tag{16.25}$$

given by

$$I \mapsto (Id - I)J(Id - I)^{-1} \tag{16.26}$$

is a diffeomorphism from the subset

$$U'(J) = \{I \in T_J\mathcal{J}_{2n} \mid Id - I \text{ is invertible }\} \tag{16.27}$$

onto the set of almost complex structures

$$U(J) = \{J' \in \mathcal{J}_{2n} \mid Id - J'J \text{ is invertible}\}. \tag{16.28}$$

Proposition 16.6. *View $(T_J\mathcal{J}_{2n}, \mathbb{J}_J)$ as an complex vector space. Then $\Phi_J : U'(J) \to U(J)$ is a holomorphic coordinate system.*

Proof. We need to show that

$$(\Phi_J)_* \circ \mathbb{J}_J = \mathbb{J} \circ (\Phi_J)_* \tag{16.29}$$

Given any $I \in T_J\mathcal{J}_{2n}$, and $v \in T_I(T_J\mathcal{J}_{2n}) \cong T_J\mathcal{J}_{2n}$, choose a path $I(t)$ such that $I(0) = I$, and $I'(0) = v$. Then differentiating and evaluating at $t = 0$ yields

$$((\Phi_J)_*|_I)(v) = -vJ(Id - I)^{-1} + (Id - I)J(Id - I)^{-1}v(Id - I)^{-1}. \tag{16.30}$$

Since $\mathbb{J}_J(v) = vJ$, we have

$$(\Phi_J)_* \circ (\mathbb{J}_J)|_I(v) = v(Id - I)^{-1} + (Id - I)J(Id - I)^{-1}vJ(Id - I)^{-1}. \tag{16.31}$$

On the other hand, we have

$$\begin{aligned}
&\mathbb{J} \circ ((\Phi_J)_*|_I)(v) = \mathbb{J}_{\Phi_J(v)}\Big(- vJ(Id - I)^{-1} + (Id - I)J(Id - I)^{-1}v(Id - I)^{-1}\Big)\\
&= \Big(- vJ(Id - I)^{-1} + (Id - I)J(Id - I)^{-1}v(Id - I)^{-1}\Big)(Id - I)J(Id - I)^{-1}\\
&= v(Id - I)^{-1} + (Id - I)J(Id - I)^{-1}vJ(Id - I)^{-1},
\end{aligned} \tag{16.32}$$

and we are done. ◻

16.3 Graph over the reals

Next, we will give another description of $\mathcal{J}(\mathbb{R}^{2n})$. Define

$$\mathcal{P}(\mathbb{R}^{2n}) = \{P \subset \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n} \mid dim_{\mathbb{C}}(P) = n, \\ P \text{ is a complex subspace satisfying } P \cap \overline{P} = \{0\}\}.$$

If we consider $\mathbb{R}^{2n} \otimes \mathbb{C}$, we note that complex conjugation is a well defined complex anti-linear map $\mathbb{R}^{2n} \otimes \mathbb{C} \to \mathbb{R}^{2n} \otimes \mathbb{C}$.

Proposition 16.7. *The space $\mathcal{P}(\mathbb{R}^{2n})$ can be explicitly identified with $\mathcal{J}(\mathbb{R}^{2n})$ by the following. If $J \in \mathcal{J}(\mathbb{R}^{2n})$ then let*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = T^{1,0}(J) \oplus T^{0,1}(J), \tag{16.33}$$

where

$$T^{0,1}(J) = \{X + iJX, X \in \mathbb{R}^{2n}\} = \{-i\}\text{-eigenspace of } J. \tag{16.34}$$

This an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and letting $T^{1,0}(J) = \overline{T^{0,1}(J)}$, we have $T^{1,0} \cap T^{0,1} = \{0\}$.

