Math 245ABC, Complex Variables and Geometry

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Contents

Introduction

This will be a year long course on holomorphic functions and complex manifolds. A very rough outline:

- Holomorphic functions of 1 complex variable.

- Holomorphic function of several complex variables.
- Newlander-Nirenberg Theorem
- Sheaf cohomology.
- Riemann surfaces.
- Higher dimensional complex manifolds.

1 Lecture 1

1.1 Cauchy's formula in one complex variable

For now, just consider $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open set. We write $z \in U$ as $z = x + iy$. Assume that f, as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$, is differentiable. This means that, for each $z \in U$, there exists a linear mapping $L_z : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{h \to 0} \frac{\| f(z+h) - f(z) - L_z h \|}{\|h\|} = 0. \tag{1.1}$$

This implies that the partial derivatives of f exist. Conversely, if the partial derivatives exist and are continuous at z, then the mapping L_z exists. Writing

$$f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}, \tag{1.2}$$

we have that

$$L_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{1.3}$$

We say that f is *complex differentiable* at $z \in U$ if there exists complex number $c_z \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{\| f(z+h) - f(z) - c_z \cdot h \|}{\|h\|} = 0. \tag{1.4}$$

This is a much stronger condition than (1.1). If such a c_z exists, then

$$c_z = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \equiv \frac{\partial f}{\partial z}. \tag{1.5}$$

Note that if f is complex differentiable at z, writing $c_z = c_1 + ic_2$ and $h = h_1 + ih_2$, we have

$$c_z h = c_1 h_1 - c_2 h_2 + i(c_2 h_1 + c_1 h_2) \tag{1.6}$$

and

$$L_z h = \begin{pmatrix} u_x h_1 + u_y h_2 \\ v_x h_1 + v_y h_2 \end{pmatrix}, \tag{1.7}$$

so we see that (1.1) is satisfied and necessarily $u_x = v_y$ and $u_y = -v_x$.

We say that f is *holomorphic* in U if it is C^1 and satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.8}$$

Defining the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

the Cauchy-Riemann equations are equivalent to $\frac{\partial}{\partial \bar{z}} f = 0$, where $f = u + iv$. If f is holomorphic, then f is complex differentiable at any $z \in U$, and we have

$$c_z = \frac{\partial}{\partial z} f. \tag{1.9}$$

Definition 1.1. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{1.10}$$

which converges absolutely and uniformly in a disc $\Delta(z_0, \epsilon)$ around z_0, for some $\epsilon > 0$.

We want to show the equivalence of holomorphicity and complex analyticity. For this, we need the following; see [GH78, page 3].

Proposition 1.2 (Cauchy-Pompieu Formula)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain in $\mathbb{C}$ with C^1 boundary. For $z \in \Omega$ and $f \in C^1(\overline{\Omega})$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)dw}{w - z} + \frac{1}{2\pi i}\int_{\Omega} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}, \tag{1.11}$$

where the boundary has the counterclockwise orientation.

Proof. The 1-form

$$\eta = \frac{1}{2\pi i}\frac{f(w)dw}{w - z}, \tag{1.12}$$

satisfies

$$d\eta = \frac{1}{2\pi i}\frac{\partial f}{\partial \bar{w}}\frac{d\bar{w} \wedge dw}{w - z}. \tag{1.13}$$

Apply Stokes' Theorem to the annular domain $\Omega \setminus \Delta(z,\epsilon)$, to get

$$\int_{\partial(\Omega\setminus\Delta(z,\epsilon))} \eta = \int_{\Omega\setminus\Delta(z,\epsilon)} d\eta. \tag{1.14}$$

The left hand side of (1.14) is

$$\int_{\partial\Omega} \eta - \int_{\partial\Delta(z,\epsilon)} \eta. \tag{1.15}$$

If $d\eta \in L^1(\Omega)$, then the right hand side of (1.14) is

$$\int_{\Omega} d\eta - \int_{\Delta(z,\epsilon)} d\eta. \tag{1.16}$$

Write $w = z + re^{i\theta}$, we estimate

$$\Big|\int_{\Delta(z,\epsilon)} d\eta\Big| \le \int_{\Delta(z,\epsilon)} |d\eta| \le C \int_{\Delta(z,\epsilon)} \Big|\frac{d\bar{w}\wedge dw}{w-z}\Big| \le C\int_0^{\epsilon}\int_0^{2\pi} dr\wedge d\theta < C'\epsilon, \tag{1.17}$$

so $d\eta$ is obviously in $L^1(\Omega)$, and taking the limit as $\epsilon \to 0$ then shows that

$$\int_{\Omega} d\eta = \int_{\partial\Omega} \eta - \lim_{\epsilon\to 0}\int_{\partial\Delta(z,\epsilon)} \eta. \tag{1.18}$$

With $w = z + \epsilon e^{i\theta}$, we compute

$$\int_{\partial\Delta(z,\epsilon)} \eta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+\epsilon e^{i\theta}))\epsilon i e^{i\theta} d\theta}{\epsilon e^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta. \tag{1.19}$$

Using the mean value theorem, we also have that

$$\Big|f(z) - \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta\Big| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z+\epsilon e^{i\theta}) - f(z)|d\theta \le C\epsilon, \tag{1.20}$$

which shows that

$$\lim_{\epsilon\to 0}\int_{\partial\Delta(z,\epsilon)} \eta = f(z), \tag{1.21}$$

and we are done. □

A consequence is the equivalence between holomorphic and analytic functions; see [GH78, page 4].

Proposition 1.3. *Let U be an open set in $\mathbb{C}$. Then f is holomorphic in U if and only if f is complex analytic in U.*

Proof. If f is holomorphic in U the Cauchy-Pompieu formula in a small disc $\Delta = \Delta(z_0, \epsilon)$ yields for $z \in \Delta$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)dw}{w-z}. \tag{1.22}$$

Then expand

$$\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \tag{1.23}$$

$$= \frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k, \tag{1.24}$$

with the sum converging absolutely and uniformly in any smaller disc. So the above yields the power series expansion

$$f(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{(w-z_0)^{k+1}}\right)(z-z_0)^k, \tag{1.25}$$

which also converges absolutely and uniformly in any smaller disc.

For the converse, if f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula without solid integral. So then f does also by uniform convergence. So we have

$$\frac{\partial}{\partial \overline{z}}f(z) = \frac{\partial}{\partial \overline{z}}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{w-z}\right) = \frac{1}{2\pi i}\int_{\partial\Delta} f(w)\left(\frac{\partial}{\partial \overline{z}}\frac{1}{w-z}\right)dw = 0. \tag{1.26}$$

For more details, see [GH78, page 4]. □

Definition 1.4. We wil let $\Omega \subset \mathbb{C}$ be a bounded domain with C^1 boundary. If u is holomorphic in an open set Ω, then we write $u \in \mathcal{O}(\Omega)$.

2 Lecture 2

2.1 Some basic results in one complex variable

First, let's recall the basic result about differentiating under an integral.

Proposition 2.1. *Let*

$$f(z) = \int_{\Omega} a(z,w)dw \wedge d\overline{w}. \tag{2.1}$$

(Note this notation does not mean that f is holomorphic in z or that a is holomorphic as a function of 2 variables!). Assume that

1. *$a(z,w) \in L^1(\Omega)$, in the w variable.*

2. $\frac{\partial a}{\partial z}$ *and* $\frac{\partial a}{\partial \bar{z}}$ *exist for all* z*, for almost every* $w \in \Omega$.

3. $\left|\frac{\partial a}{\partial z}\right| + \left|\frac{\partial a}{\partial \bar{z}}\right| \leq h(w)$*, where* $h \in L^1(\Omega)$.

Then

$$\frac{\partial f}{\partial z} = \int_\Omega \frac{\partial}{\partial z}\big(a(z,w)\big) dw \wedge d\overline{w} \tag{2.2}$$

$$\frac{\partial f}{\partial \bar{z}} = \int_\Omega \frac{\partial}{\partial \bar{z}}\big(a(z,w)\big) dw \wedge d\overline{w}. \tag{2.3}$$

Proof. Recall that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(Ref + iImf). \tag{2.4}$$

The real part of the left hand side of (2.2) is

$$Re\Big(\frac{\partial f}{\partial z}\Big) = \frac{1}{2}\Big(\frac{\partial Ref}{\partial x} + \frac{\partial Imf}{\partial y}\Big) \tag{2.5}$$

The real part of the right hand side of (2.2) is

$$\int_\Omega \frac{1}{2}\Big(\frac{\partial Re(a(x+iy,w))}{\partial x} + \frac{\partial Im(a(x+iy,w))}{\partial y}\Big) dw \wedge d\overline{w}. \tag{2.6}$$

Therefore we can consider real-valued functions, and prove for partials with respect to the real variables x and y. We have that

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\delta \to 0} \frac{f(x+\delta,y) - f(x,y)}{\delta} \tag{2.7}$$

For $\delta \neq 0$, consider

$$\frac{f(x+\delta,y) - f(x,y)}{\delta} = \int_\Omega \frac{a(x+\delta+iy,w) - a(x+iy,w)}{\delta} dw \wedge d\overline{w}. \tag{2.8}$$

By the mean value theorem, given $\delta > 0$, there exists x' on the line segment from (x,y) to $(x+\delta,y)$ such that

$$a(x+\delta+iy,w) - a(x+iy,w) = \frac{\partial a}{\partial x}(x'+iy,w)\delta, \tag{2.9}$$

so

$$\begin{aligned}\Big|\frac{a(x+\delta+iy,w) - a(x+iy,w)}{\delta}\Big| &\leq \Big|\frac{\partial a}{\partial x}(x'+iy,w)\Big| \\ &\leq \Big|\frac{\partial a}{\partial z}(x'+iy,w)\Big| + \Big|\frac{\partial a}{\partial \bar{z}}(x'+iy,w)\Big| \leq |h(w)|.\end{aligned} \tag{2.10}$$

We can do this for any sequence $\delta_n \to 0$, so the result follows from Lebesgue's dominated convergence theorem. The proof for the other derivative (2.3) is similar. □

We next go through several corollaries of the Cauchy-Pompieu formula; see [Hör90, Chapter 1] for more details.

Corollary 2.2. *(Interior derivative estimates.) Let $K \subset \Omega$ be a compact subset. Then there exist constant C_k, depending only upon K and Ω such that*

$$\sup_{z \in K} \Big| \Big(\frac{\partial}{\partial z} \Big)^k u(z) \Big| \leq C_k \|u\|_{L^1(\Omega)}, \tag{2.11}$$

for all $u \in \mathcal{O}(\Omega)$.

(Cauchy's estimate in a disc.) In the case that $u \in \mathcal{O}(\Delta(z_0, r_0))$, then there exists C_k, depending only upon k such that for any $r < r_0$, we have

$$\Big| \Big(\frac{\partial}{\partial z} \Big)^k u(z_0) \Big| \leq \frac{k!}{r^k} \|u\|_{C^0(\partial\Delta(z_0,r))}, \tag{2.12}$$

Proof. Choose a $\psi \in C_0^\infty(\Omega)$ (compact support) such that $\psi \equiv 1$ in a neighborhood of K. If $u \in \mathcal{O}(\Omega)$, then

$$\frac{\partial}{\partial \bar{z}}(\psi u) = u \frac{\partial}{\partial \bar{z}} \psi. \tag{2.13}$$

Now we apply (1.11) to ψu in Ω to get

$$\psi(z) u(z) = \frac{1}{2\pi i} \int_\Omega u(w) \frac{\partial \psi(w)}{\partial \overline{w}} \frac{dw \wedge d\bar{w}}{w - z}, \tag{2.14}$$

Now consider

$$a(z, w) = u(w) \frac{\partial \psi(w)}{\partial \overline{w}} \frac{1}{w - z}, \tag{2.15}$$

If $z \in K$, then $|w - z| > \delta > 0$, since the support of $\partial \psi / \partial \overline{w}$ is at a positive distance from K. So using Proposition 2.1, we can differentiate under the integral as many times as we like, and obtain

$$\Big(\frac{\partial}{\partial z} \Big)^k (\psi u(z)) = \frac{1}{2\pi i} \int_\Omega u(w) \frac{\partial \psi(w)}{\partial \overline{w}} \Big(\frac{\partial}{\partial z} \Big)^k \Big(\frac{1}{w - z} \Big) dw \wedge d\bar{w} \tag{2.16}$$

If $z \in K$, then ψu is equal to u in a neighborhood of z, so (2.11) follows.

Next, from (1.25), we have a power series expansion

$$u(z_0) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{u(w) dw}{(w - z_0)^{k+1}}. \tag{2.17}$$

Since $a_k = \dfrac{u^{(k)}(z_0)}{k!}$, this clearly implies (2.12). $\square$

Corollary 2.3 (Liouville)**.** *A bounded entire function on $\mathbb{C}$ is constant.*

Proof. Let $u \in \mathcal{O}(\mathbb{C})$. Choose $z_0 \in \mathbb{C}$, then for any $r > 0$, by (2.12), we have

$$|u'(z_0)| \leq Cr^{-1}, \tag{2.18}$$

which implies that $u'(z_0) = 0$, therefore u is constant. $\square$

Exercise 2.4. The fundamental theorem of algebra follows from this: If a polynomial $P(z)$ has no zeroes, then $1/P(z)$ would be a bounded entire function, and therefore constant. Details are left as an exercise.

Corollary 2.5. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \to u$ converges uniformly to u in the C^0 norm as $n \to \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Proof. Let $K \subset \Omega$, be a compact subset. Then given $\epsilon > 0$, there exist N such that

$$\sup_{z \in K} |u_m(z) - u_n(z)| < \epsilon, \tag{2.19}$$

for $m, n \geq N$. The difference $u_m - u_n \in \mathcal{O}(\Omega)$. Corollary 2.2 implies that

$$\sup_{z \in K} \left| \frac{\partial}{\partial z}(u_m - u_n)(z) \right| \leq C\epsilon. \tag{2.20}$$

This says that $\partial u_n / \partial z$ converges uniformly on K. But $\partial u_n / \partial \overline{z} = 0$, so the real partial derivatives $\partial u_n / \partial x$ and $\partial u_n / \partial y$ converge uniformly. It is an elementary result that if a sequence of functions converges uniformly, and the derivatives converge uniformly, then the limit of the derivatives is the derivative of the limit. This implies that $u \in C^1$ and

$$\frac{\partial u}{\partial \overline{z}} = \lim_{n \to \infty} \frac{\partial u_n}{\partial \overline{z}} = 0. \tag{2.21}$$

$\square$

Corollary 2.6 (Montel's Theorem)**.** *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

Proof. Corollary 2.2 yields a uniform bound on derivatives of u_n on any compact subset. By Arzela-Ascoli Theorem, some subsequence converges to a limit u uniformly on compact subsets. Then the previous corollary yields that $u \in \mathcal{O}(\Omega)$. $\square$

Corollary 2.7 (The maximum principle)**.** *If $f \in \mathcal{O}(\Omega)$, Ω is connected, and there exists a $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, then u is constant.*

Proof. The set $M \equiv \{z \in \Omega \mid |f(z)| = |f(z_0)|\}$ is closed by continuity. Take any $z' \in \Omega$ such that $|f(z')| = |f(z_0)|$. Then for any $\epsilon > 0$ such that $\Delta(z', \epsilon) \subset \Omega$, (1.11) is

$$f(z') = \frac{1}{2\pi i} \int_{\partial \Delta(z', \epsilon)} \frac{f(w)}{w - z} dw. \tag{2.22}$$

Writing $w = z_0 + \epsilon e^{i\theta}$, this is

$$f(z') = \frac{1}{2\pi}\int_0^{2\pi} f(z + \epsilon e^{i\theta})d\theta \tag{2.23}$$

which implies that $|f(z')| = |f(z + \epsilon e^{i\theta})|$ for all θ. Letting ϵ vary, we see that the set M is also open, and since Ω is connected, $M \equiv \Omega$. Therefore $|f|$ is constant in Ω, and writing $f = u + iv$, we have

$$u^2 + v^2 = \text{constant}. \tag{2.24}$$

Differentiating this yields

$$2uu_x + 2vv_x = 0 = 2uu_y + 2vv_y. \tag{2.25}$$

Using the Cauchy-Riemann equations gives

$$2uu_x - 2vu_y = 0 = 2uu_y + 2vu_x, \tag{2.26}$$

This implies that

$$2u^2u_x - 2uvu_y = 0 = 2uvu_y + 2v^2u_x, \tag{2.27}$$

and we get that

$$(u^2 + v^2)u_x = 0, \tag{2.28}$$

which implies that $u_x = 0$, so u is constant. Then f is constant. □

Corollary 2.8 (Riemann's removable singularity theorem)**.** *If $u \in \mathcal{O}(\Delta^*(z_0, r))$ where $\Delta^*(z_0, r) = \Delta(z_0, r) \setminus \{z_0\}$, satisfies*

$$u(z) = o(|z - z_0|^{-1}), \text{ as } z \to z_0 \tag{2.29}$$

Then u extends to a holomorphic function on $\Delta(z_0, r)$.

Proof. Consider the function

$$v(z) = \begin{cases} (z - z_0)^2 u(z) & z \in \Delta^*(z_0, r) \\ 0 & z = z_0 \end{cases} \tag{2.30}$$

We have

$$\lim_{z \to z_0} \frac{v(z) - v(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0)u(z) = 0 \tag{2.31}$$

by assumption, so $v(z) \in \mathcal{O}(\Delta(z_0, r))$ and therefore admits a power series expansion

$$v(z) = \sum_{k=2}^{\infty} c_k (z - z_0)^k, \tag{2.32}$$

since $v(z_0) = v'(z_0) = 0$. Then

$$u(z) = \sum_{k=0}^{\infty} c_{k+2}(z - z_0)^k \tag{2.33}$$

is the required extension. □

3 Lecture 3

3.1 The Schwarz Lemma

Lemma 3.1 (Schwarz Lemma)**.** *Let $u \in \mathcal{O}(\Delta_1(0))$, and assume that $u(0) = 0$ and $|u| \leq 1$. Then $|u'(0)| \leq 1$ and $|u(z)| \leq |z|$ for every $z \in \Delta_1(0)$. Equality holds at some z_0 if and only if $u(z) = c \cdot z$ where $c \in \mathbb{C}$ satisfies $|c| = 1$.*

Proof. Consdier $v(z) = \frac{u(z)}{z}$. Then $v \in \mathcal{O}(\Delta_1(0))$ and $v(0) = u'(0)$. Given $z \in \Delta_1(0)$, choose $r > 0$ such that $|z| < r < 1$. Then v is analytic in $\overline{\Delta_r(0)}$, and the maximum principle yields that

$$|v(z)| \leq \sup_{|w|=r} |v(w)| = \sup_{|w|=r} \frac{u(w)}{w} \leq 1/r, \tag{3.1}$$

Letting $r \to 1$, we are done. The inequality being equality at some interior point implies that v is a constant function with $|v| = 1$, equivalently $u(z) = c \cdot z$. □

Exercise 3.2. This is very useful, for example it can be used to show that any holomorphic automorphism of the unit disc must be of the form

$$\Psi(z) = e^{i\theta} \frac{z + c}{1 + \bar{c}z}, \tag{3.2}$$

where $\theta \in \mathbb{R}$ and $c \in \mathbb{C}$ satisfies $|c| < 1$. The proof is to compose with a mapping of the above form to normalize so that $\Psi(0) = 0$, and then apply the Schwarz Lemma. Details left as an exercise.

3.2 Meromorphic functions

Definition 3.3. For a domain $\Omega \subset \mathbb{C}$, we say that $f \in \mathcal{M}(\Omega)$, or f is meromorphic in Ω, if there is an open covering U_j of Ω such that $f|_{U_j} = \frac{g_j}{h_j}$, where g_j and h_j are in $\mathcal{O}(U_j)$ and h_j is not identically zero.

Note this is equivalent to saying that f has a Laurent series expansion near any $z_0 \in \Omega$ with only finitely many negative terms. That is we have

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k. \tag{3.3}$$

If some coefficient $a_k \neq 0$ for $k < 0$, then z_0 is called a *pole*. Note that the set of poles will be some discrete subset $\{w_j\}$ of Ω. The finite sum of the negative terms is called the *principal part* of f at z_0. The order of $f \in \mathcal{M}(\Omega)$ at $z_0 \in \Omega$ is the least integer $n \in \mathbb{Z}$ such that the coefficient $a_n \neq 0$ in (3.3).

Example 3.4. The function $e^{1/z}$ is holomorphic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but it is not meromorphic on $\mathbb{C}$. It has an *essential singularity* at the origin.

We will next consider a more general situation of functions which are holomorphic in an annulus $A(z_0; r_1, r_2) = \Delta(z_0, r_2) \setminus \Delta(z_0, r_1)$ with $0 < r_1 < r_2$.

Proposition 3.5. *If f is holomorphic in a neighborhood of an annulus $A(z_0; r_1, r_2)$ with $0 < r_1 < r_2$, then f admits a expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k. \tag{3.4}$$

which converges uniformly on compact subsets of $A(z_0; r_1, r_2)$. Furthermore the coefficient a_k is given by

$$a_k = \frac{1}{2\pi i} \int_{\partial\Delta(z_0,r)} \frac{f(w)}{(w - z_0)^{k+1}} dw \tag{3.5}$$

for any $r_1 \leq r \leq r_2$. In particular, the expansion (3.4) *is uniquely determined by f.*

Proof. Since f is assumed to be holomorphic in a neighborhood of the annulus, the Cauchy-Pompieu formula yields for $z \in A(z_0; r_1, r_2)$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta(z_0,r_2)} \frac{f(w)dw}{w - z} - \frac{1}{2\pi i} \int_{\partial\Delta(z_0,r_1)} \frac{f(w)dw}{w - z}. \tag{3.6}$$

If $|z - z_0| < |w - z_0|$ expand

$$\frac{1}{w - z} = \frac{1}{w - z_0 + z_0 - z} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \tag{3.7}$$

$$= \frac{1}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k, \tag{3.8}$$

and if $|z - z_0| > |w - z_0|$,

$$\frac{1}{w - z} = \frac{1}{w - z_0 + z_0 - z} = \frac{-1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}} \tag{3.9}$$

$$= \frac{-1}{z - z_0} \sum_{k=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^k, \tag{3.10}$$

Substituting these expansions into (3.6), we formally obtain

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta(z_0,r_2)} f(w) \frac{1}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k dw \tag{3.11}$$

$$+ \frac{1}{2\pi i} \int_{\partial\Delta(z_0,r_1)} f(w) \frac{1}{z - z_0} \sum_{k=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^k dw \tag{3.12}$$

If $z \in A(z_0; r_1, r_2)$ and $|w - z_0| = r_2$, then $|z - z_0| < r_2 = |w - z_0|$, so (3.7) is valid. If $|w - z_0| = r_1$, then $|z - z_0| > r_1 = |w - z_0|$, so (3.9) is valid. Both series convergenge uniformly on compact subsets of $A(z_0; r_1, r_2)$, so we have that

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \left(\int_{\partial\Delta(z_0, r_2)} \frac{f(w)}{(w - z_0)^{k+1}} dw \right) (z - z_0)^k \tag{3.13}$$

$$+ \frac{1}{2\pi i} \sum_{k=0}^{\infty} \left(\int_{\partial\Delta(z_0, r_1)} f(w)(w - z_0)^k dw \right) (z - z_0)^{-(k+1)}. \tag{3.14}$$

Note that, for any $k \in \mathbb{Z}$, the function $f(w)(w - z_0)^k$ is holomorphic in a neighborhood of the annulus, so by Stokes' Theorem we have

$$\begin{aligned} \int_{\partial\Delta(z_0, r_2)} f(w)(w - z_0)^k dw - \int_{\partial\Delta(z_0, r_1)} f(w)(w - z_0)^k dw \\ = \int_{A(z_0; r_1, r_2)} d\left(f(w)(w - z_0)^k dw \right) = 0. \end{aligned} \tag{3.15}$$

So the integral

$$\int_{\partial\Delta(z_0, r)} f(w)(w - z_0)^k dw \tag{3.16}$$

is independent of r, and we have

$$f(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial\Delta(z_0, r)} \frac{f(w)}{(w - z_0)^{k+1}} dw \right) (z - z_0)^k. \tag{3.17}$$

□

Remark 3.6. If f is holomorphic in a punctured disc $\Delta^*(z_0, r)$, then it is holomorphic in an annulus $A(z_0, r/2, r)$, so we also obtain an infinite Laurent series expansion at any isolated singularity.

Definition 3.7. If $f = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$, then the residue of f at z_0 is

$$Res_{z_0}(f) = a_{-1}. \tag{3.18}$$

Corollary 3.8 (The Residue Theorem)**.** *Assume that f is holomorphic in $\overline{\Omega} \setminus \{z_1, \ldots, z_m\}$. Then*

$$\int_{\partial\Omega} f(z) dz = 2\pi i \sum_{j=1}^{m} Res_{z_j}(f) \tag{3.19}$$

Proof. Since f is holomorphic in the complement (in Ω) of a union of balls around z_j, by Stokes's theorem (like in (3.15) above), we see that

$$\int_{\partial\Omega} f(z) dz = \sum_{j=1}^{m} \int_{\partial\Delta(z_j, \epsilon)} f(z) dz, \tag{3.20}$$

for some $\epsilon > 0$ (this is also known as Cauchy's Theorem). By Proposition 3.5,

$$\int_{\partial\Delta(z_j,\epsilon)} f(z)dz = \int_{\partial\Delta(z_j,\epsilon)} \sum_{k=-\infty}^{\infty} a_k(z-z_j)^k dz = \sum_{k=-\infty}^{\infty} a_k \int_{\partial\Delta(z_j,\epsilon)} (z-z_0)^k dz, \tag{3.21}$$

since the series converges uniformly on $\partial\Delta^*(z_j,\epsilon)$. However, letting $z = z_j + re^{i\theta}$, we see that

$$\int_{\partial\Delta(z_j,\epsilon)} (z-z_j)^k dz = \begin{cases} 2\pi i & k=-1 \\ 0 & k\neq -1 \end{cases}, \tag{3.22}$$

and we are done. □

Corollary 3.9 (The argument principle)**.** *If $f \in \mathcal{M}(\overline{\Omega})$ with no poles on $\partial\Omega$, then*

$$\int_{\partial\Delta(z_0,r)} z^q \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{d} m_j z_j^q \tag{3.23}$$

where z_j are the zeros and poles of f, with m_j is the order of f at z_j.

Proof. Let z_j be any zero or pole of f. Then we can write $f(z) = (z-z_j)^{m_j} g(z)$ where $g(z)$ is holomorphic at z_j and non-zero in a neighborhood of z_j. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}. \tag{3.24}$$

So the result follows from the residue theorem. □

4 Lecture 4

4.1 The $\overline{\partial}$-equation in domains in $\mathbb{C}$

Our goal is to prove the following result.

Theorem 4.1. *If $\Omega \subset \mathbb{C}$ is any bounded domain, and $g \in C^\infty(\Omega)$, then there exists $g \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \bar{z}} f = g$.*

If we want to solve $\frac{\partial}{\partial \bar{z}} f = g$, it is natural to guess that

$$f(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{g(w)dw \wedge d\bar{w}}{w-z}, \tag{4.1}$$

is a solution. However, letting $a(z,w) = g(w)/(w-z)$, we have

$$\frac{\partial a(z,w)}{\partial z} = g(w)\frac{-1}{(w-z)^2}, \tag{4.2}$$

so the assumptions of Proposition 2.1 are NOT satisfied, so we cannot directly differentiate under the integral sign! Another problem is that g is only assumed to be in $C^\infty(\Omega)$, so it is not in $L^1(\Omega)$ and (4.1) is not necessarily defined. We first give a preliminary result, with a stronger assumption on g.

Proposition 4.2. *If $g \in C^1(\overline{\Omega})$ then the function*

$$f(z) = \frac{1}{2\pi i}\int_\Omega \frac{g(w)dw \wedge d\bar{w}}{w-z}, \tag{4.3}$$

satisfies $f \in C^1(\overline{\Omega})$ and $\partial f/\partial\overline{z} = g$ in Ω.

Proof. Since g is assumed to be in $C^1(\overline{\Omega})$, the integral in (4.3) is well-defined. To show that $\partial f/\partial\overline{z} = g$ in Ω, we will fix any point $z_0 \in \Omega$, and show that $\partial f/\partial\overline{z}(z_0) = g(z_0)$. Choose a C^∞ cutoff function $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on $\Delta(z_0, r) \subset \Omega$, where r is sufficiently small. We then write $f = f_1 + f_2$, where

$$f_1(z) = \frac{1}{2\pi i}\int_\Omega \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w-z} \tag{4.4}$$

$$f_2(z) = \frac{1}{2\pi i}\int_\Omega \frac{(1-\psi(w))g(w)dw \wedge d\bar{w}}{w-z}. \tag{4.5}$$

For z in a small neighborhood of z_0, the integrand in f_2 does not have a singularity. We can therefore differentiate under the integral sign to see that $\partial f_2/\partial\overline{z}(z_0) = 0$. So we just need to prove that $\partial f_1/\partial\overline{z} = g$ at z_0. Since ψ has compact support, we can extend ψg to all of $\mathbb{C}$, and write

$$f_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w-z} \tag{4.6}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(\xi+z)g(\xi+z)d\xi \wedge d\bar{\xi}}{\xi}, \tag{4.7}$$

where we used the change of variables $w = \xi + z$. Note that

$$\frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial z} = \frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \xi} \tag{4.8}$$

$$\frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \overline{z}} = \frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \bar{\xi}}. \tag{4.9}$$

This shows that the z and $\overline{z}$ partials of the integrand are uniformly in L^1, so we can differentiate under the integral sign, to obtain

$$\frac{\partial f_1(z)}{\partial \overline{z}} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial\bar{\xi}}\frac{d\xi \wedge d\bar{\xi}}{\xi} \tag{4.10}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi(w)g(w))}{\partial\overline{w}}\frac{dw \wedge d\overline{w}}{w-z}. \tag{4.11}$$

Now apply the Cachy-Pompieu formula in a very large ball in $\mathbb{C}$, to conclude the right hand side is equal to $\psi(z)g(z)$ which is $g(z)$ if $z \in \Delta(z_0, r)$. □

Remark 4.3. With a little more work, the assumptions can be weakened to g being bounded and continuous on Ω; see [HL84, Theorem 1.1.3], but the derivative is interpreted as a distributional derivative.

This result does not directly help us in proving Theorem 5.3. But notice that in the proof, we also proved the following result.

Proposition 4.4. *If $g \in C_c^\infty(\Omega)$, then there exists $f \in C^\infty(\mathbb{C})$ such that $\partial f/\partial \overline{z} = g$.*

Proof. Above, we proved that there is a solution $f \in C^1(\mathbb{C})$, but the same argument allows us to differentiate f_1 infinitely many times, provided g is infinitely differentiable. □

Remark 4.5. In general, we cannot expect that f has compact support. Take $g \in C_c^\infty(\mathbb{C})$ with $\int_{\mathbb{C}} g dz \wedge d\overline{z} \neq 0$ and let f be any solution of $\partial f/\partial \overline{z} = g$. By Stokes' Theorem,

$$0 \neq \int_{\mathbb{C}} g dz \wedge d\overline{z} = \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z} = -\int_{\mathbb{C}} d(f dz) = 0, \tag{4.12}$$

if f has compact support, which is a contradiction.

Now we can prove a special case of Theorem 5.3.

Proposition 4.6. *Theorem 5.3 is true for $\Omega = \Delta(z,r)$ is a disc in $\mathbb{C}$, and $\Omega = \Delta^*(z,r)$ a punctured disc, and for $\Omega = A(z;r_1,r_2)$ an annulus.*

Proof. We first consider the case of a disc. Take a sequence $0 < r_1 < r_2 < \cdots < r$ such that $\lim_{j\to\infty} r_i = r$. Let $0 \leq \psi_k \in C_0^\infty(\Delta(z,r_{k+1}))$ and $\psi_k \equiv 1$ on $\Delta(z,r_k)$. Then $g_k = \psi_k g \in C_0^\infty(\Delta(z,r_{k+1})$, and by Proposition 4.4, we can find $f_k \in C^\infty(\mathbb{C})$ such that $\partial f_k = g_k$, which is equal to g in $\Delta(z,r_k)$.

Now there is no reason that the sequence f_k will converge to a limit, so we need to modify as follows. We claim that we can choose f_k so that

$$\sup_{z' \in \Delta(z,r_{k-1})} |f_{k+1}(z') - f_k(z')| \leq 2^{-k}. \tag{4.13}$$

Given f_2, the difference $f_3 - f_2$ is holomorphic in $\overline{\Delta(z,r_1)}$. So there exists a polynomial P_3 such that

$$\sup_{z' \in \Delta(z,r_1)} |f_3(z') - f_2(z') - P_3(z')| \leq 2^{-2}. \tag{4.14}$$

So we redefine f_3 to be $f_3 - P_3$. We then proceed by induction. Given f_k, the difference $f_{k+1} - f_k$ is holomorphic in $\overline{\Delta(z,r_{k-1})}$, so we can find a polynomial P_{k+1} such that

$$\sup_{z' \in \Delta(z,r_{k-1})} |f_{k+1}(z') - f_k(z') - P_{k+1}(z')| \leq 2^{-k}, \tag{4.15}$$

and we redefine f_{k+1} to be $f_{k+1} - P_{k+1}$.

The sequence of functions f_k will be a Cauchy sequence in any disc $\Delta(z,r')$, when $r' < r$. So there exists a uniform limit f. Fixing any m, then $f - f_m$ is then a uniform limit of holomorphic functions in $\Delta(z,r_{m-1})$, so is holomorphic by Corollary 2.5, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m/\partial \overline{z} \to \partial f/\partial \overline{z} = g, \tag{4.16}$$

and the proof is finished.

The case of a punctured disc or annulus is similar, but using truncated Laurent series expansions instead of polynomials which was proved above in Proposition 3.5 and Remark 3.6. □

5 Lecture 5

5.1 Runge's Theorem

To handle the case of an arbitrary domain $\Omega \subset \mathbb{C}$, we need some machinery.

Theorem 5.1 (Runge's approximation Theorem, first version)**.** *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z\in K} |f(z) - f_\epsilon(z)| < \epsilon, \tag{5.1}$$

and such that poles of f_ϵ are contained in $\mathbb{C} \setminus K$.

Proof. The proof is from [Sar07, Theorem IX.15], we just give an outline. From elementary arguments, there exists a simple contour γ with image in $U \setminus K$ such that $K \subset \text{Int}(\gamma) \subset U$. I.e., γ separates K from the complement of U. Note that K might have several components, and $U \setminus K$ might be disconnected, in which case γ will be disconnected. Since the countour is simple, by Cauchy's Integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw \tag{5.2}$$

for any $z \in K$. Note that this works for piecewise smooth paths because Stokes' Theorem holds for a domain with piecewise smooth boundary. By dividing the plane into a sufficiently fine grid, we can assume that γ is piecewise smooth and $\gamma = \gamma_1 + \cdots \gamma_n$, with each γ_j a line segment parallel to one of the coordinate axes. Consider each term

$$f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{5.3}$$

We can approximate this arbitrarily closely with a Riemann sum R_k, which will be of the form

$$\frac{c_1}{w_1 - z} + \cdots + \frac{c_l}{w_l - z}, \tag{5.4}$$

where the w_j are points on γ_k. Doing this for every γ_j, the proof is complete. □

In this result, the poles of f_ϵ will be quite close to K, and still in Ω. The full version of Runge's Theorem allows us to move the poles to specified points in $\mathbb{C} \setminus K$, in particular they can be moved outside of Ω.

Theorem 5.2 (Runge's approximation Theorem, second version)**.** *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Let $S \subset \mathbb{C} \setminus K$ which contains at least one point from each connected component of $\mathbb{C} \setminus K$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z\in K} |f(z) - f_\epsilon(z)| < \epsilon, \tag{5.5}$$

and such that poles of f_ϵ are contained in S. If $\mathbb{C} \setminus K$ is connected, the rational functions can be taken to be polynomials.

Proof. The proof is from [Sar07, Theorem IX.17]. In the proof of Theorem 5.1, each term in the approximation was of the form $c/(w-z)$, where $w \in \gamma$. Now choose a picewise linear path α from w to any point w_0 in the same connected component of $\mathbb{C} \setminus K$. Choose points w_i on α so that

$$|w_{i-1} - w_i| < \text{dist}(\gamma, K). \tag{5.6}$$

We show that any rational R_{j-1} function with a pole only at w_{j-1} may be uniformly approximated on K by a rational function R_j with a poles only at w_j. But this follows from consdering the Laurent series expansion of R_{j-1} centered at w_j: R_{j-1} is holomorphic in the region $U = \mathbb{C} \setminus \Delta(w_j, |w_j - w_{j-1}|)$, so the Laurent series of R_{j-1} centered at w_j converges uniformly on compact subsets of U. Then we can approximate R_{j-1} by a rational function with a pole only at w_j, uniformly on K, since K is a compact subset of U, which follows from (5.6).

If $\mathbb{C} \setminus K$ is connected, by the above argument, we can move the pole of the rational function R_j to a single point z_0 so that $K \subset \Delta(0, |z_0|)$. The Talyor series of R_j centered at any point $z \in K$ converges uniformly on K, so we can approximate by the partial sums of this Taylor series.

□

Now we can prove the general result:

Theorem 5.3. *If $\Omega \subset \mathbb{C}$ is any domain, and $g \in C^\infty(\Omega)$, then there exists $f \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \bar{z}} f = g$.*

Proof. We choose a sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \cdots$, so that $\overline{K_j} \subset \text{Int}K_{j+1}$ and $\cup K_j = \Omega$. Note that $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, and we can assume that for large j, each component of $\mathbb{C} \setminus K_j$ contains a component of $\mathbb{C} \setminus \Omega$. Let $0 \leq \psi_j \in C_0^\infty(K_{j+1})$ and $\psi_k \equiv 1$ on K_j. Then $g_j = \psi_j g \in C_0^\infty(K_{j+1})$, and by Proposition 4.4, we can find $f_j \in C^\infty(\mathbb{C})$ such that $\partial f_j = g_j$.

We claim that we can choose $f_j \in C^\infty(\Omega)$ so that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z)| \leq 2^{-j}. \tag{5.7}$$

We proceed by induction. Given f_j, the difference $f_{j+1} - f_j$ is holomorphic in

$$U \equiv \text{Int}K_j \supset K = \overline{K_{j-1}}. \tag{5.8}$$

We have $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, so by Theorem 5.2, there exists a rational function R_{j+1} such that it poles are in $\mathbb{C} \setminus \Omega$ and such that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z) - R_{j+1}(z)| \leq 2^{-j}, \tag{5.9}$$

and we redefine f_{j+1} to be $f_{j+1} - P_{j-1}$.

The sequence of functions f_j will be a Cauchy sequence in any subset K_m for fixed m. So there exists a limit f, with uniform convergence on compact subsets. Fixing any m, then $f - f_m$ is then a uniform limit of holomorphic functions in K_{m-1}, so is holomorphic by

Corollary 2.5, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m/\partial \overline{z} \to \partial f/\partial \overline{z} = g, \tag{5.10}$$

and the proof is finished.

□

6 Lecture 6

The above solution of the inhomogeneous Cauchy-Riemann equations has many corollaries, we next give a few applications. The first is Mittag-Leffler's Theorem.

Theorem 6.1 (Mittag-Leffler). *Let Ω be a domain in $\mathbb{C}$ and $\{w_j\}$ a discrete subset of Ω. Let P_j be any principal sum at w_j. Then there exists a meromorphic function $h \in \mathcal{M}(\Omega)$ such that the principal part of h at w_j is P_j and there are no other poles in Ω.*

Proof. Let ψ_j be a cutoff function supported in a small neighborhood of w_j which doesn't contain any other points in the discrete subset. Consider $g = \sum_j \psi_j P_j$. Then $\partial g/\partial \bar{z} \in C^\infty(\Omega)$. By Theorem 5.3, there exists a solution $f \in C^\infty(\Omega)$ of $\partial f/\partial \overline{z} = \partial g/\partial \bar{z}$. Then $h = g - f$ satisfies $\partial h/\partial \overline{z} = 0$, and the principal part of h at w_j is P_j. □

6.1 Logarithm of a function

We next want to answer the question of when is it possible to take to logarithm of a nowhere-zero holomorphic function.

Definition 6.2. Let U be a domain in $\mathbb{C}$. If $f \in \mathcal{O}(U)$ is nowhere-zero, then we write $f \in \mathcal{O}^*(U)$.

First, we consider a disc.

Proposition 6.3. *If $f \in \mathcal{O}^*(\Delta(z_0, r))$ then there exists $F \in \mathcal{O}(\Delta(z_0, r))$ such that $e^F = f$. Such a solution is unique up to adding an integer multiple of $2\pi i$.*

Proof. Given any $z \in \Delta(z_0, r)$, we define

$$F(z) = \int_\gamma \frac{f'(w)}{f(w)} dw, \tag{6.1}$$

where γ is the straight line path from z_0 to z. Consider a square with corners at z_0 and z, such that γ is the diagonal. Then 2 sides of the square and the diagonal form a closed triangle. By Cauchy's Theorem, the integral is the same if we integrate along the edges. Then the fundamental theorem of calculus shows that F is differentiable, and $F'(z) = f'(z)/f(z)$. Then consider the function $G(z) = fe^{-F}$. We compute

$$G'(z) = f'(z)e^{-F} - fe^{-F(z)}F'(z) = 0, \tag{6.2}$$

so G is constant. By adding the appropriate constant to F, we can therefore assume $G(z) \equiv 1$, which is $f = e^F$.

Assume that $e^{F_1} = f = e^{F_2}$. Then

$$fF_1' = e^{F_1}F_1' = e^{F_2}F_2' = fF_2', \tag{6.3}$$

and since f is nowhere vanishing, $F_1' = F_2'$, so F_1 and F_2 differ by a constant, which must be an integer multiple of $2\pi i$. □

Definition 6.4. We say that a domain U is simply connected if every simple closed curve in U bounds a disc in U.

Notice that the above proof works for any simply-connected domain. However, we will next prove this in a different way.

Definition 6.5. Let $U \in \mathbb{C}$ be a domain. We say that U has $H^1(U;\mathbb{Z}) = 0$ if any countable locally finite covering of U by discs $U_i = \Delta(z_j, r_j)$ has the following property. For any $n_{ij} \in \mathbb{Z}$ satisfying $n_{ij} = -n_{ji}$, and $n_{jk} - n_{ik} + n_{ij} = 0$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$ then there exists $n_i \in \mathbb{Z}$ such that $n_{ij} = n_j - n_i$ whenever $U_i \cap U_j \neq \emptyset$.

Remark 6.6. This definition is in fact equivalent to assuming that U is simply-connected, but this fact needs some tools from algebraic topology which we will not discuss right now.

Proposition 6.7. *Let $U \subset \mathbb{C}$ be a domain satisfying $H^1(U;\mathbb{Z}) = 0$ If $f \in \mathcal{O}^*(U)$ then there exists $F \in \mathcal{O}(U)$ such that $e^F = f$.*

Proof. Take a covering as in Definition 6.5. Then we view f as a collection of functions $f_i \in \mathcal{O}^*(U_i)$, with $f_i = f_j$ on $U_i \cap U_j$. Each U_i is a disc, so by the above, there exists $g_i \in \mathcal{O}(U_i)$ with $f_i = e^{g_i}$ on U_i. We have $\frac{f_i}{f_j} = e^{g_i - g_j} = 1$ on $U_i \cap U_j$, so there exists $n_{ij} \in \mathbb{Z}$ such that $g_i - g_j = 2\pi\sqrt{-1}n_{ij}$ there. By the assumption that $H^1(U;\mathbb{Z}) = 0$, there exists $n_i \in \mathbb{Z}$ such that $n_{ij} = n_i - n_j$ whenever $U_i \cap U_j \neq \emptyset$. We have $g_i - g_j = 2\pi\sqrt{-1}(n_i - n_j)$ on $U_i \cap U_j$. Therefore $g_i' = g_i - 2\pi\sqrt{-1}n_i$ satisfies $g_i' = g_j'$ on $U_i \cap U_j$ so defines a function $F \in \mathcal{O}(U)$ satisfying $f = e^F$. □

Remark 6.8. The function z in nowhere zero in any punctured disc $\Delta^*(0, r)$. If $z = e^F$ for $F \in \mathcal{O}(\Delta^*)$, then $F' = 1/z$ there (that is, F is a *primitive* for $1/z$). But if γ is $S^1(r/2)$, then $\int_\gamma F'(z)dz = 0$ from the fundamental theorem of calculus, but $\int_\gamma (1/z) = 2\pi\sqrt{-1}$, a contradiction.

Remark 6.9. Secretly, we are using the long exact sequence in cohomology associated to the exponential sheaf sequence

$$0 \to 2\pi\sqrt{-1} \cdot \mathbb{Z} \to \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \to 1, \tag{6.4}$$

where $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U, and similarly for $\mathcal{O}_U^*$. This gives the exact sequence

$$0 \to H^0(U;\mathbb{Z}) \to H^0(U, \mathcal{O}_U) \to H^0(U, \mathcal{O}_U^*) \to H^1(U;\mathbb{Z}). \tag{6.5}$$

If U is connected, then $H^0(U;\mathbb{Z}) \simeq \mathbb{Z}$. However, the 0th cohomology group of any sheaf is the space of global sections, so we have the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}(U) \to \mathcal{O}^*(U) \to H^1(U;\mathbb{Z}). \tag{6.6}$$

Thus if $H^1(U;\mathbb{Z}) = 0$, then the third arrow is surjective.

7 Lecture 7

7.1 Weierstrass Theorem

We give more applications of the solution of the inhomogeneous Cauchy-Riemann equations. References for this section are [Hör90, Chapter 1] and [Eps91, Section 1.6].

Definition 7.1. Let $U \subset \mathbb{C}$ be a domain. We say that U has $H^2(U;\mathbb{Z}) = 0$ if any countable locally finite covering of U by discs $U_i = \Delta(z_j, r_j)$ has the following property. For any $n_{ijk} \in \mathbb{Z}$ satisfying $n_{ijk} = -n_{jik} = -n_{ikj}$ and $n_{jkl} - n_{ikl} + n_{ijl} - n_{ijk} = 0$ whenever $U_i \cap U_j \cap U_k \cap U_l \neq \emptyset$, then there exists $n_{ij} \in \mathbb{Z}$ satisfying $n_{ij} = -n_{ji}$ such that $n_{ijk} = n_{jk} - n_{ik} + n_{ij}$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$.

Theorem 7.2 (Weierstrass)**.** *Let U be a domain in $\mathbb{C}$ with $H^2(U;\mathbb{Z}) = 0$, $\{w_j\}$ a discrete subset of U, and $n_j \in \mathbb{Z}$. Then there exists a meromorphic function $f \in \mathcal{M}(U)$ with the order of f at w_j equal to n_j and no other poles or zeroes.*

Proof. We cover U by discs $U_i = \Delta(z_i, r_i)$ such that each w_i is contained in exactly one of these discs. Define the function $f_i = (z - w_j)^{n_j}$ if $w_j \in U_i$, and let $f_i = 1$ if U_i doesn't contain any of the discrete points. On $U_i \cap U_j$, let $f_{ij} = f_i/f_j$. Then $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ is a non-vanishing holomorphic function. Since $U_i \cap U_j$ is simply-connected, we can define $g_{ij} = \log f_{ij}$. Note that $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ is only defined up to adding an integer multiple of $2\pi i$. Since

$$f_{ik} = \frac{f_i}{f_k} = \frac{f_i}{f_j}\frac{f_j}{f_k} = f_{ij} f_{jk}, \tag{7.1}$$

the g_{ij} satisfy on triple intersections $U_i \cap U_j \cap U_k$

$$g_{ij} - g_{ik} + g_{jk} = 2\pi i n_{ijk}, \tag{7.2}$$

where $n_{ijk} \in \mathbb{Z}$. The n_{ijk} satisfy the condition on intersections $U_i \cap U_j \cap U_k \cap U_l$,

$$n_{jkl} - n_{ikl} + n_{ijl} - n_{ijk} = 0. \tag{7.3}$$

By the assumption that $H^2(U;\mathbb{Z}) = 0$, there exists integers n_{ij} such that

$$n_{ijk} = n_{jk} - n_{ik} + n_{ij}, \tag{7.4}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Then $g'_{ij} = g_{ij} - 2\pi i n_{ij}$, satisfy

$$g'_{ij} - g'_{ik} + g'_{jk} = 0. \tag{7.5}$$

Choose a partition of unity ψ_i subordinate to U_i, and define

$$h_i = \sum_j g'_{ij}\psi_j, \tag{7.6}$$

which satisfies $h_i \in C^\infty(U_i)$. On $U_i \cap U_j$, we have

$$h_i - h_j = \sum_k (g'_{ik} - g'_{jk})\psi_k = \sum_k g'_{ij}\psi_k = g'_{ij}. \tag{7.7}$$

We then have that

$$\frac{\partial}{\partial \overline{z}}(h_i - h_j) = \frac{\partial}{\partial \overline{z}} g'_{ij} = 0, \tag{7.8}$$

So we can define $h \in C^\infty(U)$ by letting

$$h|_{U_i} = \frac{\partial h_i}{\partial \overline{z}} \tag{7.9}$$

By Theorem 5.3, we can solve the equation $\partial f / \partial \overline{z} = h$ for $f \in C^\infty(U)$. Then we redefine $h'_i = h_i - f$. These now satisfy $h'_i \in \mathcal{O}(U_i)$, and $h'_i - h'_j = g'_{ij}$.

So going back to the above, we define $f'_i = e^{-h'_i} f_i$. On intersections, we now have

$$\frac{f'_i}{f'_j} = \frac{e^{-h'_i} f_i}{e^{-h'_j} f_j} = e^{-h'_i + h'_j} \frac{f_i}{f_j} = e^{-g_{ij'}} \frac{f_i}{f_j} = e^{-g_{ij}} \frac{f_i}{f_j} = 1, \tag{7.10}$$

so the f'_i patch together to define $f \in \mathcal{M}(U)$. Since we only multiplied the f_i by a non-zero holomorphic function, the order of f at w_j is equal to n_j, and there are no other zeros or poles of f in U. □

Remark 7.3. Note that any domain in $\mathbb{C}$ is necessarily a non-compact 2-manifold, it will follow by Poincarè duality $H^2_{sing}(U; \mathbb{Z}) = 0$. This is equivalent to the vanishing of the Čech cohomology group $\check{H}^2(U; \mathbb{Z}) = 0$, which implies that the condition in Definition 7.1 is always satisfied for any domain $U \subset \mathbb{C}$. So the assumption that $H^2(U; \mathbb{Z}) = 0$ in actually superfluous. In contrast, if we consider a compact Riemann surface Σ, then $H^2(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}$, and Weierstrass' Theorem will not necessarily hold.

Corollary 7.4. *If $f \in \mathcal{M}(U)$, then there exists $g, h \in \mathcal{O}(U)$ such that $f = g/h$ in all of U.*

Proof. If f has poles of order n_j at w_j, then by the Weierstrass Theorem, there exists a holomorphic function $h \in \mathcal{O}(U)$ which has a zero of order n_j at w_j. Then $g = hf$ has no poles so $g \in \mathcal{O}(U)$. □

Corollary 7.5. *If U is any domain in $\mathbb{C}$, then there exists $u \in \mathcal{O}(U)$ such U cannot be extended to any larger domain.*

Proof. Take a discrete subset w_j with closure containing the boundary of Ω. By the Weierstrass Theorem, there exists $u \in \mathcal{O}(U)$ which has zeroes at the w_j, but is not identically zero. This function cannot be extended holomorphically in a neighborhood of any boundary point. This is because any holomorphic function f whose zero set has a limit point in a domain must be identically zero. (Proof: let the limit point be z_0, then we can factor $f = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$, so the zeros of a any holomorphic function are isolated; this is also known as the Identity Theorem.) □

Remark 7.6. The power series

$$F(z) = \sum_{j=0}^{\infty} z^{2^j} \tag{7.11}$$

converge on compact subsets to a holomorphic function on the unit disc, which cannot be extended past any boundary point. See [GK06, Chapter 9] and the more general Hadamard gap theorem. However, this is proved by ad hoc methods and not really related to the above proof.

8 Lecture 8

8.1 Holomorphic line bundles on domains in $\mathbb{C}$

Let $\mathfrak{U} = \{U_i\}_{i \in \mathcal{I}}$ be a countable locally finite open covering of a domain $U \subset \mathbb{C}$ by discs. For $i, j \in \mathcal{I}$, let $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfy $f_{ji} = f_{ij}^{-1}$, and

$$f_{ik} = f_{ij} \cdot f_{jk} \tag{8.1}$$

on $U_i \cap U_j \cap U_k$. We call (8.1) the *cocycle condition*, and write that $f_{ij} \in Z^1(\mathfrak{U}, \mathcal{O}_U^*)$. We call the collection $\{f_{ij}, i, j \in \mathcal{I}\}$ a *holomorphic line bundle* with respect to the open covering $\mathfrak{U}$. A collection $f_i \in \mathcal{O}(U_i)$ for $i \in \mathcal{I}$ is called a *section* of the line bundle if $f_{ij} f_j = f_i$ on $U_i \cap U_j$. The *trivial line bundle* is $f_{ij} = 1$ on all $U_i \cap U_j$. Note that there is a multiplicative structure: the product of f_{ij} and f'_{ij} is simply $f_{ij} f'_{ij}$, which clearly satisfies the cocyle condition. Define $C^0(\mathfrak{U}, \mathcal{O}_U^*)$ to be the collection of $f_i \in \mathcal{O}^*(U_i)$, which we call the space of 0-cocycles with respect to $\mathfrak{U}$. Define $\delta : C^0 \to Z^1$ by $\delta\{f_i\} = f_i/f_j$ on $U_i \cap U_j$. Then we define $H^1(\mathfrak{U}, \mathcal{O}_U^*)$ as $Z^1/\delta(C^0)$. We call this the set of *equivalence classes* of holomorphic line bundles on U, with respect to the open covering $\mathfrak{U}$. Note that two line bundles f_{ij} and f'_{ij} are equivalent if and only if there exists $f_i \in \mathcal{O}^*(U_i)$ such that $f_{ij} f_j = f'_{ij} f_i$. In particular, a line bundle is equivalent to the trivial line bundle if and only if there exists $f_i \in \mathcal{O}^*(U_i)$ such that $f_{ij} f_j = f_i$, equivalently, if and only if there exists a nowhere vanishing section.

What we have proved above can be restated as follows.

Proposition 8.1. *If $U \subset \mathbb{C}$ is a domain and $\mathfrak{U}$ is a countable locally finite covering of U by discs, then any holomorphic line bundle is equivalent to a trivial holomorphic line bundle (everything with respect to the open covering $\mathfrak{U}$).*

Proof. We simply review the above proof of the Weierstrass Theorem. In that proof, we first constructed $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfying the cocycle condition. Now we are just given $f_{ij} \in Z^1(\mathfrak{U}, \mathcal{O}_U^*)$. Then we choose $g_{ij} = \log f_{ij}$ in $\mathcal{O}(U_i \cap U_j)$. The g_{ij} satisfy on triple intersections $U_i \cap U_j \cap U_k$

$$g_{ij} - g_{ik} + g_{jk} = 2\pi i n_{ijk}, \tag{8.2}$$

where $n_{ijk} \in \mathbb{Z}$. We then used the assumption $H^2(U; \mathbb{Z}) = 0$ to find n_{ij} satisfying

$$n_{ijk} = n_{jk} - n_{ik} + n_{ij}, \tag{8.3}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Then $g'_{ij} = g_{ij} - 2\pi\sqrt{-1}n_{ij}$ satisfy

$$g'_{ij} - g'_{ik} + g'_{jk} = 0. \tag{8.4}$$

Then, using the partition of unity argument, we found $h'_i \in \mathcal{O}(U_i)$ with $g'_{ij} = h'_i - h'_j$. Define $f_i = e^{h'_i}$. Exponentiation of this yields

$$e^{g'_{ij}} = e^{g_{ij} - 2\pi\sqrt{-1}n_{ij}} = e^{g_{ij}} = f_{ij} = e^{h'_i - h'_j} = \frac{e^{h'_i}}{e^{h'_j}} \equiv \frac{f_i}{f_j}, \tag{8.5}$$

and we are done since the collection $\{f_i\}$ are a nowhere vanishing section of the line bundle. □

Remark 8.2. Similar to above, we can define $Z^1(\mathfrak{U}, \mathcal{O}_U)$ to be the collection of $g'_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfying $g'_{ij} = -g'_{ji}$, and

$$g'_{ij} - g'_{ik} + g'_{jk} = 0 \tag{8.6}$$

on triple intersections $U_i \cap U_j \cap U_k$. Note this is an additive group, as opposed to multiplicative in the case of line bundles. Define $C^0(\mathfrak{U}, \mathcal{O}_U)$ to be the collection of $h'_i \in \mathcal{O}(U_i)$ and $\delta : C^0 \to Z^1$ by $\delta\{h'_i\} = h'_j - h'_i$. Then we can define $H^1(\mathfrak{U}, \mathcal{O}_U) = Z^1/\delta(C^0)$. The argument involving a partition of unity proved that $H^1(\mathfrak{U}, \mathcal{O}_U) = 0$, which relied on Theorem 5.3, the solvability of the inhomogeneous Cauchy-Riemann equation in $C^\infty(U)$. This is a special case of a general result about the equivalence of Čech and Dolbeault cohomology.

Remark 8.3. We want to remove the dependence of the above on the open covering, we will do this later. The proof of the Weierstrass Theorem (using the partition of unity argument), yields the vanishing of the sheaf cohomology group $H^1(U, \mathcal{O}_U) = 0$. The long exact sequence in cohomology associated to the exponential sheaf sequence

$$0 \to 2\pi\sqrt{-1} \cdot \mathbb{Z} \to \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \to 1, \tag{8.7}$$

yields

$$0 = H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U^*) \to H^2(U; \mathbb{Z}) = 0, \tag{8.8}$$

from which it follows that $H^1(U, \mathcal{O}_U^*) = 0$. In other words, any holomorphic line bundle on U is (equivalent to) a trivial bundle. This will not be true in general for domains in $\mathbb{C}^n$ for $n > 1$: as an example, consider a product of punctured discs, which has $H^2(\Delta^* \times \Delta^*; \mathbb{Z}) \simeq \mathbb{Z}$. This will also not be true for a *compact* Riemann surface Σ, which satisfies $H^2(\Sigma; \mathbb{Z}) = \mathbb{Z}$.

9 Lecture 9

9.1 Power series in several variables

We review some basic facts about power series in several variables. Some good references for this material are [FG02, Chapter 1], [JP08, Chapter 1], or [KP02, Chapter 2.1], We write a point $z = (z_1, \ldots, z_n)$. The open polydisc with polyradius $r = (r_1, \ldots, r_n)$ about a point $z_0 = (z_1^0, \ldots, z_n^0)$ is the set

$$\Delta(z_0, r) = \{z \mid |z_j - z_j^0| < r_j,\ j = 1 \ldots n\}. \tag{9.1}$$

We will let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ denote a multi-index, where $\mathbb{Z}_+$ denotes the non-negative integers. Define

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \tag{9.2}$$
$$|z|^\alpha = |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \tag{9.3}$$
$$\alpha! = \alpha_1! \cdots \alpha_n! \tag{9.4}$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{9.5}$$

Definition 9.1. The series $\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha$ converges at z if some rearrangement converges, that is, give some bijection $\phi : \mathbb{Z}_+ \to (\mathbb{Z}_+)^n$, the series

$$\sum_{j=0}^{\infty} a_{\phi(j)} (z - z_0)^{\phi(j)} \tag{9.6}$$

converges. The *domain of convergence* of the power series is the interior of the set of points of convergence.

In 1 variable we know that domains of convergence are discs. Regions of convergence in several variable can be more complicated.

Example 9.2. The domain of convergence of the series $\sum_{k=0}^{\infty} z^k w^k$ is $\{(z, w) \mid |zw| < 1\}$.

Example 9.3 (Boas)**.** The series $\sum_{n=1}^{\infty} z^n w^{n!}$ converges in the 3 sets

$$U_1 = \{(z, w) \mid |w| < 1\},\ U_2 = \{(0, w)\},\ U_3 = \{(z, w) \mid |z| < 1 \text{ and } |w| = 1\}. \tag{9.7}$$

Only U_1 is an open set; the sets U_2 and U_3 are 1 dimensional, and are not domains. The domain of convergence is U_1.

Lemma 9.4 (Abel)**.** *If $\sum_\alpha a_\alpha z^\alpha$ (centered at $z = 0$) converges at the point z' then it converges uniformly and absolutely for any point z of the form $z_j = \rho_j z_j'$ where $|\rho_j| < 1$. Furthermore, a point p belongs to the domain of convergence of the power series $\sum_\alpha a_\alpha z^\alpha$ if and only if there exists a neighborhood U of p, a constant C, and $r < 1$ such that $|a_\alpha z^\alpha| \leq C r^{|\alpha|}$ for all $z \in U$.*

Proof. Since the series converges at the point z', the terms must be bounded, so there exists a constant C so that $|a_\alpha||z'|^\alpha \leq C$. Let $\rho = \max\{|\rho_1|, \ldots, |\rho_n|\} < 1$, and consider any point $z = (z_1, \ldots, z_n)$ so that $|z_j| < \rho |z_j'|$. We then have

$$|a_\alpha||z|^\alpha \leq |a_\alpha| \rho^{|\alpha|} |z'|^\alpha \leq C \rho^{|\alpha|}. \tag{9.8}$$

So given an integer $N > 0$, we have

$$\begin{aligned}\sum_{|\alpha|\leq N} |a_\alpha||z|^\alpha &= \sum_{j=0}^{N}\sum_{|\alpha|=j} |a_\alpha||z|^\alpha \\ &\leq \sum_{j=0}^{N}\sum_{|\alpha|=j} C\rho^j\end{aligned} \tag{9.9}$$

How many multi-indices of length j are there? This is counting the number of non-negative integer solutions of

$$\alpha_1 + \cdots + \alpha_n = j. \tag{9.10}$$

To see this, let $\alpha_i' = \alpha_i + 1$, then we are interested in the number of positive integer solutions to

$$\alpha_1' + \cdots + \alpha_n' = j + n. \tag{9.11}$$

So we have a total of $j + n$ integers, dividing this up into n integers is the same as putting $n - 1$ partitions somewhere in the spaces between them, so the number is

$$\binom{j+n-1}{n-1}. \tag{9.12}$$

Continuing with the above calculation,

$$\begin{aligned}\sum_{|\alpha|\leq N} |a_\alpha||z|^\alpha &\leq C\sum_{j=0}^{N}\binom{j+n-1}{n-1}\rho^j \\ &= C\sum_{j=0}^{N}\frac{(j+n-1)!}{j!(n-1)!}\rho^j \\ &= \frac{C}{(n-1)!}\sum_{j=0}^{N}(j+n-1)(j+n-2)\cdots(j+1)\rho^j \\ &\leq C_n\sum_{j=0}^{N} j^n\rho^j.\end{aligned} \tag{9.13}$$

Applying the ratio test, we have

$$\lim_{j\to\infty}\frac{(j+1)^n\rho^{j+1}}{j^n\rho^j} = \lim_{j\to\infty}\left(\frac{j+1}{j}\right)^n\rho = \rho, \tag{9.14}$$

so the series converges provided $\rho < 1$.

If p belongs to the domain of convergence, then by definition the series converges in a neighborhood of p. Then by the first part it converges in some polydisc around the origin containing z, and we follow the first part of the proof.

□

Definition 9.5. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha \tag{9.15}$$

which converges absolutely and uniformly in a polydisc $\Delta(z_0, \hat{\epsilon})$ around z_0, for some positive polyradius $\hat{\epsilon}$.

9.2 Cauchy's formula in several complex variables

Basic reference are [GH78, Hör90, Nog16].

Definition 9.6. We say that f is holomorphic in U if it is $C^1(U)$ and satisfies the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial \overline{z}^j} = 0, \ j = 1 \cdots n. \tag{9.16}$$

Proposition 9.7. *Let U be an open set in $\mathbb{C}$. Then f is holomorphic in U if and only if f is complex analytic in U.*

Proof. Consider $n = 2$, the higher-dimensional case is similar. We assume that $U = \Delta(0, r_1) \times \Delta(0, r_2)$ is a polydisc, and $f \in C^1(\overline{U})$. If f is holomorphic in U, then for fixed z_1, the slice $f(z_1, z_2)$ is a 1-variable holomorphic function for $z_2 \in \Delta(0, r_2)$. This holds similarly for the other variable, so the Cauchy-Pompieu formula applied twice yields

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi i} \int_{|w_2| = r_2} \frac{f(z_1, w_2) dw}{w_2 - z_2} \\ &= \left(\frac{1}{2\pi i} \right)^2 \int_{|w_2| = r_2} \int_{|w_1| = r_1} \frac{f(w_1, w_2) dw}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned} \tag{9.17}$$

For any $(z_1^0, z_2^0) \in U$, we expand

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0 + z_1^0 - z_1} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \frac{1}{1 - \frac{z_1 - z_1^0}{w_1 - z_1^0}} \tag{9.18}$$

$$= \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0} \right)^k \tag{9.19}$$

$$= \frac{1}{(w_1 - z_1^0)(w_2 - z_2^0)} \sum_{l=0}^{\infty} \left(\frac{z_2 - z_2^0}{w_2 - z_2^0} \right)^l \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0} \right)^k \tag{9.20}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \tag{9.21}$$

We next show that we are justified in the last step. Let $(z_1^0, z_2^0) \in \Delta(0, r_1') \times \Delta(0, r_2')$ with $r_1' < r_1$ and $r_2' < r_2$. Then we have $|w_1 - z_1^0| > r_1 - r_1'$, and $|w_2 - z_2^0| > r_2 - r_2'$. For $|z_1 - z_1^0| < (r_1 - r_1')/2$ and $|z_2 - z_2^0| < (r_2 - r_2')/2$, we then have

$$|a_{kl}| = \left| \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1}(w_2 - z_2^0)^{l+1}} \right| \leq \frac{1}{(r_1 - r_1')(r_2 - r_2')} 2^{-k} 2^{-l}, \tag{9.22}$$

so the sum converges absolutely and uniformly in any smaller polydisc by Lemma 9.4. Interchanging the integration and summation in (9.17) then yields a power series expansion for f.

The converse is similar to the 1-variable case. If f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula (9.17). So then f does also by uniform convergence. Then we can differentiate under the integral to see that f is holomorphic. For more details, see [GH78, page 6]. □

Remark 9.8. Note that the integral in (9.17) is just over a 2-dimensional torus contained in the boundary of the polydisc. The topological boundary of the polydisc is 3-dimensional, but it is not a manifold, it is $\partial(\Delta \times \Delta) = S^1 \times \Delta \cup \Delta \times S^1$, and these 2 sets intersect along the torus. The higher dimensional case is similar: the integral is over a real n-dimensional torus contained in the boundary of the polydisc.

Similar to the 1 variable case, we have the following corollaries.

