Math 245ABC, Complex Variables and Geometry

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Contents

Introduction

This will be a year long course on holomorphic functions and complex manifolds. A very rough outline:

- Holomorphic functions of 1 complex variable.
- Holomorphic function of several complex variables.
- Sheaf cohomology.
- Riemann surfaces.
- Higher dimensional complex manifolds.

1 Lecture 1

1.1 Cauchy's formula in one complex variable

For now, just consider $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open set. We write $z \in U$ as $z = x+iy$. Assume that f, as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$, is differentiable. This means that, for each $z \in U$, there exists a linear mapping $L_z : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{h \to 0} \frac{\|f(z+h) - f(z) - L_z h\|}{\|h\|} = 0. \tag{1.1}$$

This implies that the partial derivatives of f exist. Conversely, if the partial derivatives exist and are continuous at z, then the mapping L_z exists. Writing

$$f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}, \tag{1.2}$$

we have that

$$L_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{1.3}$$

We say that f is *complex differentiable* at $z \in U$ if there exists complex number $c_z \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{\|f(z+h) - f(z) - c_z \cdot h\|}{\|h\|} = 0. \tag{1.4}$$

This is a much stronger condition than (1.1). If such a c_z exists, then

$$c_z = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \equiv \frac{\partial f}{\partial z}. \tag{1.5}$$

Note that if f is complex differentiable at z, writing $c_z = c_1 + ic_2$ and $h = h_1 + ih_2$, we have

$$c_z h = c_1 h_1 - c_2 h_2 + i(c_2 h_1 + c_1 h_2) \tag{1.6}$$

and

$$L_z h = \begin{pmatrix} u_x h_1 + u_y h_2 \\ v_x h_1 + v_y h_2 \end{pmatrix}, \tag{1.7}$$

so we see that (1.1) is satisfied and necessarily $u_x = v_y$ and $u_y = -v_x$.

We say that f is *holomorphic* in U if it is C^1 and satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.8}$$

Defining the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

the Cauchy-Riemann equations are equivalent to $\frac{\partial}{\partial \bar{z}} f = 0$, where $f = u + iv$. If f is holomorphic, then f is complex differentiable at any $z \in U$, and we have

$$c_z = \frac{\partial}{\partial z} f. \tag{1.9}$$

Definition 1.1. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{1.10}$$

which converges absolutely and uniformly in a disc $\Delta(z_0, \epsilon)$ around z_0, for some $\epsilon > 0$.

We want to show the equivalence of holomorphicity and complex analyticity. For this, we need the following; see [GH78, page 3].

Proposition 1.2 (Cauchy-Pompieu Formula)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain in $\mathbb{C}$ with C^1 boundary. For $z \in \Omega$ and $f \in C^1(\overline{\Omega})$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(w) dw}{w - z} + \frac{1}{2\pi i} \int_{\Omega} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}, \tag{1.11}$$

where the boundary has the counterclockwise orientation.

Proof. The 1-form

$$\eta = \frac{1}{2\pi i} \frac{f(w) dw}{w - z}, \tag{1.12}$$

satisfies

$$d\eta = \frac{1}{2\pi i} \frac{\partial f}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w - z}. \tag{1.13}$$

Apply Stokes' Theorem to the annular domain $\Omega \setminus \Delta(z, \epsilon)$, to get

$$\int_{\partial(\Omega \setminus \Delta(z,\epsilon))} \eta = \int_{\Omega \setminus \Delta(z,\epsilon)} d\eta. \tag{1.14}$$

The left hand side of (1.14) is

$$\int_{\partial \Omega} \eta - \int_{\partial \Delta(z,\epsilon)} \eta. \tag{1.15}$$

If $d\eta \in L^1(\Omega)$, then the right hand side of (1.14) is

$$\int_{\Omega} d\eta - \int_{\Delta(z,\epsilon)} d\eta. \tag{1.16}$$

Write $w = z + re^{i\theta}$, we estimate

$$\Big|\int_{\Delta(z,\epsilon)} d\eta\Big| \leq \int_{\Delta(z,\epsilon)} |d\eta| \leq C\int_{\Delta(z,\epsilon)} \Big|\frac{d\bar{w}\wedge dw}{w-z}\Big| \leq C\int_0^{\epsilon}\int_0^{2\pi} dr\wedge d\theta < C'\epsilon, \tag{1.17}$$

so $d\eta$ is obviously in $L^1(\Omega)$, and taking the limit as $\epsilon \to 0$ then shows that

$$\int_{\Omega} d\eta = \int_{\partial\Omega}\eta - \lim_{\epsilon\to 0}\int_{\partial\Delta(z,\epsilon)}\eta. \tag{1.18}$$

With $w = z + \epsilon e^{i\theta}$, we compute

$$\int_{\partial\Delta(z,\epsilon)}\eta = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z+\epsilon e^{i\theta}))\epsilon i e^{i\theta}d\theta}{\epsilon e^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta. \tag{1.19}$$

Using the mean value theorem, we also have that

$$\Big|f(z) - \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta\Big| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z+\epsilon e^{i\theta}) - f(z)|d\theta \leq C\epsilon, \tag{1.20}$$

which shows that

$$\lim_{\epsilon\to 0}\int_{\partial\Delta(z,\epsilon)}\eta = f(z), \tag{1.21}$$

and we are done. □

A consequence is the equivalence between holomorphic and analytic functions; see [GH78, page 4].

Proposition 1.3. *Let U be an open set in $\mathbb{C}$. Then f is holomorphic in U if and only if f is complex analytic in U.*

Proof. If f is holomorphic in U the Cauchy-Pompieu formula in a small disc $\Delta = \Delta(z_0,\epsilon)$ yields for $z \in \Delta$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{w-z}. \tag{1.22}$$

Then expand

$$\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \tag{1.23}$$

$$= \frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k, \tag{1.24}$$

with the sum converging absolutely and uniformly in any smaller disc. So the above yields the power series expansion

$$f(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{(w-z_0)^{k+1}}\right)(z-z_0)^k, \tag{1.25}$$

which also converges absolutely and uniformly in any smaller disc.

For the converse, if f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula without solid integral. So then f does also by uniform convergence. So we have

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} \left(\frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)dw}{w-z} \right) = \frac{1}{2\pi i} \int_{\partial \Delta} f(w) \left(\frac{\partial}{\partial \bar{z}} \frac{1}{w-z} \right) dw = 0. \tag{1.26}$$

For more details, see [GH78, page 4]. □

Definition 1.4. We wil let $\Omega \subset \mathbb{C}$ be a bounded domain with C^1 boundary. If u is holomorphic in an open set Ω, then we write $u \in \mathcal{O}(\Omega)$.

2 Lecture 2

2.1 Some basic results in one complex variable

First, let's recall the basic result about differentiating under an integral.