For the converse, given $P \in \mathcal{P}(\mathbb{R}^{2n})$, then P may be written as a graph over $\mathbb{R}^{2n} \otimes 1$, that is

$$P = \{X' + iJX' \mid X' \in \mathbb{R}^{2n} \subset \mathbb{C}^{2n}\}, \tag{16.35}$$

with $J \in \mathcal{J}(\mathbb{R}^{2n})$, and

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \overline{P} \oplus P = T^{1,0}(J) \oplus T^{0,1}(J). \tag{16.36}$$

Proof. For the forward direction, we already know this. To see the other direction, consider the projection map Re restricted to P

$$\pi = Re : P \to \mathbb{R}^{2n}. \tag{16.37}$$

We claim this is a real linear isomorphism. Obviously, it is linear over the reals. Let $X \in P$ satisfy $\pi(X) = 0$. Then $Re(X) = 0$, so $X = iX'$ for some real $X' \in \mathbb{R}^{2n}$. But $\overline{X} = -iX' \in P \cap \overline{P}$, so by assumption $X = 0$. Since these spaces are of the same real dimension, π has an inverse, which we denote by J. Clearly then, (16.35) is satisfied. Since P is a complex subspace, given any $X = X' + iJX' \in P$, the vector $iX' = (-JX') + iX'$ must also lie in P, so

$$(-JX') + iX' = X'' + iJX'', \tag{16.38}$$

for some real X'', which yields the two equations

$$JX' = -X'' \tag{16.39}$$

$$X' = JX''. \tag{16.40}$$

applying J to the first equation yields

$$J^2X' = -JX'' = -X'. \tag{16.41}$$

Since this is true for any X', we have $J^2 = -I_{2n}$. □

Remark 16.8. We note that $J \mapsto -J$ corresponds to interchanging $T^{0,1}$ and $T^{1,0}$. If we choose $P = T^{0,1}(J) = i\mathbb{R}^n$, then Re restricted to P is not an isomorphism, for example.

Exercise 16.9. The above proposition embeds $\mathcal{J}(\mathbb{R}^{2n})$ as a subset of the complex Grassmannian $G(n, 2n, \mathbb{C})$. These spaces have the same dimension, so it is an *open* subset. Furthermore, the condition that the projection to the real part is an isomorphism is generic, so it is also dense. Show that this mapping

$$\mathcal{J}_{2n} \to \mathcal{P}_{2n} \subset G(n, 2n, \mathbb{C}) \tag{16.42}$$

is holomorphic, where $G(n, 2n, \mathbb{C})$ has its natural complex structure as a complex Grassmannian.

Exercise 16.10. Show that under the above embedding, we have that

$$\mathcal{O}_{2n} \to G_0(n, 2n, \mathbb{C}), \tag{16.43}$$

which is the Grassmannian of maximal isotropic planes in $\mathbb{C}^{2n}$, that is, n-planes P such that $(g_0)|_P = 0$, where g_0 is the Euclidean metric in $\mathbb{C}^{2n}$, extended to complex vectors by linearity.

Exercise 16.11. The previous section gives a projection $\pi : \mathcal{J}_n \to G_0(n, 2n, \mathbb{C})$. Interpret this projection in terms of the Grassmannian. That is, show that if P is an n-plane in $\mathbb{C}^{2n}$ satisfying $P \cap \overline{P} = \{0\}$, then we can associate a unique isotropic n-plane $\pi(P)$ to P. The fiber over J_0 is given by the complex Lagrangian Grassmannian,

$$Gr_L(n, 2n, \mathbb{C}) = \{P \mid \omega_0|_P = 0\}, \tag{16.44}$$

where ω_0 is the complexified standard symplectic form. Show that this projection is holomorphic.