Corollary 9.9. *If f holomorphic in Ω, then f is infinitely differentiable in Ω, and for any $z_0 \in \Omega$, f admits a power series expansion*

$$f = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha, \tag{9.23}$$

with

$$\begin{aligned} a_\alpha &= \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(z_0)}{\partial z^\alpha} \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{|w_n| = r_n} \cdots \int_{|w_1| = r_1} \frac{f(w_1, \cdots, w_n) dw_1 \cdots dw_n}{(w_1 - z_1^0)^{\alpha_1 + 1} \cdots (w_n - z_n^0)^{\alpha_n + 1}}, \end{aligned} \tag{9.24}$$

for any r such that the polydisc $\overline{\Delta(z_0, r)} \subset \Omega$.

Corollary 9.10 (The maximum principle)**.** *Let $\Omega \subset \mathbb{C}^n$ be connected, and $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$. Then $|f|$ does not assume its maximum at an interior point unless f is constant.*

Proof. We can assume that $\Omega = \Delta(0, r) = \Delta_1 \times \cdots \times \Delta_n$ is a polydisc centered at the origin, and that $|f|$ attains a maximum at the center point. The one variable function $z_1 \mapsto f(z_1, 0, \cdots, 0)$ is constant by the maximum principle in one variable, that is $f(z_1, 0, \cdots, 0) = f(0, \cdots, 0)$. Then, for any fixed z_1, the one variable function $z_2 \mapsto f(z_1, z_2, 0, \cdots, 0)$ is constant, so

$$f(z_1, z_2, 0, \cdots, 0) = f(z_1, 0, \cdots, 0) = f(0, \cdots, 0). \tag{9.25}$$

Iterating this argument shows that f is constant. □

Corollary 9.11. *Let $K \subset \Omega$ be a compact subset. Then there exist constants $C_{|\alpha|}$, depending only upon K and Ω such that*

$$\sup_{z \in K} \Big| \frac{\partial^\alpha f(z)}{\partial z^\alpha} \Big| \leq C_{|\alpha|} \sup_{z \in \Omega} |f(z)|. \tag{9.26}$$

for all $u \in \mathcal{O}(\Omega)$.

Proof. Again, we just consider the case of 2 dimensions, the higher dimensional case is similar. Fix $(z_1^0, z_2^0) \in \Omega$, and let $\Delta(z^0, (r_1, r_2)) \subset \Omega$ be a polydisc. Then for $(z_1, z_2) \in \Delta(z^0, (r_1', r_2'))$ with $r_1' < r_1$ and $r_2' < r_2$, we have

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl} (z_1 - z_1^0)^k (z_2 - z_2^0)^l, \tag{9.27}$$

where

$$a_{kl} = \left(\frac{1}{2\pi i} \right)^2 \int_{|w_2| = r_2} \int_{|w_1| = r_1} \frac{f(w_1, w_2) dw}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \tag{9.28}$$

We then get *Cauchy's inequalities*

$$\Big| \frac{\partial^{k+l} f}{\partial z_1^k \partial z_2^l} (z_1^0, z_2^0) \Big| = k! l! |a_{kl}| \leq \frac{k! l! r_1 r_2}{(r_1 - r_1')^k (r_2 - r_2')^l} \sup_{w = (w_1, w_2), |w_1| = r_1, |w_2| = r_2} |f(w)|. \tag{9.29}$$

Since the distance of K to $\partial\Omega$ is positive, the claim follows. □

The following corollaries are proved exactly as before.

Corollary 9.12. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \to u$ converges uniformly to u in the C^0 norm as $n \to \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Corollary 9.13. *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

10 Lecture 10

10.1 Hartogs' Theorem

Consider the polydisc $\Delta(0, r) = \{z \in \mathbb{C}^n \, | |z_j| < r, j = 1, \ldots, n\}$, and for $0 < r' < r$, let $A(r', r) = \Delta(r) \setminus \overline{\Delta(r')}$.

Theorem 10.1 (Hartogs)**.** *Let $n > 1$, and $0 < r' < r$. If $f \in \mathcal{O}(\overline{A(r', r)})$, then there exists a holomorphic function $F \in \mathcal{O}(\Delta(r))$ with $F = f$ on $A(r', r)$.*

Proof. Write a point in $\mathbb{C}^n$ as (z, w), where $z \in \mathbb{C}^{n-1}$. We then have

$$\begin{aligned}\big(\Delta^{n-1}(0,r) \times \Delta^1(0,r)\big) \setminus \big(\Delta^{n-1}(0,r') \times \Delta^1(0,r')\big) \\ = A^{n-1}(r',r) \times \Delta^1(0,r) \cup \Delta^{n-1}(0,r) \times A^1(r',r).\end{aligned} \tag{10.1}$$

Then for each $z \in \Delta^{n-1}(0,r')$, the cross section of $A^n(r',r)$ is a 1-dimensional annulus $A^1(r',r)$. However if $z \in A^{n-1}(r',r)$, then the cross section is a disc $\Delta^1(0,r)$. Let's write down the expression

$$F(z, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{|w|=r} \frac{f(z,w)dw}{w - z_n} \tag{10.2}$$

For $r' < |z| < r$ and $|z_n| < r$ this equals to f, by Cauchy's Theorem. This expression clearly defines a holomorphic function in all of $\Delta(r)$. The function F agrees with f whenever $r' < |z| < r$. By the identity principle, $F = f$ in $A(r', r)$, so is the required extension of f. □

In particular, while non-removable isolated singularities exist when $n = 1$, isolated singularities are always removable if $n > 1$! We also have the following corollary.

Corollary 10.2. *Let $n > 1$. If $\Omega \subset \mathbb{C}^n$ is a domain and $f \in \mathcal{O}(\Omega)$ then the zero set of f, $Z_f \equiv \{z \in \Omega \mid f(z) = 0\}$ does not contain any isolated points.*

To prove the corollary, note that $1/f$ would be a holomorphic function with an isolated singularity, which doesn't exist.

The next result will show that zero sets of holomorphic functions are large, they are always complex $(n-1)$-dimensional in a very precise sense.

10.2 Weierstrass Preparation Theorem

For $n = 1$, we know that zeros of a non-constant holomorphic function are isolated. We want to investigate the structure of zero sets of holomorphic function in $\mathbb{C}^n$. So consider $f(z_1, \ldots, z_n) \in \mathcal{O}(U)$, where U is a neighborhood of the origin, and assume that $f(0, \ldots, 0) = 0$. Consider a complex line through the origin

$$L_z = \{c(z_1, \ldots, z_n) \ |c \in \mathbb{C}\}. \tag{10.3}$$

If we restrict $f|L$, we get a 1 variable holomorphic function. If this restricted function were to vanish identically for all complex lines through the origin, then f itself would be identically zero. So if f is not identically zero, there exists a least one line on which f is not identically zero. By a rotation, we may assume that this in the nth coordinate line. I.e., for fixed $(z_1, \ldots, z_{n-1})$, we look at the holomorphic function of one variable $w \mapsto f(z_1, \ldots, z_{n-1}, w)$. By assumption $f(0, \ldots, 0, w)$ is not identically zero, so there is a unique minimal d such that $f(0, \ldots, 0, w) = w^d h$, where h does not vanish at 0. Then there exists $r > 0$ and $\delta > 0$ such that $|f(0, \ldots, 0, w)| \geq \delta$ for $|w| = r$. By continuity, there exists $\epsilon > 0$ so that $|f(z_1, \ldots, z_{n-1}, w)| \geq \delta/2$ for $|w| = r$ and $|(z_1, \ldots, z_n)| < \epsilon$. We next recall the argument principle.

Corollary 10.3 (The argument principle)**.** *If $f \in \mathcal{M}(\overline{\Omega})$ with no poles on $\partial\Omega$, then*

$$\int_{\partial\Delta(z_0,r)} w^q \frac{f'(w)}{f(w)} dw = \sum_{j=1}^{d} m_j z_j^q \tag{10.4}$$

where w_j are the zeros and poles of f, with m_j is the order of f at w_j.

To simplify notation, let $z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$. First of all, we apply the argument principle to the function $w \mapsto f(z, w)$, with $q = 0$ to get that

$$\int_{|w|=r} \frac{\partial f/\partial w(z,w)}{f(z,w)} dw = \sum_j m_j, \tag{10.5}$$

where m_j is the multiplicity of f at the zeros of the function $w \mapsto (z, w)$ inside $|w| < r$ and $|z| < \epsilon$. This is equal to d for $z = 0$. Since it is a continuous integer valued function, the function $w \mapsto f(z, w)$ also has d zeroes, counted with multiplicity for $|w| < r$ and $|z| < \epsilon$. We let $b_1(z), \cdots b_d(z)$ denote the zeros of $f(z, w)$ for $z \in \mathbb{C}^{n-1}$ fixed. The problem is that these function cannot necessarily be chosen smoothly, because there is no canonical ordering of these zeros. We define

$$g(z, w) = (w - b_1(z)) \cdots (w - b_d(z)) \tag{10.6}$$

Obviously, the zero set of g is exactly the same as the zero set of f. However, this is not obviously holomorphic due to the ordering problem. However, we may write

$$g(z, w) = w^d - \sigma_1(b(z))w^{d-1} + \sigma_2(b(z))w^{d-2} + \cdots + (-1)^d \sigma_d(b(z)) \tag{10.7}$$

where the σ_i are the ith elementary symmetric function, i.e.,

$$\sigma_1(b_1, \dots, b_d) = b_1 + \cdots + b_d \tag{10.8}$$

$$\sigma_2(b_1, \dots, b_d) = \sum_{i \neq j} b_i b_j \tag{10.9}$$

$$\vdots \tag{10.10}$$

$$\sigma_d(b_1, \dots, b_d) = b_1 \cdots b_d. \tag{10.11}$$

We have the following lemma.

Lemma 10.4. *There exists polynomials P_j with rational coefficients so that the elementary symmetric function $\sigma_i = P_i(F_1, \cdots, F_i)$, where*

$$F_q(b) = b_1^q + \cdots + b_d^q. \tag{10.12}$$

for $1 \leq i \leq d$.

Proof. Obviously $\sigma_1 = P_1$. For σ_2, we have

$$\sigma_2 = \frac{1}{2}(F_1^2 - F_2). \tag{10.13}$$

For σ_3, we have

$$\sigma_3 = \frac{1}{6}F_1^3 - \frac{1}{2}F_2F_1 + \frac{1}{3}F_3. \tag{10.14}$$

Will leave the proof of the general case as an exercise (Hint: use a induction argument, or ask Newton). □

However, if we apply the argument principle again for $q > 0$, we will get

$$F_q(b(z)) = b_1(z)^q + \cdots + b_d(z)^q = \int_{|w|=r} \frac{w^q(\partial f/\partial w)(z,w)}{f(z,w)} dw \tag{10.15}$$

Thus we see that the power sums $F_q(z) = b_1(z)^q + \cdots + b_d(z)^q$ are analytic functions of z, for $|z| < \epsilon$. By the lemma, the elementary symmetric functions are analytic there, which proves that the function $g(z,w)$ defined above is analytic.

Now we consider the function

$$h(z,w) = \frac{f(z,w)}{g(z,w)}, \tag{10.16}$$

which is analytic away from the zero set of g. By construction, for each fixed z with $|z| < \epsilon$, $g(z,w)$ has exactly the same zeros as $f(z,w)$. So the 1 variable function $w \mapsto h(z,w)$ is bounded, and thus has a removable singularity at near any zero of $g(z,w)$, so h extends to a non-zero holomorphic function for $|w| < r$. But then the Cauchy integral formula

$$h(z,w) = \frac{1}{2\pi\sqrt{-1}} \int_{|u|=r} \frac{h(z,u)du}{u-w} \tag{10.17}$$

shows that h is continuously differentiable and holomorphic in all of the variables z_j. So we have proved

Theorem 10.5 (Weierstrass Preparation Theorem)**.** *Let $f(z,w)$ be an analytic function which vanishes at the origin, but does not vanish identically on the w-axis. Then there exists a unique Weierstrass polynomial of the form*

$$g(z,w) = w^d + a_1(z)w^{d-1} + \cdots + a_d(z), \tag{10.18}$$

with $a_j(z)$ analytic with $a_j(0) = 0$, such that $f = g \cdot h$ with $h(0) \neq 0$.

This gives us a very nice description of the zero set of holomorphic functions.

Corollary 10.6. *If f is holomorphic, then locally Z_f admits a projection to a $(n-1)$ dimension polydisc Δ^{n-1}, which represents Z_f as a d-fold cover of Δ^{n-1} , which is branched over the zero locus of a holomorphic function $\mathscr{D} \in \mathcal{O}(\Delta^{n-1})$.*

Proof. This was proved above, we just need to note that the set where $w \mapsto f(z,w)$ has multiple roots is given by the vanishing of the discriminant $\mathscr{D}$, which is a polynomial in the roots. □

We also can prove another removable singularity theorem which allows for a much larger singular set.

Theorem 10.7 (Riemann Extension Theorem)**.** *Let f be holomorphic in Ω, and let Z_f denote the zero set of f. Let g be holomorphic in $\Omega \setminus Z_f$ and bounded. Then g extends to a holomorphic function $G \in \mathcal{O}(\Omega)$.*

Proof. This is clearly local, so we can assume that f is a Weierstrass polynomial in a small polydisc. Recall above, we chose $r, \epsilon > 0$ so that for $|z| < \epsilon$ and $|w| = r$, f has no zeros, so g is holomorphic there. Also, for $|z| < \epsilon$, $w \mapsto f(z,w)$ only has finite many zeroes which are isolated. Since g is bounded, for $|z| < \epsilon$, the 1 variable function $w \mapsto g(z,w)$ has removable singularities. So we can extend to $G(z,w)$. By the Cauchy integral formula, we have

$$G(z,w) = \frac{1}{2\pi\sqrt{-1}} \int_{|u|=r} \frac{g(z,u)du}{u-w} \tag{10.19}$$

is continuous differentiable and holomorphic in the z variables, so is the required extension of g. □

11 Lecture 11

11.1 Complex differential forms

Using the coordinates

$$(z^1, \ldots, z^n) = (x^1 + iy^1, \ldots, x^n + iy^n), \tag{11.1}$$

recall the definitions

$$\frac{\partial}{\partial z^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big) \tag{11.2}$$

$$\frac{\partial}{\partial \bar{z}^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big), \tag{11.3}$$

which are differential operators on complex-valued functions. We define

$$T_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{\partial/\partial x^1, \ldots, \partial/\partial x^n, \partial/\partial y^1, \ldots \partial/\partial y^n\} \tag{11.4}$$

which is a real $2n$-dimensional vector space, and define vector spaces

$$T^{1,0} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial z^j, j = 1 \ldots n\} \tag{11.5}$$

$$T^{0,1} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial \bar{z}^j, j = 1 \ldots n\} \tag{11.6}$$

$$T_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}, \tag{11.7}$$

which are complex vector spaces of complex dimension n, n, and $2n$, respectively. Note that $T_{\mathbb{C}} = T_{\mathbb{R}} \otimes \mathbb{C}$. Next define

$$\Lambda^1_{\mathbb{R}} = Hom(T_{\mathbb{R}}, \mathbb{R}), \tag{11.8}$$

to be the dual vector space to $T_{\mathbb{R}}$. We can write

$$\Lambda^1_{\mathbb{R}} = \text{span}\{dx^1, \ldots, dx^n, dy^1, \ldots, dy^n\}, \tag{11.9}$$

where the above is the dual basis to $\partial/\partial x^1, \ldots, \partial/\partial y^n$. Next, define

$$\Lambda^{1,0} = Hom(T^{1,0}, \mathbb{C}) \tag{11.10}$$
$$\Lambda^{0,1} = Hom(T^{0,1}, \mathbb{C}) \tag{11.11}$$
$$\Lambda^1_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}. \tag{11.12}$$

Similarly, we have $\Lambda^1_{\mathbb{C}} = \Lambda^1_{\mathbb{R}} \otimes \mathbb{C}$. Then we can write

$$\Lambda^{1,0} = \text{span}\{dz^j, j = 1 \ldots n\} \tag{11.13}$$
$$\Lambda^{0,1} = \text{span}\{d\bar{z}^j, j = 1 \ldots n\}, \tag{11.14}$$

where dz^j is the dual basis to $\partial/\partial z^j$ and $d\overline{z}^j$ is the dual basis to $\partial/\partial z^j$, that is,

$$\begin{aligned} dz^j(\partial/\partial z^k) &= d\bar{z}^j(\partial/\partial \bar{z}^k) = \delta^{jk}, \\ dz^j(\partial/\partial \bar{z}^k) &= d\bar{z}^j(\partial/\partial z^k) = 0. \end{aligned} \tag{11.15}$$

Define $\Lambda^k_{\mathbb{C}}$ to be the vector space

$$\Lambda^k_{\mathbb{C}} = \text{span}\{dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q} \mid i_1 < \cdots < i_p, j_1 < \cdots < j_q, p + q = k\} \tag{11.16}$$

We can then extend this notation to all indices by

$$dz^{i_1} \wedge dz^{i_2} = -dz^{i_2} \wedge dz^{i_1} \tag{11.17}$$
$$d\overline{z}^{i_1} \wedge d\overline{z}^{i_2} = -d\overline{z}^{i_2} \wedge d\overline{z}^{i_1} \tag{11.18}$$
$$dz^i \wedge d\overline{z}^j = -d\overline{z}^j \wedge dz^i. \tag{11.19}$$

This extends to a product called the wedge product:

$$\wedge : \Lambda^{k_1}_{\mathbb{C}} \oplus \Lambda^{k_2}_{\mathbb{C}} \to \Lambda^{k_1+k_2}_{\mathbb{C}} \tag{11.20}$$

which is bilinear, associative, and satisfies

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \;\; \alpha \in \Lambda^p_{\mathbb{C}}, \;\; \beta \in \Lambda^q_{\mathbb{C}}. \tag{11.21}$$

Remark 11.1. Notice we can do a similar construction and define $\Lambda^k_{\mathbb{R}}$ which will satisfy $\Lambda^k_{\mathbb{C}} = \Lambda^k_{\mathbb{R}} \otimes \mathbb{C}$.

We define $\Lambda^{p,q} \subset \Lambda^{p+q}_{\mathbb{C}}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{11.22}$$

Noting that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}, \tag{11.23}$$

we have

$$\binom{2n}{k} = \sum_{p+q=k} \binom{n}{p} \cdot \binom{n}{q}. \tag{11.24}$$

Next, suppose we are given a domain $U \subset \mathbb{C}^n$. Let

$$C^\infty_{\mathbb{C}}(U) = \{f : U \to \mathbb{C} \mid f \text{ is smooth}\}. \tag{11.25}$$

$$\mathcal{E}^k_{\mathbb{C}}(U) = C^\infty_{\mathbb{C}}(U) \otimes \Lambda^k_{\mathbb{C}} \tag{11.26}$$

$$\mathcal{E}^{p,q}(U) = C^\infty_{\mathbb{C}}(U) \otimes \Lambda^{p,q}. \tag{11.27}$$

So we have that

$$\mathcal{E}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{E}^{p,q}. \tag{11.28}$$

If $\alpha \in \mathcal{E}^{p,q}(U)$, then we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{11.29}$$

where I and J are multi-indices of length p and q, respectively, and $\alpha_{I,J} : U \to \mathbb{C}$ are smooth complex-valued functions.

The wedge product extends to

$$\wedge : \mathcal{E}^{k_1}_{\mathbb{C}} \oplus \mathcal{E}^{k_2}_{\mathbb{C}} \to \mathcal{E}^{k_1+k_2}_{\mathbb{C}} \tag{11.30}$$

which is bilinear, associative, and satisfies

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \ \ \alpha \in \mathcal{E}^p_{\mathbb{C}}, \ \ \beta \in \mathcal{E}^q_{\mathbb{C}}. \tag{11.31}$$

Remark 11.2. We can do a similar construction and define $C^\infty_{\mathbb{R}}(U)$ and $\mathcal{E}^k_{\mathbb{R}}$, which will satisfy $\mathcal{E}^k_{\mathbb{C}} = \mathcal{E}^k_{\mathbb{R}} \otimes \mathbb{C}$.

11.2 The operators ∂ and $\overline{\partial}$ in $\mathbb{C}^n$

We first define the exterior derivative operator

$$d_{\mathbb{R}} : \mathcal{E}^k_{\mathbb{R}} \to \Omega^{k+1}_{\mathbb{R}} \tag{11.32}$$

by the following. For $f \in \mathcal{E}^k_{\mathbb{R}} = C^\infty_{\mathbb{R}}(U)$, define

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i + \sum_j \frac{\partial f}{\partial j^j} dy^j. \tag{11.33}$$

Then we extend to any differential forms of any degree by

$$d(f_{IJ} dx^I \wedge dy^J) = (df_{IJ}) \wedge dx^I \wedge dy^J. \tag{11.34}$$

Proposition 11.3. *The operator* $d_{\mathbb{R}}$ *satisfies*

$$d^2 = 0 \tag{11.35}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta. \tag{11.36}$$

Proof. We leave this as an exercise. □

By complexification, this operator extends to

$$d_{\mathbb{C}} : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{11.37}$$

The following is a key proposition.

Proposition 11.4. *We have*

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}. \tag{11.38}$$

Proof. We simply check on functions that the following formula holds

$$df = \sum_k \frac{\partial f}{\partial z^k} dz^k + \sum_k \frac{\partial f}{\partial \overline{z}^k} d\overline{z}^k. \tag{11.39}$$

If $\alpha \in \mathcal{E}^{p,q}(U)$, then

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{11.40}$$

so applying d to (11.40), we obtain

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \Big) \wedge dz^I \wedge d\overline{z}^J, \tag{11.41}$$

and we are done. □

We can therefore define operators

$$\partial : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{11.42}$$

$$\overline{\partial} : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{11.43}$$

by

$$\partial \alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J \tag{11.44}$$

$$\overline{\partial} \alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J. \tag{11.45}$$

Using (17.1) and we have

$$\partial|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{11.46}$$

$$\overline{\partial}|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{11.47}$$

Corollary 11.5. *We have* $d = \partial + \overline{\partial}$ *for operators*

$$\partial : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q} \tag{11.48}$$

$$\overline{\partial} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}, \tag{11.49}$$

which satisfy

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{11.50}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{11.51}$$

If we plug in a form of type (p, q) the first term is of type $(p+2, q)$, the middle terms are of type $(p+1, q+1)$, and the last term is of type $(p, q+2)$. Since (17.1) is a direct sum, the claim follows. □

12 Lecture 12

12.1 De Rham cohomology

We can make the following definition since $d^2 = 0$.

Definition 12.1. Let $U \subset \mathbb{R}^n$ be a domain. For $0 \leq k \leq n$, the kth de Rham cohomology group is

$$H^k_{dR}(U) = \frac{\{\alpha \in \mathcal{E}^k(U) | d\alpha = 0\}}{d(\mathcal{E}^k(U))}, \tag{12.1}$$

Let $f : U \to V$ be a smooth mapping, where U and V are open subset of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then we can define the pullback of differential forms

$$f^* : \mathcal{E}^k(V) \to \mathcal{E}^k(U) \tag{12.2}$$

by

$$f^*\Big(\sum_{|I|=k} \alpha_I dy^I\Big) = \sum_{|I|=k} (\alpha_I \circ f) d(y^I \circ f) \tag{12.3}$$

Exercise 12.2. If $f : U \to V$ is smooth, then

$$d_U \circ f^* = f^* \circ d_V \tag{12.4}$$

and if $g : V \to W$ is smooth, then

$$(g \circ f)^* = f^* \circ g^*. \tag{12.5}$$

The de Rham cohomology groups enjoy the following functorality properties.

Proposition 12.3. *Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n, W \subset \mathbb{R}^l$ be domains. Let $f : U \to V$ be a smooth mapping. Then there are induced mappings*

$$f^* : H^k_{dR}(V) \to H^k_{dR}(U). \tag{12.6}$$

If $g : V \to W$ is smooth then so is $g \circ f : U \to W$ and

$$(g \circ f)^* = f^* \circ g^* : H^k_{dR}(W) \to H^k_{dR}(U). \tag{12.7}$$

In particular, if f is a diffeomorphism (one-to-one, onto, with smooth inverse), then the de Rham cohomologies of U and V are isomorphic.

Proof. We show that f^* induces a well-defined mapping on cohomology $f^* : H^k_{dR}(V) \to H^k_{dR}(U)$ by the following. If $[\alpha^k] \in H^k_{dR}(V)$ is represented by a form α^k, such that $d_V \alpha^k = 0$, then we have

$$d_U f^* \alpha^k = f^* d_V \alpha^k = f^* 0 = 0, \tag{12.8}$$

so we can define $f^*[\alpha^k] = [f^*\alpha^k]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^k + d_V \beta^{k-1}) = f^*\alpha^k + f^* d_V \beta^{k-1} = f^*\alpha^k + d_U f^* \beta^{k-1}, \tag{12.9}$$

so we have

$$[f^*(\alpha^k + d_V \beta^{k-1})] = [f^*\alpha^k + d_U f^* \beta^{k-1}] = [f^*\alpha^k]. \tag{12.10}$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \tag{12.11}$$

holds on the level of forms. Finally, if f is a diffeomorphism, then f^{-1} exists and is smooth, so we have

$$f \circ f^{-1} = id_V, \quad f^{-1} \circ f = id_U, \tag{12.12}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^k_{dR}(U)}, \quad (f^{-1})^* \circ f^* = id_{H^k_{dR}(V)}, \tag{12.13}$$

□

12.2 Dolbeault cohomology

We can make the following definition since $\overline{\partial}^2 = 0$.

Definition 12.4. Let $U \subset \mathbb{C}^n$ be a domain. For $0 \leq p, q \leq n$, the (p,q) Dolbeault cohomology group is

$$H^{p,q}_{\overline{\partial}}(U) = \frac{\{\alpha \in \mathcal{E}^{p,q}(U) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\mathcal{E}^{p,q-1}(U))}. \tag{12.14}$$

Next we define holomorphic mappings.

Definition 12.5. Let $f : U \to V$ be a C^1 mapping, where U and V are open subset of complex Euclidean spaces $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then f is *holomorphic* if writing $f = (f^1, \dots, f^m)$, then $f^j : U \to \mathbb{C}$ is holomorphic for $j = 1, \dots, m$.

Proposition 12.6. *If $f : U \to V$ is holomorphic, then f^* preserves the (p,q)-type decomposition differential forms, i.e.,*

$$f^* : \mathcal{E}^{p,q}(V) \to \mathcal{E}^{p,q}(U). \tag{12.15}$$

Proof. To see this, let $\alpha^{p,q} \in \mathcal{E}^{p,q}(V)$, of the form

$$\alpha^{p,q} = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz^I \wedge d\overline{z}^J, \tag{12.16}$$

Then by definition

$$f^*\alpha^{p,q} = \sum_{|I|=p,|J|=q} (\alpha_{IJ} \circ f) d(z^I \circ f) \wedge d(z^J \circ f). \tag{12.17}$$

But

$$d(z^j \circ f) = (\partial + \overline{\partial})(z^j \circ f) = \partial(z^j \circ f), \tag{12.18}$$

since $z^j \circ f$ is holomorphic, so this is a form of type $(1,0)$ on U. Similarly,

$$d(\overline{z}^j \circ f) = (\partial + \overline{\partial})(\overline{z^j \circ f}) = \overline{\partial}(\overline{z^j \circ f}), \tag{12.19}$$

since $\overline{z^j \circ f}$ is anti-holomorphic, so this is a form of type $(0,1)$ on U, and we are done. □

The Dolbeault cohomology groups enjoy the following functorality properties.

Proposition 12.7. *Let $U \subset \mathbb{C}^m, V \subset \mathbb{C}^n, W \subset \mathbb{C}^l$ be domains. Let $f : U \to V$ be holomorphic. Then there are induced mappings*

$$f^* : H^{p,q}(V) \to H^{p,q}(U). \tag{12.20}$$

If $g : V \to W$ is holomorphic, then so is $g \circ f : U \to W$ and

$$(g \circ f)^* = f^* \circ g^* : H^{p,q}(W) \to H^{p,q}(U). \tag{12.21}$$

In particular, if f is a biholomorphism (one-to-one, onto, with holomorphic inverse), then the Dolbeault cohomologies of U and V are isomorphic.

Proof. We know that the exterior derivative commutes with pullback,

$$d_U \circ f^* = f^* \circ d_V. \tag{12.22}$$

This is equivalent to

$$(\partial_U + \overline{\partial}_U) \circ f^* = f^* \circ (\partial_V + \overline{\partial}_V) \tag{12.23}$$

If we plug in $\alpha^{p,q} \in \Omega^{p,q}(V)$, we have 2 equations

$$\partial_U \circ f^* \alpha^{p,q} = f^* \circ \partial_V \alpha^{p,q} \tag{12.24}$$

$$\overline{\partial}_U \circ f^* \alpha^{p,q} = f^* \circ \overline{\partial}_V \alpha^{p,q} \tag{12.25}$$

The second equation implies that f^* induces a well-defined mapping on cohomology $f^* : H^{p,q}(V) \to H^{p,q}(U)$ by the following. If $[\alpha^{p,q}] \in H^{p,q}(V)$ is represented by a form $\alpha^{p,q}$, such that $\overline{\partial}_V \alpha^{p,q} = 0$, then we have

$$\overline{\partial}_U f^* \alpha^{p,q} = f^* \overline{\partial}_V \alpha^{p,q} = f^* 0 = 0, \tag{12.26}$$

so we can define $f^*[\alpha^{p,q}] = [f^* \alpha^{p,q}]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^{p,q} + \overline{\partial}_V \beta^{p,q-1}) = f^* \alpha^{p,q} + f^* \overline{\partial}_V \beta^{p,q-1} = f^* \alpha^{p,q} + \overline{\partial}_U f^* \beta^{p,q-1}, \tag{12.27}$$

so we have

$$[f^*(\alpha^{p,q} + \overline{\partial}_V \beta^{p,q-1})] = [f^* \alpha^{p,q} + \overline{\partial}_U f^* \beta^{p,q-1}] = [f^* \alpha^{p,q}]. \tag{12.28}$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \tag{12.29}$$

holds on the level of forms. Finally, if f is a biholomorphism, then f^{-1} exists and is holomorphic, so we have

$$f \circ f^{-1} = id_V, \quad f^{-1} \circ f = id_U, \tag{12.30}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^{p,q}(U)}, \quad (f^{-1})^* \circ f^* = id_{H^{p,q}(V)}, \tag{12.31}$$

□

Definition 12.8. A form $\alpha \in \mathcal{E}^{p,0}(U)$ is *holomorphic* if $\overline{\partial} \alpha = 0$, and we write that $\alpha \in \Omega^p(U)$.

Remark 12.9. We only talk about forms of type $(p, 0)$ being holomorphic, we never call a (p, q)-form holomorphic if $q > 0$. Also, we have (trivially)

$$\Omega^p(U) = H^{p,0}(U) = \{\alpha \in \mathcal{E}^{p,0}(U) \mid \overline{\partial} \alpha = 0\}. \tag{12.32}$$

Proposition 12.10. *A p-form $\alpha \in \mathcal{E}^{p,0}(U)$ is holomorphic if and only if it can be written as*

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{12.33}$$

where the $\alpha_I : U \to \mathbb{C}$ are holomorphic functions.

Proof. We have

$$\overline{\partial}\alpha = \sum_{|I|=p,k} \frac{\partial \alpha_I}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I. \tag{12.34}$$

So $\overline{\partial}\alpha = 0$ if and only if the α_I are holomorphic. □

Remark 12.11. So for $U \subset \mathbb{C}^n$ a domain, $\dim_{\mathbb{C}} H^{p,0}(U) = \infty$ is always infinite-dimensional for $0 \leq p \leq n$, in particular because any polynomial function in the z-variables is holomorphic.

Example 12.12. Let's review the case of a domain $U \subset \mathbb{C}$. First, $H^{0,0}_{\overline{\partial}}(U) = \mathcal{O}(U)$. Theorem 5.3 shows that $H^{0,1}_{\overline{\partial}}(U) = \{0\}$. The space $H^{1,0}_{\overline{\partial}}(U)$ consists of holomorphic 1-forms, but since $n = 1$, any holomorphic 1-form is of the form $f(z)dz$, where $f \in \mathcal{O}(U)$. So $H^{1,0}_{\overline{\partial}}(U) \cong \mathcal{O}(U)$. Finally,

$$H^{1,1}_{\overline{\partial}}(U) = \frac{\operatorname{Ker} \overline{\partial} : \Omega^{1,1} \to \Omega^{1,2}}{\operatorname{Im} \overline{\partial} : \Omega^{1,0} \to \Omega^{1,1}} = \frac{g dz \wedge d\overline{z}}{(\partial f / \partial \overline{z}) d\overline{z} \wedge dz} = \{0\}, \tag{12.35}$$

which also follows from Theorem 5.3.

13 Lecture 13

13.1 Jacobians

Let $f : \mathbb{C}^n \to \mathbb{C}^m$. Let the coordinates on $\mathbb{C}^n$ be given by

$$\{z^1, \dots z^n\} = \{x^1 + iy^1, \dots, x^n + iy^n\}, \tag{13.1}$$

and coordinates on $\mathbb{C}^m$ given by

$$\{w^1, \dots w^m\} = \{u^1 + iv^1, \dots, u^m + iv^m\} \tag{13.2}$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \operatorname{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{13.3}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \operatorname{span}\{\partial/\partial u^1, \dots \partial/\partial u^m, \partial/\partial v^1, \dots \partial/\partial v^m\}. \tag{13.4}$$

Then the real Jacobian of

$$f = (f^1, \dots f^{2m}) = (u^1 \circ f, u^2 \circ f, \dots, v^{2m} \circ f). \tag{13.5}$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \dots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \dots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \dots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \tag{13.6}$$

We define

$$J_{0,\mathbb{C}^n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{13.7}$$

Note that $J_{0,\mathbb{C}^n}^2 = -Id$. We have the following characterization of holomorphic mappings.

Proposition 13.1. *A C^1 mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if and only if*

$$f_* \circ J_{0,\mathbb{C}^n} = J_{0,\mathbb{C}^m} \circ f_*. \tag{13.8}$$

That is, the differential of f commutes with J_0.

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{13.9}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{13.10}$$

which is exactly the Cauchy-Riemann equations. The argument in general is similar, and is left as an exercise. □

For any differentiable f, the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^m). \tag{13.11}$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \text{span}\{\partial/\partial z^1, \dots \partial/\partial z^n, \partial/\partial \overline{z}^1, \dots \partial/\partial \overline{z}^n\}, \tag{13.12}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial w^1, \dots \partial/\partial w^m, \partial/\partial \overline{w}^1, \dots \partial/\partial \overline{w}^m\}. \tag{13.13}$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \dots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \overline{z}^1} & \dots & \frac{\partial f^1}{\partial \overline{z}^n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \dots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \overline{z}^1} & \dots & \frac{\partial f^m}{\partial \overline{z}^n} \\ \frac{\partial \overline{f}^1}{\partial z^1} & \dots & \frac{\partial \overline{f}^1}{\partial z^n} & \frac{\partial \overline{f}^1}{\partial \overline{z}^1} & \dots & \frac{\partial \overline{f}^1}{\partial \overline{z}^n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \frac{\partial \overline{f}^m}{\partial z^1} & \dots & \frac{\partial \overline{f}^m}{\partial z^n} & \frac{\partial \overline{f}^m}{\partial \overline{z}^1} & \dots & \frac{\partial \overline{f}^m}{\partial \overline{z}^n}, \end{pmatrix} \tag{13.14}$$

where $(f^1, \dots, f^m) = f$ now denotes the complex components of f. This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{13.15}$$

Notice that (13.14) is of the form

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{13.16}$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{13.17}$$

where $\mathbb{C}$-linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \tag{13.18}$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{13.19}$$

This corresponds to the embeddings

$$Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{13.20}$$

where the left $\mathbb{C}$-linear is with respect to J_0, via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{13.21}$$

Note that since the latter embedding is just a change of basis, if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB)\det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{13.22}$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{1,0}) \subset T^{1,0}. \tag{13.23}$$

13.2 Holomorphic inverse function theorem

Proposition 13.2 (Holomorphic inverse function theorem). *Let $f : U \to V$ be holomorphic where U and V are open subsets of $\mathbb{C}^n$. Let $z_0 \in U$ satisfy*

$$\det\Big(\frac{\partial f^j}{\partial z^k}\Big)(z_0) \neq 0. \tag{13.24}$$

Then there exists neighborhoods U' of z_0 and V' of $f(z_0)$ such that $f : U' \to V'$ is a biholomorphism.

Proof. By the above, since f is holomorphic, we have

$$\det(\mathcal{J}_{\mathbb{R},z_0}) = \Big| \det\Big(\frac{\partial f^j}{\partial z^k}\Big)(z_0)\Big|^2 \neq 0. \tag{13.25}$$

By the (real) smooth inverse function theorem, there exists a smooth inverse $f^{-1} : V' \to U'$, for some neighborhoods. We need to check that f^{-1} is holomorphic. For this, we differentiate the equation $z = f^{-1}(f(z))$ using the complex chain rule to get

$$0 = \frac{\partial}{\partial \overline{z}^j} f^{-1} \circ f(z) = \frac{\partial f^{-1}}{\partial w^k}\frac{\partial f^k}{\partial \overline{z}^j} + \frac{\partial f^{-1}}{\partial \overline{w}^k}\frac{\partial \bar{f}^k}{\partial \overline{z}^j} = \frac{\partial f^{-1}}{\partial \overline{w}^k}\overline{\Big(\frac{\partial f^k}{\partial z^j}\Big)} \tag{13.26}$$

The latter matrix has non-zero determinant in a neighborhood of z_0, from which we conclude that f^{-1} is holomorphic. ☐

Corollary 13.3. *Let $f : U \to \mathbb{C}$ be holomorphic, and $w \in \mathbb{C}$ be a complex regular value. That is,*

$$\nabla_{\mathbb{C}} f = \Big(\frac{\partial f}{\partial z^1}, \ldots, \frac{\partial f}{\partial z^n}\Big)(z) \neq 0 \tag{13.27}$$

for all $z \in f^{-1}(w)$. Then $f^{-1}(w)$ is a complex submanifold of U of complex codimension 1. That is, near any point $z \in f^{-1}(w)$, there exists a neighborhood U' of z and a holomorphic mapping $\Psi : V' \to U'$, where V' is a neighbohood of the origin in $\mathbb{C}^n$, such that

$$f \circ \Psi = z_n \tag{13.28}$$

and therefore $(f \circ \Psi)^{-1}(w)$ is locally a hyperplane.

Proof. This is an application of the inverse function theorem, the proof is left as an exercise. ☐

Exercise 13.4. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be given by

$$f(z_1, z_2) = z_1^2 + z_2^3 + z_1 z_2. \tag{13.29}$$

Determine the set of $w \in \mathbb{C}$ such that $f^{-1}(w)$ is a submanifold.

Exercise 13.5. Generalize Corollary 13.3 to the case of holomorphic $f : U \to \mathbb{C}^m$ for $m > 1$.

13.3 Anti-holomorphic mappings

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0, then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{13.30}$$

This corresponds to the embeddings

$$Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \tag{13.31}$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \tag{13.32}$$

We see that f is anti-holomorphic if and only if

$$f_*(T^{1,0}) \subset T^{0,1}. \tag{13.33}$$

For an arbitary mapping, not necessarily holomorphic or anti-holomorphic, we can decompose $f_* = f_*^C + f_*^A$, where

$$f_*^C = \frac{1}{2}\left(f_* - J_0 f_* J_0\right) \tag{13.34}$$

$$f_*^A = \frac{1}{2}\left(f_* + J_0 f_* J_0\right), \tag{13.35}$$

and f_*^C is holomorphic, while f_*^A is anti-holomorphic. In block matrix form, this just says that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \tag{13.36}$$

13.4 Almost complex structures

Recall that

$$T_{\mathbb{R}}(\mathbb{C}^n) = \text{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{13.37}$$

Above, we defined

$$J_0 : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^n) \tag{13.38}$$

by

$$J_{0,\mathbb{C}^n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{13.39}$$

which satisfies $J_{0,\mathbb{C}^n}^2 = -Id$. This mapping is called an *almost complex structure*. Using this mapping, we have some nice descriptions of the various spaces we previously introduced. Upon complexification, we have the decomposition.

$$T_{\mathbb{R}}(\mathbb{C}^n) \otimes \mathbb{C} = T_{\mathbb{C}}(\mathbb{C}^n) = T^{1,0} \oplus T^{0,1}. \tag{13.40}$$

We extend J_0 to a mapping

$$J_0 : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^n) \tag{13.41}$$

by complex linearity. Using J_0, we can describe the above decomposition as follows:

Proposition 13.6. *We have*

$$T^{1,0} = span\{\partial/\partial z^j, j = 1 \dots n\} = \{X - iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{13.42}$$

is the i-eigenspace of J_0 and

$$T^{0,1} = span\{\partial/\partial \bar{z}^j, j = 1 \dots n\} = \{X + iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{13.43}$$

is the $-i$-eigenspace of J_0.

Proof. We leave as an easy exercise. □

Next, recall that

$$\Lambda^1_{\mathbb{R}} = Hom(T_{\mathbb{R}}, \mathbb{R}) = \text{span}\{dx^1, \dots, dx^n, dy^1, \dots, dy^n\}, \tag{13.44}$$

is the dual vector space to $T_{\mathbb{R}}$. Upon complexification, we have a decomposition

$$\Lambda^1_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}. \tag{13.45}$$

The map J_0 also induces an endomorphism of 1-forms

$$J_0^T : \Lambda^1_{\mathbb{R}} \to \Lambda^1_{\mathbb{R}} \tag{13.46}$$

by

$$J_0^T(\omega)(v_1) = \omega(J_0 v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0^T(dx_j) = -dy_j, \quad J_0^T(dy_j) = +dx_j,$$

that is

$$J^T_{0,\mathbb{C}^n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{13.47}$$

which satisfies $(J_0^T)^2 = -Id$. We extend J_0^T to a mapping

$$J_0^T : \Lambda^1_{\mathbb{C}} \to \Lambda^1_{\mathbb{C}} \tag{13.48}$$

by complex linearity. Using J_0^T, we can describe the above decomposition as follows.

Proposition 13.7. *We have*

$$\Lambda^{1,0} = Hom(T^{1,0}, \mathbb{C}) = span\{dz^j, j = 1 \dots n\} = \{\alpha - iJ_0^T\alpha, \alpha \in \Lambda^1_{\mathbb{R}}(\mathbb{R}^{2n})\} \tag{13.49}$$

is the i-eigenspace of J_0^T, and

$$\Lambda^{0,1} = Hom(T^{0,1}, \mathbb{C}) = span\{d\bar{z}^j, j = 1 \dots n\} = \{\alpha + iJ_0^T\alpha, \alpha \in \Lambda^1_{\mathbb{R}}(\mathbb{R}^{2n})\} \tag{13.50}$$

is the $-i$-eigenspace of J_0^T.

Proof. This is left as an exercise. □

We also defined

$$\Lambda^k_{\mathbb{C}} = \Lambda^k_{\mathbb{R}} \otimes \mathbb{C} \tag{13.51}$$

and proved that there is a decomposition

$$\Lambda^k_{\mathbb{C}} = \Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{13.52}$$

with

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{13.53}$$

Finally, we can define

$$J_0^T : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C} \tag{13.54}$$

by letting

$$J_0^T(\alpha) = i^{p-q}\alpha, \tag{13.55}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J_0^T is not an almost complex structure on the space $\Lambda^k_{\mathbb{C}}$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J-invariant.

14 Lecture 14

14.1 The $\overline{\partial}$-equation for $(0,1)$-forms and Hartogs' Theorem

A reference for this section is [HL84, Section 1.2]. For $n \geq 2$, and $g \in \mathcal{E}^{0,1}(U)$, the equation $\overline{\partial} f = g$ is not always solvable. This follows from (17.14): applying $\overline{\partial}$ yields a compatibility condition $\overline{\partial} g = 0$. The following is in sharp contrast to the case $n = 1$.

Proposition 14.1. *Let $n \geq 2$ and $g \in \mathcal{E}^{0,1}_0(\mathbb{C}^n)$ (compact support) have C^∞ regularity and satisfy $\overline{\partial} g = 0$. Then there exists a smooth $f \in C^\infty_0(\mathbb{C}^n)$ (also having compact support) with $\overline{\partial} f = g$. Furthermore, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus supp(g)$.*

Proof. We write $g = \sum_{j=1}^n g_j d\overline{z}^j$. Define

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(w, z_2, \ldots, z_n)}{w - z_1} dw \wedge d\overline{w}. \tag{14.1}$$

The integral is defined since g_1 has compact support. Make the change of variable $\xi = w - z_1$, and we can write f as

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\xi + z_1, z_2, \ldots, z_n)}{\xi} d\xi \wedge d\bar{\xi}. \tag{14.2}$$

This shows that we can differentiate under the integral sign to conclude that f has C^∞ regularity. Furthermore,

$$\begin{aligned}\frac{\partial f(z_1,\ldots,z_n)}{\partial \overline{z}^1} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(\xi+z_1,z_2,\ldots,z_n)}{\partial \overline{z}^1}\frac{1}{\xi}d\xi\wedge d\bar{\xi}\\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(\xi+z_1,z_2,\ldots,z_n)}{\partial \bar{\xi}}\frac{1}{\xi}d\xi\wedge d\bar{\xi}\\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(w,z_2,\ldots,z_n)}{\partial \overline{w}}\frac{1}{w-z_1}dw\wedge d\overline{w} = g_1(z_1,\ldots,z_n),\end{aligned} \tag{14.3}$$

by the Cauchy-Pompieu formula applied to a large ball containing the support of g. The condition that $\overline{\partial} g = 0$ means that

$$0 = \sum_{j=1}^n\sum_{k=1}^n \frac{\partial g_j}{\partial \overline{z}_k} d\overline{z}^k\wedge d\overline{z}^j, \tag{14.4}$$

so

$$\frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_j} \tag{14.5}$$

for all $1\leq j,k\leq n$. Then differentiating (14.2) for $j\geq 2$, we obtain

$$\begin{aligned}\frac{\partial f(z_1,\ldots,z_n)}{\partial \overline{z}_j} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(\xi+z_1,z_2,\ldots,z_n)}{\partial \overline{z}^j}\frac{1}{\xi}d\xi\wedge d\bar{\xi}\\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_j(\xi+z_1,z_2,\ldots,z_n)}{\partial \overline{z}^1}\frac{1}{\xi}d\xi\wedge d\bar{\xi}\\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_j(w,z_2,\ldots,z_n)}{\partial \overline{w}}\frac{1}{w-z_1}dw\wedge d\overline{w} = g_j(z_1,\ldots,z_n).\end{aligned} \tag{14.6}$$

So the equation $\overline{\partial} f = g$ is satisfied everywhere. Finally, since g has compact support, it follows that f is holomorphic on the complement of a large ball $B_r(0)$ containing the support of g. But (14.1) shows that f vanishes when $\max\{|z_2|,\ldots,|z_n|\} > r$ for some r sufficiently large. Therefore f is a holomorphic function on $\mathbb{C}^n\setminus\overline{B_r(0)}$ which vanishes on the open subset $V = \{\max\{|z_2|,\ldots,|z_n|\} > r\}$. By unique continuation, $f\equiv 0$ on the unbounded component of $\mathbb{C}^n\setminus \mathrm{supp}(g)$. □

Theorem 14.2 (Hartogs)**.** *Let $n\geq 2$, U a domain, and $K\subset U$ a compact subset of U such that $U\setminus K$ is connected. Then if $u\in\mathcal{O}(U\setminus K)$, there exists $\tilde{u}\in\mathcal{O}(U)$ with $\tilde{u}|_{U\setminus K} = u$.*

Proof. Let $0\leq\chi\in C_0^\infty(U)$ and $\chi\equiv 1$ on K. Define $g = \overline{\partial}(\chi\cdot u) = \overline{\partial}((\chi-1)\cdot u)$. Since

$$\overline{\partial}(\chi u) = u\overline{\partial}(\chi) + \chi\overline{\partial}(u) = u\overline{\partial}(\chi)+0, \tag{14.7}$$

we see that g extends smoothly to U, that is, $g\in\mathcal{E}_0^{0,1}(U)$, and $\overline{\partial} g = 0$. By Proposition 14.1, there exists $f\in C_0^\infty(\mathbb{C}^n)$ with $\overline{\partial} f = g$. So then we let $\tilde{u} = (1-\chi)u + f$. This satisfies

$$\overline{\partial}\tilde{f} = -g + \overline{\partial}(f) = 0, \tag{14.8}$$

so $\tilde{u} \in \mathcal{O}(U)$. Let V denote the unbounded component of the complement of the support of χ. Since $\text{supp}(g) \subset \text{supp}(\chi)$, from Proposition 14.1, we have that $f \equiv 0$ in V, so $\tilde{u} = u$ in $U \cap V$. But since $U \setminus K$ is connected and $V \cap (U \setminus K) \neq \emptyset$, we have $\tilde{u} = u$ in $U \setminus K$ from unique continuation. □

Example 14.3. This gives another proof that point singularities are removable for $n \geq 2$. Also, polydics are removable: if u is holomorphic on $\Delta \setminus \overline{\Delta'}$, where $\overline{\Delta'} \subset \Delta$ then u extends to a holomorphic function on Δ. We also see that the same is true for balls $\overline{B_{r_1}(0)} \subset B_{r_2}(0)$ with $r_1 < r_2$.

14.2 Dolbeault cohomology of a polydisc

Some references for this section are [GH78, Section 0.2] or [Nog16, Section 3.6].

Proposition 14.4. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), and $\omega \in \mathcal{E}^{p,q}(U)$ satisfies $\overline{\partial}\omega = 0$ for $q \geq 1$, then given any polyradius $s < r$, there exists $\eta \in \mathcal{E}^{p,q-1}(\Delta(r))$ with $\overline{\partial}\eta = \omega$ satisfied in $\Delta(s)$.*

Proof. Step 1: reduce to case of $\mathcal{E}^{0,q}$. If $\omega \in \mathcal{E}^{p,q}(U)$,

$$\omega = \sum_{|I|=p,|J|=q} \omega_{IJ} dz^I \wedge d\overline{z}^J. \tag{14.9}$$

Define

$$\omega_I = \sum_{|J|=q} \omega_{IJ} d\overline{z}^J. \tag{14.10}$$

Then $\omega_I \in \mathcal{E}^{0,q}$, and $\overline{\partial}\omega_I = 0$. If $\omega_I = \overline{\partial}\eta_I$, then

$$\overline{\partial}(dz^I \wedge \eta_I) = (-1)^p dz^I \wedge \overline{\partial}\eta_I = (-1)^p dz^I \wedge \omega_I, \tag{14.11}$$

so

$$\overline{\partial}(\sum_{|I|=p} dz^I \wedge \eta_I) = (-1)^p \sum_{|I|=p} dz^I \wedge \omega_I = (-1)^p \sum_{|I|=p} \sum_{|J|=q} \omega_{IJ} dz^I \wedge d\overline{z}^J, \tag{14.12}$$

and we are done with Step 1.

Step 2. Given $s < r$, if $\omega \in \mathcal{E}^{0,q}(\Delta(r))$ and $\overline{\partial}\omega = 0$ in $\Delta(r)$, then there exists $\eta \in \mathcal{E}^{0,q-1}(\Delta(r))$ with $\overline{\partial}\eta = \omega$ satisfied in $\Delta(s)$. Choose cutoff functions $0 \leq \chi_j(t) \leq 1$ so that

$$\chi_i(t) = \begin{cases} 1 & t \leq s_j \\ 0 & t \geq r_j \end{cases}. \tag{14.13}$$

We begin with $q = 1$. Note that $\omega \in \mathcal{E}^{0,1}(\Delta(r))$, but it does not have compact support, so we proceed differently than in the proof of Proposition 14.1. Write

$$\omega = \sum_k \omega_k d\overline{z}^k, \tag{14.14}$$

and define

$$\eta_1(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|w^j| \leq r_j} \frac{\chi_1(w_1)\omega_j(w^1, z^2, \ldots, z^n)}{w^1 - z^1} dw^1 \wedge d\overline{w}^1. \tag{14.15}$$

Then $\partial \eta_1 / \partial \overline{z}^1 = \chi_1 \omega_1$, and we have

$$\overline{\partial} \eta_1 = \sum_l \frac{\partial \eta_1}{\partial \overline{z}^l} d\overline{z}^l = \chi_1 \omega_1 d\overline{z}^1 + \sum_{j>1} \frac{\partial \eta_1}{\partial \overline{z}^j} d\overline{z}^j. \tag{14.16}$$

That is, we have solved the $d\overline{z}^1$-term, modulo terms involving $d\overline{z}^j$ for $j > 1$ (we have not even used the fact that $\overline{\partial}\omega = 0$ yet!) Next, we consider the case

$$\omega = \sum_{k>1} \omega_k d\overline{z}^k, \tag{14.17}$$

Since $\overline{\partial}\omega = 0$, this tells us that $\partial \omega_2 / \partial \overline{z}^1 = 0$. Next, we define

$$\eta_2(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|w^2| \leq r_2} \frac{\chi_2(w_2)\omega_2(z^1, w^2, z^3, \ldots, z^n)}{w^2 - z^2} dw^2 \wedge d\overline{w}^2. \tag{14.18}$$

Then $\partial \eta_2 / \partial \overline{z}^2 = \chi_2 \omega_2$ and $\partial \eta_2 / \partial \overline{z}^1 = 0$, so we have

$$\overline{\partial} \eta_2 = \sum_l \frac{\partial \eta_2}{\partial \overline{z}^j} d\overline{z}^j = \chi_2 \omega_2 d\overline{z}^2 + \sum_{j>2} \frac{\partial \eta_2}{\partial \overline{z}^j} d\overline{z}^j. \tag{14.19}$$

Assume that we can solve all the terms involving $d\overline{z}^k$ for $k \leq l$, and

$$\omega = \sum_{k>l} \omega_k d\overline{z}^k. \tag{14.20}$$

Since $\overline{\partial}\omega = 0$, this tells us that $\partial \omega_{l+1} / \partial \overline{z}^j = 0$ for $j \leq l$. Then we define

$$\eta_{l+1}(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|w^{l+1}| \leq r_{l+1}} \frac{\chi_{l+1}(w_{l+1})\omega_{l+1}(z^1, \ldots, w^{l+1}, \ldots, z^n)}{w^{l+1} - z^{l+1}} dw^{l+1} \wedge d\overline{w}^{l+1}. \tag{14.21}$$

Then $\partial \eta_{l+1} / \partial \overline{z}^{l+1} = \chi_{l+1} \omega_{l+1}$ and $\partial \eta_{l+1} / \partial \overline{z}^j = 0$ for $j \leq l$, so we have

$$\overline{\partial} \eta_{l+1} = \sum_{j>l} \frac{\partial \eta_{l+1}}{\partial \overline{z}^j} d\overline{z}^j = \chi_{l+1} \omega_{l+1} d\overline{z}^{l+1} + \sum_{j>l+1} \frac{\partial \eta_{l+1}}{\partial \overline{z}^j} d\overline{z}^j. \tag{14.22}$$

By induction, we are done with the case of $q = 1$.

Next, consider the case of $q = 2$. Then

$$\omega = \sum_{1 \leq k < l} \omega_{kl} d\overline{z}^k \wedge d\overline{z}^l = \sum_{1<l} \omega_{1l} d\overline{z}^1 \wedge d\overline{z}^l + \sum_{1<k<l} \omega_{kl} d\overline{z}^k \wedge d\overline{z}^l. \tag{14.23}$$

Define $\eta = \sum_{1<k} \eta_{1k} d\overline{z}^k$, where

$$\eta_{1k}(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|w^1| \leq r_1} \frac{\chi_1(w^1)\omega_{1k}(w^1, z^2, \dots, z^n)}{w^1 - z^1} dw^1 \wedge d\overline{w}^1. \tag{14.24}$$

Then η_{1k} solves $\partial \eta_{1k} / \partial \overline{z}^1 = \chi_1 \omega_{1k}$. So then

$$\overline{\partial}\eta = \sum_{1<k} \frac{\partial \eta_{1k}}{\partial \overline{z}^l} d\overline{z}^l \wedge d\overline{z}^k = \sum_{1<k} \chi_1 \omega_{1k} d\overline{z}^1 \wedge d\overline{z}^k + R \tag{14.25}$$

where R doesn't include any $d\overline{z}^1$-s. So we have solved the terms in ω involving $d\overline{z}^1$-s. We next assume that ω is of the form

$$\omega = \sum_{1<k<l} \omega_{kl} d\overline{z}^k \wedge d\overline{z}^l = \sum_{2<l} \omega_{2l} d\overline{z}^2 \wedge d\overline{z}^l + \sum_{2<k<l} \omega_{kl} d\overline{z}^k \wedge d\overline{z}^l. \tag{14.26}$$

Let $\eta = \sum_{2<k} \eta_{2k} d\overline{z}^k$ where

$$\eta_{2k} = \frac{1}{2\pi i} \int_{|w^2| \leq r_2} \frac{\chi_2(w^2)\omega_{2k}(z^1, w^2, z^3, \dots z^n)}{w^2 - z^2} dw^2 \wedge d\overline{w}^2. \tag{14.27}$$

Then $\partial \eta_{2k} / \partial \overline{z}^2 = \chi_2 \omega_{2k}$. Furthermore, since $\overline{\partial}\omega = 0$, $\partial \eta_{2k} / \partial \overline{z}^1 = 0$. So then

$$\overline{\partial}\eta = \sum_{2<k} \overline{\partial}(\eta_{2k} d\overline{z}^k) = \sum_{2<k} \sum_{2 \leq l} \frac{\partial \eta_{2k}}{\partial \overline{z}^l} d\overline{z}^l \wedge d\overline{z}^k = \sum_{2<k} \chi_2 \omega_{2k} d\overline{z}^2 \wedge d\overline{z}^k + R, \tag{14.28}$$

where R only has terms $d\overline{z}^k \wedge d\overline{z}^l$ for $k, l \geq 3$. So we have solved as the term in ω having $d\overline{z}^1$-s or $d\overline{z}^2$-s. By a similar induction argument as in the $q = 1$ case, we can solve all terms in this manner. The case of $q > 2$ is similar, and details left as an exercise. □

We next upgrade this to have no shrinkage.

Theorem 14.5. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), then $H^{p,q}_{\overline{\partial}}(U) = \{0\}$ for $q \geq 1$.*

Proof. Choose a monotone increasing sequence of polyradii $r_1 < r_2 < \dots$ with $\lim_{j \to \infty} r_j = r$. Given $\omega \in \mathcal{E}^{0,q}(\Delta(r)$, by Step 2, we can find $\eta_j \in \mathcal{E}^{0,q-1}(\Delta(r))$ with $\overline{\partial}\eta_j = \omega$ on $\Delta(r_j)$. We do not know that the sequence η_j will converge. However, $\overline{\partial}(\eta_{j+1} - \eta_j) = 0$ in $\Delta(r_j)$. If $q \geq 2$, then by Step 2, we can find $\beta_{j+1} \in \mathcal{E}^{0,q-2}(\Delta(r_j))$ solving $\overline{\partial}(\beta_{j+1}) = \eta_{j+1} - \eta_j$ in $\Delta(r_{j-1})$. We then consider the sequence $\eta'_{j+1} = \eta_{j+1} - \overline{\partial}(\beta_{j+1})$. Then $\eta'_{j+1} \in \mathcal{E}^{0,q-2}(\Delta(r_j)$ and

$$\overline{\partial}(\eta'_{j+1}) = \overline{\partial}\eta_{j+1} - \overline{\partial}^2(\beta_{j+1}) = \omega \tag{14.29}$$

in $\Delta(r_{j-1})$, and this new sequence now obviously converges to a solution $\eta \in \mathcal{E}^{0,q}(\Delta(r)$ with $\overline{\partial}\eta = \omega$ in $\Delta(r)$.

If $q = 1$, then we prove exactly like we did in the case of $n = 1$, by approximating the difference $\eta_{j+1} - \eta_j$ by a polynomial P_{j+1} to obtain a sequence so that

$$\sup_{z \in K} |\eta_{j+1}(z) - \eta_j(z)| < 2^{-j}, \tag{14.30}$$

and we obtain a sequence converging on compact subsets to a solution. □

Remark 14.6. Using Laurent series instead of polynomials, a similar proof works to prove that Theorem 14.5 also holds for products $\Delta^*(r_1) \times \cdots \times \Delta^*(r_k) \times \Delta(r_{k+1}) \times \cdots \Delta(r_{k+l})$, that is, we can allow punctured 1-dimensional disks. With a lot more work, one can also show that Theorem 14.5 holds for $\Omega_1 \times \cdots \times \Omega_n$ with $\Omega_j \subset \mathbb{C}$ are domains. Note the result is NOT true for a punctured polydisc $\Delta(0, r) \setminus \{0\}$ for $n \geq 2$, but we cannot prove that yet.

Remark 14.7. Theorem 14.5 also holds for a ball $B(0, r) \subset \mathbb{C}^n$. However, this is difficult to prove directly. One could use the Bochner-Martinelli kernel instead of the Cauchy kernel to prove Proposition 14.4. Then one would also need to prove that the $B(0, r)$ is a Runge domain, that is, $\mathcal{O}(B(0, r))$ can be approximated by holomorphic polynomials uniformly on compact subsets. However, it seems actually easier to prove this more generally for any pseudoconvex domain (using Hörmander's L^2 methods), and then show that $B(0, r)$ is pseudoconvex.

15 Lecture 15

15.1 Almost complex manifolds

Definition 15.1. An *almost complex manifold* is a real manifold with an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$.

The following lemma shows that we can always take J to be standard at any *point*.

Lemma 15.2. *Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear mapping satisfying $J^2 = -Id$. Then there exists an invertible matrix A such that $A^{-1}JA = J_{Euc}$.*

Proof. For $X \in \mathbb{R}^{2n}$, define

$$(a + ib)X = aX + bJX. \tag{15.1}$$

Then $\mathbb{R}^{2n}$ becomes an n-dimensional complex vector space. Let $X_1, \ldots, X_n$ be a complex basis. Then $X_1, JX_1, \ldots, X_n, JX_n$ is a basis of $\mathbb{R}^{2n}$ as a real vector space, and J is obviously standard in this basis. □

Remark 15.3. The Newlander-Nirenberg Theorem deals with the following question: when can we make J standard in a *neighborhood* of a point? As we will see shortly, this cannot possibly be true for an arbitrary almost complex structure; there is an *integrability condition* which must be satisfied.

All of the linear algebra we discussed above in $\mathbb{C}^n$ can be done on an almost complex manifold (M, J). We can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \tag{15.2}$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \tag{15.3}$$

is the i-eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \tag{15.4}$$

is the $-i$-eigenspace of J.

The map J also induces an endomorphism of 1-forms by

$$J^T(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{15.5}$$

where

$$\Lambda^{1,0} = \{\alpha - iJ^T\alpha, \alpha \in T_p^*M\} \tag{15.6}$$

is the i-eigenspace of J^T, and

$$\Lambda^{0,1} = \{\alpha + iJ^T\alpha, \alpha \in T_p^*M\} \tag{15.7}$$

is the $-i$-eigenspace of J^T.

Next, we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} \tag{15.8}$$

decomposes as a direct sum.

Remark 15.4. This gives a necessary topological obstruction for existence of an almost complex structure: the bundle of complex k-forms must decompose into to a direct sum of subbundles as in (15.8).

We can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{15.9}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$. Note we can also extend J to k-forms by

$$J\alpha(X_1, \ldots, X_k) = \alpha(JX_1, \ldots, JX_k). \tag{15.10}$$

Exercise 15.5. Check that these two definitions of J on k-forms agree.

Definition 15.6. A triple (M, J, g) where J is an almost complex structure, and g is a Riemannian metric is *almost Hermitian* if

$$g(X, Y) = g(JX, JY) \tag{15.11}$$

for all $X, Y \in TM$. We also say that g is *compatible* with J.

Proposition 15.7. *Given a linear J with $J^2 = -Id$ on $\mathbb{R}^{2n}$, and a positive definite inner product g on $\mathbb{R}^{2n}$ which is compatible with J, there exist elements $\{X_1, \dots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{15.12}$$

is an ONB for $\mathbb{R}^{2n}$ with respect to g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{15.13}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of span$\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. □

Definition 15.8. To an almost Hermitian structure (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y) \tag{15.14}$$

called the *Kähler form* or *fundamental 2-form.*

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{15.15}$$

Furthermore, since

$$\omega(JX, JY) = \omega(X, Y), \tag{15.16}$$

this form is a real form of type $(1, 1)$. That is, $\omega \in \Gamma(\Lambda^{1,1}_{\mathbb{R}})$, where $\Lambda^{1,1}_{\mathbb{R}} \subset \Lambda^{1,1}$ is the real subspace of elements satisfying $\overline{\omega} = \omega$.

In Euclidean space $(\mathbb{R}^{2n}, J_0, g_{Euc})$, the fundamental 2-form is

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{15.17}$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{15.18}$$

Note that this defines an orientation on $\mathbb{C}^n$, which we will refer to as the natural orientation. Note also that

$$\omega^n_{Euc} = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{15.19}$$

Proposition 15.9. *If* (M, J) *is almost complex, then* $\dim(M)$ *is even and* M *is orientable.*

Proof. If M is of real dimension m, and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^m, \tag{15.20}$$

which implies that m is even. We will henceforth write $m = 2n$. Next, let g be any Riemannian metric on M. Then define

$$h(X, Y) = g(X, Y) + g(JX, JY). \tag{15.21}$$

Then $h(JX, JY) = h(X, Y)$ is J-invariant, so (M, J, h) is almost Hermitian. We then consider the fundamental 2-form

$$\omega(X, Y) = h(JX, Y). \tag{15.22}$$

This is a form of type $(1, 1)$, so $\omega^n \in \Lambda^{n,n}_{\mathbb{R}} \cong \Lambda^{2n}_{\mathbb{R}}$ is a top degree $2n$-form. It is nowhere-vanishing since at any point $x \in M$ by Proposition 50.1 we can assume that both $J_x = J_{Euc}$ and $g_x = g_{Euc}$, so $\omega^n(x) \neq 0$ by (15.19). Therefore, ω gives a globally defined orientation on M. □

Example 15.10. For example, $\mathbb{RP}^n$ does not admit any almost complex structure, since it is non-orientable for n even.

Definition 15.11. A smooth mapping between $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is *pseudo-holomorphic* if

$$f_* \circ J_M = J_N \circ f_* \tag{15.23}$$

We have a useful characterization of pseudo-holomorphic mappings.

Proposition 15.12. *A mapping* $f : M \to N$ *between almost complex manifolds* (M, J_M) *and* (N, J_N) *is pseudo-holomorphic if and only if*

$$f_*(T^{1,0}(M)) \subset T^{1,0}(N), \tag{15.24}$$

if and only if

$$f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M). \tag{15.25}$$

16 Lecture 16

16.1 Complex manifolds

We next define a complex manifold.

Definition 16.1. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

Example 16.2. Since holomorphic mappings are orientation-preserving by (13.22), any complex manifold is necessarily orientable. For example, $\mathbb{RP}^n$ does not admit any complex structure. Note that we knew from Example 15.10 above that there is no almost complex structure.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 16.3. *In any coordinate chart, define* $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ *by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{16.1}$$

Then $J_\alpha = J_\beta$ *on* $U_\alpha \cap U_\beta$ *and therefore gives a globally defined almost complex structure* $J : TM \to TM$ *satisfying* $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{16.2}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{16.3}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{16.4}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

The next proposition follows from the above discussion on Cauchy-Riemann equations.