Proposition 2.1. *Let*

$$f(z) = \int_{\Omega} a(z,w) dw \wedge d\bar{w}. \tag{2.1}$$

(Note this notation does not mean that f is holomorphic in z or that a is holomorphic as a function of 2 variables!). Assume that

1. $a(z,w) \in L^1(\Omega)$*, in the w variable.*
2. $\frac{\partial a}{\partial z}$ *and* $\frac{\partial a}{\partial \bar{z}}$ *exist for all z, for almost every $w \in \Omega$.*
3. $\left|\frac{\partial a}{\partial z}\right| + \left|\frac{\partial a}{\partial \bar{z}}\right| \leq h(w)$*, where $h \in L^1(\Omega)$.*

Then

$$\frac{\partial f}{\partial z} = \int_{\Omega} \frac{\partial}{\partial z}\big(a(z,w)\big) dw \wedge d\bar{w} \tag{2.2}$$
$$\frac{\partial f}{\partial \bar{z}} = \int_{\Omega} \frac{\partial}{\partial \bar{z}}\big(a(z,w)\big) dw \wedge d\bar{w}. \tag{2.3}$$

Proof. Recall that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(Ref + iImf). \tag{2.4}$$

The real part of the left hand side of (2.2) is

$$Re\Big(\frac{\partial f}{\partial z}\Big) = \frac{1}{2}\Big(\frac{\partial Ref}{\partial x} + \frac{\partial Imf}{\partial y}\Big) \tag{2.5}$$

The real part of the right hand side of (2.2) is

$$\int_\Omega \frac{1}{2}\Big(\frac{\partial Re(a(x+iy,w))}{\partial x} + \frac{\partial Im(a(x+iy,w))}{\partial y}\Big) dw \wedge d\overline{w}. \tag{2.6}$$

Therefore we can consider real-valued functions, and prove for partials with respect to the real variables x and y. We have that

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\delta\to 0} \frac{f(x+\delta,y) - f(x,y)}{\delta} \tag{2.7}$$

For $\delta \neq 0$, consider

$$\frac{f(x+\delta,y) - f(x,y)}{\delta} = \int_\Omega \frac{a(x+\delta+iy,w) - a(x+iy,w)}{\delta} dw \wedge d\overline{w}. \tag{2.8}$$

By the mean value theorem, given $\delta > 0$, there exists x' on the line segment from (x,y) to $(x+\delta,y)$ such that

$$a(x+\delta+iy,w) - a(x+iy,w) = \frac{\partial a}{\partial x}(x'+iy,w)\delta, \tag{2.9}$$

so

$$\begin{aligned} \Big|\frac{a(x+\delta+iy,w) - a(x+iy,w)}{\delta}\Big| &\leq \Big|\frac{\partial a}{\partial x}(x'+iy,w)\Big| \\ &\leq \Big|\frac{\partial a}{\partial z}(x'+iy,w)\Big| + \Big|\frac{\partial a}{\partial \overline{z}}(x'+iy,w)\Big| \leq |h(w)|. \end{aligned} \tag{2.10}$$

We can do this for any sequence $\delta_n \to 0$, so the result follows from Lebesgue's dominated convergence theorem. The proof for the other derivative (2.3) is similar. □

We next go through several corollaries of the Cauchy-Pompieu formula; see [Hör90, Chapter 1] for more details.

Corollary 2.2. *(Interior derivative estimates.) Let $K \subset \Omega$ be a compact subset. Then there exist constant C_k, depending only upon K and Ω such that*

$$\sup_{z\in K} \Big|\Big(\frac{\partial}{\partial z}\Big)^k u(z)\Big| \leq C_k \|u\|_{L^1(\Omega)}, \tag{2.11}$$

for all $u \in \mathcal{O}(\Omega)$.

(Cauchy's estimate in a disc.) In the case that $u \in \mathcal{O}(\Delta(z_0, r_0))$, then there exists C_k, depending only upon k such that for any $r < r_0$, we have

$$\Big|\Big(\frac{\partial}{\partial z}\Big)^k u(z_0)\Big| \leq \frac{k!}{r^k} \|u\|_{C^0(\partial\Delta(z_0,r))}, \tag{2.12}$$

Proof. Choose a $\psi \in C_0^\infty(\Omega)$ (compact support) such that $\psi \equiv 1$ in a neighborhood of K. If $u \in \mathcal{O}(\Omega)$, then

$$\frac{\partial}{\partial \overline{z}}(\psi u) = u \frac{\partial}{\partial \overline{z}}\psi. \tag{2.13}$$

Now we apply (1.11) to ψu in Ω to get

$$\psi(z)u(z) = \frac{1}{2\pi i}\int_\Omega u(w)\frac{\partial\psi(w)}{\partial\overline{w}}\frac{dw\wedge d\bar{w}}{w-z}, \tag{2.14}$$

Now consider

$$a(z,w) = u(w)\frac{\partial\psi(w)}{\partial\overline{w}}\frac{1}{w-z}, \tag{2.15}$$

If $z \in K$, then $|w-z| > \delta > 0$, since the support of $\partial\psi/\partial\overline{w}$ is at a positive distance from K. So using Proposition 2.1, we can differentiate under the integral as many times as we like, and obtain

$$\Big(\frac{\partial}{\partial z}\Big)^k(\psi u(z)) = \frac{1}{2\pi i}\int_\Omega u(w)\frac{\partial\psi(w)}{\partial\overline{w}}\Big(\frac{\partial}{\partial z}\Big)^k\Big(\frac{1}{w-z}\Big)dw\wedge d\bar{w} \tag{2.16}$$

If $z \in K$, then ψu is equal to u in a neighborhood of z, so (2.11) follows.

Next, from (1.25), we have a power series expansion

$$u(z_0) = \sum_{k=0}^{\infty} a_k(z-z_0)^k, \quad a_k = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{u(w)dw}{(w-z_0)^{k+1}}. \tag{2.17}$$

Since $a_k = \dfrac{u^{(k)}(z_0)}{k!}$, this clearly implies (2.12). □

Corollary 2.3 (Liouville)**.** *A bounded entire function on $\mathbb{C}$ is constant.*

Proof. Let $u \in \mathcal{O}(\mathbb{C})$. Choose $z_0 \in \mathbb{C}$, then for any $r > 0$, by (2.12), we have

$$|u'(z_0)| \leq Cr^{-1}, \tag{2.18}$$

which implies that $u'(z_0) = 0$, therefore u is constant. □

Exercise 2.4. The fundamental theorem of algebra follows from this: If a polynomial $P(z)$ has no zeroes, then $1/P(z)$ would be a bounded entire function, and therefore constant. Details are left as an exercise.

Corollary 2.5. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \to u$ converges uniformly to u in the C^0 norm as $n \to \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Proof. Let $K \subset \Omega$, be a compact subset. Then given $\epsilon > 0$, there exist N such that

$$\sup_{z\in K}|u_m(z) - u_n(z)| < \epsilon, \tag{2.19}$$

for $m, n \geq N$. The difference $u_m - u_n \in \mathcal{O}(\Omega)$. Corollary 2.2 implies that

$$\sup_{z\in K}\Big|\frac{\partial}{\partial z}(u_m-u_n)(z)\Big| \leq C\epsilon. \tag{2.20}$$

This says that $\partial u_n/\partial z$ converges uniformly on K. But $\partial u_n/\partial \bar{z} = 0$, so the real partial derivatives $\partial u_n/\partial x$ and $\partial u_n/\partial y$ converge uniformly. It is an elementary result that if a sequence of functions converges uniformly, and the derivatives converge uniformly, then the limit of the derivatives is the derivative of the limit. This implies that $u \in C^1$ and

$$\frac{\partial u}{\partial \bar{z}} = \lim_{n\to\infty} \frac{\partial u_n}{\partial \bar{z}} = 0. \tag{2.21}$$

□

Corollary 2.6 (Montel's Theorem). *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

Proof. Corollary 2.2 yields a uniform bound on derivatives of u_n on any compact subset. By Arzela-Ascoli Theorem, some subsequence converges to a limit u uniformly on compact subsets. Then the previous corollary yields that $u \in \mathcal{O}(\Omega)$. □