17 Lecture 17

17.1 Graphs over $T^{0,1}(J_0)$

Above we viewed $T^{0,1}(J)$ as a graph corresponding to the decomposition $\mathbb{C}^{2n} = \mathbb{R}^{2n} \oplus i\mathbb{R}^{2n}$. In the section we will instead view $T^{0,1}(J)$ as a graph corresponding to the decomposition $\mathbb{C}^{2n} = T^{0,1}(J_0) \oplus T^{1,0}(J_0)$. This corresponds to a mapping

$$\phi : T^{0,1}(J_0) \to T^{1,0}(J_0), \tag{17.1}$$

by writing

$$T^{0,1}(J) = \{v + \phi v \mid v \in T^{0,1}(J_0)\}. \tag{17.2}$$

Note we can view ϕ as an element of

$$Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0), \tag{17.3}$$

so we will view ϕ as an element of the latter space. In "coordinates", we can write

$$\phi = \phi_{\bar{j}}^{k} d\bar{z}^{j} \otimes \frac{\partial}{\partial z^{k}}, \tag{17.4}$$

and we will view $\phi_{\bar{j}}^{k}$ as an n by n complex matrix. We define $\overline{\phi}$ as a $\mathbb{C}$-linear mapping

$$\overline{\phi} : T^{1,0}(J_0) \to T^{0,1}(J_0), \tag{17.5}$$

by

$$\overline{\phi}(v) = \overline{\phi(\bar{v})}. \tag{17.6}$$

Consider the mapping

$$\phi + \overline{\phi} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \tag{17.7}$$

which in matrix form is

$$\phi + \overline{\phi} = \begin{pmatrix} 0 & \phi \\ \overline{\phi} & 0 \end{pmatrix}. \tag{17.8}$$

Recall from (**??**) that this is the complexification of an $\mathbb{R}$-linear mapping

$$I_{\phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \tag{17.9}$$

satisfying $I_{\phi} J_0 + J_0 I_{\phi} = 0$, which is given by

$$I_{\phi} = \begin{pmatrix} Re(\phi) & Im(\phi) \\ Im(\phi) & -Re(\phi) \end{pmatrix} \tag{17.10}$$

Note also that the composition

$$\phi\overline{\phi} : T^{1,0}(J_0) \to T^{1,0}(J_0) \tag{17.11}$$

make sense, and is given in components by

$$(\phi\overline{\phi})_{j}^{k} = \phi_{\bar{l}}^{k} (\overline{\phi})_{j}^{\bar{l}}. \tag{17.12}$$

Proposition 17.1. *If $\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0)$, then ϕ determines an almost complex structure if and only if $Id_n - \phi\overline{\phi}$ is invertible which is equivalent to $Id_{2n} \pm I_{\phi}$ being invertible. The corresponding almost complex structure is*

$$J_{\phi} = (Id + I_{\phi}))J_0(Id + I_{\phi})^{-1}. \tag{17.13}$$

Conversely, given J such that $Id - J_0 J$ is invertible, then J corresponds to a unique ϕ with $Id + I_{\phi}$ is invertible, which is given by

$$I_{\phi} = (Id - J_0 J)^{-1}(Id + J_0 J). \tag{17.14}$$

Proof. Given

$$\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0) = Hom_{\mathbb{C}}(T^{0,1}(J_0), T^{1,0}(J_0)), \tag{17.15}$$

then

$$T^{0,1}(J_\phi) = \{v + \phi v, v \in T^{0,1}(J_0)\} \tag{17.16}$$

is an n-dimensional complex subspace of $\mathbb{R}^{2n} \otimes \mathbb{C}$. If $X \in T^{0,1}_\phi \cap \overline{T^{0,1}_\phi}$, then

$$v + \phi v = w + \overline{\phi} w, \tag{17.17}$$

where $v \in T^{0,1}(J_0)$ and $w \in T^{1,0}(J_0)$. This yields the equations

$$\overline{\phi} w = v \tag{17.18}$$

$$\phi v = w. \tag{17.19}$$

This says that $\phi\overline{\phi}$ has 1 as an eigenvalue. Note this is equivalent to saying that the matrix $\phi + \overline{\phi}$ having ± 1 as an eigenvalue which is equivalent to I_ϕ having ± 1 as an eigenvalue. Next, any $\tilde{v} \in T^{0,1}(J_\phi)$ is written as

$$\begin{aligned} \tilde{v} &= v + \phi(v) \\ &= Re(v) + Re(\phi(v)) + i(Im(v) + Im(\phi(v)) \end{aligned} \tag{17.20}$$