Proposition 16.4. *If* (M, J_M) *and* (N, J_N) *are complex manifolds, then* $f : M \to N$ *is pseudo-holomorphic if and only if is a holomorphic mapping in local holomorphic coordinate systems.*

Definition 16.5. An almost complex structure J is said to be a *complex structure* if J is induced from a collection of holomorphic coordinates on M.

Proposition 16.6. *An almost complex structure* J *is a complex structure if and only if for any* $x \in M$*, there is a neighborhood* U *of* x *and a pseudo-holomorphic mapping* $\phi : (U, J) \to (\mathbb{C}^n, J_0)$ *which has non-vanishing Jacobian at* x*. Equivalently, there exist* n *pseudo-holomorphic functions* $f^j : U \to \mathbb{C}, j = 1 \dots n$*, with linearly independent differentials at* x*.*

Proof. By the inverse function theorem, ϕ gives a coordinate system in a possible smaller neighborhood of of x. The overlap mappings are pseudo-holomorphic mappings with respect to J_0, so they satisfy the Cauchy-Riemann equations, and are therefore holomorphic. The components of ϕ are functions $f^j, j = 1 \dots n$ with linearly independent differentials, and conversely, $\phi = (f^1, \dots, f^n)$ is a local coordinate system. □

Proposition 16.7. *A real 2-dimensional manifold admits an almost complex structure if and only if it is oriented.*

Proof. We have already proved the forward direction. Let M^2 be any oriented surface, and choose any Riemannian metric g on M. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric to identify $\Lambda^1 \cong TM$, we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure. □

Remark 16.8. In this case, any such J is necessarily a complex structure. This is equivalent to the problem of existence of isothermal coordinates, we will prove this soon.

16.2 The Nijenhuis tensor

When does an almost complex structure arise from a true complex structure? To answer this question, we define the following tensor associated to an almost complex structure.

Proposition 16.9. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} \tag{16.5}$$

is in $\Gamma(T^*M \otimes T^*M \otimes TM)$ *and satisfies*

$$N(Y,X) = -N(X,Y), \tag{16.6}$$
$$N(JX,JY) = -N(X,Y), \tag{16.7}$$
$$N(X,JY) = N(JX,Y) = -J(N(X,Y)). \tag{16.8}$$

Proof. Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned}
N(fX,Y) &= 2\{[J(fX),JY] - [fX,Y] - J[fX,JY] - J[J(fX),Y]\} \\
&= 2\{[fJX,JY] - [fX,Y] - J[fX,JY] - J[fJX,Y]\} \\
&= 2\{f[JX,JY] - (JY(f))JX - f[X,Y] + (Yf)X \\
&\quad - J(f[X,JY] - (JY(f))X) - J(f[JX,Y] - (Yf)JX)\} \\
&= fN(X,Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X\}.
\end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X,fY) = fN(X,Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y (16.6) is obvious, and (16.7) follows easily using $J^2 = -Id$. For (16.8)

$$N(X,JY) = -N(JX,J^2Y) = N(JX,Y), \tag{16.9}$$

and

$$
\begin{aligned}
N(X,JY) &= 2\{[JX,J^2Y]-[X,JY]-J[X,J^2Y]-J[JX,JY]\}\\
&= 2\{-[JX,Y]-[X,JY]+J[X,Y]-J[JX,JY]\}\\
&= 2J\{J[JX,Y]+J[X,JY]+[X,Y]-[JX,JY]\}\\
&= -2J\{N(X,Y)\}.
\end{aligned}
\tag{16.10}
$$

□

Proposition 16.10. *For a C^1 almost complex structure J,*

$$
N_J \in \Gamma\Big(\big\{\big(\Lambda^{2,0}\otimes T^{0,1}\big)\oplus\big(\Lambda^{0,2}\otimes T^{1,0}\big)\big\}_{\mathbb{R}}\Big). \tag{16.11}
$$

Consequently, if $\dim(M)=2n$, then the Nijenhuis tensor has $n^2(n-1)$ independent real components. In particular, if $n=1$, then $N_J\equiv 0$.

Proof. If we complexify, just using (16.6), we have

$$
\begin{aligned}
N_J &\in \Gamma((\Lambda^2\otimes TM)\otimes\mathbb{C}))\\
&= \Gamma\Big(\big(\Lambda^{2,0}\oplus\Lambda^{0,2}\oplus\Lambda^{1,1}\big)\otimes\big(T^{1,0}\oplus T^{0,1}\big)\Big).
\end{aligned}
\tag{16.12}
$$

But (16.7) says that the $\Lambda^{1,1}$ component vanishes. So we have

$$
N_J \in \Gamma\Big(\big(\Lambda^{2,0}\oplus\Lambda^{0,2}\big)\otimes\big(T^{1,0}\oplus T^{0,1}\big)\Big). \tag{16.13}
$$

Using (16.8), for $X',Y'\in\Gamma(TM)$, we have

$$
\begin{aligned}
&N_J(X'-iJX',Y'-iJY')\\
&\quad= N_J(X',Y')-N_J(JX',JY')-iN_J(JX',Y')-iN_J(X',JY')\\
&\quad= N_J(X',Y')+N_J(X',Y')+iJN_J(X',Y')+iJN_J(X',Y')\\
&\quad= 2N_J(X',Y')+2iJN_J(X',Y'),
\end{aligned}
\tag{16.14}
$$

which lies in $T^{0,1}$. This shows that the $\Lambda^{2,0}\otimes T^{1,0}$ component vanishes, so the $\Lambda^{0,2}\otimes T^{0,1}$ component also vanishes, and (16.11) follows since N_J is a real tensor. □

We have the following local formula for the Nijenhuis tensor.

Proposition 16.11. *In local coordinates, the Nijenhuis tensor is given by*

$$
N^i_{jk} = 2\sum_{h=1}^{2n}\big(J^h_j\partial_h J^i_k - J^h_k\partial_h J^i_j - J^i_h\partial_j J^h_k + J^i_h\partial_k J^h_j\big) \tag{16.15}
$$

Proof. We compute

$$
\begin{aligned}
\frac{1}{2}N(\partial_j,\partial_k) &= [J\partial_j,J\partial_k]-[\partial_j,\partial_k]-J[\partial_j,J\partial_k]-J[J\partial_j,\partial_k]\\
&= [J^l_j\partial_l,J^m_k\partial_m]-[\partial_j,\partial_k]-J[\partial_j,J^l_k\partial_l]-J[J^l_j\partial_l,\partial_k]\\
&= I+II+III+IV.
\end{aligned}
$$

The first term is

$$\begin{aligned} I &= J_j^l \partial_l (J_k^m \partial_m) - J_k^m \partial_m (J_j^l \partial_l) \\ &= J_j^l (\partial_l J_k^m) \partial_m + J_j^l J_k^m \partial_l \partial_m - J_k^m (\partial_m J_j^l) \partial_l - J_k^m J_j^l \partial_m \partial_l \\ &= J_j^l (\partial_l J_k^m) \partial_m - J_k^m (\partial_m J_j^l) \partial_l . \end{aligned}$$

The second term is obviously zero. The third term is

$$III = -J(\partial_j (J_k^l) \partial_l) = -\partial_j (J_k^l) J_l^m \partial_m . \tag{16.16}$$

Finally, the fourth term is

$$III = \partial_k (J_j^l) J_l^m \partial_m . \tag{16.17}$$

Combining these, we are done. □

Definition 16.12. If J is an almost complex structure of class C^1 satisfying $N_J \equiv 0$, then we say that J is *integrable*.

Corollary 16.13. *If (M, J) arises from a complex structure, then J is integrable.*

Proof. In local holomorphic coordinates $J = J_0$ is a constant tensor, and $N(J) = 0$ follows from Proposition 16.11. □

Next, we have an alternative characterization of the vanishing of the Nijenhuis tensor.

Proposition 16.14. *For an almost complex structure J the Nijenhuis tensor $N(J) = 0$ if and only if for any 2 vector fields $X, Y \in \Gamma(T^{1,0})$, their Lie bracket $[X, Y] \in \Gamma(T^{1,0})$.*

Proof. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \tag{16.18}$$

But this is also a $(1, 0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \tag{16.19}$$

applying J, and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \tag{16.20}$$

which is exactly the vanishing of the Nijenhuis tensor. □

17 Lecture 17

17.1 The operators ∂ and $\overline{\partial}$ for an integrable almost complex structure

Recall that on any almost complex manifold (M, J), we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{17.1}$$

We define $\mathcal{E}^k, \mathcal{E}^k_{\mathbb{C}}, \mathcal{E}^{p,q}$ to be the space of smooth sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. The real operator $d : \mathcal{E}^k_{\mathbb{R}} \to \mathcal{E}^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{17.2}$$

by complexification.

Proposition 17.1. *For a C^1 almost complex structure J*

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+2,q-1} \oplus \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1} \oplus \mathcal{E}^{p-1,q+2}, \tag{17.3}$$

and $N_J = 0$ if and only if

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}. \tag{17.4}$$

if and only if

$$d(\mathcal{E}^{1,0}) \subset \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1} \tag{17.5}$$

if and only if

$$d(\mathcal{E}^{0,1}) \subset \mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2} \tag{17.6}$$

Proof. Let $\alpha \in \mathcal{E}^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \ldots, X_r) &= \sum (-1)^j X_j \alpha(X_0, \ldots, \hat{X}_j, \ldots, X_r) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r). \end{aligned} \tag{17.7}$$

This is easily seen to vanish if more than $p + 2$ of the X_j are of type $(1, 0)$ or if more than $q + 2$ are of type $(0, 1)$, and (17.3) follows.

Next, assume that (17.6) is satisfied. Let $\alpha \in \mathcal{E}^{0,1}$, then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{17.8}$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X, Y]$. Proposition 16.14 implies that $N(J) \equiv 0$. Conversely, if $N(J) \equiv 0$, then we can reverse the steps in this argument to obtain (17.6). Equation (17.5) is just the conjugate of (17.6).

Recall that if $\alpha \in \mathcal{E}^k$ and $\beta \in \mathcal{E}^l$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{17.9}$$

The formula (17.4) then follows from this. $\square$

If $N_J = 0$, we can therefore define operators

$$\partial : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{17.10}$$

$$\overline{\partial} : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{17.11}$$

using (17.1) and

$$\partial|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{17.12}$$

$$\overline{\partial}|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{17.13}$$

Corollary 17.2. *For a C^1 almost complex structure J with $N_J = 0$, $d = \partial + \overline{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{17.14}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{17.15}$$

If we plug in a form of type (p, q) the first term is of type $(p + 2, q)$, the middle terms are of type $(p + 1, q + 1)$, and the last term is of type $(p, q + 2)$. Since (17.1) is a direct sum, the claim follows. □

Remark 17.3. If we assume that (M, J) arises from a complex structure, then we can just define these operators in local holomorphic coordinates like we did in $\mathbb{C}^n$ (and the prove that they define global operators). The point of the above is that we only assumed integrability, and did not use any local holomorphic coordinates.

17.2 Real form of the equations

Recall that for $n = 1$, any almost complex structure J satisfies $N_J = 0$, so there is no integrability condition. Let's look at various forms of the equations.

We just look in an open set in real coordinates (x, y), and then we have

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}. \tag{17.16}$$

The only condition is

$$-I = J^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} \tag{17.17}$$

If we assume that J is not too far from J_0, then $b \sim -1$ and $c \sim 1$, so we must have

$$a + d = 0, \quad a^2 + bc = -1. \tag{17.18}$$

Note that since $b \sim -1$, we can solve $c = -(1 + a^2)/b$, but we won't need to do this now. So we just consider

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix}. \tag{17.19}$$

We want to find a pseudo-holomorphic mapping

$$\phi : (U, J) \to (\mathbb{C}, J_0) \tag{17.20}$$

which has non-vanishing Jacobian at 0. So we want to solve

$$\phi_* \circ J = J_0 \circ \phi_* \tag{17.21}$$

If we write

$$\phi(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}, \tag{17.22}$$

then the pseudoholomorphic condition is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{17.23}$$

which yields the 4 equations

$$\begin{matrix} au_x + cu_y = -v_x & bu_x - au_y = -v_y \\ av_x + cv_y = u_x & bv_x - av_y = u_y \end{matrix}. \tag{17.24}$$

This looks like 4 first-order equations for 2 unknown functions, so one wouldn't expect a solution. However, the first two equations imply the second two:

$$av_x + cv_y = a(-au_x - cu_y) + c(-bu_x + au_y) = (-a^2 - bc)u_x = u_x, \tag{17.25}$$

and

$$bv_x - av_y = b(-au_x - cu_y) + a(bu_x - au_y) = (-bc - a^2)u_y = u_y, \tag{17.26}$$

using the condition that $a^2 + bc = -1$.

Example 17.4. Let's now do an example. Consider

$$J = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix}. \tag{17.27}$$

We have

$$J^2 = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17.28}$$

so this is indeed an almost complex structure.

From (17.24), the pseudoholomorphic equations are

$$2xu_x + (1 + 4x^2)u_y = -v_x \tag{17.29}$$

$$-u_x - 2xu_y = -v_y. \tag{17.30}$$

If a sufficiently smooth solution exists, then we have $v_{xy} = v_{yx}$, which yields

$$(2xu_x + (1 + 4x^2)u_y)_y = -(u_x + 2xu_y)_x \tag{17.31}$$

This can be rewritten as

$$u_{xx} + 4xu_{xy} + (1 + 4x^2)u_{yy} + 2u_y = 0. \tag{17.32}$$

Remark 17.5. This is a nice equation, because it looks like

$$\Delta_0 u + \text{lower order terms}. \tag{17.33}$$

We will return to this viewpoint later.

For now, we just notice that, by inspection, $u = x$ is obviously a solution. We then return to the pseudoholomorphic equations, and find that

$$v_x = -2x, \quad v_y = 1, \tag{17.34}$$

so we can choose $v = -x^2 + y$. So our solution is $\phi = (u, v) = (x, y - x^2)$. The Jacobian at the origin is clearly non-degenerate, so we have found a holomorphic coordinate system. Note that the mapping $\phi : \mathbb{R}^2 \to \mathbb{C}$ is defined everywhere. It is injective: if we have $(x_1, y_1 - x_1^2) = (x_2, y_2 - x_2^2)$ then the first component says that $x_1 = x_2$ and the second component then implies that $y_1 = y_2$. It is also surjective: given any $(u, v) \in \mathbb{C}$, we let $x_2 = u$, and then we need to solve $y - u^2 = v$, which obviously has a solution $y = -u^2 + v$. Thus we have found that

$$\phi : (\mathbb{R}^2, J) \to (\mathbb{C}, J_0) \tag{17.35}$$

is a global biholomorphism! Note that any function of the form $f(x, y) = h(x + i(y - x^2))$, where h is a holomorphic function with respect to J_0, is then holomorphic for J, for example

$$f(x, y) = e^x(\cos(y - x^2) + i\sin(y - x^2)). \tag{17.36}$$

18 Lecture 18

18.1 The Beltrami equation

In the basis $\{\partial/\partial x, \partial/\partial y\}$ we have J of the form

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix} \tag{18.1}$$

satisfying $a^2 + bc = -1$. Using (13.18) to change to the complex basis $\{\partial/\partial z, \partial/\partial \overline{z}\}$, then we have

$$J = \frac{1}{2}\begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix} \tag{18.2}$$

For a complex valued function w, the equation $\overline{\partial}_J w = 0$ is $\Pi_{\Lambda^{0,1}} dw = 0$, which is

$$\begin{aligned} 0 &= dw + iJdw = w_z dz + w_{\overline{z}} d\overline{z} + iJ(w_z dz + w_{\overline{z}} d\overline{z}) \\ &= w_z dz + w_{\overline{z}} d\overline{z} + i w_z J dz + i w_{\overline{z}} J d\overline{z}. \end{aligned} \tag{18.3}$$

Note that we need to use $J : \Lambda^1 \to \Lambda^1$ here, which is the transpose matrix of the above J. So we have

$$
\begin{aligned}
0 &= w_z dz + w_{\overline{z}} d\overline{z} + \frac{i}{2} w_z (i(c-b)dz + (2a + i(b+c))d\overline{z}) + \frac{i}{2} w_{\overline{z}}((2a - i(b+c))dz - i(c-b)d\overline{z}) \\
&= \Big(w_z + \frac{1}{2}(b-c)w_z + \frac{1}{2}(2ai + b + c)w_{\overline{z}} \Big) dz + \Big(w_{\overline{z}} + \frac{1}{2}(2ai - b - c)w_z + \frac{1}{2}(c-b)w_{\overline{z}} \Big) d\overline{z}.
\end{aligned}
\tag{18.4}
$$

Let's look only at the second equation which is

$$
\Big(1 + \frac{1}{2}(c-b) \Big) w_{\overline{z}} = -\frac{1}{2}(2ai - b - c)w_z. \tag{18.5}
$$

If $b - c \neq 2$, which is certainly the case if J is close to J_0, then the leading coefficient is non-zero, and we can divide to get

$$
w_{\overline{z}} = -\frac{2ai - b - c}{2 + c - b} w_z \tag{18.6}
$$

Note that the first equation is

$$
\Big(1 + \frac{1}{2}(b-c) \Big) w_z = -\frac{1}{2}(2ai + b + c)w_{\overline{z}}. \tag{18.7}
$$

If $2ai - b - c \neq 0$, then we can divide to get

$$
w_{\overline{z}} = -\frac{2 + b - c}{2ai + b + c} w_z. \tag{18.8}
$$

I claim these are the same equation. For this, we would need

$$
\frac{2ai - b - c}{2 + c - b} = \frac{2 + b - c}{2ai + b + c}, \tag{18.9}
$$

which yields

$$
(2ai - b - c)(2ai + b + c) = (2 + c - b)(2 + b - c), \tag{18.10}
$$

which is

$$
-4a^2 - (b+c)^2 = 4 - (c-b)^2. \tag{18.11}
$$

Expanding this out

$$
-4a^2 - b^2 - 2bc - c^2 = 4 - c^2 + 2bc - b^2, \tag{18.12}
$$

which is true since $a^2 + bc = -1$!

Definition 18.1. The equation

$$
w_{\overline{z}} + \mu(z, \overline{z}) w_z = 0 \tag{18.13}
$$

is called the *Beltrami equation*.

18.2 Method of characteristics

This is a general method for solving linear PDE by solving nonlinear ODEs, we just explain for the Beltrami equation. Let's solve the nonlinear ODE

$$\frac{\partial z}{\partial s} = \mu(z, s), \ z(0) = w. \tag{18.14}$$

A solution exists locally provided that $\mu(z, s)$ is a holomorphic function of 2 complex variables, or equivalently, $\mu(x, y)$ is real analytic. The solution will depend on the independent variable s and the initial conditions w, call the solution $\Phi(s, w)$, and we write

$$z = \Phi(s, w). \tag{18.15}$$

By the implicit function theorem, we can write $w = w(z, s)$ in a neighborhood of $(s, w) = (0, 0)$, provided that $\frac{\partial \Phi}{\partial w}|_{0,0} \neq 0$. But this is

$$\frac{\partial \Phi}{\partial w}\Big|_{0,0} = \lim_{h \to 0} \frac{\Phi(0, h) - \Phi(0, 0)}{h} = 1. \tag{18.16}$$

So we have

$$z = \Phi(s, w(z, s)). \tag{18.17}$$

Taking the partial derivative of (18.17) with respect to z yields

$$1 = \frac{\partial \Phi}{\partial w} \frac{\partial w}{\partial z}. \tag{18.18}$$

Taking the partial derivative of (18.17) with respect to s yields

$$0 = \frac{\partial \Phi}{\partial s} + \frac{\partial \Phi}{\partial w} \frac{\partial w}{\partial s}, \tag{18.19}$$

which is

$$0 = \mu(z, s) + \Big(\frac{\partial w}{\partial z}\Big)^{-1} \frac{\partial w}{\partial s}, \tag{18.20}$$

which is the Beltrami equation upon letting $s = \overline{z}$.

Let's return to Example (17.4), and solve using this method. Recall

$$J = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{18.21}$$

So we need to solve the Beltrami equation with

$$\mu = \frac{2ai - b - c}{2 + c - b} = \frac{4xi + 1 - 1 - 4x^2}{2 + 1 + 4x^2 + 1} = \frac{xi - x^2}{1 + x^2} = -\frac{x}{i + x}. \tag{18.22}$$

Since $x = (z + \overline{z})/2$, we have

$$\mu(z, \overline{z}) = -\frac{z + \overline{z}}{2i + z + \overline{z}}. \tag{18.23}$$

Let's solve the ODE

$$\frac{dz}{ds} = \mu(z, s), \ \ z(0) = w. \tag{18.24}$$

For our example, this is

$$\frac{dz}{ds} = -\frac{z + s}{2i + z + s}. \tag{18.25}$$

To solve this, let's make a change of variables $p = z + s$. Then

$$\frac{dp}{ds} - 1 = -\frac{p}{2i + p}, \tag{18.26}$$

which gives

$$\frac{dp}{ds} = 1 - \frac{p}{2i + p} = \frac{2i}{2i + p}, \tag{18.27}$$

or

$$(2i + p)dp = 2ids, \tag{18.28}$$

which integrates to

$$2ip + \frac{1}{2}p^2 = 2is + C, \tag{18.29}$$

which is

$$2i(z + s) + \frac{1}{2}(z + s)^2 = 2is + C. \tag{18.30}$$

Our initial conditions are $z(0) = w$, so we get

$$2i(z + s) + \frac{1}{2}(z + s)^2 = 2is + 2iw + \frac{1}{2}w^2. \tag{18.31}$$

This is

$$w^2 + 4iw - 4iz - (z + s)^2 = 0 \tag{18.32}$$

Using the quadratic formula and letting $s = \overline{z}$ yields

$$w = -2i + \sqrt{-4 + 4iz + (z + \overline{z})^2}, \tag{18.33}$$

and we take the branch of the square root satisfying $\sqrt{-4} = 2i$. Note that this does not agree with the above method, but this is because the initial conditions are different. The above solution satisfies $w(z, 0) = z$, but the solution found in the previous section was

$$w = z - ix^2 = z - \frac{i}{4}(z + \overline{z})^2, \tag{18.34}$$

which satisfies $w(z, 0) = z - \frac{i}{4}z^2$.

19 Lecture 19

19.1 Equivalence of J and μ

The following proposition gives another way to think about almost complex structures for $n = 1$.

Proposition 19.1. *If J is defined in an open set U which induces the standard orientation on U, then there exists a unique complex valued function $\mu : U \to B(0,1) \subset \mathbb{C}$ so that*

$$T_J^{0,1} = \{v + \mu\overline{v} \mid v \in T_{J_0}^{0,1}\} \subset T_{\mathbb{C}}U. \tag{19.1}$$

Explicitly, if

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{19.2}$$

with $a^2 + bc = -1$, then

$$\mu = \frac{2ai - b - c}{2 + c - b}. \tag{19.3}$$

Conversely, given a function $\mu : U \to B(0,1) \subset \mathbb{C}$, writing $\mu = f + ig$, there is a uniquely determined almost complex structure J given by

$$J = \frac{1}{1 - f^2 - g^2} \begin{pmatrix} 2g & -(1+f)^2 - g^2 \\ g^2 + (1-f)^2 & -2g \end{pmatrix} \tag{19.4}$$

which has $T_J^{0,1}$ given by the above.

Proof. Given any such J, then we have previously defined

$$T_J^{0,1} = \{X \in T_{\mathbb{C}}U \mid JX = -iX\} = \{X' + iJX' \mid X \in T_{\mathbb{R}}U\}. \tag{19.5}$$

We next claim that the projection $\pi : T_J^{0,1} \to T_{J_0}^{0,1}$ is a complex linear isomorphism. These are two 1-dimensional complex subspaces of the 2-dimensional space $TU \otimes \mathbb{C}$, so there is a complex linear projection mapping, which is given by

$$X' + iJX' \mapsto X' + iJX' + iJ_0(X' + iJX') = (X' - J_0JX') + i(J + J_0)X'. \tag{19.6}$$

Since both spaces are 1-dimensional, and π is complex linear, it is an isomorphism provided it is not the zero map. Obviously, from (19.6), if $J \neq -J_0$ then it is not the zero mapping. We may therefore write $T_J^{0,1}$ as a graph over $T_{J_0}^{0,1}$. To do this, we compute like last time: using (13.18) to change to the complex basis $\{\partial/\partial z, \partial/\partial\overline{z}\}$, then we have

$$J = \frac{1}{2} \begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix}. \tag{19.7}$$

Then a basis for the 1-dimensional space $T_J^{0,1}$ is given by

$$\frac{\partial}{\partial\bar{z}} + iJ\Big(\frac{\partial}{\partial\bar{z}}\Big) = \frac{\partial}{\partial\bar{z}} + \frac{i}{2}\Big(i(b-c)\frac{\partial}{\partial\bar{z}} + (2a + i(b+c))\frac{\partial}{\partial z}\Big) \tag{19.8}$$

$$= \Big(1 + \frac{c-b}{2}\Big)\frac{\partial}{\partial\bar{z}} + \frac{1}{2}(2ai - b - c)\frac{\partial}{\partial z}. \tag{19.9}$$

From this, we find that

$$\mu = \frac{2ai - b - c}{2 + c - b}, \tag{19.10}$$

as claimed. Using $a^2 + bc = -1$, we compute

$$|\mu|^2 = \frac{4(-1-bc) + (b+c)^2}{(2+c-b)^2} = \frac{2+b-c}{-2+b-c}. \tag{19.11}$$

To show that $|\mu| < 1$, we use the orientation condition. Notice that the condition $bc = -1 - a^2$ says that $bc < 0$, so there are 2 components to the set of almost complex structures, determined by the sign of b: if $b < 0$, then this is the component inducing the standard orientation. In this case, we have

$$\frac{2+b-c}{-2+b-c} < 1 \tag{19.12}$$

is equivalent to

$$2 + b - c > -2 + b - c, \tag{19.13}$$

which is obviously true.

Next, given any such function μ, we define

$$T_\mu^{0,1} = \text{span}\Big\{\frac{\partial}{\partial\bar{z}} + \mu\frac{\partial}{\partial z}\Big\}. \tag{19.14}$$

Define

$$T_\mu^{1,0} = \text{span}\Big\{\frac{\partial}{\partial z} + \overline{\mu}\frac{\partial}{\partial\bar{z}}\Big\}. \tag{19.15}$$

We claim that $T_\mu^{1,0} \cap T_\mu^{0,1} = \{0\}$. To see this, if the intersection was non-zero, then there would exist $\alpha \in \mathbb{C}$ so that

$$\frac{\partial}{\partial\bar{z}} + \mu\frac{\partial}{\partial z} = \alpha\Big(\frac{\partial}{\partial z} + \overline{\mu}\frac{\partial}{\partial\bar{z}}\Big). \tag{19.16}$$

This clearly implies that $\alpha = \mu$ and then $|\mu|^2 = 1$. But we have assumed that $|\mu| < 1$, so the claim follows. To find the corresponding almost complex structure J, we must have

$$\frac{\partial}{\partial\bar{z}} + \mu\frac{\partial}{\partial z} = X' + iJX', \tag{19.17}$$

for some real tangent vector X'. We then write the real and imaginary parts of the left hand side:

$$\begin{aligned}\frac{\partial}{\partial\bar{z}}+\mu\frac{\partial}{\partial z}&=\frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big)+(f+ig)\frac{1}{2}\Big(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\Big)\\&=\frac{1}{2}\Big((1+f)\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}\Big)+\frac{i}{2}\Big(g\frac{\partial}{\partial x}+(1-f)\frac{\partial}{\partial y}\Big).\end{aligned}\tag{19.18}$$

So we must have

$$J\Big((1+f)\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}\Big)=g\frac{\partial}{\partial x}+(1-f)\frac{\partial}{\partial y},\tag{19.19}$$

and since $J^2=-Id$,

$$J\Big(g\frac{\partial}{\partial x}+(1-f)\frac{\partial}{\partial y}\Big)=-\Big((1+f)\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}\Big).\tag{19.20}$$

A simple change of basis computation shows that

$$J=\frac{1}{1-f^2-g^2}\begin{pmatrix}2g & -(1+f)^2-g^2\\ g^2+(1-f)^2 & -2g\end{pmatrix}.\tag{19.21}$$

□

This gives another way to understand the Beltrami equation. Given $\mu: U\to\mathbb{C}$ with $|\mu|<1$, then since

$$T^{0,1}_{\mu}=\mathrm{span}\Big\{\frac{\partial}{\partial\bar{z}}+\mu\frac{\partial}{\partial z}\Big\},\tag{19.22}$$

a function $w: U\to\mathbb{C}$ is holomorphic if and only if

$$\Big(\frac{\partial}{\partial\bar{z}}+\mu\frac{\partial}{\partial z}\Big)w=0,\tag{19.23}$$

or

$$w_{\bar{z}}+\mu w_z=0,\tag{19.24}$$

which is exactly the Beltrami equation. We can just completely forget about the matrix version of J, and parametrize almost complex structures by a single function $\mu: U\to B(0,1)$.

Remark 19.2. This proposition also shows us that the regularity of $J: U\to\mathrm{GL}(2,\mathbb{R})$ is the same as the regularity of $\mu: U\to B(0,1)$. That is, J is $C^{k,\alpha}, C^{\infty}, C^{\omega}$ if and only if μ is also.

Remark 19.3. The complex structures inducing the reversed orientation correspond to $|\mu|>1$ together with the point at infinity, which corresponds to the complex structure $-J_0$.

19.2 Another example

This example will be crucial in proving convergence in the analytic case, and is called a *Cauchy majorant.*

Proposition 19.4. *For $\rho > 0$, let*

$$\mu^* = C\Big(\frac{1}{1-(z+\overline{z})\rho^{-1}} - 1\Big) = C\frac{z+\overline{z}}{\rho - z - \overline{z}}. \tag{19.25}$$

which is analytic in the polydisc $P(\rho/2) = \{(z,\overline{z}) \mid |z| < \rho/2, |\overline{z}| < \rho/2\}$. Then there is a solution w^* of the Beltrami equation $w^*_{\overline{z}} + \mu^*(z,\overline{z})w^*_z = 0$ satisfying $w^*(z,0) = z$, and which is analytic in the polydisc $P(\rho')$ for some $\rho' > 0$.

Proof. We first observe that

$$\frac{1}{1-(z+\overline{z})\rho^{-1}} - 1 = \sum_{k=1}^{\infty}\Big(\frac{z+\overline{z}}{\rho}\Big)^k. \tag{19.26}$$

which converges uniformly on compact subsets in $P(\rho/2)$ since

$$|z+\overline{z}| \leq |z| + |\overline{z}| < \rho/2 + \rho/2 = \rho. \tag{19.27}$$

First, if $C = 0$, then the solution is $w(z,\overline{z}) = z$. So we assume that $C \neq 0$. To find the solution w^*, we use the method of characteristics from the previous lecture. First, solve the ODE

$$\frac{dz}{ds} = \mu^*(z,s) \tag{19.28}$$

with initial condition $z(0) = w$. So we need to solve the ODE

$$\frac{dz}{ds} = C\frac{z+s}{\rho - z - s},\ z(0) = w. \tag{19.29}$$

Letting $p = z + s$, and the equation becomes

$$\frac{dp}{ds} = 1 + \frac{Cp}{\rho - p} = \frac{\rho + (C-1)p}{\rho - p},\ p(0) = w. \tag{19.30}$$

This is separable, so we rewrite as

$$\frac{\rho - p}{\rho + (C-1)p} \cdot dp = ds \tag{19.31}$$

If $C = 1$, then we have

$$(\rho - p)dp = \rho ds \tag{19.32}$$

Integrating yields

$$-\frac{1}{2}p^2 + \rho p = \rho s + C_1. \tag{19.33}$$

Plugging in the initial conditions $p(0) = z(0) = w$ gives

$$-\frac{1}{2}p^2 + \rho p = \rho s - \frac{1}{2}w^2 + \rho w. \tag{19.34}$$

Using $p = z + s$ and substituting $s = \overline{z}$, we can rewrite this as

$$w^2 - 2\rho w - (z + \overline{z})^2 + 2\rho(z + \overline{z}) - 2\rho\overline{z} = 0. \tag{19.35}$$

Using the quadratic formula, we obtain the analytic solution

$$w = \rho \pm \sqrt{\rho^2 + (z + \overline{z})^2 - 2\rho z}, \tag{19.36}$$

which is clearly analytic in a sufficiently small polydisc. To finish, we choose the branch of the square root so that

$$w(z, 0) = \rho \pm \sqrt{\rho^2 + z^2 - 2\rho z} = \rho \pm \sqrt{(z - \rho)^2} = \rho + z - \rho = z. \tag{19.37}$$

Next, if $C \neq 1$, then we can write

$$\frac{\rho - p}{\rho + (C - 1)p} = \frac{1}{C - 1}\Big(- 1 + \frac{C\rho}{\rho + (C - 1)p}\Big). \tag{19.38}$$

So the equation is

$$\Big(- 1 + \frac{C\rho}{\rho + (C - 1)p}\Big)dp = (C - 1)ds. \tag{19.39}$$

Integrating yields

$$-p + \frac{C\rho}{C - 1}\log(\rho + (C - 1)p) = (C - 1)s + C_1. \tag{19.40}$$

Plugging in the initial conditions gives

$$-p + \frac{C\rho}{C - 1}\log(\rho + (C - 1)p) = (C - 1)s - w + \frac{C\rho}{C - 1}\log(\rho + (C - 1)w). \tag{19.41}$$

In terms of $z = p - s$, this is

$$-z + \frac{C\rho}{C - 1}\log(\rho + (C - 1)(z + s)) = Cs - w + \frac{C\rho}{C - 1}\log(\rho + (C - 1)w). \tag{19.42}$$

Rewrite this as

$$w - \frac{C\rho}{C - 1}\log(\rho + (C - 1)w) = z + Cs - \frac{C\rho}{C - 1}\log(\rho + (C - 1)(z + s)). \tag{19.43}$$

Note that near $(z, s) = (0, 0)$, the right hand side is an analytic function of the variables (z, s). If we let

$$f(w) = w - \frac{C\rho}{C - 1}\log(\rho + (C - 1)w), \tag{19.44}$$

then f is analytic near $w = 0$. Also,

$$f'(0) = 1 - C \neq 0, \tag{19.45}$$

since $C \neq 1$ by assumption. By Proposition 13.2 (the holomorphic inverse function theorem), f^{-1} exists and is analytic in some neighborhood of $f(0) = 0$. So then we have

$$w = f^{-1}\Big(z + Cs - \frac{C\rho}{C-1}\log(\rho + (C-1)(z+s))\Big) \tag{19.46}$$

is analytic. Setting $s = \bar{z}$, we have

$$w = f^{-1}\Big(z + C\overline{z} - \frac{C\rho}{C-1}\log(\rho + (C-1)(z+\overline{z}))\Big) \tag{19.47}$$

which is analytic. Note also that

$$w(z,0) = f^{-1}\Big(z - \frac{C\rho}{C-1}\log(\rho + (C-1)z)\Big) = f^{-1}(f(z)) = z, \tag{19.48}$$

so the correct initial conditions are indeed satisfied. □

20 Lecture 20

20.1 The Beltrami equation: analytic case

To make the signs work out nicely, we consider instead the equation

$$w_{\overline{z}} = \mu(z,\overline{z})w_z, \;\; w(z,0) = z. \tag{20.1}$$

Note this is equivalent to our original Beltrami equation by letting $\overline{z} \mapsto -\overline{z}$, with $\mu(z,\overline{z}) \mapsto -\mu(z,-\overline{z})$. This transformation does not change the initial condition of $w(z,0) = z$. Assuming μ is analytic, we have a convergent power series expansion

$$\mu(z,\overline{z}) = \sum_{j,k} \mu_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} \sum_{j+k=l} \mu_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} \mu_l. \tag{20.2}$$

Using Lemma 15.2, we can make the ACS standard at the origin, which implies that $\mu_0 = 0$, that is, μ has no constant term. We also write

$$w = \sum_{j,k} w_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} \sum_{j+k=l} w_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} w_l. \tag{20.3}$$

We want to find a holomorphic coordinate system, so we make the assumption that $w_0 = 0$ and $w_1 = z$.

We then have

$$w_{\overline{z}} = \sum_{l=2}^{\infty} \partial_{\overline{z}} w_l \tag{20.4}$$

$$w_z = 1 + \sum_{l=2}^{\infty} \partial_z w_l. \tag{20.5}$$

We then want to solve

$$w_{\overline{z}} = \sum_{l=2}^{\infty} \partial_{\overline{z}} w_l = \mu w_z = \Big(\sum_{l=1}^{\infty} \mu_l\Big)\Big(1 + \sum_{k=2}^{\infty} \partial_z w_k\Big) = \Big(\sum_{l=1}^{\infty} \mu_l\Big) + \sum_{l=2}^{\infty} \sum_{j+k=l, j\geq 1, k\geq 2} \mu_j \partial_z w_k. \tag{20.6}$$

We then find the recursion relation

$$\partial_{\overline{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j\geq 1, k\geq 2} \mu_j \partial_z w_k. \tag{20.7}$$

Note that in the sum on the right hand side, we must have $k \leq l$, so this in indeed a recursion relation, provided that we can solve for w_{l+1}.

Fixing l, the right hand side is just a homogeneous polynomial of degree l in the variables z and $\overline{z}$. In general, if $f_l = \sum_{j+k=l, j\geq 0, k\geq 0} h_{j\bar{k}} z^j \overline{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j\geq 0, k\geq 0} \frac{1}{k+1} h_{j\bar{k}} z^j \overline{z}^{k+1} \tag{20.8}$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\overline{z}} F = f$.

Remark 20.1. Notice that our "inverse" of the $\overline{\partial}$-operator on homogeneous polynomials of degree l does not contain any terms proportional to z^{l+1}. Our inverse operator is unique with this condition. If we had not imposed this condition, one could have chosen $w_l = l! z^l + O(\overline{z})$, in which case our series would definitely not converge! Also, if we view our series as a power series in 2 complex variables, then formally $w(z, 0) = z$ exactly because of this choice of inverse to $\overline{\partial}$.

Proposition 20.2. *The coefficients $w_{j\bar{k}}$ for $j + k = l$ are a polynomial of degree $l - 1$ in the $\mu_{p\bar{q}}$ for $p + q < l$ with all coefficients non-negative rational numbers.*

Proof. Let us examine the first few steps of the iteration. We have $w_{00} = 1$, $w_{10} = 1$, and $w_{0\bar{1}} = 0$. The term w_2 is determined by

$$\partial_{\overline{z}} w_2 = \mu_1 = \mu_{1\bar{0}} z + \mu_{0\bar{1}} \overline{z}, \tag{20.9}$$

so

$$w_2 = \mu_{1\bar{0}} z\overline{z} + \frac{1}{2} \mu_{0\bar{1}} \overline{z}^2, \tag{20.10}$$

so

$$w_{2\bar{0}} = 0, \quad w_{1\bar{1}} = \mu_{1\bar{0}}, \quad w_{0\bar{2}} = \frac{1}{2} \mu_{0\bar{1}}. \tag{20.11}$$

To illustrate, let's do one more step. The term w_3 is determined by

$$\begin{aligned} \partial_{\overline{z}} w_3 = \mu_2 + \mu_1 \partial_z w_2 &= \mu_{2\bar{0}} z^2 + \mu_{1\bar{1}} z\overline{z} + \mu_{0\bar{2}} \overline{z}^2 + (\mu_{1\bar{0}} z + \mu_{0\bar{1}} \overline{z})(\mu_{1\bar{0}} \overline{z}) \\ &= \mu_{2\bar{0}} z^2 + (\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z\overline{z} + (\mu_{0\bar{2}} + \mu_{0\bar{1}} \mu_{1\bar{0}}) \overline{z}^2. \end{aligned} \tag{20.12}$$

so

$$w_3 = \mu_{2\bar{0}} z^2 \overline{z} + \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z \overline{z}^2 + \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}})\overline{z}^3. \tag{20.13}$$

so

$$w_{3\bar{0}} = 0, \quad w_{2\bar{0}} = \mu_{2\bar{0}}, \quad w_{1\bar{2}} = \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2), \quad w_{0\bar{3}} = \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}}), \tag{20.14}$$

and the claim is evidently true.

To do the general case, we prove by induction: assume the claim is true up to for $0, \ldots, l$, and we prove for $l+1$. Recall that

$$\partial_{\overline{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j\geq 1, k \geq 2} \mu_j \partial_z w_k. \tag{20.15}$$

By induction, the coefficients of w_k for $k \leq l$ are polynomials with non-negative coefficients in the $\mu_{p\bar{q}}$ with $p + \bar{q} < k < l$, so that $\partial_z w_k$ is also of this form. Then since

$$\mu_j = \sum_{k+l=j} \mu_{k\bar{l}} z^k \overline{z}^l, \tag{20.16}$$

any term $\mu_j \partial_z w_k$ is also a polynomial in the $\mu_{k\bar{l}}$ with non-negative coefficients.

To get w_{l+1}, recall that if $f_l = \sum_{j+k=l, j\geq 0, k\geq 0} h_{jk} z^j \overline{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j\geq 0, k \geq 0} \frac{1}{k+1} h_{jk} z^j \overline{z}^{k+1} \tag{20.17}$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\overline{z}} F = f$. Clearly, this preserves non-negativity of the coefficients, and we are done. □

Theorem 20.3. *If $\mu(z, \overline{z})$ is analytic in the closed polydisc $|z| \leq \rho, |\overline{z}| \leq \rho$, there there exists a unique solution of the Beltrami equation*

$$w_{\overline{z}} = \mu(z, \overline{z}) w_z \tag{20.18}$$

which is analytic in the polydisc $|z| < \rho', |\overline{z}| < \rho'$ for some $\rho' > 0$, and satisfies the Cauchy data

$$w(z, 0) = z. \tag{20.19}$$

Proof. By assumption, the series

$$\mu = \sum_{j,k} \mu_{j\bar{k}} z^j \overline{z}^k \tag{20.20}$$

converges for any point in the polydisc

$$P(\rho) = \{(z, \overline{z}) \mid |z| < \rho, |\overline{z}| < \rho\}, \tag{20.21}$$

with uniform convergence in the polydisc $\overline{P(\rho')}$, for any $\rho' < \rho$. So for any $(z, \overline{z}) \in \overline{P(\rho')}$, there exists a constant $C > 0$ so that

$$|\mu_{j\bar{k}} z^j \overline{z}^k| < C \text{ (no summation).} \tag{20.22}$$

Choosing $(z, \overline{z}) = (\rho', \rho')$, this implies that

$$|\mu_{j\bar{k}}| < C(\rho')^{-j-k}. \tag{20.23}$$

To simplify notation, let's call ρ' by ρ. Then we define

$$\mu^* = C\Big(\frac{1}{1-(z+\overline{z})\rho^{-1}} - 1\Big) = C\frac{z+\overline{z}}{\rho - z - \overline{z}}, \tag{20.24}$$

which is analytic in the polydisc $P(\rho) = \{(z, \overline{z}) \mid |z| < \rho, |\overline{z}| < \rho\}$. We have

$$\mu^* = C\sum_{j\geq 1}(z+\overline{z})^j \rho^{-j} = C \sum_{(k,l)\neq(0,0)} \rho^{-k-l}\frac{(k+l)!}{k!l!} z^k \overline{z}^l. \tag{20.25}$$

Since the multinomial coefficients are at least 1, we therefore have

$$|\mu_{j\bar{k}}| \leq C\rho^{-j-k} \leq \frac{(j+k)!}{j!k!}\rho^{-j-k} = \mu^*_{j\bar{k}}. \tag{20.26}$$

Recall from Proposition 19.4 that there is a solution w^* of the Beltrami equation for μ^* satisfying $w(z, 0) = z$ which is analytic in $P(\rho')$ for some $\rho' > 0$. Write the power series expansion for w^* as

$$w^*(z, \overline{z}) = \sum_{(j,k)\neq(0,0)} w^*_{j\bar{k}} z^j \overline{z}^k. \tag{20.27}$$

Recall that our formal power series solves

$$w_{j\bar{k}} = P_{j\bar{k}}(\mu_{*\bar{*}}), \tag{20.28}$$

where $P_{j\bar{k}}$ is a polynomial with positive coefficients depending only upon $\mu_{p\bar{q}}$ for $p+q < j+k$. Since w^* is an analytic solution of the Beltrami equation with μ^*, we must also have

$$w^*_{j\bar{k}} = P_{j\bar{k}}(\mu^*_{*\bar{*}}), \tag{20.29}$$

where $P_{j\bar{k}}$ is the *same* polynomial since $\mu^*(0, 0) = 0$ and $w^*(z, 0) = z$. We then estimate

$$|w_{j\bar{k}}| = |P_{j\bar{k}}(\mu_{*\bar{*}})| \leq P_{j\bar{k}}(|\mu_{*\bar{*}}|) \leq P_{j\bar{k}}(\mu^*_{*\bar{*}}) = w^*_{j\bar{k}}. \tag{20.30}$$

The inequalities hold since $P_{j\bar{k}}$ is a polynomial with real non-negative coefficients, and using (20.26). This shows that our power series is majorized by the power series of w^*, which implies that the power series for w also converges in the open polydisc $P(\rho')$, by the comparison test. □

This finishes the proof in the analytic case. Soon we will discuss a method to reduce the finite regularity to the analytic case, due to Malgrange.

21 Lecture 21

We will next discuss an alternate approach to this problem using elliptic theory, which only works for $n = 1$.

21.1 Reduction to an elliptic equation

Theorem 21.1. *If (M^2, J) is a real 2-dimensional almost complex manifold with J of class $C^{1,\alpha}$, then J is a complex 1-manifold. That is, near any point $x \in M$, there exists a neighborhood U and a holomorphic coordinate system $f : U \to \mathbb{C}$.*

The proof will be given over the next couple of lectures. To begin, given any point x in M, by Proposition 16.6, we need to find a function $f : U \to \mathbb{C}$ where U is a neighborhood of x satisfying $\overline{\partial}_J f = 0$ in U, and $\partial_J f(x) \neq 0$. This equation is

$$0 = \overline{\partial}_J f = df + iJ^T df. \tag{21.1}$$

Let us write $f = u + iv$, where u and v are real-valued. Then we need

$$0 = du + idv + iJ^T(du + idv) = (du - J^T dv) + i(dv + J^T du). \tag{21.2}$$

Note that applying J^T to $du = J^T dv$, results in $dv = -J^T du$, so if we solve the single equation

$$du = J^T dv, \tag{21.3}$$

then $f = u + iv$ will be pseudo-holomorphic. Note that

$$\partial f = (du + J^T dv) + i(dv - J^T du) = 2(J^T dv + idv), \tag{21.4}$$

so if $dv(x) \neq 0$, then $\partial f(x) \neq 0$. To solve (21.3), we apply the exterior derivative d to get

$$dJ^T dv = 0. \tag{21.5}$$

If v solves (21.5), then $J^T dv$ is closed, and by the Poincaré Lemma, we can solve $J^T dv = du$ in any simply-connected neighborhood U of x. To summarize, we have reduced the problem to finding a simply-connected neighborhood U of x, and a function $v : U \to \mathbb{R}$ with $dv(x) \neq 0$ solving the equation $dJ^T dv = 0$.

Let us write out the above equation (21.5) locally. With respect to the basis $\{\partial_x, \partial_y\}$, we have

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{21.6}$$

With respect to the dual basis $\{dx, dy\}$, we have

$$J^T = \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \tag{21.7}$$

So the equation is

$$dJ^T dv = d\begin{pmatrix} a & c \\ b & -a \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = d\Big((av_x + cv_y)dx + (bv_x - av_y)dy\Big) \tag{21.8}$$

$$= \Big((av_x + cv_y)_y - (bv_x - av_y)_x\Big)dy \wedge dx. \tag{21.9}$$

So the equation is equivalent to

$$(av_x + cv_y)_y - (bv_x - av_y)_x = 0 \tag{21.10}$$

Let us write this as

$$v_{xx} + v_{yy} + (av_x + (c-1)v_y)_y - ((b+1)v_x - av_y)_x = 0 \tag{21.11}$$

If $J = J_0 + o(1)$ as $z \to 0$, then we see that our equation is a perturbation of the Laplacian, and we might hope to use perturbation methods to solve.

21.2 Inverse function theorem

The following is a version of the inverse function theorem for linear operators.

Lemma 21.2. *Let $\mathscr{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded linear mapping between two Banach spaces such that $\mathscr{F}(x) = \mathscr{L}(x) + \mathscr{Q}(x)$, where $\mathscr{L}$ and $\mathscr{Q}$ are both bounded linear mappings. Assume that*

1. *$\mathscr{L}$ is an isomorphism with bounded inverse T.*
2. *$\mathscr{Q}$ satisfies $\|\mathscr{Q}\| \cdot \|T\| = \delta < 1$.*

Then $\mathscr{F}$ is also an isomorphism and

$$\|\mathscr{F}^{-1}\| \leq \frac{1}{1-\delta}\|T\|. \tag{21.12}$$

Proof. Given $f \in \mathcal{B}_2$, we want to solve the equation $\mathscr{F}x = f$ for a unique $x \in \mathcal{B}_1$ with a bound $\|x\|_{\mathcal{B}_1} \leq C\|f\|_{\mathcal{B}_2}$. Writing $x = Ty$, then the equation we want to solve becomes

$$\mathscr{F}(Ty) = (\mathscr{L} + \mathscr{Q})(Ty) = f, \tag{21.13}$$

or

$$y = f - \mathscr{Q}(Ty) \tag{21.14}$$

So we would like to find a fixed point of the operator $S : \mathcal{B}_2 \to \mathcal{B}_2$ defined by

$$Sy = f - \mathscr{Q}(Ty) \tag{21.15}$$

We next claim that under the assumptions, S is a contraction mapping from $\mathcal{B}_2$ to $\mathcal{B}_2$. To see this, we compute

$$\begin{aligned} \|Sy_1 - Sy_2\|_{\mathcal{B}_2} &= \|\mathscr{Q}(Ty_1) - \mathscr{Q}(Ty_2)\|_{\mathcal{B}_2} \\ &\leq \|\mathscr{Q}\| \cdot \|Ty_1 - Ty_2\|_{\mathcal{B}_1} \\ &\leq \|\mathscr{Q}\| \cdot \|T\| \cdot \|y_1 - y_2\|_{\mathcal{B}_2} = \delta \|y_1 - y_2\|_{\mathcal{B}_2}. \end{aligned} \tag{21.16}$$

where $\delta = \|\mathscr{Q}\|\|T\| < 1$ by assumption. We then let $y_0 = 0$, and define $y_{j+1} = Sy_j$. If $n \geq m$, we have

$$\begin{aligned} \|y_n - y_m\|_{\mathcal{B}_2} &\leq \sum_{j=m+1}^{n} \|y_j - y_{j-1}\|_{\mathcal{B}_2} \\ &= \sum_{j=m+1}^{n} \|S^{j-1}y_1 - S^{j-1}y_0\|_{\mathcal{B}_2} \\ &\leq \sum_{j=m+1}^{n} \delta^{j-1}\|y_1 - y_0\|_{\mathcal{B}_2} \\ &\leq \frac{\delta^m}{1-\delta}\|y_1 - y_0\|_{\mathcal{B}_2}. \end{aligned} \tag{21.17}$$

The right hand side limits to 0 as $m \to \infty$, so the sequence y_j is a Cauchy sequence in the Banach space $\mathcal{B}_2$, which therefore converges to a limit y_∞. Since S is continuous, we therefore have

$$Sy_\infty = S \lim_{j\to\infty} y_j = \lim_{j\to\infty} Sy_j = \lim_{j\to\infty} y_{j+1} = y_\infty. \tag{21.18}$$

Take $m = 1$ in (21.17) to get

$$\|y_n - y_1\|_{\mathcal{B}_2} \leq \frac{\delta}{1-\delta}\|y_1 - y_0\|_{\mathcal{B}_2} = \frac{\delta}{1-\delta}\|y_1\|_{\mathcal{B}_2} \tag{21.19}$$

Letting $n \to \infty$ yields

$$\|y_\infty\|_{\mathcal{B}_2} - \|y_1\|_{\mathcal{B}_2} \leq \|y_\infty - y_1\|_{\mathcal{B}_2} \leq \frac{\delta}{1-\delta}\|y_1\|_{\mathcal{B}_2}. \tag{21.20}$$

Then $x_\infty = Ty_\infty$ is a solution to $\mathscr{F}(x_\infty) = f$ and

$$\|x_\infty\|_{\mathcal{B}_1} \leq \|T\|\|y_\infty\|_{\mathcal{B}_2} \leq \frac{1}{1-\delta}\|T\| \cdot \|f\|_{\mathcal{B}_2}, \tag{21.21}$$

since $y_1 = S(y_0) = S(0) = f$, which implies (21.12). Finally, uniqueness then follows from (21.16). □

22 Lecture 22

22.1 Hölder spaces

Given a subset $D \subset \mathbb{R}^n$, $0 < \alpha < 1$, and $f : D \to \mathbb{C}$, we define

$$\|f\|_{C^0(D)} = \sup_{x \in D} |u(x)| \tag{22.1}$$

$$[f]_{\alpha;D} = \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} \tag{22.2}$$

$$\|f\|_{C^{0,\alpha}(D)} = \|f\|_{C^0(D)} + [f]_{\alpha,D}. \tag{22.3}$$

The space of continuous functions on $\overline{D}$ is denoted $C^0(\overline{D})$, and is a Banach space with the norm (22.1). The space of Hölder continuous functions $C^{0,\alpha}(\overline{D}) \subset C^0(\overline{D})$ with the norm (22.3) is a Banach space. Note that (22.2) only defines a seminorm.

Given $k \geq 0$, and a multi-index $\beta = (j_1, \ldots, j_n)$ with $j_* \geq 0$, and $|\beta| = j_1 + \cdots + j_n = k$, we define the seminorms

$$[f]_{k;D} = \sup_{x \in D, |\beta| = k} |\partial^{\beta} u(x)|, \tag{22.4}$$

$$[f]_{k,\alpha;D} = \sup_{|\beta|=k} [\partial^{\beta} f]_{\alpha,D}. \tag{22.5}$$

Then we define norms by

$$\|f\|_{C^k(D)} = \sum_{j=0}^{k} [f]_{j;D} \tag{22.6}$$

$$\|f\|_{C^{k,\alpha}(D)} = \|f\|_{C^k(D)} + [f]_{k,\alpha;D}. \tag{22.7}$$

The space of k-times continuously differentiable functions on $\overline{D}$ is denoted $C^k(\overline{D})$, and is a Banach space with the norm (22.1). The space of functions $C^{k,\alpha}(\overline{D}) \subset C^k(\overline{D})$ with the norm (22.3) is a Banach space.

Exercise 22.1. Show that if $f_1, f_2 \in C^{k,\alpha}(\overline{D})$, then so is $f_1 f_2 \in C^{k,\alpha}(\overline{D})$ and

$$\|f_1 f_2\|_{C^{k,\alpha}(\overline{D})} \leq \|f_1\|_{C^{k,\alpha}(\overline{D})} \cdot \|f_2\|_{C^{k,\alpha}(\overline{D})} \tag{22.8}$$

22.2 Solution of the Beltrami equation

Recall from above, we have reduced the problem to finding a simply-connected neighborhood U of x, and a function $v : U \to \mathbb{R}$ with $dv(x) \neq 0$ solving the equation $dJ^T dv = 0$, which is

$$v_{xx} + v_{yy} + (av_x + (c-1)v_y)_y - ((b+1)v_x - av_y)_x = 0. \tag{22.9}$$

Since $J = J_0 + o(1)$ as $(x,y) \to (0,0)$, we have that $a \to 0$, $b \to -1$ and $c \to 1$, so let us simply redefine b to be $b+1$ and c to be $c-1$, and rewrite the equation as

$$v_{xx} + v_{yy} + (av_x + cv_y)_y - (bv_x - av_y)_x = 0, \tag{22.10}$$

where we now have $a, b, c \to 0$ and $(x, y) \to (0, 0)$. Since we only need a solution in a small neighborhood, let's turn on our microscope and write for $\epsilon > 0$

$$x' = \epsilon x, y' = \epsilon y, v'(x', y') = v(\epsilon x', \epsilon y'). \tag{22.11}$$

The equation in the new coordinates is

$$v'_{xx} + v'_{yy} + (a'v'_x + c'v'_y)_y - (b'v'_x - a'v'_y)_x = 0, \tag{22.12}$$

where $a'(x', y') = a(\epsilon x', \epsilon y')$, and similarly for b', c'.

Writing $v' = y' + h$, we get

$$h_{xx} + h_{yy} + (a'h_x + c'h_y)_y - (b'h_x - a'h_y)_x = a'_x + c'_y. \tag{22.13}$$

We define

$$\mathcal{B}_1 = C^{2,\alpha}_0(\overline{B_1(0)}) = \{u \in C^{2,\alpha}(\overline{B_1(0)}) \mid u = 0 \text{ on } \partial B_1(0)\} \tag{22.14}$$

$$\mathcal{B}_2 = C^{0,\alpha}(\overline{B_1(0)}) \tag{22.15}$$

$$\mathscr{F}(h) = h_{xx} + h_{yy} + (a'h_x + c'h_y)_y - (b'h_x - a'h_y)_x \tag{22.16}$$

$$\mathscr{L}(h) = h_{xx} + h_{yy} \tag{22.17}$$

$$\mathscr{Q}(h) = (a'h_x + c'h_y)_y - (b'h_x - a'h_y)_x. \tag{22.18}$$

So we have

$$\mathscr{F} = \mathscr{L} + \mathscr{Q} : \mathcal{B}_1 \to \mathcal{B}_2. \tag{22.19}$$

Proposition 22.2. *If $a', b', c' \in C^{1,\alpha}$, then the mappings $\mathscr{F}, \mathscr{L}, \mathscr{Q}$ are bounded linear mappings.*

Proof. First, for $\mathscr{L}$, we have

$$\|\mathscr{L}h\|_{C^{0,\alpha}} = \|h_{xx} + h_{yy}\|_{C^{0,\alpha}} \leq C\|h\|_{C^{2,\alpha}}, \tag{22.20}$$

just by definition. For $\mathscr{Q}$, consider for example

$$\|a'h_{xy}\|_{C^{0,\alpha}} \leq \|a'\|_{C^{0,\alpha}} \cdot \|h_{xy}\|_{C^{0,\alpha}} \leq C\|h\|_{C^{2,\alpha}}, \tag{22.21}$$

the other terms in $\mathscr{Q}$ are treated similarly. □

Proposition 22.3. *There exists a constant $C > 0$ so that the operator norm of $\mathscr{Q}$ satisfies*

$$\|\mathscr{Q}\| \leq C\epsilon^{\alpha}. \tag{22.22}$$

Proof. Take a term in $\mathscr{Q}$ like above, that is

$$\|a'h_{xy}\|_{C^{0,\alpha}} \leq \|a'\|_{C^{0,\alpha}} \cdot \|h_{xy}\|_{C^{0,\alpha}} \leq \|a'\|_{C^{0,\alpha}} \cdot \|h\|_{C^{2,\alpha}}. \tag{22.23}$$

Since $a \in C^{0,\alpha}(B_1(0))$ with $a(0) = 0$, the Hölder assumption on a implies that

$$|a(x, y)| \leq \|a\|_{C^{\alpha}}|z|^{\alpha}. \tag{22.24}$$

So then we have

$$\|a'\|_{C^0(B_1)} = \sup_{x' \in B_1} |a(x', y')| = \sup_{x \in B_\epsilon} |a(x, y)| \leq \|a\|_{C^\alpha(B_\epsilon)} \cdot \epsilon^\alpha \tag{22.25}$$

Next, we have

$$[a']_{C^{0,\alpha}(B_1)} = \sup_{x', y' \in B_1, x' \neq y'} \frac{|a'(x') - a'(y')|}{|x' - y'|^\alpha} \tag{22.26}$$

$$= \epsilon^\alpha \sup_{x, y \in B_\epsilon, x \neq y} \frac{|a(x) - a(y)|}{|x - y|^\alpha} \tag{22.27}$$

$$\leq \epsilon^\alpha [a]_{C^{0,\alpha}(B_\epsilon)}. \tag{22.28}$$

Other terms are treated similarly to obtain

$$\|\mathscr{Q}h\|_{C^{0,\alpha}(B_1)} \leq \epsilon^\alpha (\|a\|_{C^{0,\alpha}(B_\epsilon)} + \|b\|_{C^{0,\alpha}(B_\epsilon)} + \|c\|_{C^{0,\alpha}(B_\epsilon)}) \|h\|_{C^{2,\alpha}(B_1)}. \tag{22.29}$$

Finally then, the operator norm is

$$\|\mathscr{Q}\| = \sup_{h \neq 0} \frac{\|\mathscr{Q}h\|_{C^{0,\alpha}}}{\|h\|_{C^{2,\alpha}}} \leq \epsilon^\alpha (\|a\|_{C^{0,\alpha}(B_\epsilon)} + \|b\|_{C^{0,\alpha}(B_\epsilon)} + \|c\|_{C^{0,\alpha}(B_\epsilon)}) \tag{22.30}$$

□

Now we recall the fundamental result about the Dirichlet problem for the Laplace operator.

Theorem 22.4. *The mapping $\Delta_0 : C^{2,\alpha}_0(B_1(0)) \to C^{0,\alpha}(B_1(0))$ is an isomorphism with bounded inverse, that is, there exists a constant C so that if $\Delta_0 u = f$ and $u = 0$ on the boundary, then*

$$\|u\|_{C^{2,\alpha}(B_1(0))} \leq C \|f\|_{C^{0,\alpha}(B_1(0))}. \tag{22.31}$$

Proof. We just indicate how this is proved. Any solution is unique by the maximum principle. There is actually an explicit integral formula for the solution

$$u = \int_{B_1(0)} G(x, y) f(y) dy, \tag{22.32}$$

where $G(x, y)$ is the Green's function defined by

$$G(x, y) = \begin{cases} \frac{1}{2\pi} \Big(\log|x - y| - \log\Big(|y||x - y/|y|^2|\Big) & y \neq 0 \\ \frac{1}{2\pi} \log|x| & y = 0 \end{cases}. \tag{22.33}$$

Then one can just directly verify this is a solution if $f \in C^{0,\alpha}(B_1(0))$, and also directly verify the estimate (22.31); see [GT01]. □

Now, we can use the Inverse function theorem (Lemma 21.2) to obtain that for ϵ sufficiently small, the operator $\mathscr{F} : \mathcal{B}_1 \to \mathcal{B}_2$ is an isomorphism with bounded inverse satisfying

$$\|\mathscr{F}^{-1}\| \leq C\|T\| \tag{22.34}$$

Recall that we need to solve

$$\mathcal{F}(h) = a'_x + c'_y \tag{22.35}$$

Since $\mathscr{F}$ is an isomorphism, there exists a solution h satisfying

$$\|h\|_{C^{2,\alpha}(B_1)} \leq C\|a'_x + c'_y\|_{C^\alpha(B_1)} \leq C\epsilon^\alpha\|a_x + c_y\|_{C^\alpha(B_\epsilon)} \leq C'\epsilon^\alpha \tag{22.36}$$

So then we have that $v = y + h$ satisfies

$$|v_y(0) - 1| \leq C'\epsilon^\alpha, \tag{22.37}$$

So our solution $v = y + h$ satisfies the necessary gradient condition at the origin for ϵ sufficiently small. This completes the proof of the theorem.

Unfortunately, the trick in this subsection does not help us to solve the Newlander-Nirenberg problem in higher dimensions. We will next discuss another method which is more complicated, but has the advantage that it *can* be extended to the higher dimensional case.

23 Lecture 23

23.1 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the smooth case into the analytic case [Mal69], [Nir73, Section I.4]. We want to change coordinates $\xi = \xi(z, \overline{z})$ so that such that our solution of the Beltrami equation in the z-coordinates

$$w_{\overline{z}} + \mu(z, \overline{z})w_z \tag{23.1}$$

transforms into another Beltrami equation,

$$W_{\bar{\xi}} + \tilde{U}(\xi, \bar{\xi})W_\xi = 0, \tag{23.2}$$

with $\tilde{U}$ analytic. Note that we want a real change of coordinates, so if we write $\xi = \xi_1 + i\xi_2$, we need

$$\det \begin{pmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{pmatrix} (0,0) \neq 0. \tag{23.3}$$

As we know, after a change of basis, this is

$$\det \begin{pmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial \overline{z}} \\ \frac{\partial \bar{\xi}}{\partial z} & \frac{\partial \bar{\xi}}{\partial \overline{z}} \end{pmatrix} (0,0) = \left|\frac{\partial \xi}{\partial z}(0,0)\right|^2 - \left|\frac{\partial \xi}{\partial \overline{z}}(0,0)\right|^2 \neq 0. \tag{23.4}$$

Write

$$w(z,\overline{z}) = W(\xi(z,\overline{z}), \bar{\xi}(z,\overline{z})) \tag{23.5}$$
$$\mu(z,\overline{z}) = U(\xi(z,\overline{z}), \bar{\xi}(z,\overline{z})). \tag{23.6}$$

Then

$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial \overline{z}} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial \overline{z}} \tag{23.7}$$
$$\frac{\partial w}{\partial z} = \frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}. \tag{23.8}$$

So the Beltrami equation becomes

$$\frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial \overline{z}} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial \overline{z}} = -U(\xi,\bar{\xi})\Big(\frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}\Big), \tag{23.9}$$

which we can write as

$$\frac{\partial W}{\partial \bar{\xi}} = -\left(\frac{\frac{\partial \xi}{\partial \overline{z}} + U(\xi,\bar{\xi})\frac{\partial \xi}{\partial z}}{\frac{\partial \bar{\xi}}{\partial \overline{z}} + U(\xi,\bar{\xi})\frac{\partial \bar{\xi}}{\partial z}}\right)\frac{\partial W}{\partial \xi}, \tag{23.10}$$

which is another Beltrami equation with a new right hand side

$$\tilde{U}(\xi,\bar{\xi}) = \frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}. \tag{23.11}$$

Let us try to find the coordinates so that

$$\frac{\partial}{\partial \xi}\tilde{U}(\xi,\bar{\xi}) = 0. \tag{23.12}$$

Then then new $\tilde{U}$ will be anti-holomorphic and therefore analytic by the Cauchy integral formula. From the chain rule, we have

$$\frac{\partial}{\partial \xi} = \frac{\partial z}{\partial \xi}\frac{\partial}{\partial z} + \frac{\partial \overline{z}}{\partial \xi}\frac{\partial}{\partial \overline{z}}, \tag{23.13}$$

and we have

$$\begin{aligned}\frac{\partial}{\partial \xi}\tilde{U}(\xi,\bar{\xi}) &= \frac{\partial}{\partial \xi}\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big)\\ &= \Big(z_\xi \partial_z + \overline{z}_\xi \partial_{\overline{z}}\Big)\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big).\end{aligned} \tag{23.14}$$

By the inverse function theorem, we have

$$\begin{pmatrix} z_\xi & z_{\bar{\xi}} \\ \overline{z}_\xi & \overline{z}_{\bar{\xi}} \end{pmatrix} = \begin{pmatrix} \xi_z & \xi_{\overline{z}} \\ \bar{\xi}_z & \bar{\xi}_{\overline{z}} \end{pmatrix}^{-1} = \frac{1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}\begin{pmatrix} \bar{\xi}_{\overline{z}} & -\xi_{\overline{z}} \\ -\bar{\xi}_z & \xi_z \end{pmatrix}, \tag{23.15}$$

so

$$z_\xi = \frac{1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}\bar{\xi}_{\bar{z}} \tag{23.16}$$

$$\overline{z}_\xi = \frac{-1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}\bar{\xi}_z. \tag{23.17}$$

We therefore have

$$\frac{\partial}{\partial \xi}\tilde{U}(\xi, \bar{\xi}) = \frac{1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}(\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big). \tag{23.18}$$

If we multiply through by the leading factor, we want to solve

$$0 = (\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big) \tag{23.19}$$

but keep in mind that we need to find a solution with $|\xi_z|^2(0,0) - |\xi_{\overline{z}}|^2(0,0) \neq 0$. Converting the $U(\xi, \bar{\xi})$ term back to the $(z, \overline{z})$ coordinates, we have

$$0 = (\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + \mu(z,\overline{z})\xi_z}{\bar{\xi}_{\overline{z}} + \mu(z,\overline{z})\bar{\xi}_z}\Big). \tag{23.20}$$

The equation (24.20) is *quasilinear* of the form

$$F(D^2\xi, D\xi, \xi, z, \overline{z}) = 0. \tag{23.21}$$

Definition 23.1. The linearization of F at a function ξ is given by

$$F'_\xi(h) = \frac{d}{dt}F(D^2(\xi + th), D(\xi + th), \xi + th, z, \overline{z})\Big|_{t=0}. \tag{23.22}$$

The linearization is too complicated to write down in general, but the following is all that we really need.