Corollary 2.7 (The maximum principle). *If $f \in \mathcal{O}(\Omega)$, Ω is connected, and there exists a $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, then u is constant.*

Proof. The set $M \equiv \{z \in \Omega \mid |f(z)| = |f(z_0)|\}$ is closed by continuity. Take any $z' \in \Omega$ such that $|f(z')| = |f(z_0)|$. Then for any $\epsilon > 0$ such that $\Delta(z', \epsilon) \subset \Omega$, (1.11) is

$$f(z') = \frac{1}{2\pi i} \int_{\partial\Delta(z',\epsilon)} \frac{f(w)}{w - z} dw. \tag{2.22}$$

Writing $w = z_0 + \epsilon e^{i\theta}$, this is

$$f(z') = \frac{1}{2\pi} \int_0^{2\pi} f(z + \epsilon e^{i\theta}) d\theta \tag{2.23}$$

which implies that $|f(z')| = |f(z + \epsilon e^{i\theta})|$ for all θ. Letting ϵ vary, we see that the set M is also open, and since Ω is connected, $M \equiv \Omega$. Therefore $|f|$ is constant in Ω, and writing $f = u + iv$, we have

$$u^2 + v^2 = \text{constant}. \tag{2.24}$$

Differentiating this yields

$$2uu_x + 2vv_x = 0 = 2uu_y + 2vv_y. \tag{2.25}$$

Using the Cauchy-Riemann equations gives

$$2uu_x - 2vu_y = 0 = 2uu_y + 2vu_x, \tag{2.26}$$

This implies that

$$2u^2u_x - 2uvu_y = 0 = 2uvu_y + 2v^2u_x, \tag{2.27}$$

and we get that

$$(u^2 + v^2)u_x = 0, \tag{2.28}$$

which implies that $u_x = 0$, so u is constant. Then f is constant. □

Corollary 2.8 (Riemann's removable singularity theorem). *If* $u \in \mathcal{O}(\Delta^*(z_0, r))$ *where* $\Delta^*(z_0, r) = \Delta(z_0, r) \setminus \{z_0\}$*, satisfies*

$$u(z) = o(|z - z_0|^{-1}), \text{ as } z \to z_0 \tag{2.29}$$

Then u *extends to a holomorphic function on* $\Delta(z_0, r)$*.*

Proof. Consider the function

$$v(z) = \begin{cases} (z - z_0)^2 u(z) & z \in \Delta^*(z_0, r) \\ 0 & z = z_0 \end{cases} \tag{2.30}$$

We have

$$\lim_{z \to z_0} \frac{v(z) - v(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) u(z) = 0 \tag{2.31}$$

by assumption, so $v(z) \in \mathcal{O}(\Delta(z_0, r))$ and therefore admits a power series expansion

$$v(z) = \sum_{k=2}^{\infty} c_k (z - z_0)^k, \tag{2.32}$$

since $v(z_0) = v'(z_0) = 0$. Then

$$u(z) = \sum_{k=0}^{\infty} c_{k+2} (z - z_0)^k \tag{2.33}$$

is the required extension. □

3 Lecture 3

3.1 The Schwarz Lemma

Lemma 3.1 (Schwarz Lemma). *Let* $u \in \mathcal{O}(\Delta_1(0))$*, and assume that* $u(0) = 0$ *and* $|u| \leq 1$*. Then* $|u'(0)| \leq 1$ *and* $|u(z)| \leq |z|$ *for every* $z \in \Delta_1(0)$*. Equality holds at some* z_0 *if and only if* $u(z) = c \cdot z$ *where* $c \in \mathbb{C}$ *satisfies* $|c| = 1$*.*

Proof. Consdier $v(z) = \frac{u(z)}{z}$. Then $v \in \mathcal{O}(\Delta_1(0))$ and $v(0) = u'(0)$. Given $z \in \Delta_1(0)$, choose $r > 0$ such that $|z| < r < 1$. Then v is analytic in $\overline{\Delta_r(0)}$, and the maximum principle yields that

$$|v(z)| \leq \sup_{|w|=r} |v(w)| = \sup_{|w|=r} \frac{u(w)}{w} \leq 1/r, \tag{3.1}$$

Letting $r \to 1$, we are done. The inequality being equality at some interior point implies that v is a constant function with $|v| = 1$, equivalently $u(z) = c \cdot z$. □

Exercise 3.2. This is very useful, for example it can be used to show that any holomorphic automorphism of the unit disc must be of the form

$$\Psi(z) = e^{i\theta} \frac{z + c}{1 + \bar{c}z}, \tag{3.2}$$

where $\theta \in \mathbb{R}$ and $c \in \mathbb{C}$ satisfies $|c| < 1$. The proof is to compose with a mapping of the above form to normalize so that $\Psi(0) = 0$, and then apply the Schwarz Lemma. Details left as an exercise.

3.2 Meromorphic functions

Definition 3.3. For a domain $\Omega \subset \mathbb{C}$, we say that $f \in \mathcal{M}(\Omega)$, or f is meromorphic in Ω, if there is an open covering U_j of Ω such that $f|_{U_j} = \frac{g_j}{h_j}$, where g_j and h_j are in $\mathcal{O}(U_j)$ and h_j is not identically zero.

Note this is equivalent to saying that f has a Laurent series expansion near any $z_0 \in \Omega$ with only finitely many negative terms. That is we have

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k. \tag{3.3}$$

If some coefficient $a_k \neq 0$ for $k < 0$, then z_0 is called a *pole*. Note that the set of poles will be some discrete subset $\{w_j\}$ of Ω. The finite sum of the negative terms is called the *principal part* of f at z_0. The order of $f \in \mathcal{M}(\Omega)$ at $z_0 \in \Omega$ is the least integer $n \in \mathbb{Z}$ such that the coefficient $a_n \neq 0$ in (3.3).

Example 3.4. The function $e^{1/z}$ is holomorphic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but it is not meromorphic on $\mathbb{C}$. It has an *essential singularity* at the origin.

We will next consider a more general situation of functions which are holomorphic in an annulus $A(z_0; r_1, r_2) = \Delta(z_0, r_2) \setminus \Delta(z_0, r_1)$ with $0 < r_1 < r_2$.

Proposition 3.5. *If f is holomorphic in a neighborhood of an annulus $A(z_0; r_1, r_2)$ with $0 < r_1 < r_2$, then f admits a expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k. \tag{3.4}$$

which converges uniformly on compact subsets of $A(z_0; r_1, r_2)$. Furthermore the coefficient a_k is given by

$$a_k = \frac{1}{2\pi i} \int_{\partial\Delta(z_0, r)} \frac{f(w)}{(w - z_0)^{k+1}} dw \tag{3.5}$$

for any $r_1 \leq r \leq r_2$. In particular, the expansion (3.4) is uniquely determined by f.