We compute

$$\begin{aligned} Re(\phi(v)) &= \frac{1}{2}\Big(\phi(v) + \overline{\phi(v)}\Big) \\ &= \frac{1}{2}\Big(\phi(v) + \overline{\phi}(\overline{v})\Big) \\ &= (\phi + \overline{\phi})\Big(\frac{v + \overline{v}}{2}\Big) \\ &= I_\phi(Re(v)). \end{aligned} \tag{17.21}$$

Next,

$$\begin{aligned} Im(\phi(v)) &= \frac{1}{2i}\Big(\phi(v) - \overline{\phi(v)}\Big) \\ &= \frac{1}{2i}\Big(\phi(v) - \overline{\phi}(\overline{v})\Big) \\ &= (\phi + \overline{\phi})\Big(\frac{v - \overline{v}}{2i}\Big) \\ &= I_\phi(Im(v)). \end{aligned} \tag{17.22}$$

Next, any element $v \in T^{0,1}(J_0)$ can be written as

$$v = X' + iJ_0X', \tag{17.23}$$

for $X' \in \mathbb{R}^{2n}$, so we have

$$\tilde{v} = (Id + I_\phi)X' + i(Id + I_\phi)(J_0 X'). \tag{17.24}$$

But if $\tilde{v} \in T^{0,1}(J_\phi)$, we must have

$$Im(\tilde{v}) = J_\phi Re(\tilde{v}), \tag{17.25}$$

which yields

$$(Id + I_\phi)(J_0 X') = J_\phi (Id + I_\phi)X'. \tag{17.26}$$

This implies that

$$J_\phi = (Id + I_\phi)J_0(Id + I_\phi)^{-1}. \tag{17.27}$$

The remainder of the proposition follows by solving this equation for I_ϕ. $\square$

17.2 The analogue of the Beltrami equation in higher dimensions

In this subsection, we will consider almost complex structures defined by

$$\phi^j_{\bar{k}} : U \to Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong Mat(n \times n, \mathbb{C}), \tag{17.28}$$

where U is an open subset in $\mathbb{R}^{2n}$.

Proposition 17.2. *If $\phi^j_{\bar{k}}$ defines an almost complex structure on U, then a function $f : U \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial}{\partial \overline{z}^j} f + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k} f = 0 \tag{17.29}$$

Proof. By Proposition **??**, a function f is holomorphic if and only if $Zf = 0$ for all vector fields $Z \in \Gamma(T^{0,1}_\phi)$. A local basis for $T^{0,1}_\phi$ is given by

$$Z_{\bar{j}} = \frac{\partial}{\partial \overline{z}^j} + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k}, \tag{17.30}$$

so we are done. $\square$

Proposition 17.3. *The complex structure J_ϕ is integrable if and only if*

$$\frac{\partial}{\partial \overline{z}^l} \phi^j_{\bar{k}} - \frac{\partial}{\partial \overline{z}^k} \phi^j_{\bar{l}} + \phi^m_{\bar{k}} \frac{\partial}{\partial z^m} \phi^j_{\bar{l}} - \phi^m_{\bar{l}} \frac{\partial}{\partial z^m} \phi^j_{\bar{k}} = 0. \tag{17.31}$$

Proof. By Proposition **??**, the integrability equation is equivalent to $[T^{0,1}_\phi, T^{0,1}_\phi] \subset T^{0,1}_\phi$. Writing

$$\phi = \sum \phi^j_{\bar{k}} d\overline{z}^k \otimes \frac{\partial}{\partial z^j}, \tag{17.32}$$

if J_ϕ is integrable, then we must have

$$\Big[\frac{\partial}{\partial \overline{z}^i} + \phi\Big(\frac{\partial}{\partial \overline{z}^i}\Big), \frac{\partial}{\partial \overline{z}^k} + \phi\Big(\frac{\partial}{\partial \overline{z}^k}\Big)\Big] \in T^{0,1}_\phi. \tag{17.33}$$