Proposition 23.2. *Assuming $\mu \in C^1$, then the linearization of F at $\xi = z$ is*

$$F'_z(h) = \partial_z\Big(h_{\overline{z}} + \mu(h_z - \bar{h}_{\overline{z}} - \mu\bar{h}_z)\Big) + \bar{h}_{\bar{z}}\mu_z - \bar{h}_z\mu_{\overline{z}}. \tag{23.23}$$

If $\mu(0,0) = 0$, then we have

$$F'_z(h)(0,0) = \frac{1}{4}\Delta h + c_1 h_z + c_2\overline{h}_z + c_3 h_{\overline{z}} + c_4\overline{h}_{\overline{z}}. \tag{23.24}$$

for some constants c_1, c_2, c_3, c_4. If μ has sufficiently small $C^{1,\alpha}$, norm then F'_z is an elliptic operator with Hölder coefficients bounded in C^α.

Proof. We write out

$$F(D^2(\xi+th), D(\xi+th), \xi+th, z, \overline{z}) \tag{23.25}$$

$$= \Big((\overline{\xi+th})_{\overline{z}}\partial_z - (\overline{\xi+th})_z\partial_{\overline{z}}\Big)\Big(\frac{(\xi+th)_{\overline{z}} + \mu(z,\overline{z})(\xi+th)_z}{(\overline{\xi+th})_{\overline{z}} + \mu(z,\overline{z})(\overline{\xi+th})_z}\Big). \tag{23.26}$$

Letting $\xi = z$, this becomes

$$F(D^2(z+th), D(z+th), z+th, z, \overline{z}) \tag{23.27}$$

$$= \Big((1+t\bar{h}_{\overline{z}})\partial_z - t\bar{h}_z\partial_{\overline{z}}\Big)\Big(\frac{th_{\overline{z}} + \mu(z,\bar{z})(1+th_z)}{(1+t\bar{h}_{\overline{z}}) + \mu(z,\overline{z})t\bar{h}_z}\Big). \tag{23.28}$$

We also see that

$$F_z'(h) = \partial_z\Big(h_{\overline{z}} + \mu(h_z - \bar{h}_{\overline{z}} - \mu\bar{h}_z)\Big) + \bar{h}_{\bar{z}}\mu_z - \bar{h}_z\mu_{\overline{z}}. \tag{23.29}$$

Noting that

$$\frac{\partial^2 h}{\partial z\partial\overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)h = \frac{1}{4}\Delta h, \tag{23.30}$$

the proposition follows from this. □

We next need the inverse function theorem in Banach spaces.

Lemma 23.3. *Let $\mathscr{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be a C^1-map between two Banach spaces such that $\mathscr{F}(x) = \mathscr{F}(0) + \mathscr{L}(x) + \mathscr{Q}(x)$, where the operator $\mathscr{L} : \mathcal{B}_1 \to \mathcal{B}_2$ is linear and $\mathscr{Q}(0) = 0$. Assume that*

1. *$\mathscr{L}$ is an isomorphism with inverse T satisfying $\|T\| \leq C_1$,*
2. *there are constants $r > 0$ and $C_2 > 0$ with $r < \frac{1}{3C_1C_2}$ such that*
   (a) *$\|\mathscr{Q}(x) - \mathscr{Q}(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x-y\|_{\mathcal{B}_1}$ for all $x, y \in B_r(0) \subset \mathcal{B}_1$,*
   (b) *$\|\mathscr{F}(0)\|_{\mathcal{B}_2} \leq \frac{r}{3C_1}$.*

Then there exists a unique solution to $\mathscr{F}(x) = 0$ in $\mathcal{B}_1$ such that

$$\|x\|_{\mathcal{B}_1} \leq 3C_1 \cdot \|\mathscr{F}(0)\|_{\mathcal{B}_2}. \tag{23.31}$$

Proof. Writing $x = Tf$, we can write the equation $\mathscr{F}(x) = 0$ as

$$\mathscr{F}(0) + f + \mathscr{Q}(Tf) = 0, \tag{23.32}$$

that is

$$f = -\mathscr{Q}(Tf) - \mathscr{F}(0). \tag{23.33}$$

So we would like to find a fixed point of the operator $S : \mathcal{B}_2 \to \mathcal{B}_2$ defined by

$$Sf = -\mathscr{Q}(Tf) - \mathscr{F}(0). \tag{23.34}$$

We next claim that under the assumptions, S is a contraction mapping from $B_{r/C_1}(0) \subset \mathcal{B}_2$. To see this, we compute

$$\begin{aligned}
\|Sf_1 - Sf_2\|_{\mathcal{B}_2} &= \|\mathscr{Q}(Tf_1) - \mathscr{Q}(Tf_2)\|_{\mathcal{B}_2} \\
&\leq C_2(\|Tf_1\|_{\mathcal{B}_1} + \|Tf_2\|_{\mathcal{B}_1})(\|Tf_1 - Tf_2\|_{\mathcal{B}_1} \\
&\leq C_2(2C_1 r/C_1)C_1\|f_1 - f_2\|_{\mathcal{B}_2} \leq \frac{2}{3}(\|f_1 - f_2\|_{\mathcal{B}_2}).
\end{aligned} \tag{23.35}$$

We then let $f_0 = 0$, and define $f_{j+1} = Sf_j$. If $n \geq m$, we have

$$\begin{aligned}
\|f_n - f_m\|_{\mathcal{B}_2} &\leq \sum_{j=m+1}^{n} \|f_j - f_{j-1}\|_{\mathcal{B}_2} \\
&= \sum_{j=m+1}^{n} \|S^{j-1}f_1 - S^{j-1}f_0\|_{\mathcal{B}_2} \\
&\leq \sum_{j=m+1}^{n} \left(\frac{2}{3}\right)^{j-1} \|f_1 - f_0\|_{\mathcal{B}_2} \\
&\leq \frac{(2/3)^m}{1 - 2/3}\|f_1 - f_0\|_{\mathcal{B}_2}.
\end{aligned} \tag{23.36}$$

The right hand side limits to 0 as $m \to \infty$. This proves that the sequence f_j is a Cauchy sequence in the Banach space $\mathcal{B}_2$, which therefore converges to a limit f_∞. Since S is continuous, we therefore have

$$Sf_\infty = S \lim_{j\to\infty} f_j = \lim_{j\to\infty} Sf_j = \lim_{j\to\infty} f_{j+1} = f_\infty. \tag{23.37}$$

Take $m = 1$ in (23.36) to get

$$\|f_n - f_1\|_{\mathcal{B}_2} \leq 2\|f_1 - f_0\|_{\mathcal{B}_2} = 2\|f_1\|_{\mathcal{B}_2} \tag{23.38}$$

Letting $n \to \infty$ yields

$$\|f_\infty\|_{\mathcal{B}_2} - \|f_1\|_{\mathcal{B}_2} \leq \|f_\infty - f_1\|_{\mathcal{B}_2} \leq 2\|f_1\|_{\mathcal{B}_2}. \tag{23.39}$$

Then $x_\infty = Tf_\infty$ is a solution to $\mathscr{F}(x_\infty) = 0$ and

$$\|x_\infty\|_{\mathcal{B}_1} \leq C_1\|f_\infty\|_{\mathcal{B}_2} \leq 3C_1\|\mathscr{F}(0)\|_{\mathcal{B}_2} \tag{23.40}$$

which implies (37.1). If x is any solution satisfying (37.1), then letting $f = \mathscr{L}x$, we estimate

$$\|f\|_{\mathcal{B}_2} \leq \frac{1}{C_1}\|x\|_{\mathcal{B}_1} \leq \frac{1}{C_1}3C_1\frac{r}{3C_1} = \frac{r}{C_1} \tag{23.41}$$

and uniqueness then follows from (23.35). □

24 Lecture 24

24.1 Remarks on scaling

For the original Beltrami equation $w_{\overline{z}} = \mu(z,\overline{z})w_z$, we can assume that $\mu(0,0) = 0$. By scaling the coordinates $z = \epsilon z'$, and letting $w'(z',\overline{z}') = w(\epsilon z', \epsilon \overline{z}')$, then we have

$$w'_{\overline{z}'} = \mu'(z',\overline{z}')w'_{z'}, \tag{24.1}$$

where

$$\mu'(z',\overline{z}') = \mu(\epsilon z', \epsilon\overline{z}'). \tag{24.2}$$

Lemma 24.1. *We have*

$$\|\mu'\|_{C^0(B_1)} = \|\mu\|_{C^0(B_\epsilon)} \tag{24.3}$$
$$[\mu']_{\alpha;B_1} = \epsilon^{\alpha}[\mu']_{\alpha;B_\epsilon} \tag{24.4}$$
$$\|\nabla'\mu'\|_{C^0(B_1)} = \epsilon\|\nabla\mu\|_{C^0(B_\epsilon)} \tag{24.5}$$
$$[\nabla'\mu']_{\alpha;B_1} = \epsilon^{1+\alpha}\|\nabla\mu\|_{C^0(B_\epsilon)}. \tag{24.6}$$

In general, we have

$$[\mu']_{k,\alpha;B_1} = \epsilon^{k+\alpha}[\mu]_{k,\alpha;B_\epsilon} \tag{24.7}$$

Proof. The first (24.3) is trivial. For (24.4), we have

$$\begin{aligned} [\mu']_{\alpha;B_1} &= \sup_{x',y' \in B_1, x' \neq y'} \frac{|\mu'(x') - \mu'(y')|}{|x'-y'|^{\alpha}} \\ &= \sup_{x',y' \in B_1, x' \neq y'} \frac{|\mu(\epsilon x') - \mu(\epsilon y')|}{|x'-y'|^{\alpha}} \\ &= \sup_{x,y \in B_\epsilon, x \neq y} \frac{|\mu(x) - \mu(y)|}{|x\epsilon^{-1} - y\epsilon^{-1}|^{\alpha}} \\ &= \epsilon^{\alpha}[\mu']_{\alpha;B_\epsilon}. \end{aligned} \tag{24.8}$$

For (24.5), we need to relate $\partial_{z'}\mu'$ and $\partial_z\mu$, and similarly for the barred derivative. For this we have

$$(\partial_{z'}\mu')(z',\overline{z}') = \epsilon\partial_z\mu(\epsilon z', \epsilon\overline{z}'), \tag{24.9}$$

we can write as

$$\nabla'\mu'(z',\overline{z}') = \epsilon\nabla\mu(\epsilon z', \epsilon\overline{z}'), \tag{24.10}$$

from which (24.5) follows immediately.

For (24.6), we have

$$\begin{aligned}[\nabla'\mu']_{\alpha;B_1} &= \sup_{x',y'\in B_1, x'\neq y'} \frac{|\nabla'\mu'(x') - \nabla'\mu'(y')|}{|x'-y'|^{\alpha}} \\ &= \sup_{x',y'\in B_1, x'\neq y'} \frac{|\epsilon\nabla\mu(\epsilon x') - \epsilon\nabla\mu(\epsilon y')|}{|x'-y'|^{\alpha}} \\ &= \sup_{x,y\in B_\epsilon, x\neq y} \frac{|\epsilon\nabla\mu(x) - \epsilon\nabla\mu(y)|}{|x\epsilon^{-1}-y\epsilon^{-1}|^{\alpha}} \\ &= \epsilon^{1+\alpha}\|\nabla\mu\|_{C^0(B_\epsilon)}.\end{aligned} \tag{24.11}$$

The general pattern should be obvious, and (24.7) follows from similar computations. □

Remark 24.2. It is sometimes convenient to define alternative norms as follows. First, define the semi-norms

$$[\mu]'_{k,\alpha;D} = (\mathrm{diam}(D)/2)^{k+\alpha}[\mu]_{k,\alpha;B_R}, \tag{24.12}$$

which then give the norms $\|\mu\|'_{C^{k,\alpha}(D)}$, like in the above definitions. If the domain is a ball, then the new norms are scaling invariant:

$$\|\mu'\|'_{C^{k,\alpha}(B_1)} = \|\mu\|'_{C^{k,\alpha}(B_\epsilon)}. \tag{24.13}$$

Back to our problem, we have

$$\|\mu'\|_{C^{1,\alpha}(B_1)} = \|\mu'\|'_{C^{1,\alpha}(B_1)} = \|\mu\|'_{C^{1,\alpha}(B_\epsilon)} \tag{24.14}$$

$$= \|\mu\|_{C^0(B_\epsilon)} + \epsilon\|\nabla\mu\|_{C^0(B_\epsilon)} + \epsilon^{1+\alpha}[\mu]_{1,\alpha;B_\epsilon}. \tag{24.15}$$

Since $\mu \in C^{1,\alpha}(B_\epsilon) \subset C^{0,\alpha}(B_\epsilon)$, we have that

$$|\mu(x) - \mu(y)| \leq C|x-y|^{\alpha}. \tag{24.16}$$

Since $\mu(0) = 0$, we have an estimate for $x \in B_\epsilon$,

$$|\mu(x)| \leq C|x|^{\alpha} \leq C\epsilon^{\alpha}. \tag{24.17}$$

Combining these estimates, we obtain that there exists a constant C, depending only upon $\|\mu\|_{C^{1,\alpha}(B_{r_0})}$ for some fixed radius r_0, so that

$$\|\mu'\|_{C^{1,\alpha}(B_1)} \leq C\epsilon^{\alpha}. \tag{24.18}$$

Consequently, we can just assume that our original μ is defined in B_1 and satisfies

$$\|\mu\|_{C^{1,\alpha}(B_1)} < C\epsilon^{\alpha}. \tag{24.19}$$

24.2 Solution of the Malgrange equation

We want to find a solution of the equation

$$0 = (\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + \mu(z,\overline{z})\xi_z}{\bar{\xi}_{\overline{z}} + \mu(z,\overline{z})\bar{\xi}_z}\Big). \tag{24.20}$$

with $|\xi_z|^2(0,0) - |\xi_{\overline{z}}|^2(0,0) \neq 0$. From the scaling above, we can assume that the $C^{1,\alpha}$ norm of μ in B_1 is arbitrarily small. The equation (24.20) is *quasilinear* of the form

$$F(D^2\xi, D\xi, \xi, z, \overline{z}) = 0. \tag{24.21}$$

We define

$$\begin{aligned}
\mathcal{B}_1 &= C^{2,\alpha}_0(B_1(0)) = \{u \in C^{2,\alpha}(B_1(0)) \mid u = 0 \text{ on } \partial B_1(0)\} && (24.22)\\
\mathcal{B}_2 &= C^{0,\alpha}(B_1(0)) && (24.23)\\
\mathscr{F}(h) &= F(z+h) && (24.24)\\
\mathscr{L}(h) &= F'_z(h) && (24.25)\\
\mathscr{Q}(h) &= F(z+h) - F(z) - F'_z(h) = \mathscr{F}(h) - \mathscr{F}(0) - \mathscr{L}(h). && (24.26)
\end{aligned}$$

Recall from the linearization formula in Proposition 36.2

$$F'_z(h) = \partial_z\Big(h_{\overline{z}} + \mu(h_z - \bar{h}_{\overline{z}} - \mu\bar{h}_z)\Big) + \bar{h}_{\bar{z}}\mu_z - \bar{h}_z\mu_{\overline{z}}, \tag{24.27}$$

and

$$F'_z(h)(0,0) = \frac{1}{4}\Delta h + c_1 h_z + c_2\bar{h}_z + c_3 h_{\overline{z}} + c_4\bar{h}_{\overline{z}}. \tag{24.28}$$

Since $\mu(0,0) = 0$ and μ has arbitrarily small $C^{1,\alpha}$ norm, the operator $\mathscr{L}$ is a small perturbation of the Laplacian. By the inverse function theorem for linear operators (Lemma 21.2), we can define the mapping $T : C^{0,\alpha} \to C^{2,\alpha}_0$ to be the unique solution to the Dirichlet problem

$$\mathscr{L}(Tf) = f \text{ in } B(0,1), \quad Tf = 0 \text{ on } \partial B(0,1). \tag{24.29}$$

Also from Lemma 21.2, there exists a constant C so that

$$\|Tf\|_{C^{2,\alpha}(B(0,1))} \leq C\|f\|_{C^{0,\alpha}(B(0,1))}. \tag{24.30}$$

Let us recall that

$$\begin{aligned}
\mathscr{F}(h) &= F(z+h) = F(D^2(z+th), D(z+th), z+th, z, \overline{z})\\
&= \Big((1+\bar{h}_{\bar{z}})\partial_z - \bar{h}_z\partial_{\overline{z}}\Big)\Big(\frac{h_{\overline{z}} + \mu(z,\bar{z})(1+h_z)}{(1+\bar{h}_{\overline{z}}) + \mu(z,\overline{z})\bar{h}_z}\Big).
\end{aligned} \tag{24.31}$$

In particular,

$$\mathscr{F}(0) = \partial_z\mu, \tag{24.32}$$

which has arbitrarily small norm in $\mathcal{B}_2$.

We need to estimate

$$\mathscr{Q}(h_2) - \mathscr{Q}(h_2) = F(z + h_2) - F'_z(h_2) - (F(z + h_1) - F'_z(h_1)). \tag{24.33}$$

Consider $f(t) = F(z + th_2 + (1 - t)h_1)$. Since $F(z + h) = \tilde{F}(D^2h, Dh, h, z)$, where $\tilde{F}$ is a smooth function of these variables (for C^2-norm of h sufficiently small), then using the fundamental theorem of calculus

$$F(z + h_2) - F(z + h_1) = f(1) - f(0) = \int_0^1 f'(t)dt. \tag{24.34}$$

We note that

$$\begin{aligned} f'(t) = \frac{d}{ds}F(z + sh_2 + (1 - s)h_1)|_{s=t} &= \frac{d}{ds}F(z + th_1 + (1 - t)h_2 + s(h_2 - h_1))|_{s=0} \\ &= F'_{z+th_1+(1-t)h_2}(h_2 - h_1). \end{aligned} \tag{24.35}$$

This gives the expression

$$\begin{aligned} \mathscr{Q}(h_2) - \mathscr{Q}(h_1) &= \Big(\int_0^1 F'_{z+th_1+(1-t)h_2}(h_2 - h_1)dt \Big) - F'_z(h_2 - h_1) \\ &= \Big(\int_0^1 (F'_{z+th_1+(1-t)h_2} - F'_z)dt \Big)(h_2 - h_1). \end{aligned} \tag{24.36}$$

We next claim that for any y and h, we have the estimate

$$\|(F'_{z+h} - F'_z)y\|_{C^0} \leq C\|h\|_{C^2}\|y\|_{C^2}. \tag{24.37}$$

To see this, note the linearized operator is of the form

$$\begin{aligned} &F'_u(h) = \frac{d}{dt}F(u + th)|_{t=0} \\ &= a^{ij}(D^2u, Du, u, z)D_{ij}h + b^i(D^2u, Du, u, z)D_ih + c(D^2u, Du, u, z)D_ih. \end{aligned} \tag{24.38}$$

If $\tilde{F}(D^2h, Dh, h, z)$ is C^2 in the D^2h, Dh, h variables, and continuous in the z variable, then the coefficients a^{ij}, b^i, c are C^1 as functions of D^2u, Du, u, and we have for example

$$c(D^2(z + h), D(z + h), z + h, z) = c(D^2z, Dz, z, z) + O(|D^2h| + |Dh| + |h|), \tag{24.39}$$

so the estimate (24.37) follows. This implies the quadratic estimate for the C^0-norm.

For the Hölder norm, similar to the above arguments, we see that any y and h, we have the estimate

$$\|(F'_{z+h} - F'_z)y\|_{C^\alpha} \leq C\|h\|_{C^{2,\alpha}}\|y\|_{C^{2,\alpha}}, \tag{24.40}$$

provided that $\tilde{F}$ is $C^{2,\alpha}$ in the D^2h, Dh, h variables and Hölder continuous in the z variable. This finishes the quadratic estimate.

As mentioned above, by taking ϵ sufficiently small, we can always arrange so that condition (b) is satisfied. The implicit function theorem yields a solution h with

$$\|h\|_{C^{2,\alpha}(B_1)} = o(\epsilon), \tag{24.41}$$

as $\epsilon \to 0$. Obviously, we have

$$|h(0)| = o(\epsilon), \quad |\nabla h|(0) = o(\epsilon), \tag{24.42}$$

as $\epsilon \to 0$. Then if ϵ is sufficiently small, then condition (23.4) will also be satisfied.

Remark 24.3. The minimal regularity required in the above arguments is $\mu \in C^{1,\alpha}$. One can actually get away with only assuming $\mu \in C^{0,\alpha}$, but one needs a different method to see this. The Beltrami equation can be solved locally for $\mu \in C^{0,\alpha}$ by inverting the $\partial_{\bar{z}}$ operator using the Cauchy-Pompeiu formula, and a solution can be produced by an iteration method. However, this is a bit technical (it takes around 30 pages), so we will omit. There are many great references for this method, see for example [Spi79b], [Ber58], [Ahl66].

25 Lecture 25

In the next few lectures, we will give a review of Riemannian geometry

25.1 Review of theory of vector bundles

We will next define real vector bundles, but note that everything we will say works for complex bundles, by replacing $\mathbb{R}$ with $\mathbb{C}$.

Definition 25.1. A smooth real vector bundle of rank k over a smooth manifold M^n is a topological space E together with a smooth projection

$$\pi : E \to M \tag{25.1}$$

such that

- For $p \in M$, $\pi^{-1}(p)$ is a vector space of dimension k over $\mathbb{R}$.
- There exists local trivializations, that is, there are smooth mappings

  $$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E \tag{25.2}$$

  which maps $p \times \mathbb{R}^k$ linearly onto the fiber $\pi^{-1}(p)$ for every $p \in U_\alpha$.

The transition functions of a bundle are defined as follows.

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \tag{25.3}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{25.4}$$

for $v \in \mathbb{R}^k$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the cocycle condition

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{25.5}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satisfying (38.5), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$. (It turns out this bundle is uniquely defined up to bundle equivalence, which we will define below.) If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle.

A vector bundle mapping is a mapping $F : E_1 \to E_2$ which is linear on fibers, and covers the identity map, that is, the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ M & \xrightarrow{id} & M. \end{array} \tag{25.6}$$

Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions

$$F_\alpha : U_\alpha \to Hom(\mathbb{R}^{k_1}, \mathbb{R}^{k_2}) \tag{25.7}$$

defined by

$$F_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)). \tag{25.8}$$

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$F_\alpha = \varphi^{E_2}_{\alpha\beta} F_\beta \varphi^{E_1}_{\beta\alpha}, \tag{25.9}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} F_\alpha = F_\beta \varphi^{E_1}_{\beta\alpha}. \tag{25.10}$$

We say that two bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $F : E_1 \to E_2$. This is equivalent to non-singularity of the local representatives, that is, $\det(F_\alpha) \neq 0$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions

$$f_\alpha : U_\alpha \to GL(k, \mathbb{R}) \tag{25.11}$$

such that

$$\varphi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{25.12}$$

Definition 25.2. A section of $\pi : E \to M$ is a smooth mapping $\sigma : M \to E$ such that $\pi \circ \sigma = Id_M$.

In terms of local trivializations, we define

$$s_\alpha = \pi_{\mathbb{R}^k} \circ \Phi_\alpha^{-1} \circ s : U_\alpha \to \mathbb{R}^k, \tag{25.13}$$

which yield the defining relation for a section

$$s_\alpha = \phi_{\alpha\beta} s_\beta. \tag{25.14}$$

The space of sections of a vector bundle is denoted by $\Gamma(M, E)$, which is a vector space.

25.2 Operations on bundles

Note that if $f_1 : \mathbb{R}^k \to \mathbb{R}^k$ and $f_2 : \mathbb{R}^l \to \mathbb{R}^l$ are linear maps then there is an obvious induced mapping $f_1 \oplus f_2 : \mathbb{R}^k \oplus \mathbb{R}^l \to \mathbb{R}^k \oplus \mathbb{R}^l$ defined by

$$(f_1 \oplus f_2)(v_1, v_2) \equiv f_1(v_1) \oplus f_2(v_2). \tag{25.15}$$

The direct sum $E_1 \oplus E_2$ of bundles E_1 and E_2 is a vector bundle with transition functions

$$\varphi_{\alpha\beta}^{E_1 \oplus E_2} = \varphi_{\alpha\beta}^{E_1} \oplus \varphi_{\alpha\beta}^{E_2}. \tag{25.16}$$

Note that if $f_1 : \mathbb{R}^k \to \mathbb{R}^k$ and $f_2 : \mathbb{R}^l \to \mathbb{R}^l$ are linear maps, then there is an obvious induced mapping $f_1 \otimes f_2 : \mathbb{R}^k \otimes \mathbb{R}^l \to \mathbb{R}^k \otimes \mathbb{R}^l$ defined on indecomposable tensors by

$$(f_1 \otimes f_2)(v_1 \otimes v_2) \equiv f_1(v_1) \otimes f_2(v_2), \tag{25.17}$$

and extended by linearity to the tensor product. The tensor product $E_1 \otimes E_2$ of bundles E_1 and E_2 is again a bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E_1 \otimes E_2} = \varphi_{\alpha\beta}^{E_1} \otimes \varphi_{\alpha\beta}^{E_2}. \tag{25.18}$$

The dual E^* of any bundle E, is a bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E^*} = \left((\varphi_{\alpha\beta}^{E})^{-1} \right)^T = (\varphi_{\beta\alpha}^{E})^T. \tag{25.19}$$

Note that the inverse is necessary in order to have the cocycle condition satisfied. In general, if E_1 and E_2 are vector bundles, then we define $Hom(E_1, E_2) = E_1^* \otimes E_2$, which in terms of transition functions is

$$\varphi_{\alpha\beta}^{Hom(E_1,E_2)} = \left((\varphi_{\alpha\beta}^{E_1})^{-1} \right)^T \otimes \varphi_{\alpha\beta}^{E_2}. \tag{25.20}$$

Note that for any linear map $f : \mathbb{R}^k \to \mathbb{R}^k$, there is a naturally induced mapping

$$\Lambda^p f : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \tag{25.21}$$

therefore for any vector bundle E, the pth exterior power $\Lambda^p(E)$ is defined to be the bundle with transition functions

$$\varphi_{\alpha\beta}^{\Lambda^p(E)} = \Lambda^p(\varphi_{\alpha\beta}^{E}). \tag{25.22}$$

For a complex vector bundle $\pi : E \to M$, there is another operation called the conjugate bundle $\overline{E}$ which is the complex vector bundle obtained by replacing each fiber of E with the complex conjugate vector space. The transition functions are simply

$$\varphi_{\alpha\beta}^{\overline{E}} = \overline{\varphi_{\alpha\beta}^{E}}. \tag{25.23}$$

Remark 25.3. In the above, we only defined morphisms in the category of vector bundle to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence, and is not usually used for vector bundle equivalence.

25.3 Riemannian metrics on real vector bundles

If $\pi : E \to M$ is a real vector bundle, a Riemannian metric on E is a choice of smoothly varying positive definite symmetric inner product on each fiber. That is $g \in \Gamma(E^* \otimes E^*)$ satisfying

$$g(e_1, e_2) = g(e_2, e_1), \tag{25.24}$$

$$g(e, e) > 0 \text{ for } e \neq 0. \tag{25.25}$$

Proposition 25.4. *If E is any real vector bundle, then E admits a Riemannian metric.*

Proof. Take the Euclidean metric on trivializations, and patch together using a partition of unity. □

Corollary 25.5. *For any real vector bundle E, $E^* \cong E$.*

Proof. Choose a Riemannian metric g on E. Then the mapping $\flat : E \to E^*$ defined by

$$\flat(e_1)(e_2) = g(e_1, e_2) \tag{25.26}$$

is an ismorphism on fibers, and covers the identity map. □

Remark 25.6. The inverse of $\flat$ is usually denoted by $\flat^{-1} = \sharp : E^* \to E$.

In the special case that $E = TM$ is the tangent bundle, the pair (M, g) is called a *Riemannian manifold*, with the metric $g \in \Gamma(S^2(T^*M))$. In coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i \otimes dx^j, \;\; g_{ij} = g_{ij}, \tag{25.27}$$

and $g_{ij} \gg 0$ is a positive definite symmetric matrix.

Note that a vector field $X \in \Gamma(TM)$ in coordinates $X = X^i \frac{\partial}{\partial x^i}$, then

$$\flat X = (\flat X)_i dx^i = g_{ij} X^j dx^j, \tag{25.28}$$

and similarly if $\omega = \omega_i dx^i$, then

$$\sharp\omega = (\sharp\omega)^i \frac{\partial}{\partial x^i} = g^{ij} \omega_j \frac{\partial}{\partial x^i} \tag{25.29}$$

So the flat operator lowers an index, and the sharp operator raises an index, which explains the notation for anyone familiar with reading music.

25.4 Connections on vector bundles

A connection is a mapping $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, denoted by $\nabla_X \sigma$, such that

- $\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$,
- $\nabla_X (fs) = (Xf)s + f \nabla_X s$.

Note the first condition, linearity over C^∞ functions, implies that the value of $(\nabla_X s)(p)$ only depends on the value of X_p. But the second condition means that it does depend on the values of the section in a neighborhood of p. In coordinates, letting $s_i, i = 1 \dots p$, be a local basis of sections of E,

$$\nabla_{\partial_i} s_j = \Gamma_{ij}^k s_k, \tag{25.30}$$

which defines the Christoffel symbols Γ_{ij}^k of a connection, which are not tensorial. If E carries an inner product, then ∇ is *compatible* if

$$X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle. \tag{25.31}$$

For a connection in TM, ∇ is called *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM). \tag{25.32}$$

Theorem 25.7 (Fundamental Theorem of Riemannian Geometry)**.** *There exists a unique symmetric, compatible connection in TM.*

Invariantly, the connection is defined by

$$\begin{aligned} \langle \nabla_X Y, Z \rangle = \frac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & -\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \Big). \end{aligned} \tag{25.33}$$

Letting $X = \partial_i, Y = \partial_j, Z = \partial_k$, we obtain

$$\begin{aligned} \Gamma_{ij}^l g_{lk} &= \langle \Gamma_{ij}^l \partial_l, \partial_k \rangle = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle \\ &= \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \Big), \end{aligned} \tag{25.34}$$

which yields the formula

$$\boxed{\Gamma_{ij}^k = \frac{1}{2} g^{kl} \Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \Big)} \tag{25.35}$$

for the Riemannian Christoffel symbols.

26 Lecture 26

26.1 Covariant derivatives of tensor fields

Let E and E' be vector bundles over M, with covariant derivative operators ∇, and ∇', respectively. The covariant derivative operators in $E \otimes E'$ and $Hom(E, E')$ are

$$\nabla_X (s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s') \tag{26.1}$$
$$(\nabla_X L)(s) = \nabla'_X (L(s)) - L(\nabla_X s), \tag{26.2}$$

for $s \in \Gamma(E), s' \in \Gamma(E')$, and $L \in \Gamma(Hom(E, E'))$. Note also that the covariant derivative operator in $\Lambda(E)$ is given by

$$\nabla_X(s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^{r} s_1 \wedge \cdots \wedge (\nabla_X s_i) \wedge \cdots \wedge s_r, \tag{26.3}$$

for $s_i \in \Gamma(E)$.

These rules imply that if T is an (r, s) tensor, then the covariant derivative ∇T is an $(r, s + 1)$ tensor given by

$$\nabla T(X, Y_1, \ldots, Y_s) = \nabla_X(T(Y_1, \ldots Y_s)) - \sum_{i=1}^{s} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s). \tag{26.4}$$

We next consider the above definitions in components for (r, s)-tensors. For the case of a vector field $X \in \Gamma(TM)$, ∇X is a $(1, 1)$ tensor field. By the definition of a connection, we have

$$\nabla_m X \equiv \nabla_{\partial_m} X = \nabla_{\partial_m}(X^j \partial_j) = (\partial_m X^j)\partial_j + X^j \Gamma^l_{mj} \partial_l = (\nabla_m X^i + X^l \Gamma^i_{ml})\partial_i. \tag{26.5}$$

In other words,

$$\nabla X = \nabla_m X^i (dx^m \otimes \partial_i), \tag{26.6}$$

where

$$\nabla_m X^i = \partial_m X^i + X^l \Gamma^i_{ml}. \tag{26.7}$$

However, for a 1-form ω, (26.2) implies that

$$\nabla \omega = (\nabla_m \omega_i) dx^m \otimes dx^i, \tag{26.8}$$

with

$$\nabla_m \omega_i = \partial_m \omega_i - \omega_l \Gamma^l_{im}. \tag{26.9}$$

The definition (26.1) then implies that for a general (r, s)-tensor field S,

$$\boxed{\nabla_m S^{i_1 \ldots i_r}_{j_1 \ldots j_s} \equiv \partial_m S^{i_1 \ldots i_r}_{j_1 \ldots j_s} + S^{l i_2 \ldots i_r}_{j_1 \ldots j_s} \Gamma^{i_1}_{ml} + \cdots + S^{i_1 \ldots i_{r-1} l}_{j_1 \ldots j_s} \Gamma^{i_r}_{ml} - S^{i_1 \ldots i_r}_{l j_2 \ldots j_s} \Gamma^l_{m j_1} - \cdots - S^{i_1 \ldots i_r}_{j_1 \ldots j_{s-1} l} \Gamma^l_{m j_s}.} \tag{26.10}$$

Notice the following calculation,

$$(\nabla g)(X, Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \tag{26.11}$$

from the compatbility condition. So the metric tensor is parallel, i.e., $\nabla g \equiv 0$.

26.2 Double covariant derivatives

For an (r, s) tensor field T, we will write the double covariant derivative as

$$\nabla^2 T = \nabla\nabla T, \tag{26.12}$$

which is an $(r, s+2)$ tensor.

Proposition 26.1. *If T is an (r, s)-tensor field, then the double covariant derivative satisfies*

$$\nabla^2 T(X, Y, Z_1, \ldots, Z_s) = \nabla_X(\nabla_Y T)(Z_1, \ldots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \ldots Z_s). \tag{26.13}$$

Proof. The left hand side of (26.13) is

$$\begin{aligned} \nabla^2 T(X, Y, Z_1, \ldots, Z_s) &= \nabla(\nabla T)(X, Y, Z_1, \ldots, Z_s) \\ &= \nabla_X(\nabla T(Y, Z_1, \ldots, Z_s)) - \nabla T(\nabla_X Y, Z_1, \ldots, Z_s) \\ &\quad - \sum_{i=1}^{s} \nabla T(Y, \ldots, \nabla_X Z_i, \ldots Z_s). \end{aligned} \tag{26.14}$$

The right hand side of (26.13) is

$$\begin{aligned} &\nabla_X(\nabla_Y T)(Z_1, \ldots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \ldots Z_s) \\ &= \nabla_X(\nabla_Y T(Z_1, \ldots, Z_s)) - \sum_{i=1}^{s} (\nabla_Y T)(Z_1, \ldots, \nabla_X Z_i, \ldots, Z_s) \\ &\quad - \nabla T(\nabla_X Y, Z_1, \ldots, Z_s). \end{aligned} \tag{26.15}$$

The first term on the right hand side of (26.15) is the same as first term on the right hand side of (26.14). The second term on the right hand side of (26.15) is the same as third term on the right hand side of (26.14). Finally, the last term on the right hand side of (26.15) is the same as the second term on the right hand side of (26.14). □

For illustration, let's compute an example in coordinates. If $\omega \in \Omega^1(M)$, then

$$\begin{aligned} \nabla_i \nabla_j \omega_k &= \partial_i(\nabla_j \omega_k) - \Gamma_{ij}^p \nabla_p \omega_k - \Gamma_{ik}^p \nabla_j \omega_p \\ &= \partial_i(\partial_j \omega_k - \Gamma_{jk}^l \omega_l) - \Gamma_{ij}^p(\partial_p \omega_k - \Gamma_{pk}^l \omega_l) - \Gamma_{ik}^p(\partial_j \omega_p - \Gamma_{jp}^l \omega_k). \end{aligned} \tag{26.16}$$

Expanding everything out, we can write this formally as

$$\nabla^2 \omega = \partial^2 \omega_k + \Gamma * \partial\omega + (\partial\Gamma + \Gamma * \Gamma) * \omega, \tag{26.17}$$

where $*$ denotes various tensor contractions. It appears that in general

$$\nabla_i \nabla_j \omega \neq \nabla_j \nabla_i \omega. \tag{26.18}$$

The obstruction to commuting covariant derivatives leads to the concept of curvature.

26.3 Gradient, Hessian, and Laplacian

As an example of the above, we consider the Hessian of a function. For $f \in C^1(M, \mathbb{R})$, the *gradient* is defined as

$$\nabla f = \sharp(df), \tag{26.19}$$

which is a vector field. This is standard notation, although in our notation above, $\nabla f = df$, where this ∇ denotes the covariant derivative. The *Hessian* is the $(0,2)$-tensor defined by the double covariant derivative of a function, which by Proposition 26.1 is given by

$$\nabla^2 f(X, Y) = \nabla_X(\nabla_Y f) - \nabla_{\nabla_X Y} f = X(Yf) - (\nabla_X Y)f. \tag{26.20}$$

In components, this formula is

$$\nabla^2 f(\partial_i, \partial_j) = \nabla_i \nabla_j f = \partial_i \partial_j f - \Gamma^k_{ij}(\partial_k f). \tag{26.21}$$

The symmetry of the Hessian

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X), \tag{26.22}$$

then follows easily from the symmetry of the Riemannian connection. Notice that no curvature terms appear in this formula, which happens only in this special case.

The Laplacian of a function is the trace of the Hessian when considered as an endomorphism,

$$\Delta f = tr\big(X \mapsto \sharp(\nabla^2 f(X, \cdot))\big), \tag{26.23}$$

so in coordinates is given by

$$\Delta f = g^{ij} \nabla_i \nabla_j f. \tag{26.24}$$

This turns out to be equal to

$$\boxed{\Delta f = \frac{1}{\sqrt{\det(g)}} \partial_i \big(g^{ij} \partial_j f \sqrt{\det(g)}\big).} \tag{26.25}$$

Exercise 26.2. Prove (26.25) (hint: use Jacobi's formula for the derivative of a determinant).

27 Lecture 27

27.1 Curvature in the tangent bundle

The curvature tensor is defined by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{27.1}$$

for vector fields X, Y, and Z. We define

$$Rm(X,Y,Z,W) \equiv -\langle \mathcal{R}(X,Y)Z, W\rangle. \tag{27.2}$$

We will refer to $\mathcal{R}$ as the curvature tensor of type $(1,3)$ and to Rm as the curvature tensor of type $(0,4)$.

The algebraic symmetries are:

$$\begin{aligned}
\mathcal{R}(X,Y)Z &= -\mathcal{R}(Y,X)Z && (27.3)\\
0 &= \mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y && (27.4)\\
Rm(X,Y,Z,W) &= -Rm(X,Y,W,Z) && (27.5)\\
Rm(X,Y,W,Z) &= Rm(W,Z,X,Y). && (27.6)
\end{aligned}$$

In a coordinate system we define quantities ${R_{ijk}}^l$ by

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = {R_{ijk}}^l \partial_l, \tag{27.7}$$

or equivalently,

$$\mathcal{R} = {R_{ijk}}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l. \tag{27.8}$$

Define quantities R_{ijkl} by

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l), \tag{27.9}$$

or equivalently,

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{27.10}$$

Then

$$R_{ijkl} = -\langle \mathcal{R}(\partial_i, \partial_j)\partial_k, \partial_l\rangle = -\langle {R_{ijk}}^m \partial_m, \partial_l\rangle = -{R_{ijk}}^m g_{ml}. \tag{27.11}$$

Equivalently,

$$R_{ijlk} = {R_{ijk}}^m g_{ml}, \tag{27.12}$$

that is, we lower the upper index to the *third* position. In coordinates, the algebraic symmetries of the curvature tensor are

$$\begin{aligned}
{R_{ijk}}^l &= -{R_{jik}}^l && (27.13)\\
0 &= {R_{ijk}}^l + {R_{jki}}^l + {R_{kij}}^l && (27.14)\\
R_{ijkl} &= -R_{ijlk} && (27.15)\\
R_{ijkl} &= R_{klij}. && (27.16)
\end{aligned}$$

Of course, we can write the first 2 symmetries as a $(0,4)$ tensor,

$$\begin{aligned}
R_{ijkl} &= -R_{jikl} && (27.17)\\
0 &= R_{ijkl} + R_{jkil} + R_{kijl}. && (27.18)
\end{aligned}$$

Note that using (27.16), the algebraic Bianchi identity (27.18) may be written as

$$0 = R_{ijkl} + R_{iklj} + R_{iljk}. \tag{27.19}$$

We next compute the curvature tensor in coordinates.

$$\begin{aligned}
\mathcal{R}(\partial_i, \partial_j)\partial_k &= R_{ijk}{}^{l}\partial_l \\
&= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \partial_i(\Gamma^l_{jk})\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m - \partial_j(\Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m \\
&= \Big(\partial_i(\Gamma^l_{jk}) + \Gamma^m_{jk}\Gamma^l_{im} - \partial_j(\Gamma^l_{ik}) - \Gamma^m_{ik}\Gamma^l_{jm}\Big)\partial_l,
\end{aligned} \tag{27.20}$$

which is the formula

$$\boxed{R_{ijk}{}^{l} = \partial_i(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{ik}) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}} \tag{27.21}$$

Fix a point p. Exponential coordinates around p form a normal coordinate system at p. That is $g_{ij}(p) = \delta_{ij}$, and $\partial_k g_{ij}(p) = 0$, which is equivalent to $\Gamma^k_{ij}(p) = 0$. The Christoffel symbols are

$$\Gamma^l_{jk} = \frac{1}{2}g^{lm}\Big(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\Big). \tag{27.22}$$

In normal coordinates at the point p,

$$\partial_i\Gamma^l_{jk} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} + \partial_i\partial_j g_{km} - \partial_i\partial_m g_{jk}\Big). \tag{27.23}$$

We then have at p

$$R_{ijk}{}^{l} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} - \partial_i\partial_m g_{jk} - \partial_j\partial_k g_{im} + \partial_j\partial_m g_{ik}\Big). \tag{27.24}$$

Lowering an index, we have at p

$$\begin{aligned}
R_{ijkl} &= -\frac{1}{2}\Big(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}\Big) \\
&= -\frac{1}{2}\Big(\partial^2 \owedge g\Big).
\end{aligned} \tag{27.25}$$

The $\owedge$ symbol is the Kulkarni-Nomizu product, which takes 2 symmetric $(0,2)$ tensors and gives a $(0,4)$ tensor with the same algebraic symmetries of the curvature tensor, and is defined by

$$\begin{aligned}
A \owedge B(X,Y,Z,W) =& A(X,Z)B(Y,W) - A(Y,Z)B(X,W) \\
& - A(X,W)B(Y,Z) + A(Y,W)B(X,Z).
\end{aligned}$$

To remember: the first term is $A(X,Z)B(Y,W)$, skew symmetrize in X and Y to get the second term. Then skew-symmetrize both of these in Z and W.

27.2 Sectional curvature, Ricci tensor, and scalar curvature

Let $\Pi \subset T_pM$ be a 2-plane, and let $X_p, Y_p \in T_pM$ span Π. Then

$$K(\Pi) = \frac{Rm(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{g(\mathcal{R}(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{27.26}$$

is independent of the particular chosen basis for Π, and is called the *sectional curvature* of the 2-plane Π. The sectional curvatures in fact determine the full curvature tensor:

Proposition 27.1. *Let Rm and Rm' be two curvature tensors of type $(0,4)$ which satisfy $K(\Pi) = K'(\Pi)$ for all 2-planes Π, then $Rm = Rm'$.*

From this proposition, if $K(\Pi) = k_0$ is constant for all 2-planes Π, then we must have

$$Rm(X,Y,Z,W) = k_0\Big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\Big). \tag{27.27}$$

That is

$$Rm = \frac{k_0}{2} g \owedge g. \tag{27.28}$$

In coordinates, this is

$$R_{ijkl} = k_0(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{27.29}$$

We define the *Ricci tensor* as the $(0,2)$-tensor

$$Ric(X,Y) = tr(U \to \mathcal{R}(U,X)Y). \tag{27.30}$$

We clearly have

$$Ric(X,Y) = Ric(Y,X), \tag{27.31}$$

so $Ric \in \Gamma(S^2(T^*M))$. We let R_{ij} denote the components of the Ricci tensor,

$$Ric = R_{ij} dx^i \otimes dx^j, \tag{27.32}$$

where $R_{ij} = R_{ji}$. From the definition,

$$R_{ij} = R_{lij}{}^{l} = g^{lm}R_{limj}. \tag{27.33}$$

Notice for a space of constant curvature, we have

$$R_{jl} = g^{ik}R_{ijkl} = k_0 g^{ik}(g_{ik}g_{jl} - g_{jk}g_{il}) = (n-1)k_0 g_{jl}, \tag{27.34}$$

or invariantly

$$Ric = (n-1)k_0 g. \tag{27.35}$$

The *Ricci endomorphism* is defined by

$$Rc(X) \equiv \sharp\Big(Ric(X,\cdot)\Big). \tag{27.36}$$

The *scalar curvature* is defined as the trace of the Ricci endomorphism

$$R \equiv tr(X \to \sharp Ric(X,\cdot)). \tag{27.37}$$

In coordinates,

$$R = g^{pq}R_{pq} = g^{pq}g^{lm}R_{lpmq}. \tag{27.38}$$

Note for a space of constant curvature k_0,

$$R = n(n-1)k_0. \tag{27.39}$$

Exercise 27.2. If $n = 2$, then we necessarily have

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{jk}g_{il}) \tag{27.40}$$

$$R_{ij} = Kg_{ij} \tag{27.41}$$

$$R = 2K, \tag{27.42}$$

for a function $K : M \to \mathbb{R}$, which is called the *Gaussian curvature*. So the scalar curvature determines the full curvature tensor in dimension 2.

Remark 27.3. It turns out that if $n = 3$, then the Ricci tensor determines the full curvature tensor. If $n \geq 4$, then the curvature tensor is not determined by the Ricci tensor. There is another component called the Weyl curvature, which we will discuss later.

27.3 Commuting covariant derivatives

Let $X, Y, Z \in \Gamma(TM)$, and compute using Proposition 26.1

$$\begin{aligned} \nabla^2 Z(X,Y) - \nabla^2 Z(Y,X) &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_X Y - \nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \\ &= \mathcal{R}(X,Y)Z, \end{aligned} \tag{27.43}$$

which is just the definition of the curvature tensor. In coordinates,

$$\boxed{\nabla_i \nabla_j Z^k = \nabla_j \nabla_i Z^k + R_{ijm}{}^k Z^m.} \tag{27.44}$$

We extend this to $(p,0)$-tensor fields:

$$\begin{aligned} &\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X) \\ &= \nabla_X(\nabla_Y(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_X Y}(Z_1 \otimes \cdots \otimes Z_p) \\ &\quad - \nabla_Y(\nabla_X(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_Y X}(Z_1 \otimes \ldots \otimes Z_p \\ &= \nabla_X\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p\Big) - \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_X Y} Z_i \otimes \cdots \otimes Z_p \\ &- \nabla_Y\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p\Big) + \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_Y X} Z_i \otimes \cdots \otimes Z_p. \end{aligned} \tag{27.45}$$

With a slight abuse of notation, this may be rewritten as

$$\begin{aligned}
&\nabla^2(Z_1\otimes\cdots\otimes Z_p)(X,Y)-\nabla^2(Z_1\otimes\cdots\otimes Z_p)(Y,X)\\
&=\sum_{j=1}^{p}\sum_{i=1,i\neq j}^{p} Z_1\otimes\nabla_X Z_j\otimes\cdots\nabla_Y Z_i\otimes\cdots\otimes Z_p\\
&\qquad-\sum_{j=1}^{p}\sum_{i=1,i\neq j}^{p} Z_1\otimes\nabla_Y Z_j\otimes\cdots\nabla_X Z_i\otimes\cdots\otimes Z_p\\
&\qquad\qquad+\sum_{i=1}^{p} Z_1\otimes\cdots\otimes(\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]})Z_i\otimes\cdots\otimes Z_p\\
&=\sum_{i=1}^{p} Z_1\otimes\cdots\otimes\mathcal{R}(X,Y)Z_i\otimes\cdots\otimes Z_p.
\end{aligned}\tag{27.46}$$

In coordinates, this is

$$\boxed{\nabla_i\nabla_j Z^{i_1\ldots i_p}=\nabla_j\nabla_i Z^{i_1\ldots i_p}+\sum_{k=1}^{p} R_{ijm}{}^{i_k} Z^{i_1\ldots i_{k-1} m i_{k+1}\ldots i_p}.}\tag{27.47}$$

Proposition 27.4. *For a* 1*-form* ω*, we have*

$$\nabla^2\omega(X,Y,Z)-\nabla^2\omega(Y,X,Z)=\omega(\mathcal{R}(Y,X)Z).\tag{27.48}$$

Proof. Using Proposition 26.1, we compute

$$\begin{aligned}
&\nabla^2\omega(X,Y,Z)-\nabla^2\omega(Y,X,Z)\\
&=\nabla_X(\nabla_Y\omega)(Z)-(\nabla_{\nabla_X Y}\omega)(Z)-\nabla_Y(\nabla_X\omega)(Z)-(\nabla_{\nabla_Y X}\omega)(Z)\\
&=X(\nabla_Y\omega(Z))-\nabla_Y\omega(\nabla_X Z)-\nabla_X Y(\omega(Z))+\omega(\nabla_{\nabla_X Y}Z)\\
&\quad-Y(\nabla_X\omega(Z))+\nabla_X\omega(\nabla_Y Z)+\nabla_Y X(\omega(Z))-\omega(\nabla_{\nabla_Y X}Z)\\
&=X(\nabla_Y\omega(Z))-Y(\omega(\nabla_X Z))+\omega(\nabla_Y\nabla_X Z)-\nabla_X Y(\omega(Z))+\omega(\nabla_{\nabla_X Y}Z)\\
&\quad-Y(\nabla_X\omega(Z))+X(\omega(\nabla_Y Z))-\omega(\nabla_X\nabla_Y Z)+\nabla_Y X(\omega(Z))-\omega(\nabla_{\nabla_Y X}Z)\\
&=\omega\Big(\nabla_Y\nabla_X Z-\nabla_X\nabla_Y Z+\nabla_{[X,Y]}Z\Big)+X(\nabla_Y\omega(Z))-Y(\omega(\nabla_X Z))-\nabla_X Y(\omega(Z))\\
&\quad-Y(\nabla_X\omega(Z))+X(\omega(\nabla_Y Z))+\nabla_Y X(\omega(Z)).
\end{aligned}\tag{27.49}$$

The last six terms are

$$\begin{aligned}
&X(\nabla_Y\omega(Z))-Y(\omega(\nabla_X Z))-\nabla_X Y(\omega(Z))\\
&-Y(\nabla_X\omega(Z))+X(\omega(\nabla_Y Z))+\nabla_Y X(\omega(Z))\\
&=X\Big(Y(\omega(Z))-\omega(\nabla_Y Z)\Big)-Y(\omega(\nabla_X Z))-[X,Y](\omega(Z))\\
&-Y\Big(X(\omega(Z))-\omega(\nabla_X Z)\Big)+X(\omega(\nabla_Y Z))\\
&=0.
\end{aligned}\tag{27.50}$$

□

Remark 27.5. It would have been a lot easier to assume we were in normal coordinates, and ignore terms involving first covariant derivatives of the vector fields, but we did the above for illustration.

In coordinates, this formula becomes

$$\boxed{\nabla_i \nabla_j \omega_k = \nabla_j \nabla_i \omega_k - R_{ijk}{}^{p} \omega_p.} \tag{27.51}$$

As above, we can extend this to $(0, s)$ tensors using the tensor product, in an almost identical calculation to the $(r, 0)$ tensor case. Finally, putting everything together, the analogous formula in coordinates for a general (r, s)-tensor T is

$$\boxed{\nabla_i \nabla_j T^{i_1 \dots i_r}_{j_1 \dots j_s} = \nabla_j \nabla_i T^{i_1 \dots i_r}_{j_1 \dots j_s} + \sum_{k=1}^{r} R_{ijm}{}^{i_k} T^{i_1 \dots i_{k-1} m i_{k+1} \dots i_r}_{j_1 \dots j_s} - \sum_{k=1}^{s} R_{ijj_k}{}^{m} T^{i_1 \dots i_r}_{j_1 \dots j_{k-1} m j_{k+1} \dots j_s}.} \tag{27.52}$$

28 Lecture 28

28.1 Conformal geometry and complex geometry

In real dimension 2, there is a close relation between complex geometry and conformal geometry.

Definition 28.1. Given a manifold M, Riemannian metrics g and $\tilde{g}$ are *conformal* if there exists $u : M \to \mathbb{R}$ such that $\tilde{g} = e^{-2u} g$. The *conformal class* of a Riemannian metric g is

$$[g] = \{e^{-2u} g \mid u \in C^{\infty}(M, \mathbb{R})\} \tag{28.1}$$

A mapping between manifolds $f : (M_1, g_1) \to (M_2, g_2)$ is a conformal transformation if $f^* g_2$ is conformal to g_1.

In real dimension 2, there is a close relation between complex geometry and conformal geometry.

Proposition 28.2. *A almost complex structure on a Riemann surface ($J : TM \to TM$, $J^2 = -Id$) is equivalent to an oriented conformal structure. More precisely, let (M, J) be a Riemann surface with almost complex structure J. Choose any Riemannian metric g which is compatible with J, that is, $g(JX, JY) = g(X, Y)$. Then $(M, [g])$ with the complex orientation is an oriented conformal structure. Conversely, given an oriented conformal class $(M, [g])$ then there is a unique almost complex structure J with the property that any metric in the respective conformal class is compatible with the complex structure J.*

Proof. For one direction, we need to show that any 2 compatible metrics are conformal. If $g(JX, JY) = g(X, Y)$ and $h(JX, JY) = h(X, Y)$. Choose an basis $\{e_1, e_2\}$ of $T_p M$ such that $J(e_1) = e_2$ and $J(e_2) = -e_1$. Then

$$g_{11}(p) = g(e_1, e_1) = g(Je_1, Je_1) = g(e_2, e_2) = g_{22}(p), \tag{28.2}$$

and

$$g_{12}(p) = g(e_1, e_2) = g(Je_1, Je_2) = g(e_2, -e_1) = -g_{12}(p). \tag{28.3}$$

Since $g_{11}(p) = g_{22}(p)$ and $g_{12}(p) = g_{21}(p) = 0$, we have

$$g(p) = g_{11}(p)((e^1)^2 + (e^2)^2), \tag{28.4}$$

where $\{e^1, e^2\}$ is the dual basis to $\{e_1, e_2\}$. This argument equally applies to h, to show that

$$h(p) = h_{11}(p)((e^1)^2 + (e^2)^2). \tag{28.5}$$

Since this is true at any point, it shows that g and h are conformal.

Conversely, given an oriented conformal class $[g]$, let ω be an oriented volume form, which is a nowhere-vanishing 2-form. We can choose a metric $g \in [g]$ uniquely with the property that the volume form of g, $dV_g = \omega$. Then we define $J : TM \to TM$ by

$$g(X, Y) = \omega(X, JY). \tag{28.6}$$

Note that this defines J uniquely since ω is a non-degenerate 2-form. If we do the same procedure for $\tilde{g} = e^{-2u}g$, then $dV_{\tilde{g}} = e^{-2u}g$ (since the volume form is like $\sqrt{\det(g)}dx \wedge dy$), so we obtain the same J. To see that $J^2 = -Id$, choose an oriented orthonormal basis $\{e_1, e_2\}$ for T_pM. Then $\omega = dV_g = e_1 \wedge e_2$. This implies that

$$1 = g(e_1, e_1) = \omega(e_1, Je_1) \tag{28.7}$$
$$0 = g(e_1, e_2) = \omega(e_1, Je_2) \tag{28.8}$$
$$0 = g(e_2, e_1) = \omega(e_2, Je_1) \tag{28.9}$$
$$1 = g(e_2, e_2) = \omega(e_2, Je_2), \tag{28.10}$$

which implies that $Je_1 = e_2$ and $Je_2 = -e_1$, so $J^2 = -I$ is an almost complex structure. $\square$

We also have the following result for mapping under this correspondence.

Proposition 28.3. *Complex structures (M_1, J_1) and (M_2, J_2) on Riemann surfaces are biholomorphic if and only if the corresponding conformal structures $(M, [g_1])$ and $(M_2, [g_2])$ are orientation preserving conformally equivalent.*

Proof. Given (M, J), choose a holomorphic coordinate system $\{x, y\}$. As seen in the above proof, the conformal class of a compatible metric in this coordinate system is the Euclidean conformal class $[g_{Euc}]$. So the statement reduces to proving that for $U \subset \mathbb{C}$, $f : U \to \mathbb{C}$ is holomorphic if and only if it is orientation-preserving and conformal (with respect to the Euclidean metric). Writing $f = u + iv$, then Cauchy-Riemann equations are

$$u_x = v_y, \ u_y = -v_x. \tag{28.11}$$

The equations for conformality are

$$f^* g_{Euc} = e^{-2u} g_{Euc}. \tag{28.12}$$

The Jacobian of f is given by

$$f_* = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{28.13}$$

The left hand side is

$$f^* g_{Euc}(\partial_x, \partial_x) = g_{Euc}(f_*\partial_x, f_*\partial_x) = g_{Euc}(u_x\partial_x + v_x p_y, u_x\partial_x + v_x\partial_y) = u_x^2 + v_x^2 \tag{28.14}$$
$$f^* g_{Euc}(\partial_x, \partial_y) = g_{Euc}(f_*\partial_x, f_*\partial_y) = g_{Euc}(u_x\partial_x + v_x p_y, u_y\partial_x + v_y\partial_y) = u_x u_y + v_x v_y \tag{28.15}$$
$$f^* g_{Euc}(\partial_y, \partial_y) = g_{Euc}(f_*\partial_y, f_*\partial_y) = g_{Euc}(u_y\partial_x + v_y\partial_y, u_y\partial_x + v_y\partial_y) = u_y^2 + v_y^2. \tag{28.16}$$

If f is holomorphic, then this clearly implies that f is conformal. Conversely, if f is conformal and orientation preserving, we obtain

$$u_x^2 + v_x^2 = u_y^2 + v_y^2 \tag{28.17}$$
$$u_x u_y + v_x v_y = 0 \tag{28.18}$$

This says that the vectors $\vec{v} = (u_x, v_x)$ and $\vec{w} = (u_y, v_y)$ are orthogonal and have the same norm; they must differ by a 90 degree rotation. So there are 2 possibilities:

$$(u_x, v_x) = (v_y, -u_y) \text{ or } (u_x, v_x) = (-v_y, u_y), \tag{28.19}$$

which is

$$f_* = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} \text{ or } f_* = \begin{pmatrix} u_x & u_y \\ u_y & -u_x \end{pmatrix} \tag{28.20}$$

The mapping f being orientation preserving is equivalent to the Jacobian determinant being positive, so we see that we must be in the first case, which are exactly the Cauchy-Riemann equations. □

Remark 28.4. The second case corresponds to anti-holomorphic $\partial f = 0$. That is, orientation-reversing and conformal is equivalent to anti-holomorphic.

28.2 Conformal transformation of Gauss curvature

Proposition 28.5. *Under the conformal change $\tilde{g} = e^{-2u} g$, the Christoffel symbols transform as*

$$\tilde{\Gamma}^i_{jk} = g^{il}\Big(-(\partial_j u) g_{lk} - (\partial_k u) g_{lj} + (\partial_l u) g_{jk} \Big) + \Gamma^i_{jk}. \tag{28.21}$$

Invariantly,

$$\tilde{\nabla}_X Y = \nabla_X Y - du(X)Y - du(Y)X + g(X,Y)\nabla u. \tag{28.22}$$

Proof. Using (25.35), we compute

$$\begin{aligned}
\tilde{\Gamma}^i_{jk} &= \frac{1}{2}\tilde{g}^{il}\Big(\partial_j \tilde{g}_{kl} + \partial_k \tilde{g}_{jl} - \partial_l \tilde{g}_{jk}\Big)\\
&= \frac{1}{2}e^{2u}g^{il}\Big(\partial_j(e^{-2u}g_{kl}) + \partial_k(e^{-2u}g_{jl}) - \partial_l(e^{-2u}g_{jk})\Big)\\
&= \frac{1}{2}e^{2u}g^{il}\Big(- 2e^{-2u}(\partial_j u)g_{kl} - 2e^{-2u}(\partial_k u)e^{-2u}g_{jl} + 2e^{-2u}(\partial_l u)g_{jk}\\
&\qquad + e^{-2u}\partial_j(g_{kl}) + e^{-2u}\partial_k(g_{jl}) - e^{-2u}\partial_l(g_{jk})\Big)\\
&= g^{il}\Big(- (\partial_j u)g_{kl} - (\partial_k u)g_{jl} + (\partial_l u)g_{jk}\Big) + \Gamma^i_{jk}.
\end{aligned} \tag{28.23}$$

This is easily seen to be equivalent to the invariant expression. □

Proposition 28.6. *If $n = 2$, and $\tilde{g} = e^{-2u}g$, the conformal Gauss curvature equation is*

$$\Delta_g u + K_g = \tilde{K}e^{-2u}, \tag{28.24}$$

where $\tilde{K} = K_{\tilde{g}}$.

Proof. We will use the following formulas

$$\tilde{\Gamma}^i_{jk} = g^{il}\Big(- (\partial_j u)g_{lk} - (\partial_k u)g_{lj} + (\partial_l u)g_{jk}\Big) + \Gamma^i_{jk}. \tag{28.25}$$

and

$$R_{ijk}{}^{l} = \partial_i(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{ik}) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}. \tag{28.26}$$

Recall that if $n = 2$, then we necessarily have

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{28.27}$$

Choose a normal coordinate system $\{x^1, x^2\}$ for the g metric at a point p, then

$$R_{1212} = K(p). \tag{28.28}$$

The above formulas then yield

$$K(p) = R_{1212} = R_{122}{}^{1} = \partial_1(\Gamma^1_{22}) - \partial_2(\Gamma^1_{12}). \tag{28.29}$$

Next, we have

$$\tilde{R}_{1212} = \tilde{K}(p)(\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{21}\tilde{g}_{12}) \tag{28.30}$$

But

$$\tilde{g}_{ij} = e^{-2u}g_{ij}, \tag{28.31}$$

so we get that

$$\tilde{R}_{1212} = \tilde{K}(p)e^{-4u}, \tag{28.32}$$

or

$$\tilde{K}(p) = \tilde{R}_{1212}e^{4u} \tag{28.33}$$

But from above,

$$\tilde{R}_{1212} = \tilde{g}_{1k}\tilde{R}_{122}{}^{k} = e^{-2u}\tilde{R}_{122}{}^{1}. \tag{28.34}$$

Combining these,

$$\tilde{K}(p) = e^{2u}\tilde{R}_{122}{}^{1}. \tag{28.35}$$

Now we use the above curvature formula

$$e^{-2u}\tilde{K}(p) = \tilde{R}_{122}{}^{1} = \partial_1(\tilde{\Gamma}^1_{22}) - \partial_2(\tilde{\Gamma}^1_{12}) + \tilde{\Gamma}^1_{1m}\tilde{\Gamma}^m_{22} - \tilde{\Gamma}^1_{2m}\tilde{\Gamma}^m_{12}. \tag{28.36}$$

Using the fact that the coordinates are normal for g, we compute the first term

$$\partial_1(\tilde{\Gamma}^1_{22}) = \partial_1^2 u + \partial_1\Gamma^1_{22} \tag{28.37}$$

The second term is

$$-\partial_2(\tilde{\Gamma}^1_{12}) = \partial_2^2 u - \partial_2\Gamma^1_{12}. \tag{28.38}$$

The third term is

$$\begin{aligned} \tilde{\Gamma}^1_{1m}\tilde{\Gamma}^m_{22} &= (-\partial_1 u g_{1m} - \partial_m u g_{11} + \partial_1 u g_{1m})(-\partial_2 u g_{m2} - \partial_2 u g_{m2} + \partial_m u g_{22}) && (28.39)\\ &= -\partial_m u(-\partial_2 u g_{m2} - \partial_2 u g_{m2} + \partial_m u g_{22}) && (28.40)\\ &= -(\partial_1 u)^2 + (\partial_2 u)^2. && (28.41) \end{aligned}$$

The fourth term is

$$\begin{aligned} -\tilde{\Gamma}^1_{2m}\tilde{\Gamma}^m_{12} &= -(-\partial_2 u g_{1m} - \partial_m u g_{12} + \partial_1 u g_{2m})(-\partial_1 u g_{m2} - \partial_2 u g_{m1} + \partial_m u g_{12}) && (28.42)\\ &= -(-\partial_2 u g_{1m} + \partial_1 u g_{2m})(-\partial_1 u g_{m2} - \partial_2 u g_{m1}) && (28.43)\\ &= -(\partial_2 u)^2 + (\partial_1 u)^2 && (28.44) \end{aligned}$$

Adding all these up yields

$$e^{-2u}\tilde{K}(p) = \partial_1^2 u + \partial_2^2 u + \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12}. \tag{28.45}$$

Since the coordinates are normal, we have

$$(\Delta u)(p) = g^{ij}\nabla_i\nabla_j u = g^{ij}(p)(\partial_i\partial_j u - \Gamma^k_{ij}(p)\partial_k u) = \partial_1^2 u + \partial_2^2 u, \tag{28.46}$$

which yields the formula

$$e^{-2u}\tilde{K} = \Delta u + K, \tag{28.47}$$

and we are done. □

29 Lecture 29

29.1 Structure group reduction

We give another fancier proof of the equivalence between almost complex geometry and conformal geometry in real dimension 2.