Proof. Since f is assumed to be holomorphic in a neighborhood of the annulus, the Cauchy-Pompieu formula yields for $z \in A(z_0; r_1, r_2)$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta(z_0,r_2)} \frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\int_{\partial\Delta(z_0,r_1)} \frac{f(w)dw}{w-z}. \tag{3.6}$$

If $|z-z_0| < |w-z_0|$ expand

$$\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \tag{3.7}$$

$$= \frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k, \tag{3.8}$$

and if $|z-z_0| > |w-z_0|$,

$$\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{-1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}} \tag{3.9}$$

$$= \frac{-1}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k, \tag{3.10}$$

Substituting these expansions into (3.6), we formally obtain

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta(z_0,r_2)} f(w)\frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k dw \tag{3.11}$$

$$+ \frac{1}{2\pi i}\int_{\partial\Delta(z_0,r_1)} f(w)\frac{1}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k dw \tag{3.12}$$

If $z \in A(z_0; r_1, r_2)$ and $|w-z_0| = r_2$, then $|z-z_0| < r_2 = |w-z_0|$, so (3.7) is valid. If $|w-z_0| = r_1$, then $|z-z_0| > r_1 = |w-z_0|$, so (3.9) is valid. Both series convergenge uniformly on compact subsets of $A(z_0; r_1, r_2)$, so we have that

$$f(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(\int_{\partial\Delta(z_0,r_2)} \frac{f(w)}{(w-z_0)^{k+1}}dw\right)(z-z_0)^k \tag{3.13}$$

$$+ \frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(\int_{\partial\Delta(z_0,r_1)} f(w)(w-z_0)^k dw\right)(z-z_0)^{-(k+1)}. \tag{3.14}$$

Note that, for any $k \in \mathbb{Z}$, the function $f(w)(w-z_0)^k$ is holomorphic in a neighborhood of the annulus, so by Stokes' Theorem we have

$$\begin{aligned}\int_{\partial\Delta(z_0,r_2)} f(w)(w-z_0)^k dw &- \int_{\partial\Delta(z_0,r_1)} f(w)(w-z_0)^k dw \\ &= \int_{A(z_0;r_1,r_2)} d\left(f(w)(w-z_0)^k dw\right) = 0.\end{aligned} \tag{3.15}$$

So the integral

$$\int_{\partial\Delta(z_0,r)} f(w)(w-z_0)^k dw \tag{3.16}$$

is independent of r, and we have

$$f(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial\Delta(z_0,r)} \frac{f(w)}{(w-z_0)^{k+1}} dw \right) (z-z_0)^k. \tag{3.17}$$

□

Remark 3.6. If f is holomorphic in a punctured disc $\Delta^*(z_0, r)$, then it is holomorphic in an annulus $A(z_0, r/2, r)$, so we also obtain an infinite Laurent series expansion at any isolated singularity.

Definition 3.7. If $f = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$, then the residue of f at z_0 is

$$Res_{z_0}(f) = a_{-1}. \tag{3.18}$$

Corollary 3.8 (The Residue Theorem)**.** *Assume that f is holomorphic in $\overline{\Omega} \setminus \{z_1, \dots, z_m\}$. Then*

$$\int_{\partial\Omega} f(z)dz = 2\pi i \sum_{j=1}^{m} Res_{z_j}(f) \tag{3.19}$$

Proof. Since f is holomorphic in the complement (in Ω) of a union of balls around z_j, by Stokes's theorem (like in (3.15) above), we see that

$$\int_{\partial\Omega} f(z)dz = \sum_{j=1}^{m} \int_{\partial\Delta(z_j,\epsilon)} f(z)dz, \tag{3.20}$$

for some $\epsilon > 0$ (this is also known as Cauchy's Theorem). By Proposition 3.5,

$$\int_{\partial\Delta(z_j,\epsilon)} f(z)dz = \int_{\partial\Delta(z_j,\epsilon)} \sum_{k=-\infty}^{\infty} a_k(z-z_j)^k dz = \sum_{k=-\infty}^{\infty} a_k \int_{\partial\Delta(z_j,\epsilon)} (z-z_0)^k dz, \tag{3.21}$$

since the series converges uniformly on $\partial\Delta^*(z_j, \epsilon)$. However, letting $z = z_j + re^{i\theta}$, we see that

$$\int_{\partial\Delta(z_j,\epsilon)} (z-z_j)^k dz = \begin{cases} 2\pi i & k = -1 \\ 0 & k \neq -1 \end{cases}, \tag{3.22}$$

and we are done. □

Corollary 3.9 (The argument principle)**.** *If $f \in \mathcal{M}(\overline{\Omega})$ with no poles on $\partial\Omega$, then*

$$\int_{\partial\Delta(z_0,r)} z^q \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{d} m_j z_j^q \tag{3.23}$$

where z_j are the zeros and poles of f, with m_j is the order of f at z_j.

Proof. Let z_j be any zero or pole of f. Then we can write $f(z) = (z - z_j)^{m_j} g(z)$ where $g(z)$ is holomorphic at z_j and non-zero in a neighborhood of z_j. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}. \tag{3.24}$$

So the result follows from the residue theorem. $\square$

4 Lecture 4

4.1 The $\overline{\partial}$-equation in domains in $\mathbb{C}$

Our goal is to prove the following result.

Theorem 4.1. *If $\Omega \subset \mathbb{C}$ is any bounded domain, and $g \in C^\infty(\Omega)$, then there exists $g \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \overline{z}} f = g$.*

If we want to solve $\frac{\partial}{\partial \overline{z}} f = g$, it is natural to guess that

$$f(z) = \frac{1}{2\pi i} \int_\Omega \frac{g(w) dw \wedge d\bar{w}}{w - z}, \tag{4.1}$$

is a solution. However, letting $a(z, w) = g(w)/(w - z)$, we have

$$\frac{\partial a(z, w)}{\partial z} = g(w) \frac{-1}{(w - z)^2}, \tag{4.2}$$

so the assumptions of Proposition 2.1 are NOT satisfied, so we cannot directly differentiate under the integral sign! Another problem is that g is only assumed to be in $C^\infty(\Omega)$, so it is not in $L^1(\Omega)$ and (4.1) is not necessarily defined. We first give a preliminary result, with a stronger assumption on g.

Proposition 4.2. *If $g \in C^1(\overline{\Omega})$ then the function*

$$f(z) = \frac{1}{2\pi i} \int_\Omega \frac{g(w) dw \wedge d\bar{w}}{w - z}, \tag{4.3}$$

satisfies $f \in C^1(\overline{\Omega})$ and $\partial f / \partial \overline{z} = g$ in Ω.

Proof. Since g is assumed to be in $C^1(\overline{\Omega})$, the integral in (4.3) is well-defined. To show that $\partial f / \partial \overline{z} = g$ in Ω, we will fix any point $z_0 \in \Omega$, and show that $\partial f / \partial \overline{z}(z_0) = g(z_0)$. Choose a C^∞ cutoff function $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on $\Delta(z_0, r) \subset \Omega$, where r is sufficiently small. We then write $f = f_1 + f_2$, where

$$f_1(z) = \frac{1}{2\pi i} \int_\Omega \frac{\psi(w) g(w) dw \wedge d\bar{w}}{w - z} \tag{4.4}$$

$$f_2(z) = \frac{1}{2\pi i} \int_\Omega \frac{(1 - \psi(w)) g(w) dw \wedge d\bar{w}}{w - z}. \tag{4.5}$$

For z in a small neighborhood of z_0, the integrand in f_2 does not have a singularity. We can therefore differentiate under the integral sign to see that $\partial f_2/\partial \overline{z}(z_0) = 0$. So we just need to prove that $\partial f_1/\partial \overline{z} = g$ at z_0. Since ψ has compact support, we can extend ψg to all of $\mathbb{C}$, and write

$$f_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w - z} \tag{4.6}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(\xi + z)g(\xi + z)d\xi \wedge d\bar{\xi}}{\xi}, \tag{4.7}$$

where we used the change of variables $w = \xi + z$. Note that

$$\frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial z} = \frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \xi} \tag{4.8}$$

$$\frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \overline{z}} = \frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \bar{\xi}}. \tag{4.9}$$

This shows that the z and $\overline{z}$ partials of the integrand are uniformly in L^1, so we can differentiate under the integral sign, to obtain

$$\frac{\partial f_1(z)}{\partial \overline{z}} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi(\xi + z)g(\xi + z))}{\partial \bar{\xi}} \frac{d\xi \wedge d\bar{\xi}}{\xi} \tag{4.10}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi(w)g(w))}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z}. \tag{4.11}$$

Now apply the Cachy-Pompieu formula in a very large ball in $\mathbb{C}$, to conclude the right hand side is equal to $\psi(z)g(z)$ which is $g(z)$ if $z \in \Delta(z_0, r)$. ☐

Remark 4.3. With a little more work, the assumptions can be weakened to g being bounded and continuous on Ω; see [HL84, Theorem 1.1.3], but the derivative is interpreted as a distributional derivative.