This yields

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\Big] + \Big[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] + \Big[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\Big] \in T^{0,1}_\phi \tag{17.34}$$

The first two terms are

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\Big] + \Big[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] = \sum_j \Big(\frac{\partial \phi^j_{\bar{k}}}{\partial \overline{z}^i} - \frac{\partial \phi^j_{\bar{i}}}{\partial \overline{z}^k}\Big)\frac{\partial}{\partial z^j}.$$

The third term is

$$\Big[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\Big] = \phi^j_{\bar{i}}\Big(\frac{\partial}{\partial z^j}\phi^l_{\bar{k}}\Big)\frac{\partial}{\partial z^l} - \phi^l_{\bar{k}}\Big(\frac{\partial}{\partial z^l}\phi^j_{\bar{i}}\Big)\frac{\partial}{\partial z^j}.$$

Both terms are in $T^{1,0}(J_0)$. For sufficiently small ϕ however, $T^{0,1}_\phi \cap T^{1,0}(J_0) = \{0\}$, and therefore (17.31) holds. The converse holds by reversing this argument. □

17.3 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the C^2 case into the analytic case [?, ?]. In the z coodinates, our Beltrami equation is

$$\frac{\partial w}{\partial \overline{z}^j} + \phi^k_{\bar{j}}\frac{\partial w}{\partial z^k} = 0 \tag{17.35}$$

We want to change coordinates $\xi = \xi(z, \overline{z})$ so that such that our Beltrami equation transform into another Beltrami equation with analytic coefficients. Write

$$w(z, \overline{z}) = W(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})) \tag{17.36}$$
$$\phi^k_{\bar{j}}(z, \overline{z}) = U^k_{\bar{j}}(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})). \tag{17.37}$$

Then

$$\frac{\partial w}{\partial \overline{z}^j} = \frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial \overline{z}^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \overline{z}^j} \tag{17.38}$$
$$\frac{\partial w}{\partial z^j} = \frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^j}. \tag{17.39}$$

So the Beltrami equation becomes

$$\frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial \overline{z}^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \overline{z}^j} + U^k_{\bar{j}}(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z}))\Big(\frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^k} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^k}\Big) = 0. \tag{17.40}$$

By inverting the matrix coefficients, this transforms into another Beltrami system of the form

$$\frac{\partial W}{\partial \bar{\xi}^j} + \tilde{U}^k_{\bar{j}}(\xi, \bar{\xi})\frac{\partial W}{\partial \xi^k} = 0 \tag{17.41}$$

where $\tilde{U}$ is of the form

$$\tilde{U}^k_{\bar{j}} = \Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \bar{z}^*} + U^p_* \frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)^{\bar{q}}_{\bar{j}} \Big(\frac{\partial \xi^k}{\partial \bar{z}^q} + U^p_{\bar{q}} \frac{\partial \xi^k}{\partial z^p}\Big). \tag{17.42}$$

Let us try to find coordinates so that

$$\sum_j \frac{\partial}{\partial \xi^j} \tilde{U}^k_{\bar{j}}(\xi, \bar{\xi}) = 0. \tag{17.43}$$

Since the integrability condition is independent of coordinates, we also have that

$$\frac{\partial}{\partial \bar{\xi}^l} \tilde{U}^j_{\bar{k}} - \frac{\partial}{\partial \bar{\xi}^k} \tilde{U}^j_{\bar{l}} + \tilde{U}^m_{\bar{k}} \frac{\partial}{\partial \xi^m} \tilde{U}^j_{\bar{l}} - \tilde{U}^m_{\bar{l}} \frac{\partial}{\partial \xi^m} \tilde{U}^j_{\bar{k}} = 0. \tag{17.44}$$

Let us assume that we have found our coordinate system ξ. Then if we view the coupled system (17.43)-(17.44) as an equation for $\tilde{U}^k_{\bar{j}}$, then it is not hard to see that this is an elliptic system with analytic coefficients. The Cauchy majorant method allows us to show that $\tilde{U}^k_{\bar{j}}$ are then analytic functions in the ξ coordinates.