Proposition 29.1. *A Riemann surface (M, J) with an almost complex structure J is equivalent to an oriented conformal structure $(M, [g])$.*

Proof. A complex structure is a reduction of the structure group of the frame bundle to $GL(1, \mathbb{C}) \subset GL(2, \mathbb{R})$. The explicit map is

$$a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{29.1}$$

An oriented conformal class is a reduction to $CO(2, \mathbb{R}) \subset GL(2, \mathbb{R})$, where

$$CO(2, \mathbb{R}) = \mathbb{R}_+ \times SO(2, \mathbb{R}) = \{\lambda \cdot A | \lambda \in \mathbb{R}_+, A \in SO(2, \mathbb{R})\}, \tag{29.2}$$

and it is easy to see that this is the same as image in (29.1) (hint: divide the matrix in (29.1) by $\sqrt{a^2 + b^2}$). □

29.2 Isothermal coordinates

Definition 29.2. Let (M, g) be a Riemannian surface. A coordinate system $\phi : U \to \mathbb{R}^2$ is called *isothermal* if

$$\phi_* g = e^{\lambda(x,y)}(dx^2 + dy^2), \tag{29.3}$$

for some function $\lambda : \phi(U) \to \mathbb{R}$. Equivalently, (U, g) is conformally equivalent to a domain in $\mathbb{R}^2$ with the Euclidean metric.

We can alternatively describe this as follows.

Proposition 29.3. *Fix $p \in M$. Then there exists an isothermal coordinate system around p iff and only if there exists a neighborhood U of p and $u : U \to \mathbb{R}$ such that*

$$\Delta u + K_g = 0. \tag{29.4}$$

Proof. If there exists an isothermal coordinate system, then the claim follows from the conformal transformation formula for the Gauss curvature. Conversely, given a solution of (29.4) locally, then metric $\tilde{g} = e^{-2u} g$ satisfies $K_{\tilde{g}} = 0$, that is, it is a flat metric. From basic Riemannian geometry, there exists a local chart $\phi : U \to \mathbb{R}^2$ so that $\phi_* \tilde{g} = dx^2 + dy^2$, and we are done. □

Corollary 29.4. *Let (M, J) be an almost complex structure on a oriented Riemann surface. Choose any compatible metric g. Then an oriented isothermal coordinate system for g near a point p is a holomorphic coordinate system.*

Proof. If $\phi : (U, g) \to (\mathbb{R}^2, g_{Euc})$ is isothermal, then ϕ is conformal. Then ϕ_* is pseudo-holomorphic by Proposition 28.3. □

Remark 29.5. Consequently, the Newlander-Nirenberg problem in real dimension 2 is equivalent to the local solvability of (29.4). We will discuss the solution of this next.

29.3 Negative scalar curvature

Let's first recall the inverse function theorem in Banach spaces.

Lemma 29.6. *Let $\mathscr{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be a C^1-map between two Banach spaces such that $\mathscr{F}(x) = \mathscr{F}(0) + \mathscr{L}(x) + \mathscr{Q}(x)$, where the operator $\mathscr{L} : \mathcal{B}_1 \to \mathcal{B}_2$ is linear and $\mathscr{Q}(0) = 0$. Assume that*

1. *$\mathscr{L}$ is an isomorphism with inverse T satisfying $\|T\| \leq C_1$,*
2. *there are constants $r > 0$ and $C_2 > 0$ with $r < \frac{1}{3C_1C_2}$ such that*
   (a) *$\|\mathscr{Q}(x) - \mathscr{Q}(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x - y\|_{\mathcal{B}_1}$ for all $x, y \in B_r(0) \subset \mathcal{B}_1$,*
   (b) *$\|\mathscr{F}(0)\|_{\mathcal{B}_2} \leq \frac{r}{3C_1}$.*

Then there exists a unique solution to $\mathscr{F}(x) = 0$ in $\mathcal{B}_1$ such that

$$\|x\|_{\mathcal{B}_1} \leq 3C_1 \cdot \|\mathscr{F}(0)\|_{\mathcal{B}_2}. \tag{29.5}$$

Using this, we will prove the following.

Proposition 29.7. *If (M, g) is compact, and $R < 0$, then there exists conformal metric $\tilde{g} = e^{-2u}g$ with $\tilde{R} = -1$.*

Proof. We would like to solve the equation

$$\Delta_g u + K_g = -e^{-2u}. \tag{29.6}$$

Let us first assume that there exists a C^2 solution of (29.6). Let $p \in M$ be a point where u attains a its global maximum. Then $\nabla u(p) = 0$, so

$$\Delta_g u(p) = g^{ij}(\partial_i\partial_j u - \Gamma_{ij}^k \partial_k u)(p) = g^{ij}\partial_i\partial_j u \leq 0, \tag{29.7}$$

since g^{ij} is positive definite and p is a maximum point. Then (29.6) evaluated at p becomes

$$K_g(p) \geq -e^{-2u(p)} \tag{29.8}$$

This implies that

$$u(p) \leq -\frac{1}{2}\log(-K_g(p)), \tag{29.9}$$

which gives an *a priori* upper bound on u. Similarly, by evaluating a a global minimum point q, we obtain

$$u(q) \geq -\frac{1}{2}\log(-K_g(q)), \tag{29.10}$$

which gives an a priori strictly positive lower bound on u. We have shown there exists a constant C_0 so that $\|v\|_{C^0} < C_0$. The standard elliptic estimate says that there exists a constant C, depending only on the background metric, such that (see [GT01, Chapter 4])

$$\begin{aligned} \|u\|_{C^{1,\alpha}} &\leq C(\|\Delta u\|_{C^0} + \|u\|_{C^0}) \\ &\leq C(\| - K_g - e^{-2u})\|_{C^0} + CC_0 \leq C_1, \end{aligned} \tag{29.11}$$

where C_1 depends only upon the background metric. Applying elliptic estimates again,

$$\|u\|_{C^{2,\alpha}} \leq C(\|\Delta u\|_{C^{0,\alpha}} + \|u\|_{C^{0,\alpha}}) \leq C_3, \tag{29.12}$$

where C_3 depends only upon the background metric.

Let $t \in [0,1]$, and consider the family of equations

$$\Delta u + K_g = \big((1-t)K_g - t\big)e^{-2u}. \tag{29.13}$$

Define an operator $F_t : C^{2,\alpha} \to C^{\alpha}$ by

$$F_t(u) = \Delta u + K_g - \big((1-t)K_g - t\big)e^{-2u}. \tag{29.14}$$

Let $u_t \in C^{2,\alpha}$ satisfy $F_t(u_t) = 0$. The linearized operator at u_t, $L_t : C^{2,\alpha} \to C^{\alpha}$, is given by

$$L_t(h) = \Delta h + 2\big((1-t)K_g - t\big)e^{-2u_t}h. \tag{29.15}$$

Notice that the coefficient h is strictly negative. If $L_t h = 0$, then

$$\begin{aligned} 0 &= \int_M h\Big(\Delta h + 2\big((1-t)K_g - t\big)e^{-2u_t}h\Big)dV_g \\ &= -\int_M |\nabla h|^2 dV_g + 2\int_M \big((1-t)K_g - t\big)e^{-2u_t}h^2 dV_g \end{aligned} \tag{29.16}$$

this implies that $h = 0$, so L_t is injective. Since L is formally self-adjoint, some Fredholm Theory implies that L_t is also surjective (details to be discussed later, since this needs some Sobolev space theory). Consequently, the linearized operator is invertible. Next, define

$$S = \{t \in [0,1] \mid \text{there exists a solution } u_t \in C^{2,\alpha} \text{ of } F_t(u_t) = 0\}. \tag{29.17}$$

Since the linearized operator is invertible, the implicit function theorem implies that S is open. To see this, we assume that there exists a solution $u_t \in C^{2,\alpha}(M)$ of $F_t(u_t) = 0$. Then we want to solve

$$F_{t+\epsilon}(u_{t+\epsilon}) = 0 \tag{29.18}$$

for all t sufficiently small. Let us write $u_{t+\epsilon} = u_t + h_\epsilon$, and define

$$\mathcal{F}_{t+\epsilon}(h) = F_{t+\epsilon}(u_t + h), \tag{29.19}$$

and consider as a mapping

$$\mathcal{F}_{t+\epsilon} : C^{2,\alpha}(M) \to C^{0,\alpha}(M). \tag{29.20}$$

We write out

$$\mathcal{F}_{t+\epsilon}(0) = \mathcal{F}_{t+\epsilon}(0) + \mathcal{L}(h) + \mathcal{Q}(h), \tag{29.21}$$

where $\mathcal{L}$ is the lineariozed operator. First, we have

$$\begin{aligned}\mathcal{F}_{t+\epsilon}(0) = F_{t+\epsilon}(u_t) &= \Delta(u_t) + K_g - \big((1-t-\epsilon)K_g - t - \epsilon\big)e^{-2u_t} \\ &= \epsilon(K_g + 1)e^{-2u_t}.\end{aligned} \tag{29.22}$$

Note that this has arbitrarily small $C^{0,\alpha}$ norm provided ϵ is sufficiently small. Next, we have

$$\mathcal{L}(h) = \Delta h + 2\big((1-t-\epsilon)K_g - t - \epsilon\big)e^{-2u_t}h. \tag{29.23}$$

Note that this is invertible by a similar argument to above, with elliptic estimates showing the inverse is uniformly bounded. Finally, we obtain

$$\mathcal{Q}(h) = -\big((1-t-\epsilon)K_g - t - \epsilon\big)e^{-2u_t}(e^{-2h} - 1 + 2h). \tag{29.24}$$

We need to verify that

$$\|\mathcal{Q}(h_1) - \mathcal{Q}(h_2)\|_{C^{0,\alpha}} \leq C(\|h_1\|_{C^{2,\alpha}} + \|h_1\|_{C^{2,\alpha}})\|h_1 - h_2\|_{C^{2,\alpha}}. \tag{29.25}$$

But this follows from

$$|e^{-2h} + 2h - 1| \leq Ch^2, \tag{29.26}$$

if h is sufficiently small in C^0-norm. So then we can apply the implicit function theorem from above (Lemma 29.6) to conclude that S is open.

To show that S is closed: assume u_{t_i} is a sequence of solutions with $t_i \to t_0$ as $i \to \infty$. The above elliptic estimates imply there exist a constant C_4, independent of t, such that $\|u_{t_i}\|_{C^{2,\alpha}} < C_4$. By Arzela-Ascoli, there exists $u_{t_0} \in C^{2,\alpha'}$ and a subsequence $\{j\} \subset \{i\}$ such that $u_{t_j} \to u_{t_0}$ strongly in $C^{2,\alpha'}$, for $\alpha' < \alpha$. The limit u_{t_0} is a solution at time t_0. By elliptic regularity, $u_{t_0} \in C^{2,\alpha}$. This shows that S is closed. Note that $u = 0$ solves $F_0(0) = 0$. Since the interval $[0,1]$ is connected and S is nonempty, this implies that $S = [0,1]$, and consequently there must exist a solution at $t = 1$. □

30 Lecture 30

30.1 Spaces of constant curvature

Let us recall some basic facts from Riemannian geometry. The following holds in any dimension.

Proposition 30.1. *If (M_1, g_1) and (M_2, g_2) have constant sectional curvature $K_1 = K_2 = k_0$ where k_0 is a constant, then g_1 and g_2 are locally isometric.*

Proof. By an analysis of Jacobi fields, a metric of constant curvature in radial normal coordinates the metric has the form

$$g = \begin{cases} dr^2 + r^2 g_{S^{n-1}} & k_0 = 0 \\ dr^2 + \frac{1}{k_0}\sin^2(\sqrt{k_0}\cdot r)g_{S^{n-1}} & k_0 > 0 \\ dr^2 + \frac{1}{|k_0|}\sinh^2(\sqrt{|k_0|}\cdot r)g_{S^{n-1}} & k_0 < 0 \end{cases}. \tag{30.1}$$

So then obviously any 2 such metrics are locally isometric. □

Obviously, by scaling the metric, there are really just 3 cases, $K = 0$, $K = 1$, and $K = -1$ The case of $K = 0$ is just $(\mathbb{R}^n, g_{Euc})$. The case of $K = 1$ is the round metric on the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Since $\sinh(r) \neq 0$ for $r > 0$, and $\lim_{r\to\infty} \sinh(r) = \infty$, the case of $K = -1$ is a complete metric on $\mathbb{R}^n$. We denote this as H^n and call this the hyperbolic metric.

Proposition 30.2. *Let U be a domain in $\mathbb{R}^n$. Then the metric*

$$\tilde{g} = \big(a|x|^2 + b_i x^i + c\big)^{-2} g_{Euc} \tag{30.2}$$

has constant curvature $K_0 = 4ac - |b|^2$.

Proof. This is just a calculation, and left as an exercise. □

If $K > 0$, then $\tilde{g}$ is defined on all of $\mathbb{R}^n$. The metric

$$\tilde{g} = \frac{4}{(1 + |x|^2)^2} g \tag{30.3}$$

represents the round metric with $K = 1$ on S^n under stereographic projection. If $K < 0$ then the solution is defined on a ball, or the complement of a ball, or a half space. The metric

$$\tilde{g} = \frac{4}{(1 - |x|^2)^2} g \tag{30.4}$$

is the usual ball model of hyperbolic space, and

$$\tilde{g} = \frac{1}{x_n^2} g \tag{30.5}$$

is the upper half space model of hyperbolic space.

In general, we have the following:

Theorem 30.3. *If (M, g) is complete, simply-connected, and has constant sectional curvature $K = 0, 1, -1$ then (M, g) is isometric to Euclidean space, S^n with the round metric, or hyperbolic space H^n.*

Proof. We use some basic Riemannian geometry: the exponential mapping $\exp_p : T_pM \to M$ is surjective due to completeness. If $K \leq 0$, there are no conjugate points, so $\exp_p$ has no critical points, so this mapping is a covering space. If M is simply connected, then this must be a diffeomorphism. So then the metric is given by the above model metrics globally. In the case of $K = 1$, since $K > 0$, M must be compact by Meyer's Theorem. It is not hard to show that $\exp_p$ will give a local isometry $\phi : S^n \to M$ where S^n has the round metric. This must be a covering mapping, so will be a diffeomorphism since M is simply connected. □

30.2 Uniformization on S^2

Since the conformal group of (S^2, g_S), where g_S is the round metric, is noncompact, we cannot hope to prove existence of a constant curvature metric by a compactness argument as in the $k \geq 1$ case. However, there is a trick to solve this case using only linear theory.

Theorem 30.4. *If (M, g) is a Riemann surface of genus 0, then g is conformal to (S^2, g_S).*

Proof. We remove a point p from M, and consider the manifold $(M \setminus \{p\}, g)$. We want to find a conformal factor $u : M \setminus \{p\} \to \mathbb{R}$ such that $\tilde{g} = e^{-2u} g$ is flat. The equation for this is

$$\Delta u = -K. \tag{30.6}$$

However, by the Gauss-Bonnet theorem, the right hand side has integral 4π, so this equation has no smooth solution. But we will find a solution u on $M \setminus \{p\}$ so that $u = O(\log(r))$ and $r \to 0$, where $r(x) = d(p, x)$. Let ϕ be a smooth cutoff function satisfying

$$\phi = \begin{cases} 1 & r \leq r_0 \\ 0 & r \geq 2r_0 \end{cases}, \tag{30.7}$$

and $0 \leq \phi \leq 1$, for r_0 very small. Consider the function $f = \Delta(\phi \log(r))$. Computing in normal coordinates, near p we have

$$\begin{aligned} \Delta f &= \frac{1}{\sqrt{\det(g)}} \partial_i (g^{ij} u_j \sqrt{\det(g)}) = \frac{1}{\sqrt{\det(g)}} \partial_r (u_r \sqrt{\det(g)}) \\ &= (\log(r))'' + (\log(r))' \frac{(\sqrt{\det(g)})'}{\sqrt{\det(g)}}. \end{aligned}$$

In normal coordinates, we have $\sqrt{\det(g)} = r + O(r^3)$ as $r \to 0$, so we have

$$\Delta f = -\frac{1}{r^2} + \frac{1}{r}\Big(\frac{1 + O(r^2)}{r + O(r^3)}\Big) = -\frac{1}{r^2} + \frac{1}{r^2}\Big(\frac{1 + O(r^2)}{1 + O(r^2)}\Big) = O(1) \tag{30.8}$$

as $r \to 0$.

Next, we compute

$$\int_M f dV = \lim_{\epsilon \to 0} \int_{M \setminus B(p,r)} \Delta(\phi \log(r)) dV = -\lim_{\epsilon \to 0} \int_{S(p,r)} \partial_r (\log(r)) d\sigma = -2\pi.$$

Note the minus sign is due to using the *outward* normal of the domain $M \setminus B(p, r)$. Consequently, we can solve the equation

$$\Delta(u) = -2\Delta(\phi \log(r)) - K, \tag{30.9}$$

by the Gauss-Bonnet Theorem and Fredholm Theory in L^2. Rewriting this as

$$\Delta \tilde{u} = \Delta(u + 2\phi \log(r)) = -K, \tag{30.10}$$

we see that the space $(M \setminus \{p\}, e^{-2\tilde{u}}g)$ is therefore isometric to Euclidean space, since it is clearly complete and simply connected. By the above, we can write

$$g_S = \frac{4}{(1+|x|^2)^2} e^{-2\tilde{u}} g = e^{-2v} g. \tag{30.11}$$

It is easy to see that v is a bounded solution of

$$\Delta v + K = e^{-2v} \tag{30.12}$$

on $M \setminus \{p\}$ and extends to a smooth solution on all of M by elliptic regularity, and we are done. □

30.3 Uniformization

Now we prove the following uniformization theorem.

Theorem 30.5. *Let (M, J) be a compact Riemann surface. If $k = 0$, then (M, J) is biholomorphic to the Riemann sphere S^2. If $k = 1$, then the universal cover is biholomorphic to $\mathbb{C}$. If $k > 1$, then the universal cover is biholomorphic to the unit disc $\Delta_1(0)$.*

Proof. First consider the case of genus $k \geq 2$. By the Gauss-Bonnet theorem

$$\int_M K_g dV_g = 2\pi\chi(M) = 2\pi(2-2k) < 0. \tag{30.13}$$

We first solve the equation

$$\Delta u = -K + \frac{2\pi}{Vol}(2-2k). \tag{30.14}$$

By Fredholm theory, this has a smooth solution since the right hand side has zero mean value. Consider the metric $\tilde{g} = e^{-2u}g$, From (28.24), the Gauss curvature of $\tilde{g}$ is given by

$$\tilde{K} = e^{-2u}(\Delta u + K) = e^{-2u}(\frac{2\pi}{Vol}(2-2k)) < 0, \tag{30.15}$$

since $k \geq 2$. We have found a conformal metric with strictly negative curvature, so Proposition 29.7 yields another conformal metric with constant negative curvature.

For the case $k = 1$, the equation to be solved is

$$\Delta u = -K. \tag{30.16}$$

By the Gauss-Bonnet Theorem, the right hand side has integral zero. Elementary Fredholm Theory gives existence of a unique smooth solution. The universal cover of M is isometric to $\mathbb{R}^2$ with the flat metric.

The case of $k = 0$ we proved in Theorem 30.4, so we are done. □

30.4 Automorphisms

We will also need the following:

Proposition 30.6. *If $f : \mathbb{C} \to \mathbb{C}$ is a biholomorphism, then $f(z) = az+b$ for some constants b and $a \neq 0$. If $f : \Delta_1(0) \to \Delta_1(0)$ is a biholomorphism, then*

$$f(z) = e^{i\theta} \frac{z+c}{1+\bar{c}z}, \tag{30.17}$$

where $\theta \in \mathbb{R}$ and $c \in \Delta_1(0)$.

Proof. In the first case, consider the Taylor expansion $f = \sum a_n z^n$ at the origin. If f has an essential singularity at ∞, then by the Casorati-Weierstrass Theorem, the image of the complement of any ball would be dense, which cannot happen. (To see this, if w_0 is not in the image, and there were a ball $\Delta_\epsilon(w_0)$ also not in the image, then $g(z) = (f(z) - w_0)^{-1}$ would have a removable singularity at the special point. So then $f = w_0 + g^{-1}$ would have a pole of finite order, which contradicts the essential singularity). Therefore, f is a polynomial, which must be linear by the fundamental theorem of algebra.

In the second case, one first checks that these mappings do indeed give automorphisms of the unit disc. Then given $g : \Delta_1(0) \to \Delta_1(0)$, we can compose with one of the automorphisms to assume that $g(0) = 0$. The Schwarz Lemma then implies that $|g'(0)| \leq 1$. Apply the same argument to g^{-1} yields $|(g^{-1})'(0)| \leq 1$. So we must have $|g'(0)| = 1$. Then we have equality in the Schwarz Lemma, which says that $g(z) = e^{i\theta} z$, and we are done. □

Recall that

$$S^2 = \mathbb{CP}^1 = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}\} / \sim \tag{30.18}$$

where $(z_1, z_2) \sim (z_1', z_2')$ if there exists $\lambda \neq 0$ such that $(z_1', z_2') = \lambda(z_1, z_2)$. Then we have

Proposition 30.7. *Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a biholomorphism. Then there exists $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that*

$$f[z_1, z_2] = [az_1 + bz_2, cz_1 + dz_2]. \tag{30.19}$$

Thus the automorphism group of $\mathbb{CP}^1$ is $PGL(2, \mathbb{C})$.

Proof. Obviously, any such mapping gives an automorphism of S^2. By composing with such a Mobius transformation, we can assume that $f([1, 0]) = f([1, 0])$. Then in the coordinate system $\mathbb{CP}^1 \supset \{[z, 1] \mid z \in \mathbb{C}\}$, we see that $f : \mathbb{C} \to \mathbb{C}$. Then by the above, $f(z) = az + b$ in these coordinates, and we are done. □

Corollary 30.8. *If (M, J) is a compact Riemann surface of genus 0, then (M, J) is biholomorphic to $(\mathbb{CP}^1, J_{\mathbb{CP}^1})$.*

If (M, J) is a compact Riemann surface of genus 1, then (M, J) is biholmorphic to the quotient $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$, where the group acts like translations in some lattice

$$L_{\tau_1, \tau_2} = \{m\tau_1 + n\tau_2 \mid m, n \in \mathbb{Z}\} \tag{30.20}$$

with τ_1 and τ_1 linearly independent over $\mathbb{R}$.

If (M, J) is a compact Riemann surface of genus strictly larger that 1, then (M, J) is biholmorphic to $\Delta_1(0)/\Gamma$ where Γ is a discrete co-compact subgroup of $PSL(2, \mathbb{R})$, where

$$PSL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1 \right\} / \pm I, \tag{30.21}$$

acts upon the upper half space model $H^2 = \{Im(z) > 0\}$ by fractional linear transformations

$$z \mapsto \frac{az + b}{cz + d}, \tag{30.22}$$

Proof. The first case was proved in Theorem 30.4. The only automorphisms of $\mathbb{C}$ without fixed points are the translations, so the first case follows. The second case just needs a computation: the mappings in (30.17) are exactly the claimed mappings when we identify the disc with the upper half plane. □

31 Lecture 31

Our next topic is the Newlander-Nirenberg Theorem in higher dimensions. For this, we first need to understand the space of almost complex structures, and we start with the following.

31.1 Endomorphisms

Let $End_{\mathbb{R}}(TM)$ denotes the real endomorphisms of the tangent bundle.

Proposition 31.1. *On an almost complex manifold (M, J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{+}(TM) \oplus End_{-}(TM) \tag{31.1}$$

where the first factor on the left consists of endomorphisms I commuting with J,

$$IJ = JI \tag{31.2}$$

and the second factor consists of endomorphisms I anti-commuting with J,

$$IJ = -JI \tag{31.3}$$

Proof. Given J, we define

$$I_{+} = \frac{1}{2}(I - JIJ) \tag{31.4}$$

$$I_{-} = \frac{1}{2}(I + JIJ). \tag{31.5}$$

Then

$$I_{+}J = \frac{1}{2}(IJ - JIJ^2) = \frac{1}{2}(IJ + JI),$$

and

$$JI_+ = \frac{1}{2}(JI - J^2IJ) = \frac{1}{2}(JI + IJ).$$

Next,

$$I_-J = \frac{1}{2}(IJ + JIJ^2) = \frac{1}{2}(IJ - JI),$$

and

$$JI_- = \frac{1}{2}(JI + J^2IJ) = \frac{1}{2}(JI - IJ).$$

Clearly, $I = I_+ + I_-$. To prove it is a direct sum, if $IJ = JI$ and $IJ = -JI$, then $IJ = 0$ which implies that $I = 0$ since J is invertible.

□

We write down the above in a basis. Choose a real basis $\{e_1, \ldots e_{2n}\}$ such that the complex structure J_0 is given by

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{31.6}$$

Then in matrix terms, the proposition is equivalent to the following decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A+D & B-C \\ C-B & A+D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A-D & B+C \\ B+C & D-A \end{pmatrix}. \tag{31.7}$$

So we have that $End_+(TM) \cong GL(n, \mathbb{C}) \subset GL(2n, \mathbb{C})$ with

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}, \tag{31.8}$$

and $End_-(TM) \cong \overline{GL}(n, \mathbb{C}) \subset GL(2n, \mathbb{C})$ with

$$\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A+iB \\ A-iB & 0 \end{pmatrix}. \tag{31.9}$$

31.2 The space of almost complex structures

We define

$$\mathcal{J}(\mathbb{R}^{2n}) \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n, \mathbb{R}), J^2 = -I_{2n}\}. \tag{31.10}$$

We next give some alternative descriptions of this space.

Proposition 31.2. *The space $\mathcal{J}(\mathbb{R}^{2n})$ is the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, and thus*

$$\dim(\mathcal{J}(\mathbb{R}^{2n})) = 2n^2. \tag{31.11}$$

Proof. We note that $GL(2n,\mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n,\mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$\Phi_A : J \mapsto AJA^{-1}. \tag{31.12}$$

Obviously,

$$(AJA^{-1})^2 = AJA^{-1}AJA^{-1} = AJ^2A^{-1} = -I, \tag{31.13}$$

and

$$\Phi_{AB}(J) = (AB)J(AB)^{-1} = ABJB^{-1}A^{-1} = \Phi_A\Phi_B(J), \tag{31.14}$$

so is indeed a group action (on the left). Given J and J', there exists bases

$$\{e_1,\dots,e_n,Je_1,\dots,Je_n\} \text{ and } \{e'_1,\dots,e'_n,J'e'_1,\dots,J'e'_n\}. \tag{31.15}$$

Define $S \in GL(2n,\mathbb{R})$ by $Se_k = e'_k$ and $S(Je_k) = J'e'_k$. Then $J' = SJS^{-1}$, and the action is therefore transitive. The stabilizer subgroup of J_0 is

$$Stab(J_0) = \{A \in GL(2n,\mathbb{R}) : AJ_0A^{-1} = J_0\}, \tag{31.16}$$

that is, A commutes with J_0. From (31.8) above, this is identified with $GL(n,\mathbb{C})$. □

Given $J \in \mathcal{J}_n$, let $J(t) : (-\epsilon,\epsilon) \to \mathcal{J}_{2n}$ be a smooth path with $J(0) = J$, then differentiation yields

$$-(I_{2n})' = (J \circ J)' = J' \circ J + J \circ J'. \tag{31.17}$$

So letting $J'(0) = I$, we have that

$$IJ + JI = 0. \tag{31.18}$$

Thus we can identify the tangent space at any J as

$$T_J\mathcal{J}_{2n} = \{I \in End(\mathbb{R}^n)\ |IJ + JI = 0\} = End_-(\mathbb{R}^{2n}), \tag{31.19}$$

the space of endomorphisms which anti-commute with J.

31.3 Graph over the reals

Next, we will give another description of $\mathcal{J}(\mathbb{R}^{2n})$. Define

$$\begin{aligned}\mathcal{P}(\mathbb{R}^{2n}) = \{P \subset \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n} \mid dim_{\mathbb{C}}(P) = n,\\ P \text{ is a complex subspace satisfying } P \cap \overline{P} = \{0\}\}.\end{aligned}$$

If we consider $\mathbb{R}^{2n} \otimes \mathbb{C}$, we note that complex conjugation is a well defined complex anti-linear map $\mathbb{R}^{2n} \otimes \mathbb{C} \to \mathbb{R}^{2n} \otimes \mathbb{C}$.

Proposition 31.3. *The space $\mathcal{P}(\mathbb{R}^{2n})$ can be explicitly identified with $\mathcal{J}(\mathbb{R}^{2n})$ by the following. If $J \in \mathcal{J}(\mathbb{R}^{2n})$ then let*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = T^{1,0}(J) \oplus T^{0,1}(J), \tag{31.20}$$

where

$$T^{0,1}(J) = \{X + iJX, X \in \mathbb{R}^{2n}\} = \{-i\}\text{-eigenspace of } J. \tag{31.21}$$

This an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and letting $T^{1,0}(J) = \overline{T^{0,1}(J)}$, we have $T^{1,0} \cap T^{0,1} = \{0\}$.

For the converse, given $P \in \mathcal{P}(\mathbb{R}^{2n})$, then P may be written as a graph over $\mathbb{R}^{2n} \otimes 1$, that is

$$P = \{X' + iJX' \mid X' \in \mathbb{R}^{2n} \subset \mathbb{C}^{2n}\}, \tag{31.22}$$

with $J \in \mathcal{J}(\mathbb{R}^{2n})$, and

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \overline{P} \oplus P = T^{1,0}(J) \oplus T^{0,1}(J). \tag{31.23}$$

Proof. For the forward direction, we already know this. To see the other direction, consider the projection map Re restricted to P

$$\pi = Re : P \to \mathbb{R}^{2n}. \tag{31.24}$$

We claim this is a real linear isomorphism. Obviously, it is linear over the reals. Let $X \in P$ satisfy $\pi(X) = 0$. Then $Re(X) = 0$, so $X = iX'$ for some real $X' \in \mathbb{R}^{2n}$. But $\overline{X} = -iX' \in P \cap \overline{P}$, so by assumption $X = 0$. Since these spaces are of the same real dimension, π has an inverse, which we denote by J. Clearly then, (31.22) is satisfied. Since P is a complex subspace, given any $X = X' + iJX' \in P$, the vector $iX' = (-JX') + iX'$ must also lie in P, so

$$(-JX') + iX' = X'' + iJX'', \tag{31.25}$$

for some real X'', which yields the two equations

$$JX' = -X'' \tag{31.26}$$

$$X' = JX''. \tag{31.27}$$

applying J to the first equation yields

$$J^2X' = -JX'' = -X'. \tag{31.28}$$

Since this is true for any X', we have $J^2 = -I_{2n}$. □

Remark 31.4. We note that $J \mapsto -J$ corresponds to interchanging $T^{0,1}$ and $T^{1,0}$.

Remark 31.5. If we choose $P = \text{span}_{\mathbb{C}}\{\partial/\partial x^j, j = 1 \dots n\}$. Then P is an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and Re restricted to P is not an isomorphism, for example.

Remark 31.6. The above proposition embeds $\mathcal{J}(\mathbb{R}^{2n})$ as a subset of the complex Grassmannian $G(n, 2n, \mathbb{C})$. These spaces have the same dimension, so it is an *open* subset. Furthermore, the condition that the projection to the real part is an isomorphism is generic, so it is also dense.

32 Lecture 32

32.1 Graphs over $T^{0,1}(J_0)$

Above we viewed $T^{0,1}(J)$ as a graph corresponding to the decomposition $\mathbb{C}^{2n} = \mathbb{R}^{2n} \oplus i\mathbb{R}^{2n}$. In the section we will instead view $T^{0,1}(J)$ as a graph corresponding to the decomposition $\mathbb{C}^{2n} = T^{0,1}(J_0) \oplus T^{1,0}(J_0)$. This corresponds to a mapping

$$\phi : T^{0,1}(J_0) \to T^{1,0}(J_0), \tag{32.1}$$

by writing

$$T^{0,1}(J) = \{v + \phi v \mid v \in T^{0,1}(J_0)\}. \tag{32.2}$$

Note we can view ϕ as an element of

$$Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0), \tag{32.3}$$

so we will view ϕ as an element of the latter space. In "coordinates", we can write

$$\phi = \phi^k_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z^k}, \tag{32.4}$$

and we will view $\phi^k_{\bar{j}}$ as an n by n complex matrix. We define $\overline{\phi}$ as a $\mathbb{C}$-linear mapping

$$\overline{\phi} : T^{1,0}(J_0) \to T^{0,1}(J_0), \tag{32.5}$$

by

$$\overline{\phi}(v) = \overline{\phi(\bar{v})}. \tag{32.6}$$

Consider the mapping

$$\phi + \overline{\phi} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \tag{32.7}$$

which in matrix form is

$$\phi + \overline{\phi} = \begin{pmatrix} 0 & \phi \\ \overline{\phi} & 0 \end{pmatrix}. \tag{32.8}$$

Recall from (31.9) that this is the complexification of an $\mathbb{R}$-linear mapping

$$I_\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \tag{32.9}$$

satisfying $I_\phi J_0 + J_0 I_\phi = 0$, which is given by

$$I_\phi = \begin{pmatrix} Re(\phi) & Im(\phi) \\ Im(\phi) & -Re(\phi) \end{pmatrix}. \tag{32.10}$$

Proposition 32.1. *If $\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0)$, then ϕ determines an almost complex structure if and only if I_ϕ does not have -1 as an eigenvalue. The corresponding almost complex structure is*

$$J_\phi = (Id + I_\phi) J_0 (Id + I_\phi)^{-1}. \tag{32.11}$$

Conversely, given J such that $J_0 + J$ is invertible, then J corresponds to a unique ϕ with $Id + I_\phi$ invertible, which is given by

$$I_\phi = (J_0 + J)^{-1}(J_0 - J). \tag{32.12}$$

Proof. Given

$$\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0) = Hom_{\mathbb{C}}(T^{0,1}(J_0), T^{1,0}(J_0)), \tag{32.13}$$

then

$$T^{0,1}(J_\phi) = \{v + \phi v, v \in T^{0,1}(J_0)\} \tag{32.14}$$

is an n-dimensional complex subspace of $\mathbb{R}^{2n} \otimes \mathbb{C}$. If $X \in T^{0,1}_\phi \cap \overline{T^{0,1}_\phi}$ for a non-zero vector X, then

$$X = v + \phi v = w + \overline{\phi} w, \tag{32.15}$$

where $v \in T^{0,1}(J_0)$ and $w \in T^{1,0}(J_0)$. This yields the equations

$$\overline{\phi} w = v \tag{32.16}$$

$$\phi v = w. \tag{32.17}$$

This is equivalent the matrix $\phi + \overline{\phi}$ having 1 as an eigenvalue with eigenvector (w, v). Since $\phi + \overline{\phi}$ is matrix equivalent to I_ϕ, this is equivalent to I_ϕ having 1 as an eigenvalue. But if $I_\phi V = V$, then

$$I_\phi J_0 V = -J_0 I_\phi V = -J_0 V, \tag{32.18}$$

that is $J_0 V$ is an eigenvalue of I_ϕ with eigenvalue -1. Next, any $\tilde{v} \in T^{0,1}(J_\phi)$ is written as

$$\begin{aligned} \tilde{v} &= v + \phi(v) \\ &= Re(v) + Re(\phi(v)) + i(Im(v) + Im(\phi(v)), \end{aligned} \tag{32.19}$$

for $v \in T^{0,1}(J_0)$. We compute

$$\begin{aligned} Re(\phi(v)) &= \frac{1}{2}\Big(\phi(v) + \overline{\phi(v)}\Big) \\ &= \frac{1}{2}\Big(\phi(v) + \overline{\phi}(\overline{v})\Big) \\ &= (\phi + \overline{\phi})\Big(\frac{v + \overline{v}}{2}\Big) \\ &= I_\phi(Re(v)). \end{aligned} \tag{32.20}$$

Next,

$$
\begin{aligned}
Im(\phi(v)) &= \frac{1}{2i}\Big(\phi(v) - \overline{\phi(v)}\Big)\\
&= \frac{1}{2i}\Big(\phi(v) - \overline{\phi}(\overline{v})\Big)\\
&= (\phi + \overline{\phi})\Big(\frac{v - \overline{v}}{2i}\Big)\\
&= I_{\phi}(Im(v)).
\end{aligned}
\tag{32.21}
$$

Next, any element $v \in T^{0,1}(J_0)$ can be written as

$$
v = X' + iJ_0X', \tag{32.22}
$$

for $X' \in \mathbb{R}^{2n}$, so we have

$$
\tilde{v} = (Id + I_{\phi})X' + i(Id + I_{\phi})(J_0X'). \tag{32.23}
$$

But if $\tilde{v} \in T^{0,1}(J_{\phi})$, we must have

$$
Im(\tilde{v}) = J_{\phi}Re(\tilde{v}), \tag{32.24}
$$

which yields

$$
(Id + I_{\phi})(J_0X') = J_{\phi}(Id + I_{\phi})X'. \tag{32.25}
$$

This implies that

$$
J_{\phi} = (Id + I_{\phi})J_0(Id + I_{\phi})^{-1}. \tag{32.26}
$$

The remainder of the proposition follows by solving this equation for I_{ϕ}. □

32.2 Beltrami equation in higher dimensions

Next, we let J be a continuous almost complex structure on a open set $U \subset \mathbb{R}^{2n}$ containing the origin. Then $J : U \to \mathcal{J}_n$ is a continuous function. Without loss of generality, we may assume that $J(0) = J_0$. Then $(J_0 + J)(0) = 2J_0$ is invertible, so $J_0 + J$ will also be invertible in some possibly smaller neighborhood $V \subset U$. Then by Proposition 32.1, we obtain a unique

$$
\phi^{j}_{\bar{k}} : V \to Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong Mat(n \times n, \mathbb{C}), \tag{32.27}
$$

where V is an open subset in $\mathbb{R}^{2n}$.

Proposition 32.2. *If $\phi^{j}_{\bar{k}}$ defines an almost complex structure on V, then a function $f : V \to \mathbb{C}$ is pseudo-holomorphic if and only if*

$$
\frac{\partial}{\partial \overline{z}^j} f + \phi^{k}_{\bar{j}} \frac{\partial}{\partial z^k} f = 0 \tag{32.28}
$$

for $j = 1, \ldots, n$.

Proof. It is not hard to see that a function f is pseudo-holomorphic if and only if $Zf = 0$ for all vector fields $Z \in \Gamma(T^{0,1}_{\phi})$. A local basis for $T^{0,1}_{\phi}$ is given by

$$Z_{\bar{j}} = \frac{\partial}{\partial \overline{z}^j} + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k}, \tag{32.29}$$

so we are done. □

Remark 32.3. For $n = 1$, there is only 1 component $\mu = \phi^1_{\bar{1}}$, and the pseduo-holomorphic condition is

$$\frac{\partial}{\partial \overline{z}} f + \mu \frac{\partial}{\partial z} f = 0, \tag{32.30}$$

which is of course the Beltrami equation.

33 Lecture 33

33.1 Integrability

We next interpret the vanishing of the Nijenhuis tensor as an equation on ϕ.

Proposition 33.1. *The almost complex structure J_ϕ is integrable, that is $N(J_\phi) = 0$, if and only if*

$$\frac{\partial}{\partial \overline{z}^l} \phi^j_{\bar{k}} - \frac{\partial}{\partial \overline{z}^k} \phi^j_{\bar{l}} + \phi^m_{\bar{k}} \frac{\partial}{\partial z^m} \phi^j_{\bar{l}} - \phi^m_{\bar{l}} \frac{\partial}{\partial z^m} \phi^j_{\bar{k}} = 0. \tag{33.1}$$

Proof. By Proposition 16.14, the integrability equation is equivalent to $[T^{0,1}_{\phi}, T^{0,1}_{\phi}] \subset T^{0,1}_{\phi}$. Writing

$$\phi = \sum \phi^j_{\bar{k}} d\overline{z}^k \otimes \frac{\partial}{\partial z^j}, \tag{33.2}$$

if J_ϕ is integrable, then we must have

$$\Big[\frac{\partial}{\partial \overline{z}^i} + \phi\Big(\frac{\partial}{\partial \overline{z}^i} \Big), \frac{\partial}{\partial \overline{z}^k} + \phi \Big(\frac{\partial}{\partial \overline{z}^k} \Big) \Big] \in T^{0,1}_{\phi}. \tag{33.3}$$

This yields

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi^l_{\bar{k}} \frac{\partial}{\partial z^l} \Big] + \Big[\phi^j_{\bar{i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k} \Big] + \Big[\phi^j_{\bar{i}} \frac{\partial}{\partial z^j}, \phi^l_{\bar{k}} \frac{\partial}{\partial z^l} \Big] \in T^{0,1}_{\phi} \tag{33.4}$$

The first two terms are

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi^l_{\bar{k}} \frac{\partial}{\partial z^l} \Big] + \Big[\phi^j_{\bar{i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k} \Big] = \sum_j \Big(\frac{\partial \phi^j_{\bar{k}}}{\partial \overline{z}^i} - \frac{\partial \phi^j_{\bar{i}}}{\partial \overline{z}^k} \Big) \frac{\partial}{\partial z^j}.$$

The third term is

$$\Big[\phi^j_{\bar{i}} \frac{\partial}{\partial z^j}, \phi^l_{\bar{k}} \frac{\partial}{\partial z^l} \Big] = \phi^j_{\bar{i}} \Big(\frac{\partial}{\partial z^j} \phi^l_{\bar{k}} \Big) \frac{\partial}{\partial z^l} - \phi^l_{\bar{k}} \Big(\frac{\partial}{\partial z^l} \phi^j_{\bar{i}} \Big) \frac{\partial}{\partial z^j}.$$

Both terms are in $T^{1,0}(J_0)$. For sufficiently small ϕ however, $T^{0,1}_{\phi} \cap T^{1,0}(J_0) = \{0\}$, and therefore (33.1) holds. The converse holds by reversing this argument. □

We can also derive this integrability condition in the following way. If there exists a locally defined holomorphic function f, then taking the $\overline{\partial}$-partial of (32.28) yields

$$\frac{\partial^2}{\partial\overline{z}^l\partial\overline{z}^j}f + \frac{\partial}{\partial\overline{z}^l}\Big(\phi_{\overline{j}}^k\frac{\partial}{\partial z^k}f\Big) = 0. \tag{33.5}$$

Intechanging j and l yields

$$\frac{\partial^2}{\partial\overline{z}^j\partial\overline{z}^l}f + \frac{\partial}{\partial\overline{z}^j}\Big(\phi_{\overline{l}}^k\frac{\partial}{\partial z^k}f\Big) \tag{33.6}$$

If f is C^2, then the mixed partials are equal, so subtracting these equations gives

$$\frac{\partial}{\partial\overline{z}^l}\Big(\phi_{\overline{j}}^k\frac{\partial}{\partial z^k}f\Big) - \frac{\partial}{\partial\overline{z}^j}\Big(\phi_{\overline{l}}^k\frac{\partial}{\partial z^k}f\Big) = 0. \tag{33.7}$$

Expanding this out

$$\Big(\frac{\partial}{\partial\overline{z}^l}\phi_{\overline{j}}^k - \frac{\partial}{\partial\overline{z}^j}\phi_{\overline{l}}^k\Big)\frac{\partial}{\partial z^k}f + \phi_{\overline{j}}^k\frac{\partial^2}{\partial\overline{z}^l\partial z^k}f - \phi_{\overline{l}}^k\frac{\partial^2}{\partial\overline{z}^j\partial z^k} = 0 \tag{33.8}$$

The first 2 terms are good. Using (32.28), the last 2 terms are

$$\begin{aligned}
\phi_{\overline{j}}^k\frac{\partial^2}{\partial z^k\partial\overline{z}^l}f - \phi_{\overline{l}}^k\frac{\partial^2}{\partial z^k\partial\overline{z}^j} &= -\phi_{\overline{j}}^k\frac{\partial}{\partial z^k}\Big(\phi_{\overline{l}}^p\frac{\partial}{\partial z^p}f\Big) + \phi_{\overline{l}}^k\frac{\partial}{\partial z^k}\Big(\phi_{\overline{j}}^p\frac{\partial}{\partial z^p}f\Big) &(33.9)\\
&= -\phi_{\overline{j}}^k\Big(\frac{\partial}{\partial z^k}\phi_{\overline{l}}^p\Big)\frac{\partial}{\partial z^p}f + \phi_{\overline{l}}^k\Big(\frac{\partial}{\partial z^k}\phi_{\overline{j}}^p\Big)\frac{\partial}{\partial z^p}f &(33.10)\\
&\quad - \phi_{\overline{j}}^k\phi_{\overline{l}}^p\frac{\partial^2 f}{\partial z^k\partial z^p} + \phi_{\overline{l}}^k\phi_{\overline{j}}^p\frac{\partial^2 f}{\partial z^k\partial z^p}. &(33.11)
\end{aligned}$$

The last 2 terms vanish from symmetry. So we have derived

$$0 = \Big(\frac{\partial}{\partial\overline{z}^l}\phi_{\overline{j}}^k - \frac{\partial}{\partial\overline{z}^j}\phi_{\overline{l}}^k - \phi_{\overline{j}}^p\frac{\partial}{\partial z^p}\phi_{\overline{l}}^k - \phi_{\overline{l}}^p\frac{\partial}{\partial z^p}\phi_{\overline{j}}^p\Big)\frac{\partial}{\partial z^k}f. \tag{33.12}$$

If there exists n holomorphic functions with linearly independent differentials at the origin, then this implies implies the integrability condition (33.1). This latter argument assumes that there exists holomorphic coordinates, but nevertheless still gives the correct formula for the Nijenhuis tensor.

33.2 The operator d^c

We will discuss another differential operator d^c which will be used to simplify some computations later. For an almost complex structure J with $N_J = 0$, we know that

$$d = \partial + \overline{\partial}, \tag{33.13}$$

and

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{33.14}$$

We can write these complex operators in the form

$$\overline{\partial} = \frac{1}{2}(d - id^c),\ \partial = \frac{1}{2}(d + id^c). \tag{33.15}$$

for a *real* operator $d^c : \Omega^p \to \Omega^{p+1}$ given by

$$d^c = i(\overline{\partial} - \partial), \tag{33.16}$$

which satisfies

$$d^2 = 0,\ dd^c + d^c d = 0,\ (d^c)^2 = 0. \tag{33.17}$$

We next have an alternative formula for d^c. Recall that $J : TM \to TM$ induces a dual mapping $J : T^*M \to T^*M$, and we extended to $J : \Lambda^r_{\mathbb{C}} \to \Lambda^r_{\mathbb{C}}$ by

$$J\alpha^{p,q} = i^{p-q}\alpha^{p,q}, \tag{33.18}$$

for a α a form of type (p, q). Notice that if $\alpha^r \in \Lambda^r_{\mathbb{C}}$, then

$$J^2\alpha^r = w \cdot \alpha^r,\ \text{where } w \cdot \alpha^r = (-1)^r \alpha^r, \tag{33.19}$$

since

$$J^2\alpha^{p,q} = i^{2(p-q)}\alpha^{p,q} = (-1)^{p-q}\alpha^{p,q} = (-1)^{p-q+2q}\alpha^{p,q} = (-1)^{p+q}\alpha^{p,q}. \tag{33.20}$$

Proposition 33.2. *For $\alpha \in \Lambda^r$, we have*

$$d^c\alpha = (-1)^{r+1} JdJ\alpha. \tag{33.21}$$

We also have

$$dd^c = 2i\partial\overline{\partial} = (-1)^{r+1} dJdJ\alpha. \tag{33.22}$$

Proof. For $\alpha \in \Lambda^{p,q}$, $p + q = r$, we compute

$$JdJ\alpha = i^{p-q}Jd\alpha = i^{p-q}J(\partial\alpha + \overline{\partial}\alpha) \tag{33.23}$$

$$= i^{p-q}(i^{p+1-q}\partial\alpha + i^{p-q-1}\overline{\partial}\alpha) \tag{33.24}$$

$$= i^{2(p-q)+1}\partial\alpha + i^{2(p-q)-1}\overline{\partial}\alpha \tag{33.25}$$

$$= (-1)^{p+q}(i\partial\alpha - i\overline{\partial}\alpha) = (-1)^{r+1}d^c\alpha. \tag{33.26}$$

For (33.22), using (33.14) we have

$$dd^c = (\partial + \overline{\partial})i(\overline{\partial} - \partial) = i(\partial\overline{\partial} + \overline{\partial}^2 - \partial^2 - \overline{\partial}\partial) = 2i\partial\overline{\partial}. \tag{33.27}$$

□

34 Lecture 34

To finish out the Winter quarter, we will discuss the Newlander-Nirenberg Theorem in higher dimensions.

So, we let J be a continuous almost complex structure on a open set $U \subset \mathbb{R}^{2n}$ containing the origin. Then $J : U \to \mathcal{J}_n$ is a continuous function. Without loss of generality, we may assume that $J(0) = J_0$. Then $(J_0 + J)(0) = 2J_0$ is invertible, so $J_0 + J$ will also be invertible in some possibly smaller neighborhood $V \subset U$. Then by Proposition 32.1, we obtain a unique

$$\phi^j_{\bar{k}} : V \to Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong Mat(n \times n, \mathbb{C}), \tag{34.1}$$

where V is an open subset in $\mathbb{R}^{2n}$. Since the correspondence between J-s and ϕ-s is analytic, the regularity of these will be the same. That is, if $J \in C^{k,\alpha}, C^{\omega}$, etc., then so is ϕ and vice-versa.

34.1 The analytic case

We assume that $\phi^k_{\bar{j}}$ is analytic. So there exists a power series expansion

$$\phi^k_{\bar{j}} = \sum_{I,J} (\phi^k_{\bar{j}})_{IJ} z^I \bar{z}^J. \tag{34.2}$$

Let group these terms together by homogeneity and write

$$\phi^k_{\bar{j}} = \sum_{m=0}^{\infty} (\phi^k_{\bar{j}})_m \tag{34.3}$$

where

$$(\phi^k_{\bar{j}})_m = \sum_{|I|+|J|=m} (\phi^k_{\bar{j}})_{IJ} z^I \bar{z}^J. \tag{34.4}$$

We may assume that $(\phi^k_{\bar{j}})_0 = 0$. We want to solve the equation

$$\frac{\partial}{\partial \bar{z}^j} f + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k} f = 0. \tag{34.5}$$

Let's do the same for f, we write a formal power series

$$f = \sum_{I,J} f_{IJ} z^I \bar{z}^J, \tag{34.6}$$

and group these terms together by homogeneity and write

$$f = \sum_{m=0}^{\infty} f_m \tag{34.7}$$

where

$$f_m = \sum_{|I|+|J|=m} f_{IJ} z^I \bar{z}^J. \tag{34.8}$$

By subtracting a constant, we can also assume that $f_0 = 0$. Expanding (34.5), we have

$$\overline{\partial}_0(f_1 + f_2 + \cdots) + (\phi_1 + \phi_2 + \cdots)(\partial_0 f_1 + \partial_0 f_2 + \cdots) = 0. \tag{34.9}$$

Grouping terms by homogeneity, we have

$$\begin{aligned}
\overline{\partial}_0 f_1 &= 0\\
\overline{\partial}_0 f_2 &= -\phi_1 \partial_0 f_1\\
\overline{\partial}_0 f_3 &= -\phi_1 \partial_0 f_2 - \phi_2 \partial_0 f_1\\
&\vdots
\end{aligned}$$

and we see that the general term is given by

$$\frac{\partial f_m}{\partial \overline{z}^j} = -\sum_{k+l=m, k\geq 1, l\geq 1} (\phi_{\bar{j}}^p)_k \frac{\partial f_l}{\partial z^p} \tag{34.10}$$

Proposition 34.1. *If f_j solves the above system for $j = 1, \ldots, q$, then the expression*

$$H_{q+1} = -\sum_{k+l=q+1, k\geq 1, l\geq 1} (\phi_{\bar{j}}^p)_k \frac{\partial f_l}{\partial z^p} d\overline{z}^j \tag{34.11}$$

is a form of type $(0,1)$ with respect to J_0, and satisfies $\overline{\partial}_0 H_{q+1} = 0$.

Proof. We prove this by induction. For $q = 1$, we have $\overline{\partial}_0 f_1 = 0$, so $f_1 = c_j z^j$ is a linear holomorphic function. Then

$$H_2 = -(\phi_{\bar{j}}^p)_1 c_p d\overline{z}^j. \tag{34.12}$$

For reasons of homogeneity, the integrability equation (33.1) tells us that

$$\frac{\partial}{\partial \overline{z}^l}(\phi_{\bar{j}}^p)_1 - \frac{\partial}{\partial \overline{z}^j}(\phi_{\bar{l}}^p)_1 = 0, \tag{34.13}$$

so H_2 clearly satisfies $\overline{\partial} H_2 = 0$. So assume the system is satisfied for $j = 1 \ldots q$. Then the function $f = f_1 + \cdots + f_q$ satisfies

$$\overline{\partial}_J f = H_{q+1} + O(|z|^{q+1}) \tag{34.14}$$

For the next step, we use the above fact that the integrability of J implies that the operator $\overline{\partial}_J : \Lambda^{0,1}(J) \to \Lambda^{0,2}(J)$ defined by $\overline{\partial}_J \alpha = \Pi_{\Lambda^{0,2}(J)} d\alpha$ satisfies

$$\overline{\partial}_J \overline{\partial}_J f = 0, \tag{34.15}$$

for any function f. This yields

$$0 = \overline{\partial}_J(H_{q+1} + O(|z|^{q+1}) = \overline{\partial}_J H_{q+1} + O(|z|^q). \tag{34.16}$$

Note that for $\alpha \in \Lambda^{0,1}(J)$, $J\alpha = -i\alpha$, so from Proposition 33.2, we have that

$$\overline{\partial}_J \alpha = \frac{1}{2}(d - id^c)\alpha = \frac{1}{2}(d\alpha - iJdJ\alpha) = \frac{1}{2}(d\alpha - Jd\alpha). \tag{34.17}$$

Expanding this, we obtain

$$\overline{\partial}_J \alpha = \frac{1}{2}(d\alpha - (J - J_0 + J_0)d\alpha) = \frac{1}{2}(d\alpha - J_0 d\alpha) - \frac{1}{2}(J - J_0)d\alpha. \tag{34.18}$$

From Proposition 32.1 above, the correspondence between ϕ and J is analytic, and $\phi = O(|z|)$ implies that $J - J_0 = O(|z|)$ as $z \to 0$. Now we plug in $\alpha = \overline{\partial}_J f$, and by assumption

$$0 = \overline{\partial}_J \overline{\partial}_J f = \overline{\partial}_J(H_{q+1} + O(|z|^{q+1}) = \frac{1}{2}(dH_{q+1} - J_0 dH_{q+1}) + O(|z|^q) \tag{34.19}$$

$$= \overline{\partial}_0 H_{q+1} + O(|z|^q). \tag{34.20}$$

□

Proposition 34.2. *For each $1 \leq p < \infty$, there exists $f = \sum_{j=1}^p f_j$ satisfying $\overline{\partial}_J f = O(|z|^p)$.*

Proof. We prove this by induction. For $p = 1$, we can take $f = c_p z^p$, and then

$$\overline{\partial}_J z^k = \overline{\partial}_0 z^k + \frac{i}{2}(J - J_0)dz^k = 0 + O(|z|). \tag{34.21}$$

Assume that we have found a solution for $j = 1 \dots p$. Let $f = \sum_{j=1}^p f_j$, by the induction assumption, we have

$$\overline{\partial}_J f = H_{p+1} + O(|z|^{p+1}), \tag{34.22}$$

and by the above, we need to solve the equation

$$\overline{\partial}_0 f_{p+1} = H_{p+1}. \tag{34.23}$$

From Proposition 35.4, H_{p+1} satisfies $\overline{\partial}_0 H_{p+1} = 0$. Equivalently, we can write

$$H_{p+1} = \alpha_{\bar{j}} d\overline{z}^j, \tag{34.24}$$

where the coefficients satisfy

$$\frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^l} = \frac{\partial \alpha_{\bar{l}}}{\partial \overline{z}^j},\ j, l = 1, \dots, n. \tag{34.25}$$

Define

$$f_{p+1} = \int_0^1 \sum_{j=1}^n \overline{z}^j \alpha_{\bar{j}}(z, t\overline{z}) dt. \tag{34.26}$$

Then we compute

$$
\begin{aligned}
\frac{\partial f_{p+1}}{\partial \overline{z}^k} &= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^n \overline{z}^j \frac{\partial}{\partial \overline{z}^k}\Big(\alpha_{\bar{j}}(z, t\bar{z})\Big)\Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^n \overline{z}^j \frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^k}(z, t\bar{z})t\Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^n \overline{z}^j \frac{\partial \alpha_{\bar{k}}}{\partial \overline{z}^j}(z, t\bar{z})t\Big) \\
&= \int_0^1 \frac{d}{dt}\Big(t\alpha_{\bar{k}}(z, t\bar{z})\Big) dt = \alpha_{\bar{k}}(z, \bar{z}).
\end{aligned}
\tag{34.27}
$$

□

We will discuss convergence next time.

35 Lecture 35

35.1 Convergence of formal power series solution

Last time, we constructed a formal power series solution. Today, we examine the solution in more detail. We look at the procedure in Proposition 34.2 above. We had H_{p+1} satisfying $\overline{\partial}_0 H_{p+1} = 0$. Writing

$$
H_{p+1} = \alpha_{\bar{j}} d\overline{z}^j, \tag{35.1}
$$

then the coefficients satisfy

$$
\frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^l} = \frac{\partial \alpha_{\bar{l}}}{\partial \overline{z}^j}, \ j, l = 1, \dots, n. \tag{35.2}
$$

Defining

$$
f_{p+1} = \int_0^1 \sum_{j=1}^n \overline{z}^j \alpha_{\bar{j}}(z, t\bar{z}) dt, \tag{35.3}
$$

then we showed that $\overline{\partial}_0 f_{p+1} = H_{p+1}$.

Proposition 35.1. *Writing $H_{p+1} = \alpha_{\bar{j}} d\overline{z}^j$, where*

$$
\alpha_{\bar{j}} = \sum_{|I|+|J|=p} \alpha_{\bar{j}I\bar{J}} z^I \overline{z}^J, \tag{35.4}
$$

and $f_{p+1} = \sum_{|I|+|J|=p+1} f_{I\bar{J}} z^I \overline{z}^J$. Then the coefficients $f_{I\bar{J}}$ are linear functions of the $\alpha_{\bar{j}I\bar{J}}$ with non-negative coefficients.

Proof. We plug (35.4) into (35.3), and compute

$$f_{p+1} = \int_0^1 \sum_{j=1}^n \overline{z}^j \sum_{|I|+|J|=p} \alpha_{\bar{j}I\bar{J}} z^I (t\overline{z})^J dt \tag{35.5}$$

$$= \sum_{j,|I|+|J|=p} \alpha_{\bar{j}I\bar{J}} z^I \overline{z}^J \overline{z}^j \int_0^1 t^{|J|} dt \tag{35.6}$$

$$= \sum_{j,|I|+|J|=p} \frac{1}{|J|+1} \alpha_{\bar{j}I\bar{J}} z^I \overline{z}^J \overline{z}^j, \tag{35.7}$$

and we are done. □

Remark 35.2. Our choice above is very important: there is a freedom to add an arbitrary holomorphic homogeneous polynomial to f_{p+1}, and our choice eliminates this ambiguity.

Now we return to solving the equation

$$\frac{\partial}{\partial \overline{z}^j} f + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k} f = 0. \tag{35.8}$$

We will now consider this as an equation in $\mathbb{C}^{2n}$ with coordinates $(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n)$. Note by the transformation $z^j \mapsto -z^j$, we can assume the equation is of the form

$$\frac{\partial}{\partial \overline{z}^j} f - \phi^k_{\bar{j}} \frac{\partial}{\partial z^k} f = 0. \tag{35.9}$$

Remark 35.3. Note that we cannot simply replace ϕ with $-\phi$, since the integrability equation (33.1) is not preserved under this transformation.

We have the homogeneous decompositions

$$f = \sum_{j=1}^{\infty} f_j, \;\; \phi^k_{\bar{j}} = \sum_{l=1}^{\infty} (\phi^k_{\bar{j}})_l. \tag{35.10}$$

We then found the recursive system

$$\overline{\partial} f_{q+1} = H_{q+1}, \tag{35.11}$$

where

$$H_{q+1} = \sum_{k+l=q+1, k\geq 1, l\geq 1} (\phi^p_{\bar{j}})_k \frac{\partial f_l}{\partial z^p} d\overline{z}^j \tag{35.12}$$

satisfies $\overline{\partial}_0 H_{q+1} = 0$. Then we can solve $\overline{\partial} f_{q+1} = H_{q+1}$ uniquely with the above proceudre, where $f_{q+1}(z^1, \cdots, z^n, 0, \cdots, 0) = 0$ for $q > 1$. So once $f_1 = c_j z^j$ is specified, then our procedure gives a unique formal power series solution.

Write

$$(\phi^k_{\bar{j}})_p = \sum_{|I|+|J|=p} \phi^k_{\bar{j}I\bar{J}} z^I \overline{z}^J. \tag{35.13}$$

Proposition 35.4. *The coefficients $f_{I\bar{J}}$ when $|I| + |J| = p$ are a polynomial function of degree $p-1$ in the $\phi^k_{\bar{j}K\bar{L}}$ for $|K| + |L| \leq p-1$, with all coefficients non-negative rational numbers. The polynomials are completely determined by the constants $c_1, \ldots, c_n$.*

Proof. Without loss of generality, assume that $f_1 = z^1$. Then the first nontrivial equation is

$$\frac{\partial f_2}{\partial \overline{z}^j} = (\phi^m_{\bar{j}})_1 \frac{\partial f_1}{\partial z^m} = (\phi^1_{\bar{j}})_1 = \phi^1_{\bar{j}k} z^k + \phi^1_{\bar{j}\bar{k}} \overline{z}^k. \tag{35.14}$$

Then

$$f_2 = \phi^1_{\bar{j}k} z^k \overline{z}^j + \frac{1}{2} \phi^1_{\bar{j}\bar{k}} \overline{z}^j \overline{z}^k, \tag{35.15}$$

so the claim is true for f_2. Then we proceed by induction. So assume the claim is true for $f_1, \ldots, f_p$. Then the equation for f_{p+1} is

$$\frac{\partial f_{p+1}}{\partial \overline{z}^j} = \sum_{k+l=p+1, k\geq 1, l\geq 1} (\phi^m_{\bar{j}})_k \frac{\partial f_l}{\partial z^m} \tag{35.16}$$

$$= \sum_{k+l=p+1, k\geq 1, l\geq 1} \sum_{|I|+|J|=k} \phi^m_{\bar{j}I\bar{J}} \frac{\partial f_l}{\partial z^m}. \tag{35.17}$$

From Proposition 35.1, we just need to show the claim is true for the coefficients on the right hand side. Since $l \leq p$ in the sum, by induction the claim is true for the coefficients of f_l. The operator $f_l \mapsto \partial f_l / \partial z^p$ obviously preserves non-negativity of the coefficients, so we are done. ∎

35.2 Cauchy majorant method

By assumption, the series

$$\sum_{I,J} \phi^k_{\bar{j}I\bar{J}} z^I \overline{z}^J. \tag{35.18}$$

converges for any point in the polydisc

$$P(\rho) = \{(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) \mid |z^j| < \rho, |\overline{z}^j| < \rho,\ 1 \leq j \leq n\}, \tag{35.19}$$

with uniform convergence in the polydisc $\overline{P(\rho')}$, for any $\rho' < \rho$. In particular, for any point $(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) \in \overline{P(\rho')}$, there exists a constant $C > 0$ so that

$$|\phi^k_{\bar{j}I\bar{J}} z^I \overline{z}^J| < C \ \text{ (no summation)}. \tag{35.20}$$

Choosing the point $z^j = \rho', \overline{z}^j = \rho'$ for $j = 1, \ldots, n$, this implies that

$$|\phi^k_{\bar{j}I\bar{J}}| < C(\rho')^{-(|I|+|J|)}. \tag{35.21}$$

To simplify notation, let's call ρ' by ρ. Then we define

$$\Phi(w) = C\Big(\frac{1}{1 - w\rho^{-1}} - 1\Big) = \frac{Cw}{\rho - w} \tag{35.22}$$

which is analytic in the disc $\Delta(\rho) = \{w \in \mathbb{C} \mid |w| < \rho\}$. The power series of $\Phi(w)$ is given by

$$\Phi(w) = C\sum_{j=1}^{\infty} \rho^{-j} w^j. \tag{35.23}$$

Next, we let

$$\Phi(z^1, \dots, z^n, \overline{z}^1, \dots, \overline{z}^n) = \Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n). \tag{35.24}$$

Using the multinomial theorem, we have the expansion

$$\Phi(z^1, \dots, z^n, \overline{z}^1, \dots, \overline{z}^n) = C \sum_{I,J \neq (0,0)} \rho^{-(|I|+|J|)} \frac{(|I| + |J|)!}{I!J!} z^I \overline{z}^J, \tag{35.25}$$

which converges absolutely in the polydisc

$$P = \{(z^1, \dots, z^n, \overline{z}^1, \dots, \overline{z}^n) \mid |z^j| < \rho/2n, |\overline{z}^j| < \rho/2n\}. \tag{35.26}$$

That is, the power series coefficients of $\Phi(z^1, \dots, z^n, \overline{z}^1, \dots, \overline{z}^n)$ are given by

$$\Phi_{I\bar{J}} = C\rho^{-(|I|+|J|)} \frac{(|I| + |J|)!}{I!J!}, \tag{35.27}$$

with $\Phi_{00} = 0$. Since the multinomial coeffients are at least 1, we have the inequality

$$|\phi^j_{\bar{k}I\bar{J}}| < C\rho^{-(|I|+|J|)} \leq C\rho^{-(|I|+|J|)} \frac{(|I| + |J|)!}{I!J!} = \Phi_{I\bar{J}}. \tag{35.28}$$

Next, we claim that $\Phi^k_{\bar{j}} = \Phi$ determines an integrable almost complex structure. Note we are viewing this as an $n \times n$ matrix will all entries equal. To see this, we use (33.1):

$$\begin{aligned}
&\frac{\partial}{\partial \overline{z}^l}\Phi^j_{\bar{k}} - \frac{\partial}{\partial \overline{z}^k}\Phi^j_{\bar{l}} + \Phi^m_{\bar{k}}\frac{\partial}{\partial z^m}\Phi^j_{\bar{l}} - \Phi^m_{\bar{l}}\frac{\partial}{\partial z^m}\Phi^j_{\bar{k}}\\
&= \frac{\partial}{\partial \overline{z}^l}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) - \frac{\partial}{\partial \overline{z}^k}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n)\\
&+ \sum_m \frac{\partial}{\partial z^m}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) - \sum_m \frac{\partial}{\partial z^m}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n)\\
&= 0 + 0 = 0.
\end{aligned} \tag{35.29}$$

The equation for a holomorphic function with respect to Φ is

$$\frac{\partial F}{\partial \overline{z}^j} = \Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n)\sum_m \frac{\partial F}{\partial z^m}. \tag{35.30}$$

For all $j = 1, \ldots, n$.