This result does not directly help us in proving Theorem 5.3. But notice that in the proof, we also proved the following result.

Proposition 4.4. *If $g \in C_c^\infty(\Omega)$, then there exists $f \in C^\infty(\mathbb{C})$ such that $\partial f/\partial \overline{z} = g$.*

Proof. Above, we proved that there is a solution $f \in C^1(\mathbb{C})$, but the same argument allows us to differentiate f_1 infinitely many times, provided g is infinitely differentiable. ☐

Remark 4.5. In general, we cannot expect that f has compact support. Take $g \in C_c^\infty(\mathbb{C})$ with $\int_{\mathbb{C}} g dz \wedge d\overline{z} \neq 0$ and let f be any solution of $\partial f/\partial \overline{z} = g$. By Stokes' Theorem,

$$0 \neq \int_{\mathbb{C}} g dz \wedge d\overline{z} = \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z} = -\int_{\mathbb{C}} d(f dz) = 0, \tag{4.12}$$

if f has compact support, which is a contradiction.

Now we can prove a special case of Theorem 5.3.

Proposition 4.6. *Theorem 5.3 is true for $\Omega = \Delta(z, r)$ is a disc in $\mathbb{C}$, and $\Omega = \Delta^*(z, r)$ a punctured disc, and for $\Omega = A(z; r_1, r_2)$ an annulus.*

Proof. We first consider the case of a disc. Take a sequence $0 < r_1 < r_2 < \cdots < r$ such that $\lim_{j\to\infty} r_i = r$. Let $0 \leq \psi_k \in C_0^\infty(\Delta(z, r_{k+1}))$ and $\psi_k \equiv 1$ on $\Delta(z, r_k)$. Then $g_k = \psi_k g \in C_0^\infty(\Delta(z, r_{k+1})$, and by Proposition 4.4, we can find $f_k \in C^\infty(\mathbb{C})$ such that $\partial f_k = g_k$, which is equal to g in $\Delta(z, r_k)$.

Now there is no reason that the sequence f_k will converge to a limit, so we need to modify as follows. We claim that we can choose f_k so that

$$\sup_{z'\in\Delta(z,r_{k-1})} |f_{k+1}(z') - f_k(z')| \leq 2^{-k}. \tag{4.13}$$

Given f_2, the difference $f_3 - f_2$ is holomorphic in $\overline{\Delta(z, r_1)}$. So there exists a polynomial P_3 such that

$$\sup_{z'\in\Delta(z,r_1)} |f_3(z') - f_2(z') - P_3(z')| \leq 2^{-2}. \tag{4.14}$$

So we redefine f_3 to be $f_3 - P_3$. We then proceed by induction. Given f_k, the difference $f_{k+1} - f_k$ is holomorphic in $\overline{\Delta(z, r_{k-1})}$, so we can find a polynomial P_{k+1} such that

$$\sup_{z'\in\Delta(z,r_{k-1})} |f_{k+1}(z') - f_k(z') - P_{k+1}(z')| \leq 2^{-k}, \tag{4.15}$$

and we redefine f_{k+1} to be $f_{k+1} - P_{k+1}$.

The sequence of functions f_k will be a Cauchy sequence in any disc $\Delta(z, r')$, when $r' < r$. So there exists a uniform limit f. Fixing any m, then $f - f_m$ is then a uniform limit of holomorphic functions in $\Delta(z, r_{m-1})$, so is holomorphic by Corollary 2.5, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m/\partial\overline{z} \to \partial f/\partial\overline{z} = g, \tag{4.16}$$

and the proof is finished.

The case of a punctured disc or annulus is similar, but using truncated Laurent series expansions instead of polynomials which was proved above in Proposition 3.5 and Remark 3.6. □

5 Lecture 5

5.1 Runge's Theorem

To handle the case of an arbitrary domain $\Omega \subset \mathbb{C}$, we need some machinery.

Theorem 5.1 (Runge's approximation Theorem, first version)**.** *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z\in K} |f(z) - f_\epsilon(z)| < \epsilon, \tag{5.1}$$

and such that poles of f_ϵ are contained in $\mathbb{C} \setminus K$.

Proof. The proof is from [Sar07, Theorem IX.15], we just give an outline. From elementary arguments, there exists a simple contour γ with image in $U \setminus K$ such that $K \subset \mathrm{Int}(\gamma) \subset U$. I.e., γ separates K from the complement of U. Note that K might have several components, and $U \setminus K$ might be disconnected, in which case γ will be disconnected. Since the countour is simple, by Cauchy's Integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw \tag{5.2}$$

for any $z \in K$. Note that this works for piecewise smooth paths because Stokes' Theorem holds for a domain with piecewise smooth boundary. By dividing the plane into a sufficiently fine grid, we can assume that γ is piecewise smooth and $\gamma = \gamma_1 + \cdots \gamma_n$, with each γ_j a line segment parallel to one of the coordinate axes. Consider each term

$$f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{5.3}$$

We can approximate this arbitrarily closely with a Riemann sum R_k, which will be of the form

$$\frac{c_1}{w_1 - z} + \cdots + \frac{c_l}{w_l - z}, \tag{5.4}$$

where the w_j are points on γ_k. Doing this for every γ_j, the proof is complete. □

In this result, the poles of f_ϵ will be quite close to K, and still in Ω. The full version of Runge's Theorem allows us to move the poles to specified points in $\mathbb{C} \setminus K$, in particular they can be moved outside of Ω.

Theorem 5.2 (Runge's approximation Theorem, second version)**.** *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Let $S \subset \mathbb{C} \setminus K$ which contains at least one point from each connected component of $\mathbb{C} \setminus K$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z \in K} |f(z) - f_\epsilon(z)| < \epsilon, \tag{5.5}$$

and such that poles of f_ϵ are contained in S. If $\mathbb{C} \setminus K$ is connected, the rational functions can be taken to be polynomials.

Proof. The proof is from [Sar07, Theorem IX.17]. In the proof of Theorem 5.1, each term in the approximation was of the form $c/(w - z)$, where $w \in \gamma$. Now choose a picewise linear path α from w to any point w_0 in the same connected component of $\mathbb{C} \setminus K$. Choose points w_i on α so that

$$|w_{i-1} - w_i| < \mathrm{dist}(\gamma, K). \tag{5.6}$$

We show that any rational R_{j-1} function with a pole only at w_{j-1} may be uniformly approximated on K by a rational function R_j with a poles only at w_j. But this follows from consdering the Laurent series expansion of R_{j-1} centered at w_j: R_{j-1} is holomorphic in the region $U = \mathbb{C} \setminus \Delta(w_j, |w_j - w_{j-1}|)$, so the Laurent series of R_{j-1} centered at w_j converges uniformly on compact subsets of U. Then we can approximate R_{j-1} by a rational function with a pole only at w_j, uniformly on K, since K is a compact subset of U, which follows from (5.6).