To find the coordinate system ξ, we must write out (17.43), and this becomes a second order elliptic system for ξ as a function of the original z coordinates. Viewed this way, the equation (17.43) is *quasilinear* of the form

$$F(D^2\xi, D\xi, \xi) = 0. \tag{17.45}$$

The linearization of F at a function ξ is given by

$$F'_\xi(h) = \frac{d}{dt} F(D^2(\xi + th), D(\xi + th), \xi + th)\Big|_{t=0}. \tag{17.46}$$

The linearization of F at the function $\xi = z$ is a multiple of the Laplacian (plus lower order terms), so we can write

$$F(z + h) = F(z) + \frac{1}{4}\Delta h + Q(h), \tag{17.47}$$

where the $Q(h)$ terms are higher order. Then we can apply our previous fixed point argumnet to find the solution.

18 Lecture 18

18.1 Integrability: the analytic case

We can assume that J is analytic around the origin in $\mathbb{C}^n$, then J has a convergent power series expansion

$$J = \sum_{j=0}^{\infty} J_j, \tag{18.1}$$

where J_j is a real polynomial which is homogeneous of degree j. By Lemma 2.2, we can assume that $J_0 = J_{Euc}$, after a linear change of coordinates.

For $k = 1, \dots, n$, let us try and find a function $f : U \to \mathbb{C}$, where

$$f^k = \sum_{j=1}^{\infty} f_j^k, \tag{18.2}$$

where f_j^k is homogeneous of degree j, and which satisfies

$$\overline{\partial}_J f^k = 0, \ \ f_1^k = z^k. \tag{18.3}$$

Then by the inverse function theorem, $(f^1, \dots, f^n)$ will form a holomorphic coordinate system in some possibly smaller neighborhood of the origin.

In the following, we will omit the superscript k. The equation we need to solve is

$$\begin{aligned} 0 = \overline{\partial}_J f &= \frac{1}{2}(df + iJdf) \\ &= \frac{1}{2}(df + i(J - J_0 + J_0)df) \\ &= \overline{\partial}_0 f + \frac{i}{2}(J - J_0)df. \end{aligned} \tag{18.4}$$

Writing this out term-by-term, we have the system

$$\begin{aligned} \overline{\partial}_0 f_1 &= 0 \\ \overline{\partial}_0 f_2 &= -\frac{i}{2} J_1 df_1 \\ \overline{\partial}_0 f_3 &= -\frac{i}{2}\Big(J_2 df_1 + J_1 df_2\Big), \end{aligned}$$

we see the general formula is

$$\overline{\partial}_0 f_l = -\frac{i}{2} \sum_{j+k=l} J_j df_k. \tag{18.5}$$

Proposition 18.1. *If f_j solves the above system for $j = 1, \dots, p$, then the expression*

$$H_p = -\frac{i}{2} \sum_{j+k=p} J_j df_k \tag{18.6}$$

is a form of type $(0,1)$ with respect to J_0, and satisfies $\overline{\partial}_0 H_p = 0$.

Proof. The assumption implies that $f = \sum_{j=1}^p f_j$ satisfies

$$\overline{\partial}_J f = O(|z|^p) = H_p + O(|z|^{p+1}). \tag{18.7}$$

Since $\overline{\partial}_J f$ is of type $(0,1)$ with respect to J, we have

$$J \overline{\partial}_J f = -i \overline{\partial}_J f. \tag{18.8}$$

Expanding both sides of this equation yields

$$(J_0 + J_1 + \dots)(H_p + O(|z|^{p+1}) = -i(H_p + O(|z|^{p+1}), \tag{18.9}$$

and the leading term of this equation says that

$$J_0 H_p = -i H_p, \tag{18.10}$$

so H_p is of type $(0,1)$ with respect to J_0, as claimed.