Let's assume that we can find a solution F_k of (35.30) satisying the initial conditions

$$F_k(z^1, \ldots, z^n, 0, \ldots 0) = z^k, \tag{35.31}$$

which is analytic in some polydisc $|z| < \rho', |\overline{z}| < \rho'$. Without loss of generality, we can assume that $k = 1$. Then to finish the convergence proof, recall that our formal power series solves

$$f_{I\bar{J}} = P_{I\bar{J}}(\phi^*_*), \tag{35.32}$$

where $P_{I\bar{J}}$ is a polynomial with non-negative coefficients depending only upon ϕ^*_{*KL} for $|K|+|L| < |I|+|J|$. Since F_1 is an analytic solution of the Cauchy-Riemann equations with respect to Φ, and the same initial conditions as f, we must also have

$$F_{I\bar{J}} = P_{I\bar{J}}(\Phi_{KL}), \tag{35.33}$$

where $P_{I\bar{J}}$ is the *same* polynomial since $\Phi(0,0) = 0$ and $F(z^1, \ldots, z^n, 0, \ldots 0) = z^1$ has the same initial conditions as our formal power series solution. We then estimate

$$|f_{I\bar{J}}| = |P_{I\bar{J}}(\phi^*_*)| \leq P_{I\bar{J}}(|\phi^*_*|) \leq P_{I\bar{J}}(\Phi_{K\bar{L}}) = F_{I\bar{J}}. \tag{35.34}$$

The inequalities hold since $P_{I\bar{J}}$ is a polynomial with real non-negative coefficients, and using (35.28). This shows that our power series is majorized by the power series of F, which implies that the power series for f converges in the open polydisc $P(\rho')$, by the comparison test.

35.3 Completion of convergence proof

To finish the convergence proof, we need to find a solution of

$$\frac{\partial F}{\partial \overline{z}^j} = \Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) \sum_m \frac{\partial F}{\partial z^m}, \tag{35.35}$$

for all $j = 1, \ldots, n$, satisfying the initial conditions

$$F(z^1, \ldots, z^n, 0, \ldots 0) = z^1, \tag{35.36}$$

and which is analytic in some polydisc around the origin in $\mathbb{C}^{2n}$.

Proposition 35.5. *For any choice of* $(c_1, \ldots, c_n) \in \mathbb{C}^n$*, satisfying* $c_1 + \cdots + c_n = 0$*, the function* $F = \sum c_k z^k$ *solves* (35.35).

Proof. The function F obviously makes the left hand side of (35.35) vanish for any $1 \leq j \leq n$. The right hand side of (35.35) is $\Phi \cdot (c_1 + \cdots + c_n) = 0$. □

Next, let's try and find a solution F_+ of the form

$$F_+(z^1, \dots, z^n, \overline{z}^1, \dots, \overline{z}^n) = G(z^1 + \dots + z^n, \overline{z}^1 + \dots + \overline{z}^n). \tag{35.37}$$

Let's call $z = z^1 + \dots + z^n$, $\overline{z} = \overline{z}^1 + \dots + \overline{z}^n$, and write G as a function of 2 variables $G = G(z, \overline{z})$. Then (35.35) becomes

$$\frac{\partial G}{\partial \overline{z}} = \Phi(z + \overline{z}) n \frac{\partial G}{\partial z}. \tag{35.38}$$

This is just the Beltrami equation:

$$\frac{\partial G}{\partial \overline{z}} = \frac{nC(z + \overline{z})}{\rho - z - \overline{z}} \frac{\partial G}{\partial z}. \tag{35.39}$$

But we have already found an analytic solution G for this equation, it is done in Proposition 19.4 (only the constant C has changed to nC), which satisfies the initial condition $G(z, 0) = 0$. So the corresponding solution of the n-dimensional problem satisfies

$$F_+(z^1, \dots, z^n, 0, \dots 0) = G(z^1 + \dots + z^n, 0) = z^1 + \dots + z^n. \tag{35.40}$$

Using Proposition 35.5, we see that the function

$$F = \frac{1}{n}\Big(F_+ + (z^1 - z^2) + (z^1 - z^3) + \dots + (z^1 - z^n)\Big) \tag{35.41}$$

is holomorphic with respect to Φ, is analytic in some polydisc $P(\rho')$, and satisfies the initial conditions (35.36). This finishes the proof.

36 Lecture 36

36.1 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the C^2 case into the analytic case [Mal69, Nir73]. In the z-coordinates, our holomorphic equation is

$$\frac{\partial w}{\partial \overline{z}^j} + \phi^k_{\bar{j}} \frac{\partial w}{\partial z^k} = 0 \tag{36.1}$$

We now view w as a vector-valued function to $\mathbb{C}^n$. We want to change coordinates $\xi = \xi(z, \overline{z})$ so that such that our holomorphic equation transform into another holomorphic equation with analytic coefficients. Write

$$w(z, \overline{z}) = W(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})) \tag{36.2}$$

$$\phi^k_{\bar{j}}(z, \overline{z}) = U^k_{\bar{j}}(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})). \tag{36.3}$$

Then

$$\frac{\partial w}{\partial \overline{z}^j} = \frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial \overline{z}^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \overline{z}^j} \tag{36.4}$$

$$\frac{\partial w}{\partial z^j} = \frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^j}. \tag{36.5}$$

So the holomorphic equations become

$$\frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial \bar{z}^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \bar{z}^j} + U_{\bar{j}}^k(\xi(z,\bar{z}),\bar{\xi}(z,\bar{z}))\Big(\frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^k} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^k}\Big) = 0. \tag{36.6}$$

By inverting the matrix coefficients, this transforms into another holomorphic system of the form

$$\frac{\partial W}{\partial \bar{\xi}^j} + \tilde{U}_{\bar{j}}^k(\xi,\bar{\xi})\frac{\partial W}{\partial \xi^k} = 0 \tag{36.7}$$

where $\tilde{U}$ is of the form

$$\tilde{U}_{\bar{j}}^k = \Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \bar{z}^*} + U_*^p\frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)_{\bar{j}}^{\bar{q}}\Big(\frac{\partial \xi^k}{\partial \bar{z}^q} + U_{\bar{q}}^p\frac{\partial \xi^k}{\partial z^p}\Big). \tag{36.8}$$

Let us try to find coordinates so that

$$\sum_j \frac{\partial}{\partial \xi^j}\tilde{U}_{\bar{j}}^k(\xi,\bar{\xi}) = 0. \tag{36.9}$$

To find the coordinate system ξ, we must write out (36.9), and this becomes a second order system for ξ as a function of the original z coordinates. From the chain rule, we have

$$\frac{\partial}{\partial \xi^j} = \frac{\partial z^l}{\partial \xi^j}\frac{\partial}{\partial z^l} + \frac{\partial \bar{z}^l}{\partial \xi^j}\frac{\partial}{\partial \bar{z}^l}, \tag{36.10}$$

so (36.9) becomes

$$\sum_j \Big(\frac{\partial z^l}{\partial \xi^j}\frac{\partial}{\partial z^l} + \frac{\partial \bar{z}^l}{\partial \xi^j}\frac{\partial}{\partial \bar{z}^l}\Big)\Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \bar{z}^*} + U_*^p\frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)_{\bar{j}}^{\bar{q}}\Big(\frac{\partial \xi^k}{\partial \bar{z}^q} + U_{\bar{q}}^p\frac{\partial \xi^k}{\partial z^p}\Big) = 0. \tag{36.11}$$

The inverse function theorem says that

$$\begin{pmatrix} \dfrac{\partial z^*}{\partial \xi^*} & \dfrac{\partial z^*}{\partial \bar{\xi}^*} \\ \dfrac{\partial \bar{z}^*}{\partial \xi^*} & \dfrac{\partial \bar{z}^*}{\partial \bar{\xi}^*} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \xi^*}{\partial z^*} & \dfrac{\partial \xi^*}{\partial \bar{z}^*} \\ \dfrac{\partial \bar{\xi}^*}{\partial z^*} & \dfrac{\partial \bar{\xi}^*}{\partial \bar{z}^*} \end{pmatrix}^{-1}. \tag{36.12}$$

Making this substitution, and replacing $U_{\bar{q}}^p(\xi,\bar{\xi}) = \phi_{\bar{q}}^p(z,\bar{z})$, we see that the equation (36.11) is a *quasilinear* system of the form

$$F(D^2\xi, D\xi, \xi, z, \bar{z}) = 0. \tag{36.13}$$

We recall the definition of the linearization.

Definition 36.1. The linearization of F at a function ξ is given by

$$F'_\xi(h) = \frac{d}{dt}F(D^2(\xi+th), D(\xi+th), \xi+th, z, \bar{z})\Big|_{t=0}. \tag{36.14}$$

Proposition 36.2. *Assuming $\phi \in C^1$, then the linearization of F at $\xi = z$ is*

$$F'_z(h) = -\frac{1}{4}\Delta h + (\phi + \phi^2) * \nabla^2 h + (\nabla\phi + \phi * \nabla\phi) * \nabla h. \tag{36.15}$$

If $\phi(0) = 0$, then we have

$$F'_z(h)(0) = \frac{1}{4}\Delta h + \nabla\phi * \nabla h. \tag{36.16}$$

If ϕ has sufficiently small $C^{1,\alpha}$, norm then F'_z is an elliptic operator with Hölder coefficients bounded in C^α.

Proof. We use the following formula: if $A(t)$ is a path of matrices, then

$$\frac{d}{dt}A(t)^{-1} = -A^{-1} \circ \frac{d}{dt}A \circ A^{-1}. \tag{36.17}$$

Let look at each term in (36.11)-(36.12). First, for $\xi = z$, we have

$$\begin{pmatrix} \frac{\partial \xi^*}{\partial z^*} & \frac{\partial \xi^*}{\partial \overline{z}^*} \\ \frac{\partial \bar{\xi}^*}{\partial z^*} & \frac{\partial \bar{\xi}^*}{\partial \overline{z}^*} \end{pmatrix}^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}. \tag{36.18}$$

Also,

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial (z+th)^*}{\partial z^*} & \frac{\partial (z+th)^*}{\partial \overline{z}^*} \\ \frac{\partial \overline{(z+th)}^*}{\partial z^*} & \frac{\partial \overline{(z+th)}^*}{\partial \overline{z}^*} \end{pmatrix}^{-1} \Big|_{t=0} = -\begin{pmatrix} \frac{\partial h^*}{\partial z^*} & \frac{\partial h^*}{\partial \overline{z}^*} \\ \frac{\partial \bar{h}^*}{\partial z^*} & \frac{\partial \bar{h}^*}{\partial \overline{z}^*} \end{pmatrix}. \tag{36.19}$$

Next, we look at the last matrix. At $\xi = z$, we have

$$\frac{\partial \xi^k}{\partial \overline{z}^q} + U^p_{\bar{q}} \frac{\partial \xi^k}{\partial z^p} = U^p_{\bar{q}}\delta^k_p = U^k_{\bar{q}}. \tag{36.20}$$

The linearization of this term is

$$\frac{d}{dt}\Big(\frac{\partial (z+th)^k}{\partial \overline{z}^q} + U^p_{\bar{q}}\frac{\partial (z+th)^k}{\partial z^p}\Big)\Big|_{t=0} = \frac{\partial h^k}{\partial \overline{z}^q} + U^p_{\bar{q}}\frac{\partial h^k}{\partial z^p}. \tag{36.21}$$

Next, we look at the middle matrix. At $\xi = z$, we have

$$\Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \overline{z}^*} + U^p_* \frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)^{\bar{q}}_{\bar{j}} = (I_n)^{\bar{q}}_{j}. \tag{36.22}$$

The linearization is

$$\frac{d}{dt}\Big(\Big(\frac{\partial \overline{(z+th)}^*}{\partial \overline{z}^*} + U^p_* \frac{\partial \overline{(z+th)}^*}{\partial z^p}\Big)^{-1}\Big)^{\bar{q}}_{\bar{j}}\Big|_{t=0} = -\Big(\frac{\partial \bar{h}^{\bar{q}}}{\partial \overline{z}^j} + U^p_{\bar{j}}\frac{\partial \bar{h}^{\bar{q}}}{\partial z^p}\Big) \tag{36.23}$$

Putting everything together, we obtain

$$F_z'(h) = -\sum_j \Big(\frac{\partial h^l}{\partial z^j}\frac{\partial}{\partial z^l} + \frac{\partial \bar{h}^l}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{z}^l}\Big)\Big(\delta_{\bar{j}}^{\bar{q}} U_{\bar{q}}^k\Big) - \sum_j \frac{\partial}{\partial z^j}\Big(\frac{\partial \bar{h}^{\bar{q}}}{\partial \overline{z}^j} + U_{\bar{j}}^p \frac{\partial \bar{h}^{\bar{q}}}{\partial z^p}\Big) U_{\bar{q}}^k \tag{36.24}$$

$$- \sum_j \frac{\partial}{\partial z^j}\delta_{\bar{j}}^{\bar{q}}\Big(\frac{\partial h^k}{\partial \overline{z}^q} + U_{\bar{q}}^p \frac{\partial h^k}{\partial z^p}\Big) \tag{36.25}$$

$$= -\sum_j \Big(\frac{\partial h^l}{\partial z^j}\frac{\partial}{\partial z^l} + \frac{\partial \bar{h}^l}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{z}^l}\Big) U_{\bar{j}}^k - \sum_j \frac{\partial}{\partial z^j}\Big(\frac{\partial \bar{h}^{\bar{q}}}{\partial \overline{z}^j} + U_{\bar{j}}^p \frac{\partial \bar{h}^{\bar{q}}}{\partial z^p}\Big) U_{\bar{q}}^k \tag{36.26}$$

$$- \sum_j \frac{\partial}{\partial z^j}\Big(\frac{\partial h^k}{\partial \overline{z}^j} + U_{\bar{j}}^p \frac{\partial h^k}{\partial z^p}\Big). \tag{36.27}$$

This is of the form

$$F_z'(h) = -\frac{1}{4}\Delta h + \nabla h * \nabla U + \nabla^2 h * U + \nabla h * U * \nabla U + \nabla^2 h * U^2. \tag{36.28}$$

At the origin, all of the terms with U vanish by assumption, so we have

$$F_z'(h)(0) = -\frac{1}{4}\Delta h + \nabla h * \nabla U \tag{36.29}$$

□

From the above discussion on coordinate changes, let $\xi = \epsilon^{-1} z$. If $\xi \in B_1(0)$ then $z \in B_\epsilon(0)$. With $\phi_{\bar{j}}^k(z, \overline{z}) = U_{\bar{j}}^k(\epsilon^{-1} z, \epsilon^{-1}\overline{z})$, then we have

$$\tilde{U}_{\bar{j}}^k(\xi, \bar{\xi}) = \epsilon \cdot \epsilon^{-1} \cdot U_{\bar{j}}^k(\xi, \bar{\xi}) = U_{\bar{j}}^k(\xi, \bar{\xi}). \tag{36.30}$$

Note that

$$\|\tilde{U}\|_{C^0(B_1(0))} = \|\phi\|_{C^0(B_\epsilon(0))}, \tag{36.31}$$

But since $\phi \in C^1(B_\epsilon(0)) \subset C^{1,\alpha}(B_\epsilon(0))$, by the mean value theorem we have

$$|\phi(x) - \phi(y)| \leq C|x - y|. \tag{36.32}$$

Letting $y = 0$, since $\phi(y) = 0$ by assumption, we have

$$|\phi(x)| \leq C|x|. \tag{36.33}$$

Therefore, we have

$$\|\tilde{U}\|_{C^0(B_1(0))} \leq C\epsilon. \tag{36.34}$$

Next, with a slight abuse of notation, we have

$$\|\nabla\tilde{U}\|_{C^0(B_1(0))} = \sup_{\xi\in B_1(0)} \Big|\frac{\partial\tilde{U}(\xi,\bar{\xi})}{\partial\xi}\Big| + \Big|\frac{\partial\tilde{U}(\xi,\bar{\xi})}{\partial\bar{\xi}}\Big| \tag{36.35}$$

$$= \sup_{\xi\in B_1(0)} \Big|\frac{\partial\phi(\epsilon\xi,\epsilon\bar{\xi})}{\partial\xi}\Big| + \Big|\frac{\partial\phi(\epsilon\xi,\epsilon\bar{\xi})}{\partial\bar{\xi}}\Big| \tag{36.36}$$

$$= \epsilon \sup_{z\in B_\epsilon(0)} \Big|\frac{\partial\phi(z,\overline{z})}{\partial z}\Big| + \Big|\frac{\partial\phi(z,\overline{z})}{\partial\overline{z}}\Big| \tag{36.37}$$

$$= \epsilon\cdot\|\nabla\phi\|_{C^0(B_\epsilon(0))}. \tag{36.38}$$

Also, we compute

$$\sup_{x,y\in B_1(0),x\neq y} \frac{|\nabla\tilde{U}(x)-\nabla\tilde{U}(y)|}{|x-y|^\alpha} = \sup_{x,y\in B_\epsilon(0),x\neq y} \frac{\epsilon\nabla_z\phi(x,\bar{x})-\epsilon\nabla_z\phi(y,\bar{y})}{\epsilon^{-\alpha}|x-y|^\alpha} \tag{36.39}$$

$$= \epsilon^{1+\alpha} \sup_{x,y\in B_\epsilon(0),x\neq y} \frac{\nabla_z\phi(x,\bar{x})-\nabla_z\phi(y,\bar{y})}{|x-y|^\alpha}. \tag{36.40}$$

Since the $C^{1,\alpha}$ norm is the sum of these 3 parts, we can assume without loss of generality that

$$\|\phi\|_{C^{1,\alpha}(B_1(0))} < \epsilon. \tag{36.41}$$

for any $\epsilon > 0$.

Proposition 36.3. *For ϵ sufficiently small, the linearized operator*

$$F'_z : C^{2,\alpha}_0(B_1(0)) \to C^{0,\alpha}(B_1(0)) \tag{36.42}$$

is invertible with bounded inverse (independent of ϵ), where the domain satisfies Dirichlet boundary conditions.

Proof. The leading term in F'_z is just the vector Laplacian, which is completely uncoupled. So by standard elliptic elliptic theory for the Laplacian (on functions), we know that the leading term is invertible, with bounded inverse (with Dirichlet boundary conditions on each component). We next show that for ϵ sufficiently small, F'_z will be an arbitrarily small perturbation of the Laplacian in operator norm. To see this, we write

$$F'_z(h) = -\frac{1}{4}\Delta h + Qh, \tag{36.43}$$

where Q are the lower order terms. Let $\mathcal{B}_1 = C^{2,\alpha}(B_1(0))_0$ and $\mathcal{B}_2 = C^{0,\alpha}(B_1(0))$. Recall that for the Hölder norms, we have

$$\|fg\|_{C^{k,\alpha}} \leq \|f\|_{C^{k,\alpha}}\cdot\|g\|_{C^{k,\alpha}}, \tag{36.44}$$

so we estimate

$$\begin{aligned}
\|Qh\|_{\mathcal{B}_2} &= \|(\phi+\phi^2)*\nabla^2 h + (\nabla\phi + \phi * \nabla\phi) * \nabla h\|_{\mathcal{B}_2} && (36.45)\\
&\leq (\|\phi\|_{\mathcal{B}_2} + \|\phi\|_{\mathcal{B}_2}^2)\cdot\|\nabla^2 h\|_{\mathcal{B}_2} + (\|\nabla\phi\|_{\mathcal{B}_2} + \|\phi\|_{\mathcal{B}_2}\|\nabla\phi\|_{\mathcal{B}_2})\cdot\|\nabla h\|_{\mathcal{B}_2} && (36.46)\\
&\leq (\epsilon+\epsilon^2)\cdot\|h\|_{\mathcal{B}_1}. && (36.47)
\end{aligned}$$

So the operator norm of Q is estimated

$$\sup_{0\neq h\in\mathcal{B}_1}\frac{\|Qh\|_{\mathcal{B}_2}}{\|h\|_{\mathcal{B}_1}} \leq \epsilon+\epsilon^2. \tag{36.48}$$

So by the above inverse function theorem, Lemma 21.2, F'_z is also invertible with bounded inverse if ϵ is sufficiently small. □

Remark 36.4. Note that the above proof reduced everything to invertibility of the Laplacian on functions, we did not need to quote any results about elliptic systems of PDEs.

Proposition 36.5. *If ϵ is sufficiently small then*

$$\|F(z)\|_{\mathcal{B}_2} < C\epsilon. \tag{36.49}$$

Proof. From the above computations, we have

$$F(z) = \sum_j \frac{\partial}{\partial z^j}\phi^k_{\bar{j}}, \tag{36.50}$$

so we have

$$\|F(z)\|_{C^{0,\alpha}(B_1(0))} \leq C\|\nabla\phi\|_{C^{0,\alpha}(B_1(0))} \leq C\|\phi\|_{C^{1,\alpha}(B_1(0))} \leq C\epsilon. \tag{36.51}$$

□

37 Lecture 37

37.1 Inverse function theorem

We recall once again the inverse function theorem.

Lemma 37.1. *Let $\mathscr{F}:\mathcal{B}_1\to\mathcal{B}_2$ be a C^1-map between two Banach spaces such that $\mathscr{F}(x) = \mathscr{F}(0)+\mathscr{L}(x)+\mathscr{Q}(x)$, where the operator $\mathscr{L}:\mathcal{B}_1\to\mathcal{B}_2$ is linear and $\mathscr{Q}(0)=0$. Assume that*

1. *$\mathscr{L}$ is an isomorphism with inverse T satisfying $\|T\|\leq C_1$,*
2. *there are constants $r>0$ and $C_2>0$ with $r<\frac{1}{3C_1C_2}$ such that*
   (a) *$\|\mathscr{Q}(x)-\mathscr{Q}(y)\|_{\mathcal{B}_2}\leq C_2\cdot(\|x\|_{\mathcal{B}_1}+\|y\|_{\mathcal{B}_1})\cdot\|x-y\|_{\mathcal{B}_1}$ for all $x,y\in B_r(0)\subset\mathcal{B}_1$,*
   (b) *$\|\mathscr{F}(0)\|_{\mathcal{B}_2}\leq\frac{r}{3C_1}$.*

Then there exists a unique solution to $\mathscr{F}(x) = 0$ in $\mathcal{B}_1$ such that

$$\|x\|_{\mathcal{B}_1} \leq 3C_1 \cdot \|\mathscr{F}(0)\|_{\mathcal{B}_2}. \tag{37.1}$$

To use the inverse function theorem, Lemma 37.1, it remains to verify the estimate on the non-linear terms. Recall $\mathcal{B}_1 = C_0^{2,\alpha}(B_1(0), \mathbb{R}^{2n})$ and $\mathcal{B}_2 = C^{0,\alpha}(B_1(0), \mathbb{R}^{2n})$. Note that F and ξ are vector-valued, but for simplicity of notation in the following discussion, we will assume they are scalar-valued. Let write our nonlinear operator as $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ as

$$\mathcal{F}(\xi) = F(D^2\xi, D\xi, \xi, z, \overline{z}), \tag{37.2}$$

where $F : \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R} \times B_1(0) \to \mathbb{R}$. Write these variables as (r_{ij}, p_i, u, x). From (36.11), we have that for any fixed $x \in B_1(0)$, F is analytic in the r_{ij}, p_i, and u variables, and

$$F, \nabla_{r,p,u}F, \nabla^2_{r,p,u}F \in C^{0,\alpha}(\mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R} \times B_1(0)). \tag{37.3}$$

Note that we are slightly abusing notation, since this is only true on the subset for which the inverse matrix in (36.12) exists. Define

$$H : C^{2,\alpha}(B_1(0), \mathbb{R}) \to C^{0,\alpha}(B_1(0), \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R}) \tag{37.4}$$

by

$$u \mapsto (\nabla^2 u, \nabla u, u). \tag{37.5}$$

Then we can write

$$\mathcal{F}(u) = G \circ H(u), \tag{37.6}$$

where $G : C^{0,\alpha}(B_1(0), \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R}) \to C^{0,\alpha}(B_1(0), \mathbb{R})$ is defined by

$$G(r_{ij}(x), p_i(x), u(x)) = F(r_{ij}(x), p_i(x), u(x), x). \tag{37.7}$$

We want to show the estimate on the nonlinear terms

$$\|\mathcal{F}(u_2) - \mathcal{F}(u_1) - \mathcal{F}'_z(u_2 - u_1)\|_{\mathcal{B}_2} \leq C(\|u_1\|_{\mathcal{B}_1} + \|u_2\|_{\mathcal{B}_1}) \cdot \|u_1 - u_2\|_{\mathcal{B}_1}. \tag{37.8}$$

From the chain rule, we have

$$\mathcal{F}'_z(h) = G'_{H(z)} \circ H'_z(h). \tag{37.9}$$

But H is a bounded linear operator, because

$$\|H(u)\|_{C^{0,\alpha}} = \|(\nabla^2 u, \nabla u, u)\|_{C^{0,\alpha}} \leq \|u\|_{C^{2,\alpha}}. \tag{37.10}$$

So if we show for $a_1, a_2 \in C^{0,\alpha}(B_1(0), \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R})$ that

$$\|G(a_2) - G(a_1) - G'_{H(z)}(a_2 - a_1)\|_{\mathcal{B}_2} \leq C(\|a_1\|_{C^{0,\alpha}} + \|a_2\|_{C^{0,\alpha}}) \cdot \|a_2 - a_1\|_{C^{0,\alpha}}, \tag{37.11}$$

then since H is linear,

$$\begin{aligned}\|\mathcal{F}(u_2)-\mathcal{F}(u_1)-\mathcal{F}'_z(u_2-u_1)\|_{\mathcal{B}_2} &= \|G\circ H(u_2)-G\circ H(u_1)-G'_{H(z)}\circ H'_z(u_2-u_1)\|_{\mathcal{B}_2}\\ &= \|G\circ H(u_2)-G\circ H(u_1)-G'_{H(z)}\circ(Hu_2-Hu_1)\|_{\mathcal{B}_2}\\ &\leq C(\|H(u_2)\|_{\mathcal{B}_2}+\|H(u_1)\|_{\mathcal{B}_2})\cdot\|H(u_2)-H(u_1)\|_{\mathcal{B}_2}\\ &\leq C'(\|u_2\|_{\mathcal{B}_2}+\|u_1\|_{\mathcal{B}_2})\cdot\|u_2-u_1\|_{\mathcal{B}_2}.\end{aligned} \tag{37.12}$$

So we just need to show an estimate on G. Again, the fact that the domain of G is vector-valued functions doesn't matter, so for simplicity, we just assume that we have $G : C^{0,\alpha}(B_1(0),\mathbb{R}) \to C^{0,\alpha}(B_1(0),\mathbb{R})$ where $G(u(x)) = F(u(x),x)$, and $F : \mathbb{R}\times B_1(0) \to \mathbb{R}$, with

$$F, F_u, F_{uu} \in C^{0,\alpha}(\mathbb{R}\times B_1(0)). \tag{37.13}$$

The linearized operator of G at a function u_0 is simply

$$G'_{u_0}(h) = \frac{d}{dt}F((u_0+th)(x),x)|_{t=0} = F_u(u_0(x),x)h. \tag{37.14}$$

By considering the function $G(u_0+u)$ instead, which satisfies the same properties as the original G, we can assume that $u_0 = 0$. We let

$$f(t) = G((1-t)u_1+tu_2). \tag{37.15}$$

The fundamental theorem of calculus says

$$f(1)-f(0) = \int_0^1 f'(t)dt, \tag{37.16}$$

which is

$$G(u_2)-G(u_1) = \int_0^1 G'_{(1-t)u_1+tu_2}(u_2-u_1)dt. \tag{37.17}$$

We rewrite this as

$$G(u_2)-G(u_1)-G'_0(u_2-u_1) = \int_0^1 \Big(G'_{(1-t)u_1+tu_2}-G'_0\Big)(u_2-u_1)dt. \tag{37.18}$$

So we need to prove the estimate

$$\|(G'_u-G'_0)h\|_{C^{0,\alpha}(B_1(0))} \leq C\|h\|_{C^{0,\alpha}(B_1(0))}\cdot\|u\|_{C^{0,\alpha}(B_1(0))}. \tag{37.19}$$

But we have

$$(G'_u-G'_0)h = (F_u(u(x),x)-F_u(0,x))h, \tag{37.20}$$

which implies

$$\|(G'_u - G'_0)h\|_{C^{0,\alpha}(B_1(0))} \leq \|F_u(u(x),x) - F_u(0,x)\|_{C^{0,\alpha}(B_1(0))}\|h\|_{C^{0,\alpha}(B_1(0))}, \tag{37.21}$$

and we just need to prove the estimate

$$\|F_u(u(x),x) - F_u(0,x)\|_{C^{0,\alpha}(B_1(0))} \leq C\|u\|_{C^{0,\alpha}(B_1(0))}. \tag{37.22}$$

Now we let $f(t) = F_u(tu(x),x)$. The fundamental theorem of calculus gives

$$F_u(u(x),x) - F_u(0,x) = \int_0^1 F_{uu}(tu(x),x)u(x)dt. \tag{37.23}$$

First, we estimate the C^0-norm

$$\begin{aligned} \|F_{uu}(tu(x),x)u(x)\|_{C^0(B_1(0))} &\leq \|F_{uu}(tu(x),x)\|_{C^0(B_1(0))}\|u(x)\|_{C^0(B_1(0))} \\ &\leq C\|u(x)\|_{C^0(B_1(0))}, \end{aligned} \tag{37.24}$$

as long as u is small enough so that $u(x)$ is in the domain of definition of F_{uu}.

Next, we estimate the C^α semi norm. Note that

$$\begin{aligned} (f\cdot g)(x) - (f\cdot g)(y) &= f(x)g(x) - f(y)g(y) \\ &= f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y) \\ &\leq (f(x) - f(y))g(x) + f(y)(g(x) - g(y)), \end{aligned} \tag{37.25}$$

which implies the estimate

$$[fg]_\alpha \leq [f]_\alpha \|g\|_{C^0} + \|f\|_{C^0}[g]_\alpha. \tag{37.26}$$

So we estimate

$$[F_{uu}(tu(x),x)u(x)]_\alpha \leq [F_{uu}(tu(x),x)]_\alpha \|u\|_{C^0} + \|F_{uu}(tu(x),x)\|_{C^0}[u]_\alpha. \tag{37.27}$$

We have that

$$|F_{uu}(tu(x),x) - F_{uu}(tu(y),y))| \leq [F_{uu}]_\alpha(|tu(x) - tu(y)| + |x-y|)^\alpha \tag{37.28}$$

$$\leq [F_{uu}]_\alpha(t[u]_\alpha|x-y|^\alpha + |x-y|)^\alpha \leq C[F_{uu}]_\alpha, \tag{37.29}$$

as long as $[u]_\alpha$ is bounded. Putting all this together, we have

$$\begin{aligned} \|F_u(u(x),x) - F_u(0,x)\|_{C^{0,\alpha}(B_1(0))} &\leq C\|F_{uu}(tu(x),x)u(x)\|_{C^{0,\alpha}(B_1(0))} \\ &\leq C\|u\|_{C^{0,\alpha}(B_1(0))}. \end{aligned} \tag{37.30}$$

Returning to our problem, we have

$$\mathcal{F}(h) = F(z+h), \tag{37.31}$$

satisfying

$$\|\mathcal{F}(0)\|_{\mathcal{B}_2} \leq C\epsilon, \tag{37.32}$$

and $\mathcal{F}'_0 : \mathcal{B}_1 \to \mathcal{B}_2$ is invertible with bounded inverse independent of ϵ, for sufficiently small ϵ. Letting

$$\mathcal{F}(h) = \mathcal{F}(0) + \mathcal{F}'_0(h) + \mathcal{Q}(h), \tag{37.33}$$

we have proved that there exists a constant C so that

$$\|\mathcal{Q}(x) - \mathcal{Q}(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x - y\|_{\mathcal{B}_1} \tag{37.34}$$

So by Lemma 23.3, there exists a solution to $\mathcal{F}(h) = 0$ satisfying

$$\|h\|_{\mathcal{B}_1} \leq C\|\mathcal{F}(0)\|_{\mathcal{B}_2} \leq C\epsilon. \tag{37.35}$$

Then the vector-valued function $z^j + h^j$ satisfies

$$\nabla_i(z^j + h^j) = \delta_i^j + \nabla_i h_j, \tag{37.36}$$

so for ϵ sufficiently small the Jacobian at 0 is invertible, and we have therefore found the required coordinate system, which is of class $C^{2,\alpha}$.

37.2 Analyticity

We begin this subsection with the following observation.

Proposition 37.2. *Let (M_2, J) be an almost complex manifold, and $f : M_1 \to M_2$ a C^1 diffeomorphism. Then $N_{f^*J} = f^* N_J$. Consequently, the equations of integrability are independent of the coordinate system.*

Proof. Recall the definition of the Nijenhuis tensor

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\}. \tag{37.37}$$

Let $X, Y \in \Gamma(TM_2)$. Then $(f^*J)(X) = f_*^{-1} J f_* X$. Also, for a diffeomorphism, we have $f_*[X, Y] = [f_*X, f_*Y]$; see [Spi79a]. Therefore

$$\begin{aligned} N_{f^*J}(X, Y) &= 2\{[f_*^{-1}Jf_*X, f_*^{-1}Jf_*Y] - [X, Y] \\ &\qquad - f_*^{-1}Jf_*[X, f_*^{-1}Jf_*Y] - f_*^{-1}Jf_*[f_*^{-1}Jf_*X, Y]\} \\ &= 2\{f_*^{-1}[Jf_*X, Jf_*Y] - f_*^{-1}[f_*X, f_*Y] - f_*^{-1}J[f_*X, Jf_*Y] - f_*^{-1}J[Jf_*X, f_*Y]\} \\ &= 2f_*^{-1}\{[Jf_*X, Jf_*Y] - [f_*X, f_*Y] - J[f_*X, Jf_*Y] - J[Jf_*X, f_*Y]\} \\ &= f_*^{-1}N_J(f_*X, f_*Y) = f^*N_J. \end{aligned} \tag{37.38}$$

□

By assumption, the complex structure J corresponding to ϕ is integrable, so by Proposition 33.1, we have

$$\frac{\partial}{\partial \overline{z}^l}\phi_{\overline{k}}^j - \frac{\partial}{\partial \overline{z}^k}\phi_{\overline{l}}^j + \phi_{\overline{k}}^m \frac{\partial}{\partial z^m}\phi_{\overline{l}}^j - \phi_{\overline{l}}^m \frac{\partial}{\partial z^m}\phi_{\overline{k}}^j = 0. \tag{37.39}$$

Let $f : U \to V$ be the change of coordinates mapping from the ξ-coordinates to the z-coordinates. Clearly f^*J is associated to $f^*\phi$, and the above derivation shows that the components of $f^*\phi$ in the ξ-coordinates are given by $\tilde{U}$. By Proposition 37.2, the integrability condition is independent of coordinates, so we have that

$$\frac{\partial}{\partial \bar{\xi}^l}\tilde{U}^j_{\bar{k}} - \frac{\partial}{\partial \bar{\xi}^k}\tilde{U}^j_{\bar{l}} + \tilde{U}^m_{\bar{k}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{l}} - \tilde{U}^m_{\bar{l}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{k}} = 0. \tag{37.40}$$

Above, we have found the coordinates ξ so that

$$\sum_j \frac{\partial}{\partial \xi^j}\tilde{U}^k_{\bar{j}}(\xi, \bar{\xi}) = 0. \tag{37.41}$$

Now we view the coupled system (37.40)-(37.41) as an equation for $\tilde{U}^k_{\bar{j}}$ in the new ξ-coordinates.

Proposition 37.3. *If $\|\tilde{U}\|_{C^0}$ is sufficiently small, then the system* (37.40)-(37.41) *is an overdetermined elliptic first-order system with analytic coefficients.*

Proof. We need to linearize at $\tilde{U}$, but under the assumptions, it is clearly equivalent to proving ellipticity for the system

$$\frac{\partial}{\partial \bar{z}^l}\phi^j_{\bar{k}} - \frac{\partial}{\partial \bar{z}^k}\phi^j_{\bar{l}} = 0 \tag{37.42}$$

$$\sum_j \frac{\partial}{\partial z^j}\phi^k_{\bar{j}} = 0. \tag{37.43}$$

For $(\xi, \bar{\xi})$ a complex cotangent vector, the symbol is

$$\phi \mapsto \Big(\bar{\xi}_l\phi^j_{\bar{k}} - \bar{\xi}_{\bar{k}}\phi^j_{\bar{l}}, \sum_j \xi_j \phi^k_{\bar{j}}\Big) \tag{37.44}$$

If the right hand side vanishes, then we have

$$0 = \sum_k \xi_k\bar{\xi}_l\phi^j_{\bar{k}} - \sum_k \xi_k\bar{\xi}_{\bar{k}}\phi^j_{\bar{l}} = -|\xi|^2\phi^j_{\bar{l}}. \tag{37.45}$$

So if $\xi \neq 0$, then the symbol mapping is injective. □

Remark 37.4. In other words, applying $\partial/\partial\xi^k$ to the first equations, and using the second equation yields

$$-\frac{\partial}{\partial \xi^k}\frac{\partial}{\partial \bar{\xi}^k}\tilde{U}^j_{\bar{l}} + \frac{\partial}{\partial \xi^k}\Big(\tilde{U}^m_{\bar{k}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{l}} - \tilde{U}^m_{\bar{l}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{k}}\Big) = 0, \tag{37.46}$$

which is a determined second-order elliptic system, if $\|\tilde{U}\|_{C^0}$ is sufficiently small.

Since the PDE system is analytic, a classical result implies that $\tilde{U}^k_{\bar{j}}$ are then analytic functions in the ξ coordinates; see [Mor66, Chapter 6.7]. So we have proved:

Theorem 37.5. *If (M, J) satisfies $J \in C^{1,\alpha}$ and J is an integrable complex structure, then there exists a $C^{2,\alpha}$ coordinate system defined in a neighborhood of any point such that J is real analytic in these coordinates. Consequently, there exists a $C^{2,\alpha}$ holomorphic coordinate system for J.*

Remark 37.6. By further analysis of the Malgrange system, Hill-Taylor have reduced the regularity assumption; [HT03]. For example, it suffices to assume $J \in W^{1,p}$, for $p > 2n$.

38 Lecture 38

Let's recall again some basics of vector bundles. Previously, we discussed real vector bundles, but now we will discuss complex vector bundles. Most things are the same, although there are some important differences such as holomorphic vector bundles and Hermitian metrics.

38.1 Complex vector bundles

We begin with the following definition.

Definition 38.1. A complex vector bundle of rank k over a smooth manifold M^n is a topological space E together with a smooth projection

$$\pi : E \to M \tag{38.1}$$

such that

- For $p \in M$, $\pi^{-1}(p)$ is a vector space of dimension k over $\mathbb{C}$.
- There exists local trivializations, that is, there are smooth mappings

  $$\Phi_\alpha : U_\alpha \times \mathbb{C}^k \to E \tag{38.2}$$

  which maps $p \times \mathbb{C}^k$ linearly onto the fiber $\pi^{-1}(p)$ for every $p \in U_\alpha$.

The transition functions of a bundle are defined as follows.

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C}) \tag{38.3}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{38.4}$$

for $v \in \mathbb{C}^k$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the cocycle condition

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{38.5}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satisfying (38.5), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$. (It turns out this bundle is uniquely defined up to bundle equivalence, which we will define below.)

Definition 38.2. Assume that the base M is a smooth manifold. If the transition function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth complex vector bundle.

Definition 38.3. Assume that the base M is a complex manifold. If the transition function $\varphi_{\alpha\beta}$ are holomorphic (where $GL(k, \mathbb{C})$ is given the complex structure as an open subset of $\mathbb{C}^{k^2}$), then we say that E is a holomorphic vector bundle. If $k = 1$ we say that E is a holomorphic line bundle.

A vector bundle mapping is a mapping $F : E_1 \to E_2$ which is linear on fibers, and covers the identity map, that is, the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ M & \xrightarrow{id} & M. \end{array} \tag{38.6}$$

Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions

$$F_\alpha : U_\alpha \to Hom(\mathbb{C}^{k_1}, \mathbb{C}^{k_2}) \tag{38.7}$$

defined by

$$F_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)). \tag{38.8}$$

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$F_\alpha = \varphi^{E_2}_{\alpha\beta} F_\beta \varphi^{E_1}_{\beta\alpha}, \tag{38.9}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} F_\alpha = F_\beta \varphi^{E_1}_{\beta\alpha}. \tag{38.10}$$

We say that two bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $F : E_1 \to E_2$. This is equivalent to non-singularity of the local representatives, that is, $\det(F_\alpha) \neq 0$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions

$$f_\alpha : U_\alpha \to GL(k, \mathbb{C}) \tag{38.11}$$

such that

$$\varphi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{38.12}$$

Definition 38.4. A section of $\pi : E \to M$ is a smooth mapping $\sigma : M \to E$ such that $\pi \circ \sigma = Id_M$.

In terms of local trivializations, we define

$$s_\alpha = \pi_{\mathbb{C}^k} \circ \Phi_\alpha^{-1} \circ s : U_\alpha \to \mathbb{C}^k, \tag{38.13}$$

which yield the defining relation for a section

$$s_\alpha = \phi_{\alpha\beta} s_\beta. \tag{38.14}$$

The space of sections of a vector bundle is denoted by $\Gamma(M, E)$, which is a vector space. In the smooth category, this is infinite dimensional, but if M is compact then the space of holomorphic sections is always finite-dimensional.

Example 38.5. Let the base $M = \mathbb{P}^1$. The equivalence class of a point is denoted by $[z_0, z_1]$. We can cover $\mathbb{P}^1$ by two open sets $U_0 = \{[1, z] \mid z \in \mathbb{C}\}$ and $U_1 = \{[z, 1] \mid z \in \mathbb{C}\}$. The intersection $U_0 \cap U_1$ can be identified with $\mathbb{C}^*$. Since there is only 1 non-trivial intersection, to define a holomorphic line bundle over $\mathbb{P}^1$, we only need to specify a single holomorphic function

$$\phi_{12} : \mathbb{C}^* \simeq U_0 \cap U_1 \to GL(1, \mathbb{C}) \simeq \mathbb{C}^* \tag{38.15}$$

For any $k \in \mathbb{Z}$, take $\phi_{01} = (z_1/z_0)^k$. This defines a holomorphic line bundle over $\mathbb{P}^1$, which we denote as $\pi : \mathcal{O}_{\mathbb{P}^1}(k) \to \mathbb{P}^1$.

Later, we will prove that $\mathcal{O}_{\mathbb{P}^1}(k)$ for $k \in \mathbb{Z}$ is the complete list of holomorphic line bundles on $\mathbb{P}^1$ (up to equivalence), but for now you can assume this for the following exercise.

Exercise 38.6. The function $\phi_{01} = e^z$ defines a bundle on $\mathbb{P}^1$. For which k is this isomorphic to $\mathcal{O}_{\mathbb{P}^1}(k)$?

The follows proposition tells us the space of holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(k)$.

Proposition 38.7. *The space of holomorphic section is*

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = \begin{cases} \mathbb{C}^{k+1} & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{38.16}$$

Proof. A section is given by functions $s_0 : U_0 \to \mathbb{C}$ and $s_1 : U_1 \to \mathbb{C}$ satisfying $s_0 = \phi_{01} s_1 = (z_1/z_0)^k s_1$. If we pull back everything to the U_1 coordinates, this will be

$$s_0(z) = z^k s_1(1/z). \tag{38.17}$$

Then s_0 is an entire function on $\mathbb{C}$ satisfying $s_0 = O(z^k)$ as $z \to \infty$. By Liouville's theorem, s_0 is a polynomial of degree k if $k \geq 0$, but vanishes identically if $k < 0$. □

We can also define a family of line bundles on $\mathbb{P}^n$.

Example 38.8. Let $\mathbb{P}^n = [z_0, \dots, z_n]$. This is covered by $n + 1$ copies of $\mathbb{C}^n$ by letting $U_j = [z_0, \dots, 1_j, \dots, z_n]$. On $U_i \cap U_j$, we defines the transition function as

$$\phi_{ij} = (z_j/z_i)^m. \tag{38.18}$$

We have

$$\phi_{ij}\phi_{jk} = (z_j/z_i)^m (z_k/z_j)^m = (z_k/z_i)^m = \phi_{ik}, \tag{38.19}$$

so these satisfy the cocycle condition, and therefore defines a line bundle on $\mathbb{P}^n$ which we denote by $\pi : \mathcal{O}_{\mathbb{P}^n}(k) \to \mathbb{P}^n$.

Exercise 38.9. Prove that

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \begin{cases} \mathbb{C}^{\binom{m+n}{n}} & m \geq 0 \\ 0 & m < 0 \end{cases} \tag{38.20}$$

(Hint: for $m \geq 0$, this is the dimension of homogeneous polynomials of degree m in $n + 1$ variables.)

We will show later that $\mathcal{O}_{\mathbb{P}^n}(m)$ for $m \in \mathbb{Z}$ is the complete list of holomorphic line bundles on $\mathbb{P}^n$ up to equivalence.

38.2 Operations on bundles

Note that if $f_1 : \mathbb{C}^k \to \mathbb{C}^k$ and $f_2 : \mathbb{C}^l \to \mathbb{C}^l$ are linear maps then there is an obvious induced mapping $f_1 \oplus f_2 : \mathbb{C}^k \oplus \mathbb{C}^l \to \mathbb{C}^k \oplus \mathbb{C}^l$ defined by

$$(f_1 \oplus f_2)(v_1, v_2) \equiv f_1(v_1) \oplus f_2(v_2). \tag{38.21}$$

The direct sum $E_1 \oplus E_2$ of bundles E_1 and E_2 is a vector bundle with transition functions

$$\varphi_{\alpha\beta}^{E_1 \oplus E_2} = \varphi_{\alpha\beta}^{E_1} \oplus \varphi_{\alpha\beta}^{E_2}. \tag{38.22}$$

Note that if $f_1 : \mathbb{C}^k \to \mathbb{C}^k$ and $f_2 : \mathbb{C}^l \to \mathbb{C}^l$ are linear maps, then there is an obvious induced mapping $f_1 \otimes f_2 : \mathbb{C}^k \otimes \mathbb{C}^l \to \mathbb{C}^k \otimes \mathbb{C}^l$ defined on indecomposable tensors by

$$(f_1 \otimes f_2)(v_1 \otimes v_2) \equiv f_1(v_1) \otimes f_2(v_2), \tag{38.23}$$

and extended by linearity to the tensor product. The tensor product $E_1 \otimes E_2$ of bundles E_1 and E_2 is again a bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E_1 \otimes E_2} = \varphi_{\alpha\beta}^{E_1} \otimes \varphi_{\alpha\beta}^{E_2}. \tag{38.24}$$

The dual $E^* = Hom(E, \mathbb{C})$ of any bundle E, is a bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E^*} = \left((\varphi_{\alpha\beta}^{E})^{-1}\right)^T = (\varphi_{\beta\alpha}^{E})^T. \tag{38.25}$$

Note that the inverse is necessary in order to have the cocycle condition satisfied. In general, if E_1 and E_2 are vector bundles, then we define $Hom(E_1, E_2) = E_1^* \otimes E_2$, which in terms of transition functions is

$$\varphi_{\alpha\beta}^{Hom(E_1,E_2)} = \left((\varphi_{\alpha\beta}^{E_1})^{-1}\right)^T \otimes \varphi_{\alpha\beta}^{E_2}. \tag{38.26}$$

Note that for any linear map $f : \mathbb{C}^k \to \mathbb{C}^k$, there is a naturally induced mapping

$$\Lambda^p f : \Lambda^p(\mathbb{C}^k) \to \Lambda^p(\mathbb{C}^k) \tag{38.27}$$

therefore for any vector bundle E, the pth exterior power $\Lambda^p(E)$ is defined to be the bundle with transition functions

$$\varphi_{\alpha\beta}^{\Lambda^p(E)} = \Lambda^p(\varphi_{\alpha\beta}^{E}). \tag{38.28}$$

For a complex vector bundle $\pi : E \to M$, there is another operation called the conjugate bundle $\overline{E}$ which is the complex vector bundle obtained by replacing each fiber of E with the complex conjugate vector space. The transition functions are simply

$$\varphi_{\alpha\beta}^{\overline{E}} = \overline{\varphi_{\alpha\beta}^{E}}. \tag{38.29}$$

Remark 38.10. In the above, we only defined morphisms in the category of vector bundle to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence, and is not usually used for vector bundle equivalence, we will discuss this later.

39 Lecture 39

39.1 Hermitian metrics on complex vector bundles

If $\pi : E \to M$ is a real vector bundle, then we saw previously that E always admits a Riemannian metric (using a partition of unity), which gave an isomorphism $E \simeq E^*$. For complex vector bundles, there is a difference, which we will explain next. Let $\pi : E \to M$ is a complex vector bundle, a Hermitian metric h on E is a choice of smoothly varying Hermitian inner product on each fiber. That is, h is $\mathbb{C}$-linear in the first argument, $\mathbb{C}$-antilinear in the second argument, and satisfies

$$h(v_1, v_2) = \overline{h(v_2, v_1)} \tag{39.1}$$

$$h(v, v) > 0 \text{ for } v \neq 0. \tag{39.2}$$

We can also view h as a section $\underline{h} \in \Gamma(E^* \otimes \overline{E}^*)$ as follows. For $v_1 \in E$ and $v_2 \in \overline{E}$, define

$$\underline{h}(v_1, v_2) = h(v_1, \overline{v_2}). \tag{39.3}$$

Then $\underline{h}$ is $\mathbb{C}$-linear in both arguments, and satisfies for $v_1 \in E$ and $v_2 \in \overline{E}$,

$$\underline{h}(v_1, v_2) = h(v_1, \overline{v_2}) = \overline{h(\overline{v_2}, v_1)} = \overline{\underline{h}(\overline{v_2}, \overline{v_1})} \tag{39.4}$$

and for $0 \neq v \in E$, $h(v, \overline{v}) > 0$.

Proposition 39.1. *If E is any complex vector bundle, then E admits a hermitian metric.*

Proof. Take the Euclidean metric on $\mathbb{C}^n$, i.e.,

$$h_{Euc}(v, w) = \sum v_j \overline{w}_j \tag{39.5}$$

on trivializations, and patch together using a partition of unity. □

Remark 39.2. Even if the bundle E is holomorphic, a Hermitian metric is never to be considered as a holomorphic object.

Corollary 39.3. *For any complex vector bundle E, we have $\overline{E}^* \cong E$. Equivalently, $\overline{E} \cong E^*$.*

Proof. Choose a hermitian metric h on E. Define the mapping $\flat : E \to \overline{E}^*$ by

$$\flat(v_1)(v_2) = h(v_1, \overline{v_2}) = \underline{h}(v_1, v_2) \tag{39.6}$$

Then $\flat$ is a $\mathbb{C}$-linear mapping. We show that $\flat$ is injective. Given $0 \neq v \in E$, then

$$\flat(v)(\overline{v}) = h(v, v) \neq 0, \tag{39.7}$$

which shows that $\flat v$ is not zero. Consequently, $\flat$ is an isomorphism. □

Remark 39.4. In the special case of a complex line bundle $\pi : L \to M$, we have that

$$L^* \otimes \overline{L}^* \simeq L^* \otimes L \tag{39.8}$$

is a trivial bundle (since the transition function for L^* are just the inverse of those for L). Therefore a choice of Hermitian metric h on a complex line bundle, which is equivalent to choice of $\underline{h}$, is equivalent to a choice of smooth positive real-valued function on M.

Proposition 39.5. *If M is compact, then for any holomorphic vector bundle $\pi : E \to M$, the space of holomorphic sections $\Gamma(M, E)$ is finite-dimensional.*

Proof. Choose a Hermitian metric h on E. Then

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) dV, \tag{39.9}$$

where dV is any choice of volume element. This gives a Hermitian inner product on $\Gamma(M, E)$, and makes $\Gamma(M, E)$ into a Hilbert space. Let s be any holomorphic section with $\|s\| = 1$. Let $U_i = \Delta_{r_i}(p_i)$ be an open cover of M by polydiscs which trivialize E, and so that any section is given by $s_i : U_i \to \mathbb{C}^k$ satisfying the compatibility condition $s_i = \phi_{ij} s_j$. Since s_i is holomorphic, it admits a power series expansion in U_i, $s_i = \sum_I c_I (z - z_0)^I$ which converges uniformly in any slightly smaller polydisc. Note that

$$\int_{\Delta_r(0)} z^I \bar{z}^J dV_{Euc} = \begin{cases} 0 & I \neq J \\ (2\pi)^n \prod_{k=1}^n \frac{1}{2j_k+2} r^{2j_k+2} & I = J \end{cases} \tag{39.10}$$

We then have

$$(2\pi)^n \sum_I |c_I|^2 \prod_{k=1}^n \frac{1}{2i_k + 2} r^{2I} r^2 = \int_{\Delta_r(p_i)} |s|^2 dV_{Euc} \leq C' \|s\|_{L^2} \leq C', \tag{39.11}$$

since the volume element is comparable to the Euclidean metric on U_i. So each coefficient satisfies

$$|c_I| \leq C' \frac{r^I r}{(2\pi)^{n/2}} \prod_{k=1}^n (2i_k + 2). \tag{39.12}$$

Then for $z \in \Delta_{r_1}$ with $r_1 < r - \epsilon$, we have

$$|s(z)| \leq \sum_I |c_I| r_1^{|I|} \leq C'' \sum_I \left(\frac{r_1}{r} \right)^{|I|} \leq C''', \tag{39.13}$$

so s is uniformly bounded in any slightly smaller ball.

Next, take any sequence s^j of holomorphic sections with $\|s^j\| = 1$. By the above, there is a uniform C^0 bound on s^j. So by Montel's theorem, by shrinking U_i slightly if necessary, there is a subsequence $s_i^{j'}$ which converges to a holomorphic limit on U_i. Since M is compact, the covering is finite, so passing to further subsequences we can obtain convergence on all of M. Then it is a basic result that if the unit ball in a Hilbert space is compact, then the Hilbert space must be finite dimensional. □

39.2 Sub-bundles and quotient bundles

Definition 39.6. Given vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ over the same base space M, and assume that $E_1 \subset E_2$. We say that E_1 is a subbundle of E_2, if each fiber $\pi_1^{-1}(x) \subset \pi_2^{-1}(x)$ is a vector subspace.

Definition 39.7. If $E_1 \subset E_2$ is a subbundle, then the quotient bundle E_2/E_1 is the vector bundle with fiber $\pi_2^{-1}(x)/\pi_1^{-1}(x)$ over x.

Exercise 39.8. Prove directly that the quotient bundle is a vector bundle. That is, find local trivializations for E_2/E_1.

Note the following corollary.

Corollary 39.9. *If $E_1 \subset E$ is a sub-bundle, then there exists a subbundle $E_2 \subset E$ such that*

$$E \cong E_1 \oplus E_2. \tag{39.14}$$

Furthermore, the quotient bundle $(E/E_1) \cong E_2$.

Proof. Choose a Hermitian metric h on E, and let $E_2 = (E_1)^{\perp}$. Use Gram-Schmidt to construct local trivializations for $(E_1)^{\perp}$ to show this is indeed a subbundle. The rest follows from linear algebra. □

39.3 Exact sequences

Let $\pi_j : E_j \to M$ be vector bundles over M. Assume that there are vector bundle mappings

$$E^1 \xrightarrow{F^1} E^2 \xrightarrow{F^2} E^3 \ . \tag{39.15}$$

We say that (39.15) is exact at E^2 if the induced mappings on fibers

$$E^1_x \xrightarrow{F^1_x} E^2_x \xrightarrow{F^2_x} E^3_x \ . \tag{39.16}$$

satisfies $\mathrm{Im}(F^1_x) = \mathrm{Ker}(F^2_x)$ for every $x \in M$.

If we have

$$0 \longrightarrow E^1_x \xrightarrow{F^1_x} E^2_x \xrightarrow{F^2_x} E^3_x \longrightarrow 0 \tag{39.17}$$

is exact at E^1_x, E^2_x and E^3_x for every $x \in M$, then we say that

$$0 \longrightarrow E^1 \xrightarrow{F^1} E^2 \xrightarrow{F^2} E^3 \longrightarrow 0 \tag{39.18}$$

is a short exact sequence of vector bundles. From above, this implies that $F^1(E^1)$ is a subbundle of E_2, and there is an isomorphism $E^3 \simeq E^2/F^1(E^1)$.

Conversely, if we assume that $E \subset F$ is a subbundle, there there is a short exact sequence

$$0 \longrightarrow E \xrightarrow{\iota} F \xrightarrow{\pi} E/F \longrightarrow 0, \tag{39.19}$$

where $\iota : E^1 \to E^2$ is the inclusion mapping, and $\pi : F \to E/F$ is the projection mapping.

40 Lecture 40

40.1 Pull-back bundles

We will next discuss the pull-back operation on bundle. Everything below holds for both the smooth and holomorphic categories. If M and N are smooth manifolds, and $\pi_N : E \to N$ is a vector bundle over N, then given a smooth mapping $f : M \to N$, define

$$f^*E = \{(p, v) \in M \times E \mid f(p) = \pi_N(v)\}. \tag{40.1}$$

Proposition 40.1. *The pullback f^*E is a vector bundle over M, with projection given by $\pi_1(p, v) = p$, and the fiber f^*E over $p \in M$ is identified with the fiber $E_{f(p)}$, i.e., the following diagram commutes*

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\pi_2} & E \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{40.2}$$

Proof. Let $\Phi : U \times \mathbb{R}^k \to \pi_N^{-1}(U)$ be a local trivialization for E. The set $f^{-1}(U)$ is open since f is continuous, and define

$$f^*\Phi : f^{-1}(U) \times \mathbb{R}^k \to \pi_1^{-1}(f^{-1}(U)) \tag{40.3}$$

by

$$f^*\Phi(x, v) = (x, \Phi(f(x), v)). \tag{40.4}$$

The reader can verfiy that this is a local trivialization for f^*E. ☐

Remark 40.2. Note that if U_α is an open cover of N with transition function $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$, then the transition functions for f^*E with respect to the covering $V_\alpha = f^{-1}(U_\alpha)$ are

$$\varphi_{\alpha\beta}^{f^*E} = \varphi_{\alpha\beta} \circ f. \tag{40.5}$$

Next we note that sections can be pulled back to sections of the pullback bundle.

Definition 40.3. Let $f : M \to N$ be a smooth mapping between smooth manifolds, and $\pi : E \to N$ be a vector bundle over N. If $\sigma : N \to E$ is a section of E, then $(\sigma \circ f)(x) = (x, \sigma(f(x)))$ is a section of $\pi_1 : f^*E \to M$ and is called the pullback of σ under f.

The fact that this is a section of the pullback bundle is almost obvious, we just need to check that

$$\pi_1(\sigma \circ f)(x) = \pi_1(x, \sigma(f(x))) = x. \tag{40.6}$$

Also, vector bundle mappings can be pulled back.

Definition 40.4. If $F : E_1 \to E_2$ is a bundle mapping for vector bundles over N, and $f : M \to N$, then $f^*F : f^*E_1 \to f^*E_2$ is defined by $f^*F(p, v) = (p, F(v))$ and is a bundle mapping.

Since $f(p) = \pi_1(v)$, we have that $\pi_2 F(v) = f(p)$, so f^*F indeed maps f^*E_1 to f^*E_2, and it is clearly a bundle mapping.

Next, we have the following, which says that the pullback functor is exact in the category of vector bundles. That is, it preserves short exact sequences.

Proposition 40.5. *Let*

$$0 \longrightarrow E^1 \xrightarrow{F^1} E^2 \xrightarrow{F^2} E^3 \longrightarrow 0 \tag{40.7}$$

be a short exact sequence of vector bundles on N*, and* $f : M \to N$*. Then*

$$0 \longrightarrow f^*E^1 \xrightarrow{f^*F^1} f^*E^2 \xrightarrow{f^*F^2} f^*E^3 \longrightarrow 0 \tag{40.8}$$

is a short exact sequence of vector bundles on M*.*

Proof. One just needs to check exactness on fibers over $p \in M$, which is just exactness of the original sequence over the point $f(p)$. □

40.2 Remarks on bundle equivalence

Above, we defined bundles to be equivalent if $F : E_1 \to E_2$ is a mapping which is an isomorphism on fibers and covers the identity mapping, that is, the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ M & \xrightarrow{id} & M. \end{array} \tag{40.9}$$

Let us consider the more general situation where F is a mapping which is an isomorphism on fibers and covers a diffeomorphism $f : M \to M$,

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ M & \xrightarrow{f} & M. \end{array} \tag{40.10}$$

Proposition 40.6. *In this setting, the bundle* $\pi_1 : E_1 \to M$ *is isomorphic to* f^*E_2*.*

Proof. We need to find a mapping H, which is an isomorphism on fibers, such that the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{H} & f^*E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ M & \xrightarrow{id} & M. \end{array} \tag{40.11}$$

Define $H : E_1 \to f^* E_2$ by

$$H(e_1) = (\pi_{E_1}(e_1), F(e_1)). \tag{40.12}$$

Then H covers the identity map, and is an isomorphism on fibers. □

So if we had defined bundle equivalence using the coarser notion of covering a diffeomorphism, then we also need to mod out the first notion of equivalence by the pull-back operation.

Example 40.7. (Line bundles over $\mathbb{P}^1 \times \mathbb{P}^1$). Consider the product $\mathbb{P}^1 \times \mathbb{P}^1$, with projection mappings π_1 and π_2. For $m, n \in \mathbb{Z}$, we can form the line bundle

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(m) \oplus \pi_2^* \mathcal{O}_{\mathbb{P}^1}(n). \tag{40.13}$$

It turns out that this is the complete list of holomorphic line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$, up to equivalence (meaning covering the identity mapping), i.e., $Pic(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$. If we had allowed arbitrary mappings of the base, then we would get a coarser equivalence. To see this, let $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, defined by $f(p_1, p_2) = (p_2, p_1)$. Then we have the extra equivalence

$$\begin{aligned} f^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n) &= f^* \Big(\pi_1^* \mathcal{O}_{\mathbb{P}^1}(m) \oplus \pi_2^* \mathcal{O}_{\mathbb{P}^1}(n) \Big) \\ &= f^* \pi_1^* \mathcal{O}_{\mathbb{P}^1}(m) \oplus f^* \pi_2^* \mathcal{O}_{\mathbb{P}^1}(n)) \\ &= (\pi_1 \circ f)^* \mathcal{O}_{\mathbb{P}^1}(m) \oplus (\pi_2 \circ f)^* \mathcal{O}_{\mathbb{P}^1}(n) \\ &= \pi_2^* \mathcal{O}_{\mathbb{P}^1}(m) \oplus \pi_1^* \mathcal{O}_{\mathbb{P}^1}(n) \\ &= \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(n, m), \end{aligned} \tag{40.14}$$

so we would only get the symmetric product of $\mathbb{Z}$ and $\mathbb{Z}$.

We will never, never, ever, ever, never use this coarser notion of bundle equivalence.

40.3 The Euler sequence

We can define a line bundle L on $\mathbb{P}^n$ as follows:

$$L = \{([p], v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in [p]\}, \tag{40.15}$$

with $\pi : L \to \mathbb{P}^n$ given by $\pi([p], v) = [p]$, which is called the tautological line bundle over $\mathbb{P}^n$.

Proposition 40.8. *There is a bundle isomorphism $L \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$.*

Proof. Over the subset $U_j = \{[z_1, \ldots, 1_j, \ldots z_n]\}$, we can define a non-zero section $\sigma_j : U_j \to L$ by

$$\sigma_j([z_1, \ldots, 1_j, \ldots z_n]) = ([z_1, \ldots, 1_j, \ldots z_n], (z_1, \ldots, 1_j, \ldots z_n)) \tag{40.16}$$

This gives a local trivialization of L, $\Phi_j : U_j \times \mathbb{C} \to \pi^{-1}(U_j)$ by

$$\Phi_j : ([z_1, \ldots, 1_j, \ldots z_n], z) \mapsto \Big([z_1, \ldots, 1_j, \ldots z_n], z \cdot (z_1, \ldots, 1_j, \ldots z_n) \Big). \tag{40.17}$$

Recall that the transition functions

$$\phi_{ij} : U_i \cap U_j \to GL(1, \mathbb{C}) = \mathbb{C}^* \tag{40.18}$$

are defined by

$$\phi_{ij}([p])(v) = \pi_2(\Phi_i^{-1} \circ \Phi_j([p], v)). \tag{40.19}$$

Letting $[p] = [z_0, \ldots, z_n]$, we compute

$$\phi_{ij}([z_0, \ldots, z_n])(v) = \pi_2 \Phi_i^{-1}\Big([p], v\Big(\frac{z_0}{z_j}, \ldots, 1_j, \ldots, \frac{z_n}{z_j}\Big)\Big) \tag{40.20}$$

$$= \pi_2 \Phi_i^{-1}\Big([p], v \cdot \frac{z_i}{z_j}\Big(\frac{z_0}{z_i}, \ldots, 1_i, \ldots, \frac{z_n}{z_i}\Big)\Big) = v \cdot \frac{z_i}{z_j}. \tag{40.21}$$

Since the transition functions of $\mathcal{O}_{\mathbb{P}^n}(m)$ were $(z_j/z_i)^m$, this proves that $m = -1$. □

Next, we have

Proposition 40.9 (The Euler sequence on $\mathbb{P}^n$)**.** *There is an exact sequence of vector bundles*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T^{1,0}(\mathbb{P}^n) \to 0 \tag{40.22}$$

Proof. Consider the projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Above, we proved that $\mathcal{O}_{\mathbb{P}^n}(1)$ is the dual of the bundle L discussed above. So a vector $v \in \mathcal{O}_{\mathbb{P}^n}(1)_{[p]}$ is identified with a linear function $l : [p] \to \mathbb{C}$. Given such a v, we can define a vector field on $[p] \subset \mathbb{C}^{n+1}$ by

$$(X_{v,j})_x = v(x)\frac{\partial}{\partial z_j}, \tag{40.23}$$

for $x \in [p]$ and $j = 0, \ldots n+1$. It is easy to check that (exercise)

$$\pi_*(X_{v,j})_x = \pi_*(X_{v,j})_{\lambda \cdot x} \tag{40.24}$$

for any $\lambda \in \mathbb{C}^*$. So the vector field $X_{v,j}$ (defined on the line $[p]$) descends to a vector in $T^{1,0}_{[p]}\mathbb{P}^n$. So then we can define a bundle mapping by

$$\mathcal{E} : \overbrace{\mathcal{O}_{\mathbb{P}^n}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(1)}^{n+1} \to T^{1,0}(\mathbb{P}^n) \tag{40.25}$$

by

$$\mathcal{E}(v_0, \ldots, v_n) \mapsto \sum_j v_j \frac{\partial}{\partial z_j}. \tag{40.26}$$

Clearly this is surjective with 1-dimensional kernel spanned by the vector field on $\mathbb{C}^{n+1}$ given by

$$X_{Euler} = \sum_j z_j \frac{\partial}{\partial z_j}, \tag{40.27}$$

which finishes the proof. □

We have the following important corollary.