If $\mathbb{C} \setminus K$ is connected, by the above argument, we can move the pole of the rational function R_j to a single point z_0 so that $K \subset \Delta(0, |z_0|)$. The Talyor series of R_j centered at any point $z \in K$ converges uniformly on K, so we can approximate by the partial sums of this Taylor series.

□

Now we can prove the general result:

Theorem 5.3. *If $\Omega \subset \mathbb{C}$ is any domain, and $g \in C^\infty(\Omega)$, then there exists $f \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \bar{z}} f = g$.*

Proof. We choose a sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \cdots$, so that $\overline{K_j} \subset \mathrm{Int} K_{j+1}$ and $\cup K_j = \Omega$. Note that $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, and we can assume that for large j, each component of $\mathbb{C} \setminus K_j$ contains a component of $\mathbb{C} \setminus \Omega$. Let $0 \leq \psi_j \in C_0^\infty(K_{j+1})$ and $\psi_k \equiv 1$ on K_j. Then $g_j = \psi_j g \in C_0^\infty(K_{j+1})$, and by Proposition 4.4, we can find $f_j \in C^\infty(\mathbb{C})$ such that $\partial f_j = g_j$.

We claim that we can choose $f_j \in C^\infty(\Omega)$ so that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z)| \leq 2^{-j}. \tag{5.7}$$

We proceed by induction. Given f_j, the difference $f_{j+1} - f_j$ is holomorphic in

$$U \equiv \mathrm{Int} K_j \supset K = \overline{K_{j-1}}. \tag{5.8}$$

We have $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, so by Theorem 5.2, there exists a rational function R_{j+1} such that it poles are in $\mathbb{C} \setminus \Omega$ and such that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z) - R_{j+1}(z)| \leq 2^{-j}, \tag{5.9}$$

and we redefine f_{j+1} to be $f_{j+1} - P_{j-1}$.

The sequence of functions f_j will be a Cauchy sequence in any subset K_m for fixed m. So there exists a limit f, with uniform convergence on compact subsets. Fixing any m, then $f - f_m$ is then a uniform limit of holomorphic functions in K_{m-1}, so is holomorphic by Corollary 2.5, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m / \partial \overline{z} \to \partial f / \partial \overline{z} = g, \tag{5.10}$$

and the proof is finished.

□

6 Lecture 6

The above solution of the inhomogeneous Cauchy-Riemann equations has many corollaries, we next give a few applications. The first is Mittag-Leffler's Theorem.

Theorem 6.1 (Mittag-Leffler). *Let Ω be a domain in $\mathbb{C}$ and $\{w_j\}$ a discrete subset of Ω. Let P_j be any principal sum at w_j. Then there exists a meromorphic function $h \in \mathcal{M}(\Omega)$ such that the principal part of h at w_j is P_j and there are no other poles in Ω.*

Proof. Let ψ_j be a cutoff function supported in a small neighborhood of w_j which doesn't contain any other points in the discrete subset. Consider $g = \sum_j \psi_j P_j$. Then $\partial g/\partial \bar{z} \in C^\infty(\Omega)$. By Theorem 5.3, there exists a solution $f \in C^\infty(\Omega)$ of $\partial f/\partial \bar{z} = \partial g/\partial \bar{z}$. Then $h = g - f$ satisfies $\partial h/\partial \overline{z} = 0$, and the principal part of h at w_j is P_j. □

6.1 Logarithm of a function

We next want to answer the question of when is it possible to take to logarithm of a nowhere-zero holomorphic function.

Definition 6.2. Let U be a domain in $\mathbb{C}$. If $f \in \mathcal{O}(U)$ is nowhere-zero, then we write $f \in \mathcal{O}^*(U)$.

First, we consider a disc.

Proposition 6.3. *If $f \in \mathcal{O}^*(\Delta(z_0, r))$ then there exists $F \in \mathcal{O}(\Delta(z_0, r))$ such that $e^F = f$. Such a solution is unique up to adding an integer multiple of $2\pi i$.*

Proof. Given any $z \in \Delta(z_0, r)$, we define

$$F(z) = \int_\gamma \frac{f'(w)}{f(w)} dw, \tag{6.1}$$

where γ is the straight line path from z_0 to z. Consider a square with corners at z_0 and z, such that γ is the diagonal. Then 2 sides of the square and the diagonal form a closed triangle. By Cauchy's Theorem, the integral is the same if we integrate along the edges. Then the fundamental theorem of calculus shows that F is differentiable, and $F'(z) = f'(z)/f(z)$. Then consider the function $G(z) = fe^{-F}$. We compute

$$G'(z) = f'(z)e^{-F} - fe^{-F(z)}F'(z) = 0, \tag{6.2}$$

so G is constant. By adding the appropriate constant to F, we can therefore assume $G(z) \equiv 1$, which is $f = e^F$.

Assume that $e^{F_1} = f = e^{F_2}$. Then

$$fF_1' = e^{F_1}F_1' = e^{F_2}F_2' = fF_2', \tag{6.3}$$

and since f is nowhere vanishing, $F_1' = F_2'$, so F_1 and F_2 differ by a constant, which must be an integer multiple of $2\pi i$. □

Definition 6.4. We say that a domain U is simply connected if every simple closed curve in U bounds a disc in U.

Notice that the above proof works for any simply-connected domain. However, we will next prove this in a different way.

Definition 6.5. Let $U \in \mathbb{C}$ be a domain. We say that U has $H^1(U;\mathbb{Z}) = 0$ if any countable locally finite covering of U by discs $U_i = \Delta(z_j, r_j)$ has the following property. For any $n_{ij} \in \mathbb{Z}$ satisfying $n_{ij} = -n_{ji}$, and $n_{jk} - n_{ik} + n_{ij} = 0$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$ then there exists $n_i \in \mathbb{Z}$ such that $n_{ij} = n_j - n_i$ whenever $U_i \cap U_j \neq \emptyset$.

Remark 6.6. This definition is in fact equivalent to assuming that U is simply-connected, but this fact needs some tools from algebraic topology which we will not discuss right now.