For the next step, we use the assumption of integrability of J which implies that the operator $\overline{\partial}_J : \Lambda^{0,1}(J) \to \Lambda^{0,2}(J)$ defined by $\overline{\partial}_J \alpha = \Pi_{\Lambda^{0,2}(J)} d\alpha$ satisfies

$$\overline{\partial}_J \overline{\partial}_J f = 0, \tag{18.11}$$

for any function f.

Note that for $\alpha \in \Lambda^{0,1}(J)$, $J\alpha = -i\alpha$, so from Proposition 4.9, we have that

$$\overline{\partial}_J \alpha = \frac{1}{2}(d - id^c)\alpha = \frac{1}{2}(d\alpha - iJdJ\alpha) = \frac{1}{2}(d\alpha - Jd\alpha) \tag{18.12}$$

Expanding this, we obtain

$$\overline{\partial}_J \alpha = \frac{1}{2}(d\alpha - (J - J_0 + J_0)d\alpha) = \frac{1}{2}(d\alpha - J_0 d\alpha) - \frac{1}{2}(J - J_0)d\alpha. \tag{18.13}$$

Now we plug in $\alpha = \overline{\partial}_J f$, and by assumption

$$0 = \overline{\partial}_J \overline{\partial}_J f = \overline{\partial}_J (H_p + O(|z|^{p+1}) = \frac{1}{2}(dH_p - J_0 dH_p) + O(|z|^p). \tag{18.14}$$

But from the first part of the proof, H_p is of type $(0,1)$ with respect to J_0, so we conclude that

$$0 = \frac{1}{2}(dH_p - J_0 dH_p) = \overline{\partial}_0 H_p. \tag{18.15}$$

□

Proposition 18.2. *For each $1 \leq p < \infty$, there exists $f = \sum_{j=1}^{p} f_j$ satisfying $\overline{\partial}_J f = O(|z|^p)$.*

Proof. We prove this by induction. For $p = 1$, we have $f = z^k$, and then

$$\overline{\partial}_J z^k = \overline{\partial}_0 z^k + \frac{i}{2}(J - J_0)dz^k = 0 + O(|z|), \tag{18.16}$$

Assume that we have found a solution for $j = 1 \dots p$. Let $f = \sum_{j=1}^{p} f_j$, by the induction assumption, we have

$$\overline{\partial}_J f = H_p + O(|z|^{p+1}), \tag{18.17}$$

and by the above, we need to solve the equation

$$\overline{\partial}_0 f_{p+1} = H_p = -\frac{i}{2}\sum_{j+k=p} J_j df_j. \tag{18.18}$$

From Proposition 18.1, H_p is a form of type $(0,1)$ with respect to J_0, and satisfies $\overline{\partial}_0 H_p = 0$. We can therefore write

$$H_p = \alpha_{\bar{j}} d\overline{z}^j, \tag{18.19}$$

where

$$\frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^l} = \frac{\partial \alpha_{\bar{l}}}{\partial \overline{z}^j},\ j, l = 1, \ldots, n. \tag{18.20}$$

Define

$$f_{p+1} = \int_0^1 \sum_{j=1}^n \overline{z}^j \alpha_{\bar{j}}(z, t\bar{z}) dt. \tag{18.21}$$

Then we compute

$$\begin{aligned}
\frac{\partial f_{p+1}}{\partial \overline{z}^k} &= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^n \overline{z}^j \frac{\partial}{\partial \overline{z}^k}\Big(\alpha_{\bar{j}}(z, t\bar{z})\Big)\Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^n \overline{z}^j \frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^k}(z, t\bar{z}) t\Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^n \overline{z}^j \frac{\partial \alpha_{\bar{k}}}{\partial \overline{z}^j}(z, t\bar{z}) t\Big) \\
&= \int_0^1 \frac{d}{dt}\Big(t\alpha_{\bar{k}}(z, t\bar{z})\Big) dt = \alpha_{\bar{k}}(z, \bar{z}).
\end{aligned} \tag{18.22}$$

□

18.2 Convergence

Use Cauchy's method of majorants. Alternatively, we can use ODE methods. TO BE COMPLETED.

References

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