Corollary 40.10. *We have an isomorphism* $K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Proof. The dual of the Euler sequence is

$$0 \to \Lambda^{1,0}(\mathbb{P}^n) \to L^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0. \tag{40.28}$$

For any short exact sequence

$$0 \to A \to B \to C \to 0, \tag{40.29}$$

it holds that

$$\Lambda^{\dim(B)}(B) \cong \Lambda^{\dim(A)}(A) \otimes \Lambda^{\dim(C)}(C), \tag{40.30}$$

and the corollary follows from this, since the top exterior power of the direct sum of line bundles is isomorphic to the tensor product of the line bundles. □

Remark 40.11. The bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is usually denoted by H in algebraic geometry, and called the *hyperplane bundle*. This is because the zero locus of any non-trivial holomorphic section is a linearly embedded hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$. So the Euler sequence is usually written

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to H^{n+1} \to T^{1,0}(\mathbb{P}^n) \to 0 \tag{40.31}$$

This sequence splits differentiably, that is

$$H^{n+1} \cong_{C^\infty} \mathcal{O}_{\mathbb{P}^n} \oplus T^{1,0}(\mathbb{P}^n) \tag{40.32}$$

but not holomorphically. To see this, we just consider $n = 1$. If this were true, it would say that

$$\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \tag{40.33}$$

where we used the above corollary, which says that $K_{\mathbb{P}^1} = \Lambda^{1,0}(\mathbb{P}^1) \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, which upon taking the dual says that $T^{1,0}(\mathbb{P}^1) \simeq \mathcal{O}_{\mathbb{P}^1}(2)$. This cannot be true holomorphically since any holomorphic section of the left hand side must vanish somewhere, but the right hand side obviously has a non-vanishing section coming from the trivial bundle.

41 Lecture 41

41.1 The need for sheaves

Given a vector bundle mapping $F : E_1 \to E_2$, we would like to define a "kernel" so that

$$0 \longrightarrow \mathrm{Ker}(F) \longrightarrow E_1 \xrightarrow{F} E_2 \tag{41.1}$$

is exact, and also define a "cokernel" so that

$$E_1 \xrightarrow{F} E_2 \longrightarrow \mathrm{Coker}(F) \longrightarrow 0 \tag{41.2}$$

is exact.

Example 41.1. Let $E = \mathcal{O}_{\mathbb{C}}$ be the trivial line bundle over $\mathbb{C}$. That is $E = \mathbb{C} \times \mathbb{C}$ with $\pi(z, v) = z$. Define $F : E \to E$ by $F(z, v) = (z, z \cdot v)$. The induced mapping $F_z : E_z \to E_z$ is $v \mapsto z \cdot v$. If $z \neq 0$, then $\mathrm{Ker}(F_z) = 0$, $\mathrm{Im}(F_z) = \mathbb{C}$, but if $z = 0$, then $\mathrm{Ker}(F_0) = \mathbb{C}$, $\mathrm{Im}(F_0) = 0$. Therefore both $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ cannot be defined as vector bundles.

In general, we can only say the following.

Proposition 41.2. *The bundles* $\mathrm{Ker}(F)$ *and* $\mathrm{Coker}(F)$ *exist if and only if* F *has constant rank.*

Proof. The proof is left as an exercise. □

So to talk about kernel and cokernels of arbitrary vector bundle mappings, we need to enlarge our category of objects, this is exactly what sheaves do.

41.2 Presheaves

We next begin our study of sheaf theory. Great references are Lee's book [Lee24, Chapter 5], [War83, Chapter 5], and also Tu's notes on sheaf theory. We begin with the definition.

Definition 41.3. Let X be a topological space. A *presheaf* of abelian groups on X is a mapping from open sets in X to abelian groups $\mathscr{F} : U \mapsto \mathscr{F}(U)$ with the following properties. For each $U \subset V$ open sets, there is a restriction mapping $r^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$ satisfying

1. r^V_U is a homomorphism of Abelian groups,
2. If $U \subset V \subset W$, then $r^W_U = r^V_U \circ r^W_V$,
3. $r^U_U = Id$.

Remark 41.4. An element $\sigma \in \mathscr{F}(U)$ is call a section of $\mathscr{F}$ over U, and we also write $\sigma \in \Gamma(U, \mathscr{F})$. An element $\sigma \in \mathscr{F}(X)$ is called a global section.

Next, we define the morphisms between objects.

Definition 41.5. A morphism $f : \mathscr{F} \to \mathscr{G}$ between presheaves is a collection of homomorphisms $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ \downarrow{\scriptstyle r^U_V} & & \downarrow{\scriptstyle r^U_V} \\ \mathscr{F}(V) & \xrightarrow{f_V} & \mathscr{G}(V) \end{array} \tag{41.3}$$

commutes.

Example 41.6. The *zero presheaf* is $\mathscr{F}(U) = 0$ for all U with all restriction mappings the zero mapping.

Example 41.7. For any space X and abelian group G, the *constant presheaf* with values in G, denoted also by G, is $G(U) = G$ for all U and all restriction mappings the identity mapping of G.

Example 41.8. For any space X and abelian group G, the *locally constant presheaf* with values in G, denoted by $\underline{G}$ is

$$\underline{G}(U) = \{f : U \to G \mid f \text{ locally constant } \}, \tag{41.4}$$

with $r_U^V(\{f : V \to G\}) = f|_U$. Note that since a constant function is locally constant there is a natural morphism of presheaves $G \to \underline{G}$.

Definition 41.9. If $\mathscr{F}$ is a presheaf on X and $p \in X$, then the collection of abelian groups $\{F(U)\}_{p\in U}$ forms a direct system of abelian group, and we define the *stalk* of $\mathscr{F}$ at p to be the abelian group

$$\mathscr{F}_p = \varinjlim \mathscr{F}(U), \tag{41.5}$$

where the direct limit is taken over all open set U with $p \in U$.

For convenience, let us recall the definition of the above direct limit. If $p \in U \cap V$, $\sigma_U \in \mathscr{F}(U)$, $\sigma_V \in \mathscr{F}(V)$, we say that $\sigma_U \sim \sigma_V$ if there exists an open set $W \subset U \cap V$ such that $r_W^U \sigma_U = r_W^V \sigma_V$. Then

$$\varinjlim \mathscr{F}(U) = \coprod_{U \ni p} \mathscr{F}(U)/\sim, \tag{41.6}$$

where the notation means the disjoint union over all open sets with $p \in U$.

Example 41.10. If M is a smooth manifold, then $\mathscr{E}^0$ is the presheaf of C^∞ functions, that is, for $U \subset M$,

$$\mathscr{E}^0(U) = \{f : U \to \mathbb{C} \mid f \text{ is } C^\infty\} \tag{41.7}$$

For $p \in M$, the stalk $\mathscr{E}^0_p$ is called the space of germs of C^∞ functions at p.

41.3 Sheaves

Definition 41.11. A *sheaf* $\mathscr{F}$ is a presheaf which satisfies the following. For all $U \subset X$ and any open cover $\{U_i\}$ of U, we have

1. (Uniqueness) If $\sigma, \tau \in \mathscr{F}(U)$ satisfy $\sigma|_{U_i} = \tau|_{U_i}$ for every i, then $\sigma = \tau$.
2. (Gluing) Given $\sigma_i \in \mathscr{F}(U_i)$ for every i satisfying $\sigma_i|_{U_i \cap U_j} = \sigma_j|U_i \cap U_j$ for every i and j, then there exists $\sigma \in \mathscr{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for every i.

Example 41.12. Consider $\mathbb{R}^n$ with the constant presheaf $\mathbb{Z}$. Let B_1 and B_2 be any two disjoint balls. Let $1 = \sigma_1 \in \mathbb{Z}(B_1)$ and $2 = \sigma_2 \in \mathbb{Z}(B_2)$. Then there is no section $\sigma \in \mathbb{Z}(B_1 \cup B_2$ with $\sigma|_{B_1} = 1$ and $\sigma|_{B_2} = 2$, so this presheaf does not satisfy the gluing axiom, and is therefore not a sheaf. However, if we consider $\mathbb{R}^n$ with the *locally constant presheaf* $\underline{\mathbb{Z}}$, then this does satisfy the uniqueness and gluing axioms, so is a sheaf.

Example 41.13. Let M be a manifold, and G be any nontrivial abelian group. Define a presheaf $\mathscr{S}$ by $\mathscr{S}(X) = G$, but $\mathscr{S}(U) = 0$ for any open set $U \neq X$. Write $M = \cup_i U_i$ where U_i are open sets with $U_i \neq X$. If $0 \neq \gamma \in \mathscr{S}(X) = G$, then $\gamma|_{U_i} = 0$ for every i, so this does not satisfy the uniqueness axiom. Note also that $\mathscr{S}_p = 0$ for all $p \in X$, that is, all the stalks vanish, but $\mathscr{S}$ is not the trivial presheaf.

Given a presheaf of abelian groups $\mathscr{F}(U)$ on a topological space X, we define a topological space

$$\mathscr{F}^+ = \coprod_{p \in X} \mathscr{F}_p. \tag{41.8}$$

There is a obvious projection $\pi : \mathscr{F}^+ \to X$. If $U \subset X$ is open and $\sigma \in \mathscr{F}^+(U)$, denote by σ_p the germ of σ at p. Then $U \ni p \mapsto \sigma_p$ determines a section of $\pi : \mathscr{F}^+ \to X$. The sets $\sigma(U)$ for U open in X is then a basis for a topology on $\mathscr{F}^+$. The space $\mathscr{F}^+$ is called the etale space of the presheaf.

Definition 41.14. Given a presheaf of abelian group $\mathscr{F}(U)$ on a topological space X, define a presheaf $\mathscr{F}^+(U)$ by letting

$$\mathscr{F}^+(U) = \{\sigma : U \to \mathscr{F}^+ \mid \sigma \text{ is continuous, and } \pi \circ \sigma = Id_U\}. \tag{41.9}$$

We called $\mathscr{F}^+(U)$ the *sheafification* of $\mathscr{F}(U)$.

We note that there is a morphism of presheaves $\theta : \mathscr{F}(U) \to \mathscr{F}^+(U)$ defined by

$$\sigma \in \mathscr{F}(U) \mapsto \{U \ni p \mapsto \mathscr{F}_p\}. \tag{41.10}$$

The name is justified by the following.

Proposition 41.15. *For any presheaf $\mathscr{F}(U)$, then $\mathscr{F}^+(U)$ is a sheaf which satisfies $\mathscr{F}^+_p = \mathscr{F}_p$ for every $p \in X$. Furthermore, for every morphism of presheaves $\phi : \mathscr{F}(U) \to \mathscr{G}(U)$, there exists a unique presheaf morphism $\phi^+ : \mathscr{F}^+(U) \to \mathscr{G}(U)$ such that*

$$\begin{array}{ccc} \mathscr{F}^+ & & \\ \theta \uparrow & \overset{\phi^+}{\searrow} & \\ \mathscr{F} & \xrightarrow{\phi} & \mathscr{G} \end{array} \tag{41.11}$$

Proof. The proof is left as an exercise. ∎

Exercise 41.16. If the presheaf $\mathscr{F}$ is a sheaf, then $\mathscr{F}^+ = \mathscr{F}$.

Example 41.17. For the presheaf $\mathscr{S}$ above with $\mathscr{S}(X) = G$ and $\mathscr{S}(U) = 0$ for $U \neq X$, the sheafification $\mathscr{S}^+$ is the zero sheaf.

Example 41.18. For the constant presheaf G, the sheafification $G^+ = \underline{G}$ is the sheaf of locally constant functions.

42 Lecture 42

42.1 Kernel, image, and quotient sheaves

We next define the kernel and image of a morphism of sheaves.

Definition 42.1. A morphism $\phi : \mathscr{F} \to \mathscr{G}$ of sheaves $\mathscr{F}$ and $\mathscr{G}$ over the same topological space X is just a morphism of the corresponding presheaves. The kernel of ϕ is the sheaf

$$\mathrm{Ker}(\phi)(U) = \mathrm{Ker}\{\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)\} \tag{42.1}$$

The image of ϕ is the sheafification $\mathrm{Im}^+(\phi)$ of the image presheaf

$$\mathrm{Im}(\phi)(U) = \mathrm{Im}\{\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)\} \tag{42.2}$$

It is easy to see that $\mathrm{Ker}(\phi)$ is always a sheaf. Also, the following example shows the need for the sheafification when defining the image sheaf.

Example 42.2. Consider $X = S^1$, let $\mathcal{E}^0$ be the sheaf of germs of C^∞ functions and $\mathcal{E}^1$ be the sheaf of germs of C^∞ 1-forms. Then $d : \mathcal{E}^0 \to \mathcal{E}^1$ is a morphism of sheaves. The image presheaf is

$$\mathrm{Im}(\phi)(U) = \{df \mid f : U \to \mathbb{C} \text{ is } C^\infty\}. \tag{42.3}$$

Cover S^1 by subsets U_i which are diffeomoprhic to intervals, and consider the 1-form $d\theta$. By the Poincar'e Lemma, $d\theta|_{U_i} = df_i$ on U_i. But there is no C^∞ function $f : S^1 \to \mathbb{C}$ with $df = d\theta$ since if there were,

$$2\pi = \int_{S^1} d\theta = \int_{S^1} df = 0, \tag{42.4}$$

by Stokes' Theorem, which is a contradiction. This means the gluing axiom is violated, and the image presheaf is not a sheaf.

Definition 42.3. A morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is *injective* if $\mathrm{Ker}(\phi) = 0$, and is *surjective* is $\mathrm{Im}(\phi) = \mathscr{G}$. We say that sheaves $\mathscr{F}$ and $\mathscr{G}$ on X are *isomorphic* if there is a morphism $\phi : \mathscr{F} \to \mathscr{G}$ which is both injective and surjective.

Warning: sheaves being isomorphic does not mean that the corresponding presheaves are isomorphic.

Proposition 42.4. *A morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is injective if and only if the corresponding presheaf morphism $\phi : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open set $U \subset X$.*

The morphism ϕ is surjective if and only if given $U \subset X$ and $\sigma \in \mathscr{G}(U)$, there exists an open subset $V \subset U$ and $\sigma \in \mathscr{F}(V)$ such that $r^U_V \sigma = \phi(\sigma)$.

Proof. Since there was no sheafification involved in defining kernels, the first part is obvious. The second part is an exercise. □

Definition 42.5. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For $p \in X$, the *stalk mapping* $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ is defined as follows. If $\sigma \in \mathscr{F}(U)$ is a section representing the germ $s_p \in \mathscr{F}_p$, then $\phi_p(s_p)$ is the germ of $\phi(s) \in \mathscr{G}_p$.

It is easy to see this is well-defined. We next have the VIP (very important proposition).

Proposition 42.6. *A morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is injective (surjective) if and only if the stalk mapping $\phi_p : \mathscr{F} \to \mathscr{G}$ is injective (surjective) for every $p \in X$.*

Proof. This follows from Proposition 42.4. □

Proposition 42.7. *Let $\mathscr{F}$ be a sheaf for which the stalk $\mathscr{F}_p = 0$ for every $p \in X$. Then $\mathscr{F} = 0$ is the zero sheaf.*

Proof. Consider the morphism $\phi : \mathscr{F} \to 0$ defined by $\phi(\sigma) = 0$ for $\sigma \in \mathscr{F}(U)$. The induced mapping on stalks is $\phi_p : \mathscr{F}_p = 0 \to 0_p = 0$ is both injective and surjective. From Proposition 42.6, ϕ itself is both injective and surjective, so is an isomorphism of sheaves. □

Next, we define subsheaves and quotient sheaves.

Definition 42.8. We say that $\mathscr{F}$ is a subsheaf of $\mathscr{G}$ if for each $U \subset X$, $\mathscr{F}(U)$ is a subgroup of $\mathscr{G}(U)$ and the inclusion mapping $i : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves. If $\mathscr{F}$ is a subsheaf of $\mathscr{G}$ then the quotient sheaf $\mathscr{G}/\mathscr{F}$ is the sheafification of the presheaf

$$U \mapsto \mathscr{G}(U)/\mathscr{F}(U). \tag{42.5}$$

Exercise 42.9. Let $\mathcal{E}^0(S^1)$ denote the sheaf of germs of smooth functions on the circle. Then the constant sheaf $\underline{\mathbb{C}}$ is a of subsheaf $\mathcal{E}^0(S^1)$, and we write

$$0 \to \underline{\mathbb{C}} \to \mathcal{E}^0(S^1). \tag{42.6}$$

Show that the quotient presheaf is not a sheaf (hint: it fails the gluing axiom). This shows the need for taking sheafifications when defininig the quotient sheaf.

42.2 Exact sequences of sheaves

Definition 42.10. We say that the sequence of sheaves on X

$$\mathscr{F}^1 \xrightarrow{\phi^1} \mathscr{F}^2 \xrightarrow{\phi^2} \mathscr{F}^3 \tag{42.7}$$

is exact at $\mathscr{F}^2$ if $\mathrm{Im}(\phi^1) = Ker(\phi^2)$.

We have the following proposition.

Proposition 42.11. *The sequence* (42.7) *is exact at $\mathscr{F}^2$ if and only if the associated sequence on stalks*

$$\mathscr{F}^1_p \xrightarrow{\phi^1_p} \mathscr{F}^2_p \xrightarrow{\phi^2_p} \mathscr{F}^3_p \tag{42.8}$$

is exact at $\mathscr{F}^2_p$ for every $p \in X$.

Equivalently, the sequence (42.7) *is exact at $\mathscr{F}^2$ if and only the following 2 conditions are satisfied:*

1. *For all $U \subset X$ open, and all $\sigma^1 \in \mathscr{F}^1(U)$, we have $\phi^2(\phi^1(\sigma^1)) = 0$,*
2. *For all $U \subset X$ open, and $\sigma^2 \in \mathscr{F}^2(U)$, with $\phi^2(U) = 0$, there is an open set $V \subset U$ and $\sigma^1 \in \mathscr{F}^1(V)$ such that $\phi^1(\sigma^1) = r^U_V \sigma^2$.*

Proof. This follows from combining the results in the previous subsection, details are left to the reader. □

Definition 42.12. We say that the sequence

$$0 \longrightarrow \mathscr{F}^1 \xrightarrow{\phi^1} \mathscr{F}^2 \xrightarrow{\phi^2} \mathscr{F}^3 \longrightarrow 0. \tag{42.9}$$

is a *short exact sequence* if it is exact at $\mathscr{F}^1, \mathscr{F}^2$, and $\mathscr{F}^3$. In other words, ϕ^1 is injective, $\mathrm{Im}(\phi^1) = \mathrm{Ker}(\phi^2)$, and ϕ^2 is surjective.

Proposition 42.13. *If $\mathscr{F}^1$ is a subsheaf of $\mathscr{F}^2$, then there is a short exact sequence*

$$0 \longrightarrow \mathscr{F}^1 \xrightarrow{i} \mathscr{F}^2 \xrightarrow{\phi^2} \mathscr{F}^2/\mathscr{F}^1 \longrightarrow 0, \tag{42.10}$$

where $\iota : \mathscr{F}^1 \to \mathscr{F}^2$ is the inclusion mapping. Also, for any short exact sequence of sheaves

$$0 \longrightarrow \mathscr{F}^1 \xrightarrow{\phi^1} \mathscr{F}^2 \xrightarrow{\phi^2} \mathscr{F}^3 \longrightarrow 0. \tag{42.11}$$

there is a natural sheaf isomorphism $\mathscr{F}^3 \simeq \mathscr{F}^2/\mathrm{Im}(\phi^1)$.

Proof. Again, follows by combining results in the previous section, details left to the reader. □

Example 42.14 (The exponential sequence)**.** Let M be a complex manifold, $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M (with group operation given by addition of functions) and $\mathcal{O}^*(M)$, the sheaf of germs of nowhere-zero holomorphic functions on M (with group operation given by multiplication of functions), and $\underline{\mathbb{Z}}$ the sheaf of locally constant integer valued functions on M. Then

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O}_M \xrightarrow{e^{2\pi\sqrt{-1}\cdot}} \mathcal{O}^*_M \longrightarrow 1 \tag{42.12}$$

is a short exact sequence, where ι is the natural inclusion. The only issue is surjectivity, but this follows since any non-zero holomorphic function on a simply-connected domain has a logarithm, which is uniquely defined up to adding multiples of $2\pi\sqrt{-1}$.

Note that in general, the associated presheaf mapping is not surjective. To see this, let $M = \mathbb{C}^*$, and $z \in \mathcal{O}_{\mathbb{C}^*}(\mathbb{C}^*)$. But $\log(z)$ does not exist in this domain.

By replacing $\mathcal{O}$ with $\mathcal{E}$, we also obtain the smooth exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{E}_M \xrightarrow{e^{2\pi\sqrt{-1}\cdot}} \mathcal{E}^*_M \longrightarrow 1 \tag{42.13}$$

We will see later that these two sequences have quite different properties!

43 Lecture 43

43.1 Sheaves of modules

Let X be a topological space, and fix a sheaf of commutative rings $\mathscr{R}$.

Definition 43.1. A sheaf of modules $\mathscr{F}$ over $\mathscr{R}$ on X is a sheaf of abelian group such that for each open set $U \subset X$, $\mathscr{F}(U)$ is an $\mathscr{R}(U)$-module. Furthermore, the module structure is compatible with restriction in the following sense. If $U \subset V$ are open sets $s \in \mathscr{R}(V)$ and $\sigma \in \mathscr{F}(V)$, then $r^V_U(s \cdot \sigma) = r^V_U s \cdot r^V_U \sigma$.

Everything above can be generalized to sheaves of $\mathscr{R}$-modules, i.e., where sheaf morphisms are also required to be compatible with the module structure. First, we consider direct sums.

Definition 43.2. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups on X, then the direct sum $\mathscr{F} \oplus \mathscr{G}$ is the sheaf for open $U \subset X$

$$(\mathscr{F} \oplus \mathscr{G})(U) = \mathscr{F}(U) \oplus \mathscr{G}(U). \tag{43.1}$$

It is easy to verify that this is a sheaf, and if both $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{R}$-modules, then so is $\mathscr{F} \oplus \mathscr{G}$. Next, we define the tensor product.

Definition 43.3. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathscr{R}$-modules on X. Then $\mathscr{F} \otimes_{\mathscr{R}} \mathscr{G}$ is the sheaf associated to the presheaf

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{R}(U)} \mathscr{G}(U). \tag{43.2}$$

Example 43.4. Let M be a topological manifold, $\pi : E \to M$ a topological vector bundle, and let $\mathcal{C}_M$ denote the sheaf of germs of continuous functions on M. Then $\mathcal{C}_M(E)$, the sheaf of germs of continuous sections of E is a $\mathcal{C}_M$ module.

Example 43.5. Let M be a smooth manifold, $\pi : E \to M$ a smooth vector bundle, and let $\mathcal{E}_M$ denote the sheaf of germs of C^∞ functions on M. Then $\mathcal{E}_M(E)$, the sheaf of germs of smooth section of E is an $\mathcal{E}_M$ module.

Example 43.6. Let M be a complex manifold, $\pi : E \to M$ a holoomorphic vector bundle, and let $\mathcal{O}_M$ denote the sheaf of germs of holomorphic functions on M. Then $\mathcal{O}_M(E)$, the sheaf of germs of holomorphic sections of E is an $\mathcal{O}_M$ module.

Example 43.7. This is to show that the sheafification is necessary when defining the tensor product. Consider the line bundle $L = \mathcal{O}_{\mathbb{P}^1}(1)$, which has a 2-dimensional space of global sections. The dual bundle is $L^* = \mathcal{O}_{\mathbb{P}^1}(-1)$, which has no global sections. Since $L \otimes L^*$ is a trivial line bundle, we have

$$\mathcal{O}_{\mathbb{P}^1}(L) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(L^*) = \mathcal{O}_{\mathbb{P}^1}. \tag{43.3}$$

But on the presheaf level, letting $U = \mathbb{P}^1$, the left hand side is

$$\Gamma(\mathbb{P}^1, L) \otimes_{\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)} \Gamma(\mathbb{P}^1, L^*) \simeq \{0\} \otimes_{\mathbb{C}} \mathbb{C}^2 \simeq \{0\}, \tag{43.4}$$

but the right hand side is $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C}$, so the presheaves are not the same. (But their sheafifications are, which we will see next!)

Proposition 43.8. *Let M be a complex manifold, and $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be holomorphic vector bundles. Then*

$$\mathcal{O}_M(E_1 \otimes E_2) \simeq \mathcal{O}_M(E_1) \otimes_{\mathcal{O}_M} \mathcal{O}_M(E_2), \tag{43.5}$$

as $\mathcal{O}_M$-modules. That is, the sheaf of germs of holomorphic sections of the tensor product is isomorphic to the sheaf tensor product $\mathcal{O}_M(E_1) \otimes_{\mathcal{O}_M} \mathcal{O}_M(E_2)$..

Proof. See [Lee24, Proposition 5.12]. We define a morphism of presheaves as follows. Let $U \subset X$ be an open set. Let

$$\sigma \in \Big(\mathcal{O}_M(E_1) \otimes_{\mathcal{O}_M} \mathcal{O}_M(E_2)\Big)(U) = \mathcal{O}_M(E_1)(U) \otimes_{\mathcal{O}_M(U)} \mathcal{O}_M(E_2)(U). \tag{43.6}$$

Then σ can be written as a finite sum

$$\sigma = \sum_j \sigma_j^1 \otimes \sigma_j^2, \tag{43.7}$$

where $\sigma_j^1 \in \mathcal{O}_M(E_1)(U)$, and $\sigma_j^2 \in \mathcal{O}_M(E_2)(U)$. The right hand side is naturally a section of $\mathcal{O}_M(E_1 \otimes E_2)(U)$, so this defines the sheaf morphism of $\mathcal{O}_M$ modules

$$F_U : \Big(\mathcal{O}_M(E_1) \otimes_{\mathcal{O}_M} \mathcal{O}_M(E_2)\Big)(U) \to \mathcal{O}_M(E_1 \otimes E_2)(U), \tag{43.8}$$

Given $p \in X$, for any neighborhood U_p of p such that $E_1|_{U_p}$ and $E_2|_{U_p}$ are trivial, we will show that F_{U_p} is an isomorphism. This will imply that F is a sheaf isomorphism since it is then an isomorphism on stalks.

Let e_k^1 and e_l^2 be local holomorphic frames over U_p for E_1 and E_2, respectively. Then any σ as in (43.6) can be written as

$$\sigma = \sum_{j,k,l} f_j^k e_k^1 \otimes g_j^l e_l^2 = \sum_{j,k,l} f_j^k g_j^l e_k^1 \otimes e_l^2, \tag{43.9}$$

where f_j^k and g_j^l are holomorphic functions on U. Note that $e_k^1 \otimes e_l^2$ are a local basis of holomorphic sections of $E_1 \otimes E_2$ over U. Therefore, if $F_U(\sigma) = 0$, then $\sum_j f_j^k g_j^l \equiv 0$, which then obviously implies that $\sigma = 0$. So F_{U_p} is injective. To show surjectivity, if $\sigma \in \mathcal{O}_M(E_1 \otimes E_2)(U_p)$, then

$$\sigma = \sum h^{kl} e_k^1 \otimes e_l^2, \tag{43.10}$$

where h^{kl} are holomorphic functions on U. So then

$$\sigma = F_U(\sum_k \sigma_k^1 \otimes \sigma_k^2), \tag{43.11}$$

where

$$\sigma_k^1 = e_k^1, \ \sigma_k^2 = \sum_l h^{kl} e_l^2. \tag{43.12}$$

□

43.2 Locally free sheaves

Definition 43.9. Let $\mathscr{F}$ be a sheaf of abelian groups on X and $U \subset X$ an open set. Then the restriction of $\mathscr{F}$ to U, denoted $\mathscr{F}|_U$ is the sheaf

$$\mathscr{F}|_U(V) = \mathscr{F}(V) \tag{43.13}$$

for all open subsets $V \subset U$ (OK since V is necessarily open in X).

Definition 43.10. A sheaf of $\mathscr{F}$ of $\mathscr{R}$-modules on X is called *locally free* of rank k if for each point $p \in X$, there is an open neighborhood U of p so that $\mathscr{F}|_U$ is $\mathscr{R}$-module isomorphic to $\mathscr{R}|_U^k \equiv \mathscr{R}|_U \oplus \cdots \oplus \mathscr{R}|_U$, the k-fold direct sum.

The following is a VIP.

Proposition 43.11. *Let M be a complex manifold, and $\pi : E \to M$ a holomorphic vector bundle of rank k. Then $\mathcal{O}_M(E)$ is locally free of rank k. Conversely, if $\mathscr{F}$ is a locally free sheaf of $\mathcal{O}_M$-modules of rank k on M, then there is a rank k holmorphic vector bundle $\pi : E \to M$, which is uniquely determined up to bundle equivalence, so that $\mathscr{F} \simeq \mathcal{O}_M(E)$ as $\mathcal{O}_M$-modules.*

Proof. See [Lee24, Proposition 5.14]. One direction is easy. Let $U \subset X$ be any open set on which U is trivial. Then there exists a local basis of holomorphic sections $e_j \in \mathcal{O}_M(E)(U), j = 1 \dots k$. For any $V \subset U$, define a mapping $F_V : \mathcal{O}_V^k \to \mathcal{O}_M(E)(V)$ by

$$F_U(f_1, \dots, f_k) = \sum_j f_j e_j. \tag{43.14}$$

This obviously gives an sheaf isomorphism between $\mathcal{O}_U^k$ and $\mathcal{O}_M(E)|_U$.

For the converse, let $\mathscr{F}$ be a locally free sheaf of rank k on M. There is a covering U_α of X and sheaf isomorphisms

$$\Phi_\alpha : \mathscr{F}|_{U_\alpha} \to \mathcal{O}_{U_\alpha}^k. \tag{43.15}$$

On $U_\alpha \cap U_\beta$, we have

$$\Phi_\alpha \circ \Phi_\beta^{-1} : \mathcal{O}_{U_\alpha \cap U_\beta}^k \to \mathcal{O}_{U_\alpha \cap U_\beta}^k, \tag{43.16}$$

which is an isomorphism of $\mathcal{O}_{U_\alpha \cap U_\beta}$-modules. This means that

$$\Phi_\alpha \circ \Phi_\beta^{-1}(f_1, \dots, f_k) = \Big(\sum_j a_1^j f_j, \dots, \sum_j a_j^k f_j \Big), \tag{43.17}$$

where a_i^j are $GL(k, \mathbb{C})$-valued holomorphic functions. These obviously satisfy the cocycle condition, and define a holomorphic vector bundle E over X.

The proof that $\mathscr{F}$ is isomorphic to $\mathcal{O}_M(E)$ and that E is uniquely determined up to bundle equivalence are left to the reader. $\square$

43.3 Skyscraper sheaves

Example 43.12 (Skyscraper sheaf)**.** Let X be a topological space, $p \in X$, and S any non-trivial abelian group. Define a presheaf on X by

$$\mathscr{S}(U) = \begin{cases} S & p \in U \\ 0 & p \notin U \end{cases} \tag{43.18}$$

It is easy to verify that this is a sheaf.

Example 43.13. Let M be a complex manifold, $p \in M$, let $S = \mathbb{C}^k$. Then the skyscraper sheaf is denoted by $\mathbb{C}^k_p$. Note that this is a sheaf of $\mathcal{O}_M$-modules: for $U \subset M$ open, $f \in \mathcal{O}_M(U)$, and $\sigma \in \mathbb{C}^k_p(U)$, define

$$(f \cdot \sigma)(U) = \begin{cases} f(p)\sigma & p \in U \\ 0 & p \notin U. \end{cases} \tag{43.19}$$

Remark 43.14. With this structure as an $\mathcal{O}_M$-module, $\mathbb{C}^k_p$ is an example of a *torsion* sheaf. That is, for any $0 \neq v \in \mathbb{C}^k$, and open U with $p \in U$, there exists a section $\sigma_v \in \mathbb{C}^k_p(U)$ with $(\sigma_v)_q = 0$ for $q \neq p$ but $(\sigma_v)_p = v$. Then if $f \in \mathcal{O}_U$ is any nontrivial holomorphic function on U with $f(p) = 0$, we have that $f\sigma_v = 0$.

Example 43.15. Define a sheaf morphism $z : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$ where $z_U : \mathcal{O}_{\mathbb{C}}(U) \to \mathcal{O}_{\mathbb{C}}(U)$ is $z_U(f) = z \cdot f$. Then we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{\ z\ } \mathcal{O}_{\mathbb{C}} \longrightarrow \mathbb{C}_0 \longrightarrow 0. \tag{43.20}$$

This solves our earlier problem: the mapping z gives a vector bundle mapping from the trivial bundle $\mathbb{C} \times \mathbb{C}$ over $\mathbb{C}$ to itself by $(z, v) \mapsto (z, zv)$. This mapping does not have constant rank, so the kernel and cokernel do not exist as vector bundles. But they do exist as sheaves!

Example 43.16. Let M be a complex manifold, and let E be a holomorphic vector bundle over M. For $p \in M$, consider the fiber E_p as a skyscraper sheaf concentrated at p. Then we have a surjective morphism of sheaves

$$\mathcal{O}_M(E) \xrightarrow{\ ev_p\ } E_p \longrightarrow 0 \tag{43.21}$$

as follows: for $\sigma \in \mathcal{O}_M(E)(U)$

$$(ev_p)_U \sigma = \begin{cases} \sigma(p) \in E_p & p \in U \\ 0 \in E_p & p \notin U \end{cases} \tag{43.22}$$

The kernel of ev_p is given by

$$\mathrm{Ker}(ev_p)(U) = \{\sigma \in \mathcal{O}_U(E) \mid \sigma(p) = 0\}. \tag{43.23}$$

Exercise 43.17. Let $p \in M = \mathbb{P}^1$, and with $E = \mathcal{O}_{\mathbb{P}^1}$ the trivial line bundle. Prove that there is an isomorphism $\mathrm{Ker}(ev_p) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, and we therefore have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\ \otimes s_0\ } \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\ ev_p\ } \mathbb{C}_p \longrightarrow 0, \tag{43.24}$$

where s_0 is a certain global section of $\mathcal{O}_{\mathbb{P}^1}(1)$ and we use the isomorphism

$$\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1}. \tag{43.25}$$

44 Lecture 44

44.1 Čech cohomology of an open cover

Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X, and $\mathscr{U} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an open covering. We define Čech cochains as follows. First, define

$$C^0(\mathscr{U}, \mathscr{F}) = \prod_{\alpha \in \mathcal{I}} \Gamma(U_\alpha, \mathscr{F}) \tag{44.1}$$

We write an element $f \in C^0(\mathscr{U}, \mathscr{F})$ as $f_\alpha \in \mathscr{F}(U_\alpha)$ for each $\alpha \in \mathcal{I}$. Next, define

$$C^1(\mathscr{U}, \mathscr{F}) = \prod_{\alpha_0, \alpha_1 \in \mathcal{I}} \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathscr{F}) \tag{44.2}$$

We write an element $f \in C^1(\mathscr{U}, \mathscr{F})$ as $f_{\alpha_0\alpha_1} \in \mathscr{F}(U_{\alpha_0} \cap U_{\alpha_1})$ for $\alpha_0, \alpha_1 \in \mathcal{I}$. We require the skew-symmetry condition $f_{\alpha_0\alpha_1} = -f_{\alpha_1\alpha_0}$. In general, we let

$$C^k(\mathscr{U}, \mathscr{F}) = \prod_{\alpha_0, \dots, \alpha_k \in \mathcal{I}} \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}, \mathscr{F}). \tag{44.3}$$

We write an element $f \in C^k(\mathscr{U}, \mathscr{F})$ as $f_{\alpha_0 \cdots \alpha_k} \in \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$, and require that f be skew-symmetric in all indices [1].

The *coboundary operator* $\delta^k : C^k(\mathscr{U}, \mathscr{F}) \to C^{k+1}(\mathscr{U}, \mathscr{F})$ is defined by

$$(\delta^k f)_{\alpha_0 \cdots \alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j f_{\alpha_0 \cdots \hat{\alpha_j} \cdots \alpha_{k+1}}, \tag{44.4}$$

where we have omitted the restriction mappings to simplify notation. For example,

$$(\delta^0 f)_{\alpha_0\alpha_1} = f_{\alpha_1} - f_{\alpha_0} \tag{44.5}$$

$$(\delta^1 f)_{\alpha_0\alpha_1\alpha_2} = f_{\alpha_1\alpha_2} - f_{\alpha_1\alpha_2} + f_{\alpha_0\alpha_1} \tag{44.6}$$

$$(\delta^2 f)_{\alpha_0\alpha_1\alpha_2\alpha_2} = f_{\alpha_1\alpha_2\alpha_3} - f_{\alpha_0\alpha_2\alpha_3} + f_{\alpha_0\alpha_1\alpha_3} - f_{\alpha_0\alpha_1\alpha_2}, \tag{44.7}$$

etc.

Proposition 44.1. *We have $\delta^k \circ \delta^{k-1} = 0$ for $k \geq 1$.*

Proof. For $k = 1$, we check that

$$(\delta^1 \circ \delta^0 f)_{\alpha\beta\gamma} = (\delta^1(f_\beta - f_\alpha))_{\alpha\beta\gamma} = f_\gamma - f_\beta - f_\gamma + f_\alpha + f_\beta - f_\alpha = 0, \tag{44.8}$$

and for $k = 2$,

$$\begin{aligned} (\delta^2 \circ \delta^1 f)_{\alpha\beta\gamma\delta} &= (\delta^1 f)_{\beta\gamma\delta} - (\delta^1 f)_{\alpha\gamma\delta} + (\delta^1 f)_{\alpha\beta\delta} - (\delta^1 f)_{\alpha\beta\gamma} \\ &= f_{\gamma\delta} - f_{\beta\delta} + f_{\beta\gamma} - (f_{\gamma\delta} - f_{\alpha\delta} + f_{\alpha\gamma}) \\ &\quad + f_{\beta\delta} - f_{\alpha\delta} + f_{\alpha\beta} - (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}) = 0. \end{aligned} \tag{44.9}$$

The general case is left as an exercise. □

[1] If one does not require skew-symmetry, then the resulting cohomology is the same, so it doesn't really matter; see Gunning Volume III

Definition 44.2. We let $\operatorname{Ker}\delta^k \equiv Z^k(\mathscr{U},\mathscr{F})$, $\operatorname{Im}\delta^{k-1} \equiv B^k(\mathscr{U},\mathscr{F})$, and

$$\check{H}^k(\mathscr{U},\mathscr{F}) \equiv \frac{Z^k(\mathscr{U},\mathscr{F})}{B^k(\mathscr{U},\mathscr{F})} \tag{44.10}$$

Elements in $Z^k(\mathscr{U},\mathscr{F})$ are called *cocycles* and elements in $B^k(\mathscr{U},\mathscr{F})$ are called *coboundaries* and

Example 44.3. If $k = 0$, $B^0(\mathscr{U},\mathscr{F}) = 0$, so then $\check{H}^0(\mathscr{U},\mathscr{F}) = Z^0(\mathscr{U},\mathscr{F})$. If $f_\alpha \in Z^0(\mathscr{U},\mathscr{F})$, then on $U_\alpha \cap U_\beta$ we have $f_\alpha = f_\beta$. Consequently by the gluing condition,

$$\check{H}^0(\mathscr{U},\mathscr{F}) = \mathscr{F}(X) = \Gamma(X,\mathscr{F}) \tag{44.11}$$

is the space of global sections of $\mathscr{F}$ over X.

Example 44.4. If $k = 1$, then $f \in Z^1(\mathscr{U},\mathscr{F})$ satisfies

$$f_{\alpha_1\alpha_2} - f_{\alpha_0\alpha_2} + f_{\alpha_0\alpha_1} = 0 \tag{44.12}$$

The coboundaries are

$$f_{\alpha_0\alpha_1} = f_{\alpha_1} - f_{\alpha_0}. \tag{44.13}$$

If we instead use multiplicative notation, this is

$$f_{\alpha_0\alpha_2} = f_{\alpha_0\alpha_1} f_{\alpha_1\alpha_2}, \tag{44.14}$$

and a coboundary is

$$f_{\alpha_0\alpha_1} = f_{\alpha_1} f_{\alpha_0}^{-1} \tag{44.15}$$

Note that these are exactly analogous to the cocycle condition of a vector bundle, and triviality of the bundle.

Example 44.5 (Case of S^1)**.** Let $X = S^1$ and $\mathscr{F} = \underline{\mathbb{R}}$. Consider the open covering $U_0 = X$. Then $C^0(\mathscr{U},\mathscr{F}) = \mathbb{R}$ and $C^j(\mathscr{U},\mathscr{F}) = \{0\}$ for $j \geq 1$, so we have

$$\check{H}^0(\mathscr{U},\mathscr{F}) = \mathbb{R}, \quad \check{H}^1(\mathscr{U},\mathscr{F}) = 0. \tag{44.16}$$

Next, if $\mathscr{U}$ is a covering with 2 intervals so that the intersection is 2 intervals, then

$$C^0(\mathscr{U},\mathscr{F}) = \mathbb{R}^2, \quad C^1(\mathscr{U},\mathscr{F}) = \mathbb{R}^2 \tag{44.17}$$

and $C^j = 0$ for $j \geq 2$. Since δ^0 obviously has rank 1, we conclude that

$$\check{H}^0(\mathscr{U},\mathscr{F}) = \mathbb{R}, \quad \check{H}^1(\mathscr{U},\mathscr{F}) = \mathbb{R}. \tag{44.18}$$

Next, if cover by 3 intervals $\{U_0, U_1, U_2\}$, so that the intersections are connected intervals, we then have

$$C^0(\mathscr{U},\mathscr{F}) = \mathbb{R}^3, \quad C^1(\mathscr{U},\mathscr{F}) = \mathbb{R}^3. \tag{44.19}$$

If we choose a basis for C^1 indexed by the open sets U_{01}, U_{02}, U_{12}, where $U_{ij} = U_i \cap U_j$, then matrix of δ^1 is given by

$$\delta^1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{44.20}$$

It is easy to see that this has rank 2, and we conclude that

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = \mathbb{R}, \quad \check{H}^1(\mathscr{U}, \mathscr{F}) = \mathbb{R}. \tag{44.21}$$

We obtain the same answer for a covering by k intervals, so the answer seems to stabilize.

Example 44.6. Let $X = S^2 = \mathbb{P}^1$, and let $\mathscr{F} = \underline{\mathbb{R}}$. Choose an open covering of S^2 by 2 open set by slightly enlarging the northern and southern hemispheres, call these U_0 and U_1. Then $U_0 \cap U_1$ is a tubular neighborhood of the equator. We have

$$C^0(\mathscr{U}, \mathscr{F}) = \mathbb{R}^2, \quad C^1(\mathscr{U}, \mathscr{F}) = \mathbb{R}, \tag{44.22}$$

and $C^j(\mathscr{U}, \mathscr{F}) = 0$ for $j \geq 2$. Consequently,

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = \mathbb{R}, \quad \check{H}^1(\mathscr{U}, \mathscr{F}) = 0, \tag{44.23}$$

and $\check{H}^j(\mathscr{U}, \mathscr{F}) = 0$ for $j \geq 2$.

Next, we can find a good cover of S^2 by slightly enlarging the faces of tetrahedron, call these U_0, U_1, U_2, U_3, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. We see that

$$C^0(\mathscr{U}, \mathscr{F}) = \mathbb{R}^4, \quad C^1(\mathscr{U}, \mathscr{F}) = \mathbb{R}^6, \quad C^2(\mathscr{U}, \mathscr{F}) = \mathbb{R}^4. \tag{44.24}$$

We compute that

$$\begin{aligned} \operatorname{Ker} \delta^1 = \{ & f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23} \mid \\ & f_{12} - f_{02} + f_{01} = 0, f_{13} - f_{03} + f_{01} = 0, f_{23} - f_{03} + f_{02} = 0, f_{23} - f_{13} + f_{12} = 0\}. \end{aligned} \tag{44.25}$$

This shows that a kernel element is completely determined by f_{01}, f_{02}, f_{03}, and $\operatorname{rank}(\delta^1) = 3$. A similar computation proves that $\operatorname{rank}(\delta^0) = 3$. We conclude that

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = \mathbb{R}, \quad \check{H}^1(\mathscr{U}, \mathscr{F}) = 0, \quad \check{H}^2(\mathscr{U}, \mathscr{F}) = \mathbb{R}, \tag{44.26}$$

and $\check{H}^j(\mathscr{U}, \mathscr{F}) = 0$ for $j \geq 3$.

Instead of a tetrahedron, one could also use a hexahedron (cube), octahedron, icosahedron, or dodecahedron, to construct a covering of S^2. We leave it to the interested student to compute the Čech cohomology of this covers (answer: you get the same answer for the cohomology groups as for the tetrahedron).

45 Lecture 45

45.1 Cochain complexes

Let us formalize some of the above notions. A collection A^p of abelian groups for $p \geq 0$ and homomorphisms $\delta_A^p : A^p \to A^{p+1}$ for $p \geq 0$ satisfying $\delta_A^{p+1}\delta_A^p = 0$ is called a *cochain complex*.

$$\cdots \xrightarrow{\delta_A^{p-2}} A^{p-1} \xrightarrow{\delta_A^{p-1}} A^p \xrightarrow{\delta_A^p} A^{p+1} \xrightarrow{\delta_A^{p+1}} \cdots . \tag{45.1}$$

Since $\mathrm{Im}(\delta^{p-1}) \subset \mathrm{Ker}(\delta^p)$, and the groups are abelian, we can make the following definition.

Definition 45.1. The pth cohomology group of a cochain complex is the abelian group

$$H^p(A^*) = \frac{\mathrm{Ker}\{\delta_A^p : A^p \to A^{p+1}\}}{\mathrm{Im}\{\delta_A^{p-1} : A^{p-1} \to A^p\}} \tag{45.2}$$

Definition 45.2. A morphism $\alpha : A^* \to B^*$ of cochain complexes is a collection of homomorphisms $\alpha^p : A^p \to B^p$ such that $\delta_B^p \alpha^p = \alpha^{p+1}\delta_A^p$ for $p \geq 0$. In other words, $\alpha : A^* \to B^*$ is a morphism if the following diagram commutes

$$\begin{array}{ccc} A^p & \xrightarrow{\delta_A^p} & A^{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ B^p & \xrightarrow{\delta_B^p} & B^{p+1}. \end{array} \tag{45.3}$$

Proposition 45.3. *Morphisms satisfy the following properties:*

- *Composition of morphisms: If $\alpha : A^* \to B^*$ and $\beta : B^* \to C^*$ are morphisms of chain complexes, then $\beta \circ \alpha : A^* \to C^*$ is a morphism.*
- *Associativity: If $\gamma : C^* \to D^*$ is another morphism, then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.*

Consequently, the collection of cochain complexes and morphisms of cochain complexes forms a category.

Proof. The diagram looks like

$$\begin{array}{ccc} A^p & \xrightarrow{d_A^p} & A^{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ B^p & \xrightarrow{d_B^p} & B^{p+1}. \\ \downarrow{\scriptstyle \beta^p} & & \downarrow{\scriptstyle \beta^{p+1}} \\ C^p & \xrightarrow{d_C^p} & C^{p+1}. \end{array} \tag{45.4}$$

We want to show that

$$\beta^{p+1} \circ \alpha^{p+1} \circ d_A^p = d_C^p \circ \beta^p \circ \alpha^p. \tag{45.5}$$

Using commutativity of the top square, the left hand side of (45.5) is

$$\beta^{p+1} \circ \alpha^{p+1} \circ d_A^p = \beta^{p+1} \circ d_B^p \circ \alpha^p. \tag{45.6}$$

Using commutativity of the bottom square, the right hand side of (45.5) is

$$d_C^p \circ \beta^p \circ \alpha^p = \beta^{p+1} \circ d_B^p \circ \alpha^p, \tag{45.7}$$

which proves (45.5).

Associativity is clear: $\gamma^p \circ (\beta^p \circ \alpha^p) = (\gamma^p \circ \beta^p) \circ \alpha^p$ holds for every $p \geq 0$ since composition of mappings is associative. □

Proposition 45.4. *A morphism of cochain complexes $\alpha : A^* \to B^*$ induces mappings $H^p\alpha : H^p(A) \to H^p(B)$. Furthermore, if $\beta : B^* \to C^*$ is another morphism of cochain complexes, then*

$$H^p(\beta \circ \alpha) = H^p\beta \circ H^p\alpha. \tag{45.8}$$

Proof. Given $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$, we have

$$\delta_B^p \alpha^p a^p = \alpha^{p+1} \delta_A^p a^p = 0, \tag{45.9}$$

therefore we can define $(H^p\alpha^p)[a^p] = [\alpha^p a^p]$. To check that this is well-defined,

$$[\alpha^p(a^p + \delta_A^{p-1} a^{p-1})] = [\alpha^p a^p + \alpha^p \delta_A^{p-1} a^{p-1}] = [\alpha^p a^p + \delta_B^{p-1} \alpha^{p-1} a^{p-1}] = [\alpha^p a^p]. \tag{45.10}$$

Next, for $[a^p] \in H^p(A)$ represented by $a^p \in A^p$, we have

$$H^p(\beta \circ \alpha)[a^p] = [(\beta^p \circ \alpha^p) a^p] = [\beta^p(\alpha^p(a^p))] = H^p\beta^p[\alpha^p(a^p)] = H^p\beta(H^p\alpha[a^p]). \tag{45.11}$$

□

Definition 45.5. For $p \geq 0$, the pth cohomology functor H^p is the mapping between the category of cochain complexes to the category of abelian groups given by $A \mapsto H^p(A)$.

Proposition 45.6. *The functor H^p is a covariant functor.*

Proof. The functor H^p maps objects to objects, just by mapping the cochain complex C^* to the abelian group $H^p(C)$. Also for each morphism $\alpha : A^* \to B^*$ between cochain complexes, we associate the morphism $H^p\alpha : H^p(A) \to H^p(B)$. The covariant property is (45.8). □

Next, we make the following definition.

Definition 45.7. Let $\alpha : A^* \to B^*$, and $\beta : A^* \to B^*$ be morphisms of cochain complexes. We say that α is *cochain homotopic* to β if there exists mappings $S^p : A^p \to B^{p-1}$ such that

$$\alpha^p - \beta^p = \delta_B^{p-1} S^p + S^{p+1} \delta_A^p \tag{45.12}$$

for every $p \geq 0$.

Proposition 45.8. *If α is cochain homotopic to β then*

$$H^p\alpha = H^p\beta : H^p(A) \to H^p(B). \tag{45.13}$$

Proof. Consider the mapping $H^p\alpha - H^p\beta$, and take $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$. Then

$$\begin{aligned} (H^p\alpha - H^p\beta)[a^p] &= (H^p(\alpha - \beta))[a^p] = [(H^p(\alpha - \beta))a^p] \\ &= [\delta_B^{p-1} S^p a^p + S^{p+1} \delta_A^p a^p = [\delta_B^{p-1} S^p a^p] = 0. \end{aligned} \tag{45.14}$$

□

45.2 Čech cohomology

As seen in the above examples, the Čech cohomology depends on the choice of open cover. We next want to define a cohomology theory which does not depend on the cover.

Definition 45.9. Given an open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$, another open cover $\mathscr{V} = \{U_\beta\}_{\beta \in B}$ is call a *refinement* of $\mathscr{U}$ if for each $\beta \in B$, there is $\alpha \in A$ such that $V_\beta \subset U_\alpha$, and we say that any mapping $\rho : B \to A$ such that $V_\beta \subset U_{\phi(\beta)}$ for every $\beta \in B$ is a *refining map*.

If $\mathscr{V}$ is a refinement of $\mathscr{U}$ with refining map ρ, then we can define a mapping

$$\rho^k : C^k(\mathscr{U}, \mathscr{F}) \to C^k(\mathscr{V}, \mathscr{F}) \tag{45.15}$$

by

$$\rho^k(f_{\beta_0 \cdots \beta_k}) = f_{\rho(\beta_0) \cdots \rho(\beta_k)}|_{V_{\beta_0} \cap \cdots \cap V_{\beta_k}}. \tag{45.16}$$

Proposition 45.10. *The mapping $\rho^k : C^k(\mathscr{U}, \mathscr{F}) \to C^k(\mathscr{V}, \mathscr{F})$ induces a mapping*

$$\rho^k : \check{H}^k(\mathscr{U}, \mathscr{F}) \to \check{H}^k(\mathscr{V}, \mathscr{F}), \tag{45.17}$$

which is independent of the refining mapping ρ.

Proof. For the first claim, it is clear that

$$\delta^k_{\mathscr{V}} \rho^k = \rho^{k+1} \delta^k_{\mathscr{U}}, \tag{45.18}$$

for all k which says that ρ is a morphism of cochain complexes, so induces a well-defined mapping on cohomology by Proposition 45.4.

Next, assume we have two refining mappings $\rho : B \to A$ and $\tilde{\rho} : B \to A$. The proof of the second part involves constructing a cochain homotopy between ρ^k and $\tilde{\rho}^k$. That is, a homomorphism

$$\theta^k : C^k(\mathscr{U}, \mathscr{F}) \to C^{k-1}(\mathscr{V}, \mathscr{F}) \tag{45.19}$$

which satisfies

$$\theta^{k+1} \delta^k + \delta^{k-1} \theta^k = \rho^k - \tilde{\rho}^k. \tag{45.20}$$

Then the proposition follows from Proposition (45.8). The definition of θ is as follows. For $c^k \in C^k(\mathscr{U}, \mathscr{F})$, let

$$(\theta^k c^k)_{\beta_0 \cdots \beta_{k-1}} = \sum_{j=0}^{k-1} (-1)^j c_{\rho(\beta_0) \cdots \rho(\beta_j) \tilde{\rho}(\beta_{j+1}) \cdots \tilde{\rho}(\beta_{k-1})}|_{V_{\beta_0} \cap \cdots \cap V_{\beta_{k-1}}}. \tag{45.21}$$

and we refer the reader to [Lee24, Lemma 6.5] for the "messy computation" to prove (45.20) (there is a nice telescoping sum). □

Let $\mathscr{U}, \mathscr{V}, \mathscr{W}$ be open covers such that $\mathscr{W}$ is a refinement of $\mathscr{V}$ with refining mapping $\rho_{\mathscr{W}\mathscr{V}}$ and $\mathscr{V}$ is a refinement of $\mathscr{U}$ with refining mapping $\rho_{\mathscr{V}\mathscr{U}}$. Then $\rho_{\mathscr{W}\mathscr{U}} = \rho_{\mathscr{V}\mathscr{U}} \circ \rho_{\mathscr{W}\mathscr{V}}$ is a refinining map from $\mathscr{W}$ to $\mathscr{U}$. Consequently, the group $H^k(\mathscr{U}, \mathscr{F})$ with the well-defined refinement mappings in cohomology form a directed system of abelian groups, and we can therefore make the following definition.

Definition 45.11. Let $\mathscr{F}$ be a sheaf of abelian groups on the topological space X. Then

$$\check{H}^k(X, \mathscr{F}) \equiv \varinjlim \check{H}^k(\mathscr{U}, \mathscr{F}), \tag{45.22}$$

where the directed limit is taken over all open covers of X.

Remark 45.12. There is a problem because the collection of all open covers of a space is not necessarily a set. One can deal with this by only allowing open cover indexed by a sufficiently "large" set, such as the power set of X. Alternatively, one can use a trick of Godemont.

Next, we have the following VIP.

Proposition 45.13. *Let X be a topological space and $\mathscr{F}$ and $\mathscr{G}$ sheaves of abelian groups on X. If $F : \mathscr{F} \to \mathscr{G}$ is a sheaf morphism, then there is an induced homomorphism*

$$F^k : \check{H}^k(X, \mathscr{F}) \to \check{H}^k(X, \mathscr{G}), \tag{45.23}$$

which is functorial. That is, if $G : \mathscr{G} \to \mathscr{Q}$ is another sheaf morphism, then

$$(G \circ F)^k = G^k \circ F^k. \tag{45.24}$$

Proof. Given any open cover $\mathscr{U}$ of X, then we can define

$$F^k : C^k(\mathscr{U}, \mathscr{F}) \to C^k(\mathscr{U}, \mathscr{G}) \tag{45.25}$$

by

$$F^k(f_{\alpha_0 \cdots \alpha_k})) = F(f_{\alpha_0 \cdots \alpha_k}). \tag{45.26}$$

Clearly, this satisfies $F^k \circ \delta^{k-1} = \delta^k \circ F^{k-1}$, so is a morphism of cohcain complexes, and by Proposition 45.4 induces a mapping

$$F^k : \check{H}^k(\mathscr{U}, \mathscr{F}) \to \check{H}^k(\mathscr{U}, \mathscr{G}). \tag{45.27}$$

If $\mathscr{V}$ is another open cover which is a refinement of $\mathscr{U}$ with refining map ρ, then

$$\rho^k_{\mathscr{G}} \circ F^k_{\mathscr{U}} = F^k_{\mathscr{V}} \circ \rho^k_{\mathscr{F}}, \tag{45.28}$$

which implies that F^k descends to the direct limit to give (45.23). The functorality (45.24) follows, since this is true on the cochain level. □

46 Lecture 46

46.1 Exact sequences of cochain complexes

Definition 46.1. A sequence of abelian groups A, B, C, with homomorphisms $\alpha : A \to B$, $\beta : B \to C$

$$0 \xrightarrow{0} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{0} 0 \tag{46.1}$$

is called *exact* if the kernel of each mapping is equal to the image of the previous mapping. That is $Ker(\alpha) = \{0\}$ if and only if α is injective. Next, $Ker(\beta) = Im(\alpha)$. Finally, $Im(\beta) = C$, if and only if β is surjective.

Let C_i be a cocomplex of abelian groups for $i = 1, 2, 3$.

$$\cdots \xrightarrow{\delta_i^{p-2}} C_i^{p-1} \xrightarrow{\delta_i^{p-1}} C_i^p \xrightarrow{\delta_i^p} C_i^{p+1} \xrightarrow{\delta_i^{p+1}} \cdots . \tag{46.2}$$

with $\delta^2 = 0$. A morphism from C_i to C_j are mappings $\alpha^k : C_i^k \to C_j^k$ such that the following diagram commutes for every p

$$\begin{array}{ccc} C_i^p & \xrightarrow{\delta_i^p} & C_i^{p+1} \\ \downarrow{\alpha^p} & & \downarrow{\alpha^{p+1}} \\ C_j^p & \xrightarrow{\delta_j^p} & C_j^{p+1} \end{array} \tag{46.3}$$

For cocomplexes C_1, C_2, C_3, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of cocomplexes

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{46.4}$$

is *short exact* if the sequence

$$0 \xrightarrow{0} C_1^p \xrightarrow{\alpha^p} C_2^p \xrightarrow{\beta^p} C_3^p \xrightarrow{0} 0 \tag{46.5}$$

is exact for every p.

Lemma 46.2 (The zig-zag lemma for cochain complexes)**.** *If*

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{46.6}$$

is a short exact sequence of cocomplexes, then there exist connecting homomorphisms

$$\psi^p : H^p(C_3) \to H^{p+1}(C_1) \tag{46.7}$$

for every p such that the sequence

$$\cdots \xrightarrow{\psi^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\psi^p} H^{p+1}(C_1) \longrightarrow \cdots \tag{46.8}$$

is exact.

Proof. We look at the huge commutative diagram

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1^{p-1} & \xrightarrow{\alpha^{p-1}} & C_2^{p-1} & \xrightarrow{\beta^{p-1}} & C_3^{p-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_1^{p-1}} & & \downarrow{\scriptstyle \delta_2^{p-1}} & & \downarrow{\scriptstyle \delta_3^{p-1}} & & \\
0 & \longrightarrow & C_1^{p} & \xrightarrow{\alpha^{p}} & C_2^{p} & \xrightarrow{\beta^{p}} & C_3^{p} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_1^{p}} & & \downarrow{\scriptstyle \delta_2^{p}} & & \downarrow{\scriptstyle \delta_3^{p}} & & \\
0 & \longrightarrow & C_1^{p+1} & \xrightarrow{\alpha^{p+1}} & C_2^{p+1} & \xrightarrow{\beta^{p+1}} & C_3^{p+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
\tag{46.9}
$$

which has all horizontal rows exact.

To define the connecting homomorphism, take $c_3^p \in C_3^p$ with $\delta_3^p c_3^p = 0$. By exactness of the middle row, β_p is surjective, so $c_3^p = \beta^p(c_2^p)$ for some $c_2^p \in C_2^p$. Then since the diagram commutes, we have

$$\beta^{p+1}\delta_2^p c_2^p = \delta_3^p \beta^p c_2^p = \delta_3^p c_3^p = 0. \tag{46.10}$$

By exactness of the bottow row, we have $\delta_2^p c_2^p = \alpha^{p+1} c_1^{p+1}$ for some $c_1^{p+1} \in C_1^{p+1}$. Since C_1 is a cocomplex, and by commutativity of the diagram, we have

$$0 = \delta_2^{p+1}\delta_2^p c_2^p = \delta_2^{p+1}\alpha^{p+1} c_1^{p+1} = \alpha^{p+2}\delta_1^{p+1} c_1^{p+1}, \tag{46.11}$$

which implies that $\delta_1^{p+1} c_1^{p+1} = 0$, since α^{p+2} is injective. So we define $\psi^p(c_3^p) = [c_1^{p+1}]$, the cohomology class of c_1^{p+1} in $H^{p+1}(C_1)$.

To prove this mapping is well-defined: assume that we started with $c_p^3 \in C_p^3$ which was of the form $c_p^3 = \delta_3^{p-1} c_3^{p-1}$. Then we can write $c_3^{p-1} = \beta^{p-1} c_2^{p-1}$, and the element $\tilde{c}_2^p = \delta_2^{p-1} c_2^{p-1}$ satisfies $\beta^p(\tilde{c}_2^p) = c_3^p$. But this element is exact, so the next step clearly gives zero. Independence of the choice of c_2^p is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing. □

Exercise 46.3. Prove that the sequence (46.8) is exact.

46.2 Long exact sequence

Theorem 46.4. *Let X be a paracompact Hausdorff topological space, $\mathscr{F}, \mathscr{G}, \mathscr{H}$ sheaves of abelian groups on X, and let*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\gamma} \mathscr{H} \longrightarrow 0 \tag{46.12}$$

be a short exact sequence of sheaves. Then there is a long exact sequence of abelian groups

$$\cdots \xrightarrow{\psi^{p-1}} \check{H}^p(X,\mathscr{F}) \xrightarrow{\phi^p} \check{H}^p(X,\mathscr{G}) \xrightarrow{\gamma^p} \check{H}^p(X,\mathscr{H}) \xrightarrow{\psi^p} \check{H}^{p+1}(X,\mathscr{F}) \longrightarrow \cdots \tag{46.13}$$

Proof. For any open covering $\mathscr{U} = \{U_i\}_{i\in\mathcal{I}}$, there is an exact sequence

$$0 \longrightarrow C^p(\mathscr{U},\mathscr{F}) \xrightarrow{\phi^p} C^p(\mathscr{U},\mathscr{G}) \xrightarrow{\gamma^p} C^p(\mathscr{U},\mathscr{H}) \tag{46.14}$$

Note that the mapping γ^p is not necessarily surjective. If we define $C^p_a(\mathscr{U},\mathscr{H}) = \operatorname{Im}\gamma^p$, then we have a short exact sequence

$$0 \longrightarrow C^p(\mathscr{U},\mathscr{F}) \xrightarrow{\phi^p} C^p(\mathscr{U},\mathscr{G}) \xrightarrow{\gamma^p} C^p_a(\mathscr{U},\mathscr{H}) \longrightarrow 0. \tag{46.15}$$

If $\mathscr{V} = \{V_j\}_{j\in\mathcal{J}}$ is a refinement of $\mathscr{U}$ and $\rho : \mathcal{J} \to \mathcal{I}$ is a refining map, then we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^p(\mathscr{U},\mathscr{F}) & \xrightarrow{\phi^p} & C^p(\mathscr{U},\mathscr{G}) & \xrightarrow{\gamma^p} & C^p_a(\mathscr{U},\mathscr{H}) & \longrightarrow & 0. \\ & & \downarrow{\scriptstyle \rho^p} & & \downarrow{\scriptstyle \rho^p} & & \downarrow{\scriptstyle \rho^p} & & \\ 0 & \longrightarrow & C^p(\mathscr{V},\mathscr{F}) & \xrightarrow{\phi^p} & C^p(\mathscr{V},\mathscr{G}) & \xrightarrow{\gamma^p} & C^p_a(\mathscr{V},\mathscr{H}) & \longrightarrow & 0. \end{array} \tag{46.16}$$

Using the zig-zag lemma, we obtain a commutative diagram

$$\begin{array}{ccccccccc} \cdots \xrightarrow{\psi^{p-1}} & \check{H}^p(\mathscr{U},\mathscr{F}) & \xrightarrow{\phi^p} & \check{H}^p(\mathscr{U},\mathscr{G}) & \xrightarrow{\gamma^p} & \check{H}^p_a(\mathscr{U},\mathscr{H}) & \xrightarrow{\psi^p} & \check{H}^{p+1}(\mathscr{U},\mathscr{F}) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \rho^p} & & \downarrow{\scriptstyle \rho^p} & & \downarrow{\scriptstyle \rho^p} & & \downarrow{\scriptstyle \rho^p} & \\ \cdots \xrightarrow{\psi^{p-1}} & \check{H}^p(\mathscr{V},\mathscr{F}) & \xrightarrow{\phi^p} & \check{H}^p(\mathscr{V},\mathscr{G}) & \xrightarrow{\gamma^p} & \check{H}^p_a(\mathscr{V},\mathscr{H}) & \xrightarrow{\psi^p} & \check{H}^{p+1}(\mathscr{V},\mathscr{F}) & \longrightarrow \cdots \end{array} \tag{46.17}$$

This means we can pass to the direct limit, and obtain a long exact sequence

$$\cdots \xrightarrow{\psi^{p-1}} \check{H}^p(X,\mathscr{F}) \xrightarrow{\phi^p} \check{H}^p(X,\mathscr{G}) \xrightarrow{\gamma^p} \check{H}^p_a(X,\mathscr{H}) \xrightarrow{\psi^p} \check{H}^{p+1}(X,\mathscr{F}) \longrightarrow \cdots \tag{46.18}$$

Next, we will prove that $\check{H}^p_a(X,\mathscr{H}) \simeq \check{H}^p(X,\mathscr{H})$, which will complete the proof. To see this, for each open cover $\mathscr{U}$, consider the cochain complex

$$0 \longrightarrow C^p_a(\mathscr{U},\mathscr{H}) \xrightarrow{\iota^p} C^p(\mathscr{U},\mathscr{H}) \xrightarrow{q^p} C^p_q(\mathscr{U},\mathscr{H}) \longrightarrow 0, \tag{46.19}$$

where ι is the inclusion mapping, and C^p_q is the quotient group. This sequence is compatible with refinement mappings, so by taking direct limits, we obtain a long exact sequence

$$\cdots \xrightarrow{\psi^{p-1}} \check{H}^p_a(X,\mathscr{H}) \xrightarrow{\iota^p} \check{H}^p(X,\mathscr{H}) \xrightarrow{q^p} \check{H}^p_q(X,\mathscr{H}) \xrightarrow{\psi^p} \check{H}^{p+1}_a(X,\mathscr{H}) \longrightarrow \cdots \tag{46.20}$$

Therefore, if we prove that $\check{H}^p_q(X,\mathscr{H}) \simeq 0$ for all $p \geq 0$, we will be done. This vanishing will follow from the claim: If $c^p \in C^p(\mathscr{U},\mathscr{H})$ is a cochain, then there exists a refinement $\mathscr{V}$ of $\mathscr{U}$ and a refining map $\rho : \mathcal{J} \to \mathcal{I}$ such that $\rho^p c^p \in C^p_a(\mathscr{V},\mathscr{H})$. For this, we can assume that $\mathscr{U}$ is locally finite (using paracompact Hausdorff assumption), and we choose a locally finite refinement $\mathscr{V}$ indexed by points $x \in X$, such each V_x is contained in the intersection of all the U_i-s containing x, and on each $(p+1)$-fold intersection for

$$c_{i_0 \dots i_p} \in C^p(U_{i_0} \cap \dots \cap U_{i_p}, \mathscr{H}), \tag{46.21}$$

we have that $c^p|_{V_x} = \gamma^p(c^p_a)$, where

$$c^p_a \in C^p(V_x, \mathscr{H}). \tag{46.22}$$

From this it follows that $\rho^p c^p \in C^p_a(\mathscr{V}, \mathscr{H})$, and we are done. □

46.3 Acyclic resolutions

Let X be a paracompact Hausdorff space, and $\mathscr{F}$ be a sheaf of abelian groups on X.