Proposition 6.7. *Let $U \subset \mathbb{C}$ be a domain satisfying $H^1(U;\mathbb{Z}) = 0$ If $f \in \mathcal{O}^*(U)$ then there exists $F \in \mathcal{O}(U)$ such that $e^F = f$.*

Proof. Take a covering as in Definition 6.5. Then we view f as a collection of functions $f_i \in \mathcal{O}^*(U_i)$, with $f_i = f_j$ on $U_i \cap U_j$. Each U_i is a disc, so by the above, there exists $g_i \in \mathcal{O}(U_i)$ with $f_i = e^{g_i}$ on U_i. We have $\frac{f_i}{f_j} = e^{g_i - g_j} = 1$ on $U_i \cap U_j$, so there exists $n_{ij} \in \mathbb{Z}$ such that $g_i - g_j = 2\pi\sqrt{-1}n_{ij}$ there. By the assumption that $H^1(U;\mathbb{Z}) = 0$, there exists $n_i \in \mathbb{Z}$ such that $n_{ij} = n_i - n_j$ whenever $U_i \cap U_j \neq \emptyset$. We have $g_i - g_j = 2\pi\sqrt{-1}(n_i - n_j)$ on $U_i \cap U_j$. Therefore $g_i' = g_i - 2\pi\sqrt{-1}n_i$ satisfies $g_i' = g_j'$ on $U_i \cap U_j$ so defines a function $F \in \mathcal{O}(U)$ satisfying $f = e^F$. □

Remark 6.8. The function z in nowhere zero in any punctured disc $\Delta^*(0,r)$. If $z = e^F$ for $F \in \mathcal{O}(\Delta^*)$, then $F' = 1/z$ there (that is, F is a *primitive* for $1/z$). But if γ is $S^1(r/2)$, then $\int_\gamma F'(z)dz = 0$ from the fundamental theorem of calculus, but $\int_\gamma (1/z) = 2\pi\sqrt{-1}$, a contradiction.

Remark 6.9. Secretly, we are using the long exact sequence in cohomology associated to the exponential sheaf sequence

$$0 \to 2\pi\sqrt{-1} \cdot \mathbb{Z} \to \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \to 1, \tag{6.4}$$

where $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U, and similarly for $\mathcal{O}_U^*$. This gives the exact sequence

$$0 \to H^0(U;\mathbb{Z}) \to H^0(U, \mathcal{O}_U) \to H^0(U, \mathcal{O}_U^*) \to H^1(U;\mathbb{Z}). \tag{6.5}$$

If U is connected, then $H^0(U;\mathbb{Z}) \simeq \mathbb{Z}$. However, the 0th cohomology group of any sheaf is the space of global sections, so we have the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}(U) \to \mathcal{O}^*(U) \to H^1(U;\mathbb{Z}). \tag{6.6}$$

Thus if $H^1(U;\mathbb{Z}) = 0$, then the third arrow is surjective.

7 Lecture 7

7.1 Weierstrass Theorem

We give more applications of the solution of the inhomogeneous Cauchy-Riemann equations. References for this section are [Hör90, Chapter 1] and [Eps91, Section 1.6].

Definition 7.1. Let $U \subset \mathbb{C}$ be a domain. We say that U has $H^2(U;\mathbb{Z}) = 0$ if any countable locally finite covering of U by discs $U_i = \Delta(z_j, r_j)$ has the following property. For any $n_{ijk} \in \mathbb{Z}$ satisfying $n_{ijk} = -n_{jik} = -n_{ikj}$ and $n_{jkl} - n_{ikl} + n_{ijl} - n_{ijk} = 0$ whenever $U_i \cap U_j \cap U_k \cap U_l \neq \emptyset$, then there exists $n_{ij} \in \mathbb{Z}$ satisfying $n_{ij} = -n_{ji}$ such that $n_{ijk} = n_{jk} - n_{ik} + n_{ij}$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$.

Theorem 7.2 (Weierstrass)**.** *Let U be a domain in $\mathbb{C}$ with $H^2(U;\mathbb{Z}) = 0$, $\{w_j\}$ a discrete subset of U, and $n_j \in \mathbb{Z}$. Then there exists a meromorphic function $f \in \mathcal{M}(U)$ with the order of f at w_j equal to n_j and no other poles or zeroes.*

Proof. We cover U by discs $U_i = \Delta(z_i, r_i)$ such that each w_i is contained in exactly one of these discs. Define the function $f_i = (z - w_j)^{n_j}$ if $w_j \in U_i$, and let $f_i = 1$ if U_i doesn't contain any of the discrete points. On $U_i \cap U_j$, let $f_{ij} = f_i/f_j$. Then $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ is a non-vanishing holomorphic function. Since $U_i \cap U_j$ is simply-connected, we can define $g_{ij} = \log f_{ij}$. Note that $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ is only defined up to adding an integer multiple of $2\pi i$. Since

$$f_{ik} = \frac{f_i}{f_k} = \frac{f_i}{f_j}\frac{f_j}{f_k} = f_{ij} f_{jk}, \tag{7.1}$$

the g_{ij} satisfy on triple intersections $U_i \cap U_j \cap U_k$

$$g_{ij} - g_{ik} + g_{jk} = 2\pi i n_{ijk}, \tag{7.2}$$

where $n_{ijk} \in \mathbb{Z}$. The n_{ijk} satisfy the condition on intersections $U_i \cap U_j \cap U_k \cap U_l$,

$$n_{jkl} - n_{ikl} + n_{ijl} - n_{ijk} = 0. \tag{7.3}$$

By the assumption that $H^2(U;\mathbb{Z}) = 0$, there exists integers n_{ij} such that

$$n_{ijk} = n_{jk} - n_{ik} + n_{ij}, \tag{7.4}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Then $g'_{ij} = g_{ij} - 2\pi i n_{ij}$, satisfy

$$g'_{ij} - g'_{ik} + g'_{jk} = 0. \tag{7.5}$$

Choose a partition of unity ψ_i subordinate to U_i, and define

$$h_i = \sum_j g'_{ij}\psi_j, \tag{7.6}$$

which satisfies $h_i \in C^\infty(U_i)$. On $U_i \cap U_j$, we have

$$h_i - h_j = \sum_k (g'_{ik} - g'_{jk})\psi_k = \sum_k g'_{ij}\psi_k = g'_{ij}. \tag{7.7}$$

We then have that

$$\frac{\partial}{\partial \overline{z}}(h_i - h_j) = \frac{\partial}{\partial \overline{z}} g'_{ij} = 0, \tag{7.8}$$

So we can define $h \in C^\infty(U)$ by letting

$$h|_{U_i} = \frac{\partial h_i}{\partial \overline{z}} \tag{7.9}$$

By Theorem 5.3, we can solve the equation $\partial f / \partial \overline{z} = h$ for $f \in C^\infty(U)$. Then we redefine $h'_i = h_i - f$. These now satisfy $h'_i \in \mathcal{O}(U_i)$, and $h'_i - h'_j = g'_{ij}$.

So going back to the above, we define $f'_i = e^{-h'_i} f_i$. On intersections, we now have

$$\frac{f'_i}{f'_j} = \frac{e^{-h'_i} f_i}{e^{-h'_j} f_j} = e^{-h'_i + h'_j} \frac{f_i}{f_j} = e^{-g_{ij'}} \frac{f_i}{f_j} = e^{-g_{ij}} \frac{f_i}{f_j} = 1, \tag{7.10}$$

so the f'_i patch together to define $f \in \mathcal{M}(U)$. Since we only multiplied the f_i by a non-zero holomorphic function, the order of f at w_j is equal to n_j, and there are no other zeros or poles of f in U. □

Remark 7.3. Note that any domain in $\mathbb{C}$ is necessarily a non-compact 2-manifold, it will follow by Poincarè duality $H^2_{sing}(U; \mathbb{Z}) = 0$. This is equivalent to the vanishing of the Čech cohomology group $\check{H}^2(U; \mathbb{Z}) = 0$, which implies that the condition in Definition 7.1 is always satisfied for any domain $U \subset \mathbb{C}$. So the assumption that $H^2(U; \mathbb{Z}) = 0$ in actually superfluous. In contrast, if we consider a compact Riemann surface Σ, then $H^2(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}$, and Weierstrass' Theorem will not necessarily hold.