Theorem 46.5. *Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \cdots \tag{46.23}$$

be an exact sequence of sheaves on X. Assume that $H^q(X, \mathscr{F}^j) = 0$ for all $j \geq 0$ and $q \geq 1$. Then

$$\check{H}^0(X, \mathscr{F}) \simeq \Gamma(X, \mathrm{Ker}\{d^0 : \Gamma(X, \mathscr{F}^0) \to \Gamma(X, \mathscr{F}^1)\}), \tag{46.24}$$

and

$$\check{H}^q(X, \mathscr{F}) \simeq \frac{\mathrm{Ker}\{d^q : \Gamma(X, \mathscr{F}^q) \to \Gamma(X, \mathscr{F}^{q+1})\}}{\mathrm{Im}\{d^{q-1} : \Gamma(X, \mathscr{F}^{q-1}) \to \Gamma(X, \mathscr{F}^q)\}} \tag{46.25}$$

if $q \geq 1$.

Proof. Let $\mathscr{Z}^j = \mathrm{Ker}\, d^j$, which is a subsheaf of $\mathscr{F}^j$. By assumption,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{Z}^1 \longrightarrow 0 \tag{46.26}$$

is exact. The long exact sequence in sheaf cohomology then implies that

$$\check{H}^1(X, \mathscr{F}) \simeq \frac{\check{H}^0(X, \mathscr{Z}^1)}{d(H^0(X, \mathscr{F}^0))}, \tag{46.27}$$

and

$$\check{H}^q(X, \mathscr{Z}^1) \simeq \check{H}^{q+1}(X, \mathscr{F}) \tag{46.28}$$

for $q \geq 1$. We also have that

$$0 \longrightarrow \mathscr{Z}^j \longrightarrow \mathscr{F}^j \xrightarrow{d^j} \mathscr{Z}^{j+1} \longrightarrow 0 \tag{46.29}$$

is exact for $j \geq 1$. The long exact sequence yields

$$\check{H}^1(X, \mathscr{Z}^j) \simeq \frac{\check{H}^0(X, \mathscr{Z}^{j+1})}{d(\check{H}^0(X, \mathscr{F}^j))} \tag{46.30}$$

and

$$\check{H}^q(X, \mathscr{Z}^{j+1}) \simeq \check{H}^{q+1}(X, \mathscr{Z}^j), \tag{46.31}$$

for $j \geq 1$. To finish the argument, we have

$$\check{H}^q(X, \mathscr{F}) \simeq \check{H}^{q-1}(X, \mathscr{Z}^1) \simeq \check{H}^{q-2}(X, \mathscr{Z}^2) \simeq \cdots \simeq \check{H}^1(X, \mathscr{Z}^{q-1}) \simeq \frac{\check{H}^0(X, \mathscr{Z}^q)}{d(\check{H}^0(X, \mathscr{Z}^{q-1}))}. \tag{46.32}$$

□

47 Lecture 47

From now on, we let X be a paracompact Hausdorff space.

Definition 47.1. Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X. We say that $\mathscr{U} = \{U_i\}_{i \in \mathcal{I}}$ is a *good cover* of X for $\mathscr{F}$ if

$$\check{H}^q(U_{i_0} \cap \cdots \cap U_{i_k}, \mathscr{F}) = 0 \tag{47.1}$$

for all $q \geq 1$ and for all $k \geq 0$.

The great thing about good covers is that they can be used to compute the sheaf cohomology:

Proposition 47.2. *Assume that $\mathscr{F}$ admits an acyclic resolution*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \cdots \tag{47.2}$$

wiith $H^q(X, \mathscr{F}^j) = 0$ for all $j \geq 0$ and $q \geq 1$. Furthermore, assume that (48.11) *restricted to any open subset of X is also an acyclic resolution. If $\mathscr{U}$ is a good cover of X for $\mathscr{F}$, then*

$$\check{H}^q(\mathscr{U}, \mathscr{F}) \simeq \check{H}^q(X, \mathscr{F}) \tag{47.3}$$

for all $q \geq 0$.

Proof. By assumption, the sequence (48.11) restricted to any intersection $U = U_{i_0} \cap \cdots \cap U_{i_k}$ is an acyclic resolution of $\mathscr{F}|_U$. By Theorem 46.5, we have

$$\check{H}^q(U, \mathscr{F}) \simeq \frac{\operatorname{Ker}\{d^q : \Gamma(U, \mathscr{F}^q) \to \Gamma(U, \mathscr{F}^{q+1})\}}{\operatorname{Im}\{d^{q-1} : \Gamma(U, \mathscr{F}^{q-1}) \to \Gamma(U, \mathscr{F}^q)\}} \tag{47.4}$$

if $q \geq 1$. Since $\mathscr{U}$ is a good cover, this vanishes, so

$$\Gamma(U, \operatorname{Ker} d^q) = d^{q-1}\big(\Gamma(U, \mathscr{F}^{q-1})\big). \tag{47.5}$$

Letting $\mathcal{Z}^j \equiv \operatorname{Ker}(d^j) \subset \mathscr{F}^j$, we have the exact sequence of Čech cochains

$$0 \longrightarrow C^q(\mathscr{U}, \mathcal{Z}^j) \longrightarrow C^q(\mathscr{U}, \mathscr{F}^j) \xrightarrow{d^j} C^q(\mathscr{U}, \mathcal{Z}^{j+1}) \longrightarrow 0 \tag{47.6}$$

for all $q \geq 0$ and $j \geq 0$.

The long exact sequence in cohomology implies that

$$\check{H}^1(\mathscr{U}, \mathcal{Z}^j) \simeq \frac{\check{H}^0(\mathscr{U}, \mathcal{Z}^{j+1})}{d^j \check{H}^0(\mathscr{U}, \mathscr{F}^j)} \simeq \frac{\check{H}^0(X, \mathcal{Z}^{j+1})}{d^j \check{H}^0(X, \mathscr{F}^j)} \simeq \check{H}^{j+1}(X, \mathscr{F}), \tag{47.7}$$

by Theorem 46.5, and

$$\check{H}^q(\mathscr{U}, \mathcal{Z}^{j+1}) \simeq \check{H}^{q+1}(\mathscr{U}, \mathcal{Z}^j) \tag{47.8}$$

for $q \geq 1$.

To finish the argument, we have

$$\check{H}^q(\mathscr{U}, \mathscr{F}) \simeq \check{H}^{q-1}(\mathscr{U}, \mathcal{Z}^1) \simeq \check{H}^{q-2}(\mathscr{U}, \mathcal{Z}^2) \simeq \cdots \simeq \check{H}^1(\mathscr{U}, \mathcal{Z}^{q-1}) \simeq \check{H}^q(X, \mathscr{F}). \tag{47.9}$$

□

The following will be a key tool in finding acyclic resolutions of various sheaves.

Proposition 47.3. *Let M be a smooth manifold, and let $\mathscr{F}$ be a sheaf of $\mathcal{E}_M(\mathcal{C}_M)$ modules, where $\mathcal{E}_M(\mathcal{C}_M)$ is the sheaf of germs of C^∞ (continuous) functions on M. Then $\check{H}^q(M, \mathscr{F}) = 0$ for $q \geq 1$.*

Proof. Given a locally finite open cover $\mathscr{U} = \{U_i\}_{i \in \mathcal{I}}$, let $\{\phi_i\}$ be a partition of unity subordinate to $\mathscr{U}$. Define $\theta : C^p(\mathscr{U}, \mathscr{F}) \to C^{p-1}(\mathscr{U}, \mathscr{F})$ by

$$(\theta c)_{i_0 \cdots i_{p-1}} = \sum_i \phi_i c_{i i_0 \cdots i_{p-1}}. \tag{47.10}$$

Since the cover is locally finite, the sum is finite near any point $x \in M$. We claim that on $C^p(\mathscr{U}, \mathscr{F})$, we have

$$\delta\theta + \theta\delta = Id. \tag{47.11}$$

To prove this, we compute

$$(\delta\theta c)_{i_0 \cdots i_p} = \sum_{j=0}^{p} (-1)^j (\theta c)_{i_0 \cdots \hat{i}_j \cdots i_p} = \sum_{j=0}^{p} \sum_i (-1)^j \phi_i c_{i i_0 \cdots \hat{i}_j \cdots i_p}. \tag{47.12}$$

The next term is

$$(\theta\delta c)_{i_0 \cdots i_p} = \sum_i \phi_i (\delta c)_{i i_0 \cdots i_p} = \sum_i \phi_i c_{i_0 \cdots i_p} + \sum_i \sum_{j=0}^{p} (-1)^{j+1} \phi_i c_{i_0 \cdots \hat{i}_j \cdots i_p}. \tag{47.13}$$

Summing these together, we obtain (47.11) since $\sum_i \phi_i = 1$. To finish the proof, if $c^p \in Z^p(\mathscr{U}, \mathscr{F})$ is a cocycle, then $\delta\theta c = c$ is a coboundary. We have shown that $\check{H}^q(\mathscr{U}, \mathscr{F}) = 0$ for $q \geq 1$ for any open cover, so the directed limit also vanishes. □

47.1 de Rham cohomology

Let M be a smooth manifold, and $\mathcal{E}^p_M$ be the sheaf of germs of smooth p-forms on M. Then

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{E}^0_M \xrightarrow{d^0} \mathcal{E}^1_M \xrightarrow{d^1} \cdots, \tag{47.14}$$

where d is the exterior derivative, is exact by the Poincaré Lemma. By Proposition 47.3, this is an acyclic resolution on any open subset $U \subset M$. Consequently, by Proposition 47.2

$$\check{H}^p(M, \underline{\mathbb{R}}) \simeq H^p_{dR}(M), \tag{47.15}$$

for all $p \geq 0$, where $H^*_{dR}(M)$ denotes the de Rham cohomology of M. In particular, the de Rham cohomology is a topological invariant of M.

Proposition 47.4. *If M is smooth manifold, then M admits a good cover for $\underline{\mathbb{R}}$.*

Proof. Note that by endowing M with a Riemannian metric, we can take an open cover of M by geodesically convex open sets. The intersection of any 2 geodesically convex open sets in geodesically convex. Furthermore, using the exponential map, any geodesically covex set is diffeomorphic to a star-shaped domain in $\mathbb{R}^n$, which has vanishing higher degree de Rham cohomology. □

Remark 47.5. Another way to prove this is to slightly enlarge all the vertices, edges, faces, etc., of any triangulation of M.

47.2 Dolbeault cohomology

Let M be a complex manifold, Ω^p be the sheaf of germs of holomorphic $(p, 0)$ forms, and $\mathcal{E}^{p,q}$ be the sheaf of germs of smooth (p, q)-forms on M. Then

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \cdots \tag{47.16}$$

is exact by the $\overline{\partial}$-Poincaré Lemma which we proved a long time ago (the "easy" version, which allows for shrinkage). By Proposition 47.3, this is an acyclic resolution on any open subset $U \subset M$. Consequently, by Proposition 47.2

$$\check{H}^q(M, \Omega^p) \simeq H^{p,q}_{\overline{\partial}}(M), \tag{47.17}$$

for all $p, q \geq 0$, where $H^*_{\overline{\partial}}(M)$ denotes the Dolbeault cohomology of M.

Proposition 47.6. *If M is a complex manifold, then M admits a good cover for Ω^p.*

Proof. We can cover M by U_i which are which are biholomorphic to balls in $\mathbb{C}^n$. In particular, the U_i are pseudoconvex. The intersections of pseudoconvex manifolds are pseudoconvex. On any pseudoconvex manifold U, we have $H^q(U, \Omega^p) = 0$ for all $q > 0$, as can be seen from Hormander's L^2 theory, so this will give a good cover. □

Remark 47.7. The above is true more generally for any coherent analytic sheaf on any complex manifold.

48 Lecture 48

48.1 Dolbeault cohomology of $\mathbb{P}^n$

First, let $X = \mathbb{P}^1 = [u, v]$. There is an open cover $\mathscr{U} = \{U, V\}$, with $U = [u, 1]$ and $V = [1, v]$, which are both copies of $\mathbb{C}$. The intersection $U \cap V$ is biholomorphic to $\mathbb{C}^*$. Since we know that any domain in $\mathbb{C}$ has vanishing Dolbeault groups $H^{p,q}$ for $q \geq 1$, this is a good cover for $\Omega^p_{\mathbb{P}^1}$, so by Proposition 47.2, we can use this to compute the Dolbeault cohomology groups of $\mathbb{P}^1$.

Let $c^0 \in C^0(\mathscr{U}, \mathcal{O})$, the c^0 has 2 components,

$$c^0_0 = -\sum_{j=0}^{\infty} a_j u^j, \quad c^0_1 = \sum_{j=0}^{\infty} b_j v^j. \tag{48.1}$$

Then if c^0 is a cocycle, we must have

$$0 = (\delta c^0)_{01} = c^0_1(u^{-1}) - c^0_0(u) = \sum_{j=0}^{\infty} b_j u^{-j} + \sum_{j=0}^{\infty} a_j u^j, \tag{48.2}$$

which says that $b_j = a_j = 0$ for $j \neq 0$ and $a_0 + b_0 = 0$. Consequently, $\check{H}^0(\mathscr{U}, \mathcal{O}) \simeq \mathbb{C}$. Alternatively, one can use the Dolbeault isomorphism to see that $H^{0,0}_{\bar{\partial}}(\mathbb{P}^1) = \mathbb{C}$ since any holomorphic function on a compact complex manifold is constant by the maximum principle.

Let $c^1 \in C^1(\mathscr{U}, \mathcal{O})$. Then c has only 1 component corresponding to the single interesection $U \cap V$, and can therefore be viewed as a holomorphic function $h \in \mathcal{O}_{\mathbb{C}^*}$. We know that any such function has a Laurent expansion

$$h = \sum_{j=-\infty}^{\infty} a_j u^j. \tag{48.3}$$

Let

$$f = -\sum_{j=0}^{\infty} a_j u^j, \quad g = \sum_{j=1}^{\infty} a_{-j} v^n \tag{48.4}$$

and define a cochain $c^0 \in C^0(\mathscr{U}, \mathscr{U})$ by $c^0_0 = f$, and $c^0_1 = g$. Then

$$(\delta c^0)_{01} = g(1/u) - f(u) = h, \tag{48.5}$$

which proves that c^1 is a coboundary, and therefore $\check{H}^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Next, let $c^0 \in C^0(\mathscr{U}, \Omega^1)$. Then

$$c^0_0 = \Big(\sum_{j=0}^{\infty} a_j u^j\Big) du, \quad c^0_1 = \Big(\sum_{j=0}^{\infty} b_j v^j\Big) dv, \tag{48.6}$$

Then if c^0 is a cocycle, we must have

$$0 = (\delta c^0)_{01} = c^0_1(u^{-1}) - c^0_0(u) = \Big(\sum_{j=0}^{\infty} b_j u^{-j}\Big)\Big(\frac{-du}{u^2}\Big) - \sum_{j=0}^{\infty} a_j u^j du \tag{48.7}$$

$$= -\Big(\sum_{j=0}^{\infty} b_j u^{-j-2} + \sum_{j=0}^{\infty} a_j u^j\Big) du, \tag{48.8}$$

which implies that $a_j = b_j = 0$ for all j, so $\check{H}^0(\mathbb{P}^1, \Omega^1) = 0$.

Next, let $c^1 \in C^1(\mathscr{U}, \Omega^1)$. Then

$$c^1_{01} = \Big(\sum_{j=-\infty}^{\infty} a_j u^j\Big) \tag{48.9}$$

From the above computation, we see that all terms are coboundaries, except for the term $a_{-1}u^{-1}$. Consequently, $\check{H}^0(\mathbb{P}^1, \Omega^1) = \mathbb{C}$.

Exercise 48.1. In a similar fashion, using the good cover of $\mathbb{P}^n$ given by $U_0, \dots, U_n$, show that

$$H^q(\mathbb{P}^1, \Omega^p) = \begin{cases} \mathbb{C} & p = q \leq n \\ 0 & \text{otherwise} \end{cases}. \tag{48.10}$$

48.2 Mayer-Vietoris sequence

Proposition 48.2. *Assume that M is a smooth manifold, and let $\mathscr{F}$ be a sheaf on M which admits a resolution*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \cdots \tag{48.11}$$

with $\mathscr{F}^j$ a module over $\mathcal{E}_M$. If $\mathscr{U} = \{U, V\}$ is any open covering of X, then there is a long exact sequence

$$\cdots \xrightarrow{\psi^{p-1}} \check{H}^p(X, \mathscr{F}) \xrightarrow{\phi^p} \check{H}^p(U, \mathscr{F}) \oplus \check{H}^p(U, \mathscr{F}) \xrightarrow{\gamma^p} \check{H}^p(U \cap V, \mathscr{F}) \xrightarrow{\psi^p} \check{H}^{p+1}(X, \mathscr{F}) \longrightarrow \cdots \tag{48.12}$$

Proof. For each $j \geq 0$, we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}^j) \xrightarrow{\phi^j} \Gamma(U, \mathscr{F}^j) \oplus \Gamma(V, \mathscr{F}^j) \xrightarrow{\gamma^j} \Gamma(U \cap V, \mathscr{F}^j) \longrightarrow 0. \tag{48.13}$$

where $\phi^j(s) = (r^X_U s, r^X_V s)$, and $\gamma^j(s_U, s_V) = r^U_{U\cap V} s_U - r^V_{U \cap V} s_V$. Note that ϕ^j is injective by the uniqueness property for sheaves. Obviously, $\operatorname{Im} \phi^j \subset \operatorname{Ker} \gamma^j$. If $(s_U, s_v) \in \operatorname{Ker}(\gamma^j)$ then these sections agree on the intersection, so by the gluing property for sheaves $\operatorname{Ker} \gamma^j \subset \operatorname{Im} \phi^j$. Finally, to see that γ^j is surjective, let $s_{U\cap V} \in \Gamma(U \cap V, \mathscr{F}^j)$. Then $\phi_U s_{U \cap V}$ is supported on a closed subset of V, so upon extension by zero, can be considered as an element of $\Gamma(V, \mathscr{F}^j)$. Similarly $\phi_V s_{U\cap V} \in \Gamma(U, \mathscr{F}^j)$. Then

$$\gamma^j(\phi_V s_{U \cap V}, -\phi_U s_{U\cap V}) = (\phi_U + \phi_V) s_{U \cap V} = s_{U \cap V}. \tag{48.14}$$

The claim then follows from the associated long exact sequence in cohomology and Proposition 47.2. ☐

48.3 Cohomology of S^n

The following lemma will be useful.

Lemma 48.3. *If*

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_{k-1} \longrightarrow V_k \longrightarrow 0. \tag{48.15}$$

is exact, then

$$0 = \dim(V_1) - \dim(V_2) + \dim(V_3) + \cdots + (-1)^{k-1}\dim(V_k). \tag{48.16}$$

Proof. Induction. □

Cover S^n with 2 open sets U, V, with $U \cong \mathbb{R}^n \cong V$ and $U \cap V \cong S^{n-1}$, we use the Mayer-Vietoris sequence and induction to get

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases}. \tag{48.17}$$

To see this, consider first the case of S^1.

$$\begin{aligned} 0 \longrightarrow H^0_{dR}(S^1) &\xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \\ H^1_{dR}(S^1) &\xrightarrow{\beta^1} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^1} H^1_{dR}(U \cap V) \longrightarrow 0. \end{aligned} \tag{48.18}$$

But $U \cap V$ is contractible to 2 points, so this is

$$\begin{aligned} 0 \longrightarrow \mathbb{R} &\xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta^0} \\ H^1_{dR}(S^1) &\xrightarrow{\beta^1} 0. \end{aligned} \tag{48.19}$$

The lemma then says that $H^1_{dR}(S^1) \cong \mathbb{R}$.

Next, for $n > 1$, look at the beginning of the Mayer-Vietoris sequence

$$0 \longrightarrow H^0_{dR}(S^n) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots \tag{48.20}$$

But now $U \cap V$ is connected, so this is

$$0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \xrightarrow{\delta^0} \cdots . \tag{48.21}$$

Since β is injective, the kernel of α^0 is 1-dimensional. But α^0 has a 2-dimensional domain, so the image of α^0 is 1-dimensional, that is α^0 is surjective. So we can move to the next level and get

$$0 \longrightarrow H^1_{dR}(S^n) \xrightarrow{\beta^0} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^0} H^1_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots , \tag{48.22}$$

Since U and V are contractible, this says that $H^1_{dR}(S^n) = 0$ for $n \geq 2$.

Next, we look at the upper portion of the Mayer-Vietoris sequence

$$\begin{aligned} &\cdots \longrightarrow H^{n-2}_{dR}(U) \oplus H^{n-2}_{dR}(V) \xrightarrow{\alpha^{n-2}} H^{n-2}_{dR}(S^{n-1}) \\ &\xrightarrow{\delta^{n-2}} H^{n-1}_{dR}(S^n) \xrightarrow{\beta^p} 0 \xrightarrow{\alpha^p} H^{n-1}_{dR}(S^{n-1}) \\ &\xrightarrow{\delta^{n-1}} H^n_{dR}(S^n) \xrightarrow{\beta^{p+1}} 0. \end{aligned} \tag{48.23}$$

This yields

$$H^n_{dR}(S^n) \cong H^{n-1}_{dR}(S^{n-1}) \cong \mathbb{R}, \tag{48.24}$$

and

$$H^k_{dR}(S^n) \cong H^{k-1}_{dR}(S^{n-1}) = 0, \tag{48.25}$$

for $2 \leq k \leq n-1$, so this finishes the proof.

48.4 de Rham cohomology of $\mathbb{P}^n$

Theorem 48.4. *The de Rham cohomology of $\mathbb{P}^n$ is given by*

$$H^k_{dR}(\mathbb{P}^n) = \begin{cases} \mathbb{R} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}. \tag{48.26}$$

Proof. We note that $\P^n = U \cup V$, where U is diffeomorphic to $\mathbb{C}^n$, V is a tubular neighborhood of $\P^{n-1}$ and $U \cap V$ deformation retracts onto S^{2n-1}. The Mayer-Vietoris sequence gives

$$\begin{aligned} &0 \longrightarrow H^0(\mathbb{P}^n) \longrightarrow H^0_{dR}(\mathbb{C}^n) \oplus H^0_{dR}(\mathbb{P}^{n-1}) \longrightarrow H^0_{dR}(S^{2n-1}) \\ &\longrightarrow H^1_{dR}(\mathbb{P}^n) \longrightarrow H^1_{dR}(\mathbb{P}^{n-1}) \longrightarrow H^1_{dR}(S^{2n-1}) \\ &\longrightarrow H^2_{dR}(\mathbb{P}^n) \longrightarrow H^2_{dR}(\mathbb{P}^{n-1}) \longrightarrow H^2_{dR}(S^{2n-1}) \longrightarrow \cdots \end{aligned} \tag{48.27}$$

The theorem follows by induction since we know the de Rham cohomology of S^{2n-1} is non-zero only in degrees 0 and $2n-1$. $\square$

Remark 48.5. Note that $b^{2k} = b^{k,k} = 1$ with all other Hodge numbers vanishing. Thus the Hodge numbers seem to be detecting the topology. This will be explained later with the Kähler identities.

49 Lecture 49

49.1 The sheaf of not necessarily continuous sections

Several of the results in the previous section hold under even weaker assumptions, but this needs a little more machinery, which we briefly discuss. Any sheaf admits a canonical resolution called the Godemont resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \cdots, \tag{49.1}$$

where $\mathscr{F}^j$ is *flasque* (also called *flabby*). That is, for any open set U, the restriction mapping $r^X_U : \Gamma(X, \mathscr{F}^j) \to \Gamma(X, \mathscr{F}^j)$ is surjective. The resolution is constructed by taking the sheaf of germs of *not necessarily continuous* sections of the associated étale space, taking the quotient sheaf, and then repeating to get a resolution. Since any section can be extended discontinuously to the entire space, the canonical resolution is obviously flasque. Furthermore, it is not hard to show that any flasque sheaf is acyclic (exercise). In particular, Theorem 46.5 then holds for the canonical resolution. Also Proposition 47.2 about good covers then holds without any assumptions by using the canonical resolution. Finally, the Mayer-Vietoris sequence in Proposition 48.2 holds by choosing the resolution to be the canonical one, because (48.13) is clearly exact from the flasque condition.

49.2 The $\overline{\partial}$ operator on holomorphic vector bundles

The above $\overline{\partial}$ operator on vector fields is a special case of a general construction on holomorphic vector bundles. Recall that the transition functions of a complex vector bundle are locally defined functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$, satisfying

$$\phi_{\alpha\beta} = \phi_{\alpha\gamma}\phi_{\gamma\beta}. \tag{49.2}$$

Definition 49.1. A vector bundle $\pi : E \to M$ is a *holomorphic vector bundle* if in complex coordinates the transitition functions $\phi_{\alpha\beta}$ are holomorphic.

Recall that a section of a vector bundle is a mapping $\sigma : M \to E$ satisfying $\pi \circ \sigma = Id_M$. In local coordinates, a section satisfies

$$\sigma_\alpha = \phi_{\alpha\beta}\sigma_\beta, \tag{49.3}$$

and conversely any locally defined collection of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}^m$ satisfying (49.3) defines a global section. A section is *holomorphic* if in complex coordinates, the σ_α are holomorphic.

Proposition 49.2. *If $\pi : E \to M$ is a holomorphic vector bundle, then there are first order differential operators*

$$\overline{\partial} : \Gamma(\Lambda^{p,q} \otimes E) \to \Gamma(\Lambda^{p,q+1} \otimes E), \tag{49.4}$$

satisfying the following properties:

- *A section $\sigma \in \Gamma(E)$ is holomorphic if and only if $\overline{\partial}(\sigma) = 0$.*
- *For a function $f : M \to \mathbb{C}$, and a section $\sigma \in \Gamma(\Lambda^{p,q} \otimes E)$,*
$$\overline{\partial}(f \cdot \sigma) = (\overline{\partial} f) \wedge \sigma + f \cdot \overline{\partial}\sigma. \tag{49.5}$$
- $\overline{\partial} \circ \overline{\partial} = 0$.

Proof. Let σ_j be a local basis of holomorphic sections of E in U_α, and write any section $\sigma \in \Gamma(U_\alpha, \Lambda^{p,q} \otimes E)$ as

$$\sigma = \sum s_j \otimes \sigma_j, \tag{49.6}$$

where $s_j \in \Gamma(U_\alpha, \Lambda^{p,q})$. Then define

$$\overline{\partial}\sigma = \sum (\overline{\partial} s_j) \otimes \sigma_j. \tag{49.7}$$

We need to check that this is independent of the choice of local basis. Let σ_j' be another local basis of holomorphic sections of E in U_α, and write σ as

$$\sigma = \sum s_j' \otimes \sigma_j'. \tag{49.8}$$

We can write $\sigma_j' = (\phi^{-1})_{jl}\sigma_l$, where ϕ_{**} is a nonsingular matrix of holomorphic functions. Then

$$\sigma = \sum s_j' \otimes \sigma_j' = \sum s_j' \otimes (\phi^{-1})_{jl}\sigma_l = \sum (\phi^{-1})_{jl} s_j' \otimes \sigma_l, \tag{49.9}$$

so it follows that

$$s_j' = \phi_{jl} s_l, \tag{49.10}$$

and we can write

$$\sigma = \sum \phi_{jl} s_l \otimes \sigma_j'. \tag{49.11}$$

Consequently

$$\begin{aligned} \overline{\partial}\sigma &= \sum (\overline{\partial} s_j') \otimes \sigma_j' = \sum \overline{\partial}(\phi_{jk} s_k) \otimes \sigma_j' \\ &= \sum \phi_{jk} \overline{\partial}(s_k) \otimes \sigma_j' = \sum (\overline{\partial} s_k) \otimes \phi_{jk}\sigma_j' = \sum (\overline{\partial} s_k) \otimes \sigma_k, \end{aligned}$$

and the operator $\overline{\partial}_E$ is well-defined. The other properties follow immediately from the definition. □

Definition 49.3. For a holomorphic vector bundle $\pi : E \to M$, the (p, q) Dolbeault cohomology group with coefficients in E is

$$H^{p,q}_{\overline{\partial}}(M, E) = \frac{\{\alpha \in \Lambda^{p,q}(M, E) \mid \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M, E))}. \tag{49.12}$$

We have an acyclic resolution

$$0 \longrightarrow \Omega^p(E) \longrightarrow \mathcal{E}^{p,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,1}(E) \xrightarrow{\overline{\partial}_E} \cdots \tag{49.13}$$

Note that this is exact by the $\overline{\partial}$-Poincaré Lemma and the fact that E is locally holomorphically trivial. These are all modules over $\mathcal{E}_M$, so similar to above, we can conclude that

$$H^{p,q}_{\overline{\partial}}(M, E) \cong H^q(M, \Omega^p(E)). \tag{49.14}$$

50 Lecture 50

50.1 Hermitian metrics

We next consider (M, J, g) where g is a Riemannian metric, and we assume that g and J are compatible. That is,

$$g(X, Y) = g(JX, JY). \tag{50.1}$$

The metric g is called an almost-Hermitian metric. If J is also integrable, then g is called Hermitian. We extend g by complex linearity to a symmetric inner product on $T \otimes \mathbb{C}$. The following will be useful later.

Proposition 50.1. *There exist elements $\{X_1, \dots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{50.2}$$

is an ONB for $\mathbb{R}^{2n}$ with respect to g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{50.3}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of $span\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. □

To a Hermitian metric $(\mathbb{R}^{2n}, J, g)$ we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{50.4}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{50.5}$$

Since

$$\omega(JX, JY) = \omega(X, Y), \tag{50.6}$$

this form is a real form of type $(1, 1)$, and is called the *Kähler form* or *fundamental 2-form*.

In Euclidean space, this form is

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{50.7}$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{50.8}$$

Note that this defines an orientation on $\mathbb{C}^n$, which we will refer to as the natural orientation. Note also that

$$\omega^n = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{50.9}$$

Corollary 50.2. *Any almost complex manifold (M, J) is orientable.*

Proof. Given J, there always exists an almost-Hermitian metric h with respect to J. To see this, let g be any Riemannian metric, and let

$$h(X, Y) = g(X, Y) + g(JX, JY). \tag{50.10}$$

Let $\omega = h(JX, Y)$ be the Kähler form, then $\omega^n \in \Lambda^{n,n}_{\mathbb{R}} \cong \Lambda^{2n}_{\mathbb{R}}$ is a nowhere vanishing n-form. It is nowhere-vanishing since at any point, we can assume we are Euclidean by Proposition 50.1. □

The following proposition gives a fundamental relation between the covariant derivative of J, the exterior derivative of ω and the Nijenhuis tensor.

Proposition 50.3. *Let (M, g, J) be an almost Hermitian manifold. Then*

$$2g((\nabla_X J)Y, Z) = -d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX). \tag{50.11}$$

Proof. The covariant derivative of an endomorphism is given by

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y) \tag{50.12}$$

so we have

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) - g(J(\nabla_X Y), Z). \tag{50.13}$$

Since g is J-invariant, and $J^2 = -Id$, it follows that

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) + g(\nabla_X Y, JZ). \tag{50.14}$$

The Riemannian connection is defined by

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &-\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\Big).\end{aligned} \tag{50.15}$$

Now apply formula (50.15) to both terms on the right hand side of (50.14) to obtain

$$\begin{aligned}2g((\nabla_X J)Y, Z) = {}& Xg(JY, Z) + JYg(X, Z) - Zg(X, JY) \\ &- g(JY, [X, Z]) - g(Z, [JY, X]) + g(X, [Z, JY]) \\ &+ Xg(Y, JZ) + Yg(JZ, X) - JZg(X, Y) \\ &- g(Y, [X, JZ]) - g(JZ, [Y, X]) + g(X, [JZ, Y]).\end{aligned} \tag{50.16}$$

Next, using (17.7), we compute

$$
\begin{aligned}
d\omega(X, JY, JZ) &= X\omega(JY, JZ) - JY\omega(X, JZ) + JZ\omega(X, JY) \\
&\quad - \omega([X, JY], JZ) + \omega([X, JZ], JY) - \omega([JY, JZ], X]) \\
&= Xg(J^2Y, JZ) - JYg(JX, JZ) + JZg(JX, JY) \\
&\quad - g(J[X, JY], JZ) + g(J[X, JZ], JY) - g(J[JY, JZ], X]) \\
&= -Xg(Y, JZ) - JYg(X, Z) + JZg(X, Y) \\
&\quad - g([X, JY], Z) + g([X, JZ], Y) + g([JY, JZ], JX]).
\end{aligned}
\tag{50.17}
$$

The next term is

$$
\begin{aligned}
d\omega(X, Y, Z) &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\
&\quad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X]) \\
&= Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) \\
&\quad - g(J[X, Y], Z) + g(J[X, Z], Y) - g(J[Y, Z], X]).
\end{aligned}
\tag{50.18}
$$

The last term is

$$
\begin{aligned}
&\frac{1}{2}g(N(Y, Z), JX) \\
&\quad = g([JY, JZ], JX) - g([Y, Z], JX) - g(J[Y, JZ], JX) - g(J[JY, Z], JX) \\
&\quad = g([JY, JZ], JX) - g([Y, Z], JX) - g([Y, JZ], X) - g([JY, Z], X)
\end{aligned}
\tag{50.19}
$$

We then obtain the right hand side of (50.11) is

$$
\begin{aligned}
&- d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX) \\
&= Xg(Y, JZ) + JYg(X, Z) - JZg(X, Y) \\
&\quad + g([X, JY], Z) - g([X, JZ], Y) - g([JY, JZ], JX]) \\
&+ Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) \\
&\quad - g(J[X, Y], Z) + g(J[X, Z], Y) - g(J[Y, Z], X]) \\
&+ g([JY, JZ], JX) - g([Y, Z], JX) - g([Y, JZ], X) - g([JY, Z], X).
\end{aligned}
\tag{50.20}
$$

The first two terms of the last line cancel out with terms on the previous lines, so this simplifies to

$$
\begin{aligned}
&- d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX) \\
&= Xg(Y, JZ) + JYg(X, Z) - JZg(X, Y) + g([X, JY], Z) - g([X, JZ], Y) \\
&+ Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) - g(J[X, Y], Z) + g(J[X, Z], Y) \\
&- g([Y, JZ], X) - g([JY, Z], X),
\end{aligned}
\tag{50.21}
$$

and each of these 12 terms appears exactly once in (50.16). □

Corollary 50.4. *If (M, g, J) is Hermitian, then $d\omega = 0$ if and only if J is parallel.*

Proof. Since $N = 0$, the forward implication follows immediately from (50.11). For the converse, If J is parallel, then ω is also (since g is parallel).. The exterior derivative $d : \Omega^p \to \Omega^{p+1}$ can be written in terms of covariant differentiation.

$$d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{50.22}$$

which follows immediately from (13.15) using normal coordinates around a point. This shows that a parallel form is closed, and we are done. □

Corollary 50.5. *If (M, g, J) is almost Hermitian, $\nabla J = 0$ implies that $d\omega = 0$ and $N = 0$. I.e., $\nabla J = 0$ implies that (M, g, J) is Kähler.*

Proof. In the previous corollary, we already proved that J parallel implies that $d\omega = 0$. Then (50.11) implies that $N = 0$. □

Definition 50.6. An almost Hermitian manifold (M, g, J) is *Kähler* if J is integrable and $d\omega = 0$, or equivalently, if $\nabla J = 0$.

Note that if (M, g, J) is Kähler, then ω is a parallel $(1, 1)$-form.

Proposition 50.7. *There are the following equivalences:*

- *M^{2n} is almost complex if and only if the structure group of the principal frame bundle can be reduced from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$.*
- *(M^{2n}, g, J) is almost Hermitian if and only if the structure group of the bundle of orthonormal frames can be reduced from $O(2n)$ to $\mathrm{U}(n)$.*
- *(M^{2n}, g, J) is Kähler if and only if the holonomy group is contained in $\mathrm{U}(n)$.*

Remark 50.8. If (M, g, J) is almost Hermitian, then $h(X, Y) = g(X, \overline{Y})$ is a Hermitian inner product, which is conjugate linear in the second argument.

50.2 Complex tensor notation

Choosing any real basis of the form $\{X_1, JX_1, \ldots, X_n, JX_n\}$, let us abbreviate

$$Z_\alpha = \frac{1}{2}\Big(X_\alpha - iJX_\alpha\Big) \tag{50.23}$$

$$Z_{\overline{\alpha}} = \frac{1}{2}\Big(X_\alpha + iJX_\alpha\Big), \tag{50.24}$$

and define

$$g_{\alpha\beta} = g(Z_\alpha, Z_\beta) \tag{50.25}$$

$$g_{\overline{\alpha}\overline{\beta}} = g(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) \tag{50.26}$$

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) \tag{50.27}$$

$$g_{\overline{\alpha}\beta} = g(Z_{\overline{\alpha}}, Z_\beta). \tag{50.28}$$

Notice that

$$
\begin{aligned}
g_{\alpha\beta} = g(Z_\alpha, Z_\beta) &= \frac{1}{4} g(X_\alpha - iJX_\alpha, X_\beta - iJX_\beta) \\
&= \frac{1}{4}\Big(g(X_\alpha, X_\beta) - g(JX_\alpha, JX_\beta) - i(g(X_\alpha, JX_\beta) + g(JX_\alpha, X_\beta))\Big) \\
&= 0,
\end{aligned}
$$

since g is J-invariant, and $J^2 = -Id$. Similarly,

$$g_{\overline{\alpha}\overline{\beta}} = 0, \tag{50.29}$$

Also, from symmetry of g, we have

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) = g(Z_{\overline{\beta}}, Z_\alpha) = g_{\overline{\beta}\alpha}. \tag{50.30}$$

However, applying conjugation, since g is real we have

$$\overline{g_{\alpha\overline{\beta}}} = \overline{g(Z_\alpha, Z_{\overline{\beta}})} = g(Z_{\overline{\alpha}}, Z_\beta) = g(Z_\beta, Z_{\overline{\alpha}}) = g_{\beta\overline{\alpha}}, \tag{50.31}$$

which says that $g_{\alpha\overline{\beta}}$ is a Hermitian matrix.

We repeat the above for the fundamental 2-form ω, and define

$$\omega_{\alpha\beta} = \omega(Z_\alpha, Z_\beta) = i g_{\alpha\beta} = 0 \tag{50.32}$$

$$\omega_{\overline{\alpha}\overline{\beta}} = \omega(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = -i g_{\overline{\alpha}\overline{\beta}} = 0 \tag{50.33}$$

$$\omega_{\alpha\overline{\beta}} = \omega(Z_\alpha, Z_{\overline{\beta}}) = i g_{\alpha\overline{\beta}} \tag{50.34}$$

$$\omega_{\overline{\alpha}\beta} = \omega(Z_{\overline{\alpha}}, Z_\beta) = -i g_{\overline{\alpha}\beta}. \tag{50.35}$$

The first 2 equations are just a restatement that ω is of type $(1,1)$. Also, note that

$$\omega_{\alpha\overline{\beta}} = i g_{\alpha\overline{\beta}}, \tag{50.36}$$

defines a skew-Hermitian matrix.

On a complex manifold, the fundamental 2-form in holomorphic coordinates takes the form

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta. \tag{50.37}$$

Remark 50.9. Note that for the Euclidean metric, we have $g_{\alpha\overline{\beta}} = \frac{1}{2}\delta_{\alpha\beta}$, so

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{50.38}$$

Proposition 50.10. *(M, g, J) is Kähler if and only if in any local holomorphic coordinate system,*

$$\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} = \frac{\partial g_{k\overline{\beta}}}{\partial z^\alpha}, \tag{50.39}$$

Proof. If (M, g, J) is Kähler, then

$$
\begin{aligned}
0 = d\omega &= i \sum_{\alpha,\beta=1}^{n} (dg_{\alpha\overline{\beta}}) \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta=1}^{n} (\partial g_{\alpha\overline{\beta}} + \overline{\partial} g_{\alpha\overline{\beta}}) \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta=1}^{n} \Big\{ \sum_{k} \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} dz^k \Big) + \sum_{k} \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^k} d\overline{z}^k \Big) \Big\} \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} dz^k \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} + i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^{\alpha} \wedge d\overline{z}^{\beta}.
\end{aligned}
\tag{50.40}
$$

However, the first term is a form of type $(2, 1)$, and the second term is a form of type $(1, 2)$ so both sums must vanish, which is equivalent to (50.39). The converse follows by reversing the above calculation. □

We also see that the Kähler condition on a Hermitian manifold is equivalent to $\overline{\partial}\omega = 0$, which is also equivalent to $\partial\omega = 0$, since ω is real.

51 Lecture 51

51.1 Existence of local Kähler potential

First, let's recall a special case of the $\overline{\partial}$-Poincaré lemma.

Lemma 51.1. *If α is a smooth $(0, 1)$-form in a closed ball $\overline{B} \subset \mathbb{C}^n$ satisfying $\overline{\partial}\alpha = 0$, then there exists $f : B \to \mathbb{C}$ such that $\alpha = \overline{\partial} f$.*

Proof. Write $\alpha = \sum_{j=1}^{n} \alpha_{\overline{j}} d\overline{z}^j$. Then

$$
0 = \overline{\partial}\alpha = \sum_{j,k=1}^{n} \frac{\partial \alpha_{\overline{j}}}{\partial \overline{z}^k} d\overline{z}^k \wedge d\overline{z}^j. \tag{51.1}
$$

This implies that

$$
\frac{\partial \alpha_{\overline{j}}}{\partial \overline{z}^k} = \frac{\partial \alpha_{\overline{k}}}{\partial \overline{z}^j} \tag{51.2}
$$

for all $1 \leq j, k \leq n$.

We want to find f such that $\partial f = \alpha$, which in components is

$$
\frac{\partial f}{\partial \overline{z}^k} = \alpha_{\overline{k}} \tag{51.3}
$$

for all $1 \leq k \leq n$.

Recall from one complex variable that if $B \subset \mathbb{C}$, and $g : \overline{B} \to \mathbb{C}$ is smooth, then there exists $f : B \to \mathbb{C}$ such that $\frac{\partial}{\partial \overline{z}} f = g$. The solution can be written explicitly as

$$f(z) = \frac{1}{2\pi i} \int_B g(w) \frac{dw \wedge d\overline{w}}{w - z}. \tag{51.4}$$

So we define

$$f(z^1, \dots z^n) = \frac{1}{2\pi i} \int_B \alpha_{\overline{1}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1}. \tag{51.5}$$

By the above remark, we have $\partial_{\overline{1}} f = \alpha_{\overline{1}}$. Next, for $k > 1$,

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}^k}(z^1, \dots z^n) &= \frac{1}{2\pi i} \int_B \frac{\partial}{\partial \overline{z}^k} \alpha_{\overline{1}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1} \\ &= \frac{1}{2\pi i} \int_B \frac{\partial}{\partial \overline{z}^1} \alpha_{\overline{k}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1} \\ &= \alpha_{\overline{k}}(z^1, \dots, z^n), \end{aligned} \tag{51.6}$$

and we are done. □

We will prove the following very special property of Kähler metrics.

Proposition 51.2. *If (M, g, J) is Kähler then for each $p \in M$, there exists an open neighborhood U of p and a function $u : U \to \mathbb{R}$ such that $\omega = i\partial\overline{\partial}u$.*

Proof. Choose local homorphic coordinates z^j around p. Then in a ball B in these coordinates, since ω is a real closed 2-form, from the usual Poincaré lemma, there exists a real 1-form α such that $\omega = d\alpha$ in B. Next, write $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where $\alpha^{1,0}$ is a 1-form of type $(1, 0)$, and $\alpha^{0,1}$ is a 1-form of type $(0, 1)$. Since α is real, $\overline{\alpha^{1,0}} = \alpha^{0,1}$. Next,

$$\begin{aligned} \omega = d\alpha &= \partial\alpha + \overline{\partial}\alpha \\ &= \partial\alpha^{1,0} + \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0} + \overline{\partial}\alpha^{0,1} \end{aligned} \tag{51.7}$$

The first and last terms on the right hand side are forms of type $(2, 0)$ and $(0, 2)$, respectively. Since ω is of type $(1, 1)$, we must have $\overline{\partial}\alpha^{0,1} = 0$. Since we are in a ball in $\mathbb{C}^n$, the $\overline{\partial}$−Poincaré Lemma 51.1 says that there exists a function $f : B \to \mathbb{C}$ such that $\alpha^{0,1} = \overline{\partial} f$ in B. Substituting this into (51.7), we obtain

$$\omega = \partial\overline{\partial} f + \overline{\partial}\partial\overline{f} = i\partial\overline{\partial}(2Im(f)). \tag{51.8}$$

□

Proposition 51.3. *(M, g, J) is Kähler if and only if for each $p \in M$, there exists a holomorphic coordinate system around p such that*

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} (\delta_{jk} + O(|z|^2)_{jk}) dz^j \wedge d\overline{z}^k, \tag{51.9}$$

as $|z| \to 0$.

Proof. If this is true then $d\omega(p) = 0$ for any point p, so $d\omega \equiv 0$. Conversely, we can assume that $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$. From Proposition 51.2, we can find $u : B \to \mathbb{R}$ so that

$$u = c_0 + Re(c_{1j}z^j) + Re(c_{2ij}z^i z^j + c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3), \tag{51.10}$$

and $\omega = i\partial\overline{\partial}u$. But the first terms on the left hand side are in the kernel of the $\partial\overline{\partial}$-operator, so by subtracting these terms, we can assume that

$$u = Re(c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3). \tag{51.11}$$

Then since $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$, we have that

$$u = \frac{1}{2}|z|^2 + Re\{a_{jkl}z^j z^k z^l + b_{jkl}\overline{z}^j z^k z^l)\} + O(|z|^4). \tag{51.12}$$

Consider the coordinate change

$$z^k = w^k + \sum c_{klm}w^l w^m. \tag{51.13}$$

This will eliminate the b_{jkl} terms in the expansion of u, and the remaining cubic terms are annihilated by the $\partial\overline{\partial}$-operator, so by subtracting those terms, we can arrange that

$$u = \frac{1}{2}|w|^2 + O(|w|^4), \tag{51.14}$$

and (51.9) follows. □

51.2 L^2 adjoints

For the real operator $d : \Gamma(\Lambda^p) \to \Gamma(\Lambda^{p+1})$, the formal L^2-adjoint d^* is defined by

$$\int_M \langle d^*\alpha, \beta\rangle dV = \int_M \langle \alpha, d\beta\rangle dV, \tag{51.15}$$

where $\alpha \in \Gamma(\Lambda^p(M))$, and $\beta \in \Gamma(\Lambda^{p-1}(M))$, and where $\langle\cdot,\cdot\rangle = g(\cdot,\cdot)$, and dV is the oriented Riemannian volume element.

The Riemannian inner product on forms extends by complex linearity to an inner product on complex valued forms. For α and β sections of $\Lambda^k_{\mathbb{C}}$, we define the Hermitian inner product of α and β to be

$$(\alpha, \beta) = g(\alpha, \overline{\beta}). \tag{51.16}$$

The formula (51.15) holds for complex valued forms. Replacing β with $\overline{\beta}$, we have

$$\int_M \langle d^*\alpha, \overline{\beta}\rangle dV = \int_M \langle \alpha, d\overline{\beta}\rangle dV. \tag{51.17}$$

But since d is a real operator, $d\overline{\beta} = \overline{d\beta}$, so we can write this as

$$\int_M (d^*\alpha, \beta) dV = \int_M (\alpha, d\beta) dV. \tag{51.18}$$

That is, d^* is the L^2 adjoint of d with respect to the Hermitian inner product.

We next want to compute the formal L^2 adjoints of other operators. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{p,q+1}), \tag{51.19}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p,q+1}) \xrightarrow{\overline{\partial}^*} \Gamma(\Lambda^{p,q}), \tag{51.20}$$

is defined as follows. For $\alpha \in \Gamma(\Lambda^{p,q+1})$ and $\beta \in \Gamma(\Lambda^{p,q})$, we have

$$\int_M (\alpha, \overline{\partial}\beta) dV = \int_M (\overline{\partial}^* \alpha, \beta) dV, \tag{51.21}$$

where dV denotes the Riemannian volume element. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\partial} \Gamma(\Lambda^{p+1,q}), \tag{51.22}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p+1,q}) \xrightarrow{\overline{\partial}^*} \Gamma(\Lambda^{p,q}), \tag{51.23}$$

is defined similarly.

The Hodge Laplacian is $\Delta_H : \Gamma(\Lambda^p) \to \Gamma(\Lambda^p)$ defined by

$$\Delta_H = d^* d + d d^*. \tag{51.24}$$

We also have the following Laplacians on (p, q)-forms

$$\Delta_\partial : \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p,q}) \tag{51.25}$$

$$\Delta_{\overline{\partial}} : \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p,q}) \tag{51.26}$$

are defined by

$$\Delta_\partial = \partial^* \partial + \partial \partial^* \tag{51.27}$$

$$\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + \overline{\partial} \overline{\partial}^*. \tag{51.28}$$

Remark 51.4. By definition, Δ_∂ and $\Delta_{\overline{\partial}}$ preserve the type, but we do not know whether Δ_H maps $\Gamma(\Lambda^{p,q})$ to $\Gamma(\Lambda^{p,q})$ i.e., there is no obvious reason why it should preserve the type.

51.3 Hodge star operator

For a real oriented Riemannian manifold of dimension m, the Hodge star operator is a algebraic mapping

$$* : \Lambda^p \to \Lambda^{m-p} \tag{51.29}$$

defined by

$$\alpha \wedge * \beta = g_{\Lambda^p}(\alpha, \beta) dV_g, \tag{51.30}$$

for $\alpha, \beta \in \Lambda^p$, where dV_g is the oriented Riemannian volume element. Note that

$$*^2 = (-1)^{p(m-p)} Id_{\Lambda^p}. \tag{51.31}$$

Since $*$ is an pointwise operator, it induces a mapping on sections, which is denoted by the smae notation $* : \Gamma(\Lambda^p) \to \Gamma(\Lambda^{m-p})$. The Hodge star operator yields an explicit formula for d^*.

Proposition 51.5. *On a compact oriented Riemannian manifold (M, g) of real dimension m, for $\alpha \in \Gamma(\Lambda^p(M))$, we have*

$$d^*\alpha = (-1)^{m(p+1)+1} * d * \alpha. \tag{51.32}$$

Proof. For $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$, we compute

$$\begin{aligned}
\int_M \langle \alpha, d\beta \rangle dV &= \int_M d\beta \wedge *\alpha \\
&= \int_M \Big(d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big) \\
&= \int_M (-1)^{p+(m-p+1)(p-1)} \beta \wedge * * d * \alpha \\
&= \int_M \langle \beta, (-1)^{m(p+1)+1} * d * \alpha \rangle dV \\
&= \int_M \langle \beta, d^* \alpha \rangle dV.
\end{aligned} \tag{51.33}$$

□

If M is a complex manifold of complex dimension $n = m/2$, and g is a Hermitian metric, then the Hodge star extends to the complexification

$$* : \Lambda^p \otimes \mathbb{C} \to \Lambda^{2n-p} \otimes \mathbb{C}. \tag{51.34}$$

Proposition 51.6. *If (M, J, g) is Hermitian, then*

$$* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}. \tag{51.35}$$

Proof. Recall the following formula for the volume form in $\mathbb{C}^n$:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{51.36}$$

By an easy computation, it follows that the claim holds on $\mathbb{C}^n$, therefore it holds any any point of a Hermitian manifold (it is not a differential operator). □

Consequently, the operator

$$\overline{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \tag{51.37}$$

defined by

$$\bar{*}\alpha = \overline{*\alpha} \tag{51.38}$$

is a $\mathbb{C}$-antilinear mapping and satisfies

$$\alpha \wedge \bar{*}\beta = g_{\Lambda^p}(\alpha, \overline{\beta}) dV_g. \tag{51.39}$$

for $\alpha, \beta \in \Lambda^p \otimes \mathbb{C}$.

Proposition 51.7. *The L^2-adjoints of $d, \overline{\partial}, \overline{\partial}^*$ are given by*

$$d^* = -\bar{*}\ d\ \bar{*} \tag{51.40}$$

$$\partial^* = -\bar{*}\ \partial\ \bar{*} \tag{51.41}$$

$$\overline{\partial}^* = -\bar{*}\ \overline{\partial}\ \bar{*}, \tag{51.42}$$

Proof. The dimension of an almost complex manifold is even, so know that $d^* = - * d *$. Taking a conjugate of this equation yields the first formula. Apply the first formula to $d = \partial + \overline{\partial}$, we have

$$\partial^* + \overline{\partial}^* = d^* = -\bar{*}\ d\ \bar{*} = -\bar{*}\ \partial\ \bar{*} - \bar{*}\ \overline{\partial}\ \bar{*} \tag{51.43}$$

Considering the degrees of the operators on the right hand side yields the last 2 formulas. □

52 Lecture 52

52.1 Serre duality

Corollary 52.1. *On a Hermitian manifold, we have*

$$\Delta_{\overline{\partial}} \bar{*} = \bar{*} \Delta_{\overline{\partial}} \tag{52.1}$$

Proof. We compute on $\Lambda^k_{\mathbb{C}}$,

$$\Delta_{\overline{\partial}} \bar{*} = (\overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*)\bar{*} = (-\bar{*}\overline{\partial}\bar{*}\overline{\partial} - \overline{\partial}\bar{*}\overline{\partial}\bar{*})\bar{*} = -\bar{*}\overline{\partial}\bar{*}\overline{\partial}\bar{*} + (-1)^{k+1}\overline{\partial}\bar{*}\overline{\partial} \tag{52.2}$$

On the other hand,

$$\bar{*}\Delta_{\overline{\partial}} = \bar{*}(-\bar{*}\overline{\partial}\bar{*}\overline{\partial} - \overline{\partial}\bar{*}\overline{\partial}\bar{*}) = (-1)^{k+1}\overline{\partial}\bar{*}\overline{\partial} - \bar{*}\overline{\partial}\bar{*}\overline{\partial}\bar{*}. \tag{52.3}$$

□

Letting

$$\mathbb{H}^{p,q}(M, g) = \{\alpha \in \Lambda^{p,q} | \Delta_{\overline{\partial}}\alpha = 0\}, \tag{52.4}$$

Hodge theory tells us that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathbb{H}^{p,q}(M, g), \tag{52.5}$$

is finite-dimensional, and that

$$\Lambda^{p,q} = \mathbb{H}^{p,q}(M,g) \oplus Im(\Delta_{\overline{\partial}}) \tag{52.6}$$

$$= \mathbb{H}^{p,q}(M,g) \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^*), \tag{52.7}$$

with this being an orthogonal direct sum in L^2. Consequently,

$$\mathbb{H}^{p,q}(M,J,g) \cong H^{p,q}_{\overline{\partial}}(M,J). \tag{52.8}$$

Let $h^{p,q} \equiv \dim H^{p,q}_{\overline{\partial}}(M,J)$.

Corollary 52.2. *Let (M,J) be a compact complex manifold of complex dimension n. Then*

$$H^{p,q}_{\overline{\partial}}(M) \cong (H^{n-p,n-q}_{\overline{\partial}}(M))^*, \tag{52.9}$$

and therefore

$$h^{p,q}(M) = h^{n-p,n-q}(M) \tag{52.10}$$

Proof. From Corollary 52.1, the mapping $\overline{*}$ preserves the space of harmonic forms, and is invertible. The result then follows from Hodge theory. The dual appears since the operator $\overline{*}$ is $\mathbb{C}$-antilinear. □

A similar argument works with forms taking values in a holomorphic bundle. Choosing a hermitian metric h on E, we have a Hodge star operator

$$\overline{*}_E : \Lambda^{p,q}(E) \to \Lambda^{n-p,n-q}(E^*) \tag{52.11}$$

which is defined by the following. For $\alpha \in \Lambda^{p,q}(E)$. of the form $\alpha_1 \otimes s_1$, where $\alpha_1 \in \Lambda^{p,q}$ and $s_1 \in \Gamma(E)$, and $\gamma \in \Lambda^{n-p,n-q}(E^*)$ of the form $\gamma_1 \otimes s_2$, where $\gamma_1 \in \Lambda^{n-p,n-q}$, and $s_2 \in \Gamma(E^*)$, then

$$\alpha \wedge \gamma \equiv (\alpha_1 \wedge \gamma_1)(s_2(s_1)) \in \Lambda^{n,n}. \tag{52.12}$$

Therefore, we can define an operator

$$\overline{*}_E : \Lambda^{p,q}(E) \to \Lambda^{n-p,n-q}(E^*) \tag{52.13}$$

by insisting that for $\alpha, \beta \in \Lambda^{p,q}(E)$,

$$\alpha \wedge \overline{*}_E \beta = g_{\Lambda^{p,q}(E)}(\alpha, \overline{\beta}) dV_g, \tag{52.14}$$

where dV_g is the Riemannian volume element.

Corollary 52.3. *If $E \to M$ is a holomorphic vector bundle, then*

$$H^{p,q}_{\overline{\partial}}(M,E) \cong \Big(H^{n-p,n-q}(M,E^*)\Big)^* \tag{52.15}$$

Proof. We define $\overline{\partial}_E$ to be the L^2 adjoint of $\overline{\partial}_E$, which is again given by

$$\overline{\partial}_E^* = -\bar{*}_E\ \overline{\partial}_E\ \bar{*}_E, \tag{52.16}$$

and let

$$\Delta_{\overline{\partial}_E} = \overline{\partial}_E\overline{\partial}_E^* + \overline{\partial}_E^*\overline{\partial}_E. \tag{52.17}$$

One can verify easily that

$$\Delta_{\overline{\partial}_E}\bar{*}_E = \bar{*}_E\Delta_{\overline{\partial}_E} \tag{52.18}$$

so $\bar{*}_E$ gives a conjugate linear isomorphism between the corresponding spaces of harmonic form. The results follows from Hodge theory. □

Serre duality is often stated in the following way:

$$H^p(M, \mathcal{O}(E)) \cong \big(H^{n-p}(M, \mathcal{O}(K \otimes E^*))\big)^*, \tag{52.19}$$

where $K = \Lambda^{n,0}$ is the canonical bundle, and these are the cohomologys groups of the sheaf of holomorphic sections of the corresponding bundles. To see this, note by the Dolbeault Theorem,

$$H^p(M, \Omega^q(E)) \cong H^{q,p}_{\overline{\partial}_E}(M, E), \tag{52.20}$$

We then have

$$\begin{aligned} H^p(M, \mathcal{O}(E)) &\cong H^{0,p}_{\overline{\partial}_E}(M, E) \cong \big(H^{n,n-p}_{\overline{\partial}_E}(M, E^*)\big)^* \\ &\cong \big(H^{n-p}(M, \Omega^n(E^*))\big)^* = \big(H^{n-p}(M, \mathcal{O}(K \otimes E^*))\big)^*. \end{aligned} \tag{52.21}$$

52.2 The Laplacian on a Kähler manifold

Let L denote the mapping

$$L : \Lambda^{p,q} \to \Lambda^{p+1,q+1} \tag{52.22}$$

given by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form. Define

$$\Lambda \equiv L^* : \Lambda^{p,q} \to \Lambda^{p-1,q-1}. \tag{52.23}$$

Proposition 52.4. *If $(M, J.g)$ is Kähler then*

$$[\Lambda, \partial] = i\overline{\partial}^*,\ [\Lambda, \overline{\partial}] = -i\partial^*,\ [\Lambda, d] = -(d^c)^* \tag{52.24}$$

$$[L, \partial^*] = i\overline{\partial},\ [L, \overline{\partial}^*] = -i\partial,\ [L, d^*] = -d^c. \tag{52.25}$$

Proof. Note that the second identity is the conjugate of the first. Therefore, if the first identity is true,

$$[\Lambda, d] = [\Lambda, \partial + \overline{\partial}] = [\Lambda, \partial] + [\Lambda, \overline{\partial}] = i\overline{\partial}^* - i\partial^* = (-i(\overline{\partial} - \partial))^* = -(d^c)^*, \tag{52.26}$$

then the third identity follows. The last three identities are just the adjoints of the first three.

So to prove all of these identities, we only need to prove the first. To prove the first identity, one proves this for $\mathbb{C}^n$ with the standard Kähler form. The proof is a 2 page calculation, and is left as an exercise. Then for an arbitrary Kähler manifold, the identity follows by using Kähler normal coordinates at any point, and the fact that the identity only depends on the metric and its first derivatives at the point. □

On a Kähler manifold, we have the following very special occurrence.

Proposition 52.5. *For $\alpha \in \Gamma(\Lambda^{p,q})$, if (M, J, g) is Kähler, then*

$$\Delta_H \alpha = 2\Delta_{\partial}\alpha = 2\Delta_{\overline{\partial}}\alpha. \tag{52.27}$$

Proof. We first show that

$$\Delta_H = \Delta_{\partial} + \Delta_{\overline{\partial}}. \tag{52.28}$$

To see this

$$\begin{aligned} \Delta_H = dd^* + d^*d = &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= \partial\partial^* + \partial^*\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} + \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial} \\ &= \Delta_{\partial} + \Delta_{\overline{\partial}} + \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial}. \end{aligned} \tag{52.29}$$

Using Proposition 52.4,

$$\begin{aligned} i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) &= \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ &= \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial \\ &= \partial\Lambda\partial - \partial\Lambda\partial = 0. \end{aligned} \tag{52.30}$$

The sum of the last two terms in (52.29) also vanishes, just by taking the conjugate of the above computation, and (52.28) follows.

To finish the proof, we show that

$$\Delta_{\partial} = \Delta_{\overline{\partial}} \tag{52.31}$$

To see this, we again use Proposition 52.4, to compute

$$\begin{aligned} i\Delta_{\partial} = i\partial\partial^* + i\partial^*\partial &= \partial(-[\Lambda, \overline{\partial}]) - [\Lambda, \overline{\partial}]\partial \\ &= \partial\overline{\partial}\Lambda - \partial\Lambda\overline{\partial} - \Lambda\overline{\partial}\partial + \overline{\partial}\Lambda\partial. \end{aligned} \tag{52.32}$$

Also, we compute

$$\begin{aligned} i\Delta_{\overline{\partial}} = i\overline{\partial}\overline{\partial}^* + i\overline{\partial}^*\overline{\partial} &= \overline{\partial}([\Lambda, \partial]) + [\Lambda, \partial]\overline{\partial} \\ &= \overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial} \\ &= \overline{\partial}\Lambda\partial + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial}, \end{aligned} \tag{52.33}$$

from which (52.31) follows.

□

Using Hodge theory, we get the following structure on the cohomology of a Kähler manifold.

Proposition 52.6. *If* (M, J, g) *is a compact Kähler manifold, then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M), \tag{52.34}$$

and

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{q,p}_{\overline{\partial}}(M)^*. \tag{52.35}$$

Consequently,

$$b^k(M) = \sum_{p+q=k} h^{p,q}(M) \tag{52.36}$$

$$h^{p,q}(M) = h^{q,p}(M). \tag{52.37}$$

Proof. This follows because if a harmonic k-form is decomposed as

$$\phi = \phi^{p,0} + \phi^{p-1,1} + \cdots + \phi^{1,p-1} + \phi^{0,p}, \tag{52.38}$$

then

$$0 = \Delta_H \phi = 2\Delta_{\overline{\partial}}\phi^{p,0} + 2\Delta_{\overline{\partial}}\phi^{p-1,1} + \cdots + 2\Delta_{\overline{\partial}}\phi^{1,p-1} + 2\Delta_{\overline{\partial}}\phi^{0,p}, \tag{52.39}$$

therefore

$$\Delta_{\overline{\partial}}\phi^{p-k,k} = 0, \tag{52.40}$$

for $k = 0 \dots p$.

Next,

$$\overline{\Delta_{\overline{\partial}}\phi} = \Delta_{\partial}\overline{\phi}, \tag{52.41}$$

so conjugation sends harmonic forms to harmonic forms. □

This yields a topologicial obstruction for a complex manifold to admit a Kähler metric:

Corollary 52.7. *If* (M, J, g) *is a compact Kähler manifold, then the even betti numbers are non-zero, and the odd Betti numbers of* M *are even.*

Proof. We recall the following formula for the volume form on $\mathbb{C}^n$

$$\omega^n_{Euc} = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{52.42}$$

Consequently, on any Kähler manifold, we have

$$\omega^n = n! dV_g \tag{52.43}$$

Consider the the $2q$-form ω^q. If $\omega^q = d\alpha$, then

$$n!Vol(g) = \int_M \omega^n = \int_M d\alpha \wedge \omega^{n-q} = \int_M d(\alpha \wedge \omega^{n-q}) = 0, \tag{52.44}$$

by Stokes' theorem, which is a contradiction. This proves the first statement.

For the second statement, then we use Proposition 52.6. If r is odd, then

$$b^r(M) = \sum_{p+q=r} h^{p,q} = h^{r,0} + h^{r-1,1} + \cdots + h^{1,r-1} + h^{0,r}, \tag{52.45}$$

which is clearly even for r odd, using the symmetry $h^{p,q} = h^{q,p}$. □

For a complex curve, we have the following corollary.

Corollary 52.8. *Let (M, J, g) be a compact connected Riemann surface. Then*

$$b^{1,1} = h^{1,1} = 1, \tag{52.46}$$

$$b^{0,1} = b^{1,0} = \tau, \tag{52.47}$$

where τ is the genus of M.

Proof. Since ω is a $(1,1)$-form, it is automatically closed, so g is Kähler. The rest follows from the above. □

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