Corollary 7.4. *If $f \in \mathcal{M}(U)$, then there exists $g, h \in \mathcal{O}(U)$ such that $f = g/h$ in all of U.*

Proof. If f has poles of order n_j at w_j, then by the Weierstrass Theorem, there exists a holomorphic function $h \in \mathcal{O}(U)$ which has a zero of order n_j at w_j. Then $g = hf$ has no poles so $g \in \mathcal{O}(U)$. □

Corollary 7.5. *If U is any domain in $\mathbb{C}$, then there exists $u \in \mathcal{O}(U)$ such U cannot be extended to any larger domain.*

Proof. Take a discrete subset w_j with closure containing the boundary of Ω. By the Weierstrass Theorem, there exists $u \in \mathcal{O}(U)$ which has zeroes at the w_j, but is not identically zero. This function cannot be extended holomorphically in a neighborhood of any boundary point. This is because any holomorphic function f whose zero set has a limit point in a domain must be identically zero. (Proof: let the limit point be z_0, then we can factor $f = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$, so the zeros of a any holomorphic function are isolated; this is also known as the Identity Theorem.) □

Remark 7.6. The power series

$$F(z) = \sum_{j=0}^{\infty} z^{2^j} \tag{7.11}$$

converge on compact subsets to a holomorphic function on the unit disc, which cannot be extended past any boundary point. See [GK06, Chapter 9] and the more general Hadamard gap theorem. However, this is proved by ad hoc methods and not really related to the above proof.

8 Lecture 8

8.1 Holomorphic line bundles on domains in $\mathbb{C}$

Let $\mathfrak{U} = \{U_i\}_{i\in\mathcal{I}}$ be a countable locally finite open covering of a domain $U \subset \mathbb{C}$ by discs. For $i, j \in \mathcal{I}$, let $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfy $f_{ji} = f_{ij}^{-1}$, and

$$f_{ik} = f_{ij} \cdot f_{jk} \tag{8.1}$$

on $U_i \cap U_j \cap U_k$. We call (8.1) the *cocycle condition*, and write that $f_{ij} \in Z^1(\mathfrak{U}, \mathcal{O}_U^*)$. We call the collection $\{f_{ij}, i, j \in \mathcal{I}\}$ a *holomorphic line bundle* with respect to the open covering $\mathfrak{U}$. A collection $f_i \in \mathcal{O}(U_i)$ for $i \in \mathcal{I}$ is called a *section* of the line bundle if $f_{ij} f_j = f_i$ on $U_i \cap U_j$. The *trivial line bundle* is $f_{ij} = 1$ on all $U_i \cap U_j$. Note that there is a multiplicative structure: the product of f_{ij} and f'_{ij} is simply $f_{ij} f'_{ij}$, which clearly satisfies the cocyle condition. Define $C^0(\mathfrak{U}, \mathcal{O}_U^*)$ to be the collection of $f_i \in \mathcal{O}^*(U_i)$, which we call the space of 0-cocycles with respect to $\mathfrak{U}$. Define $\delta : C^0 \to Z^1$ by $\delta\{f_i\} = f_i/f_j$ on $U_i \cap U_j$. Then we define $H^1(\mathfrak{U}, \mathcal{O}_U^*)$ as $Z^1/\delta(C^0)$. We call this the set of *equivalence classes* of holomorphic line bundles on U, with respect to the open covering $\mathfrak{U}$. Note that two line bundles f_{ij} and f'_{ij} are equivalent if and only if there exists $f_i \in \mathcal{O}^*(U_i)$ such that $f_{ij} f_j = f'_{ij} f_i$. In particular, a line bundle is equivalent to the trivial line bundle if and only if there exists $f_i \in \mathcal{O}^*(U_i)$ such that $f_{ij} f_j = f_i$, equivalently, if and only if there exists a nowhere vanishing section.

What we have proved above can be restated as follows.

Proposition 8.1. *If $U \subset \mathbb{C}$ is a domain and $\mathfrak{U}$ is a countable locally finite covering of U by discs, then any holomorphic line bundle is equivalent to a trivial holomorphic line bundle (everything with respect to the open covering $\mathfrak{U}$).*

Proof. We simply review the above proof of the Weierstrass Theorem. In that proof, we first constructed $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfying the cocycle condition. Now we are just given $f_{ij} \in Z^1(\mathfrak{U}, \mathcal{O}_U^*)$. Then we choose $g_{ij} = \log f_{ij}$ in $\mathcal{O}(U_i \cap U_j)$. The g_{ij} satisfy on triple intersections $U_i \cap U_j \cap U_k$

$$g_{ij} - g_{ik} + g_{jk} = 2\pi i n_{ijk}, \tag{8.2}$$

where $n_{ijk} \in \mathbb{Z}$. We then used the assumption $H^2(U; \mathbb{Z}) = 0$ to find n_{ij} satisfying

$$n_{ijk} = n_{jk} - n_{ik} + n_{ij}, \tag{8.3}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Then $g'_{ij} = g_{ij} - 2\pi\sqrt{-1}n_{ij}$ satisfy

$$g'_{ij} - g'_{ik} + g'_{jk} = 0. \tag{8.4}$$

Then, using the partition of unity argument, we found $h'_i \in \mathcal{O}(U_i)$ with $g'_{ij} = h'_i - h'_j$. Define $f_i = e^{h'_i}$. Exponentiation of this yields

$$e^{g'_{ij}} = e^{g_{ij} - 2\pi\sqrt{-1}n_{ij}} = e^{g_{ij}} = f_{ij} = e^{h'_i - h'_j} = \frac{e^{h'_i}}{e^{h'_j}} \equiv \frac{f_i}{f_j}, \tag{8.5}$$

and we are done since the collection $\{f_i\}$ are a nowhere vanishing section of the line bundle. □

Remark 8.2. Similar to above, we can define $Z^1(\mathfrak{U}, \mathcal{O}_U)$ to be the collection of $g'_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfying $g'_{ij} = -g'_{ji}$, and

$$g'_{ij} - g'_{ik} + g'_{jk} = 0 \tag{8.6}$$

on triple intersections $U_i \cap U_j \cap U_k$. Note this is an additive group, as opposed to multiplicative in the case of line bundles. Define $C^0(\mathfrak{U}, \mathcal{O}_U)$ to be the collection of $h'_i \in \mathcal{O}(U_i)$ and $\delta : C^0 \to Z^1$ by $\delta\{h'_i\} = h'_j - h'_i$. Then we can define $H^1(\mathfrak{U}, \mathcal{O}_U) = Z^1/\delta(C^0)$. The argument involving a partition of unity proved that $H^1(\mathfrak{U}, \mathcal{O}_U) = 0$, which relied on Theorem 5.3, the solvability of the inhomogeneous Cauchy-Riemann equation in $C^\infty(U)$. This is a special case of a general result about the equivalence of Čech and Dolbeault cohomology.

Remark 8.3. We want to remove the dependence of the above on the open covering, we will do this later. The proof of the Weierstrass Theorem (using the partition of unity argument), yields the vanishing of the sheaf cohomology group $H^1(U, \mathcal{O}_U) = 0$. The long exact sequence in cohomology associated to the exponential sheaf sequence

$$0 \to 2\pi\sqrt{-1} \cdot \mathbb{Z} \to \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \to 1, \tag{8.7}$$

yields

$$0 = H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U^*) \to H^2(U; \mathbb{Z}) = 0, \tag{8.8}$$

from which it follows that $H^1(U, \mathcal{O}_U^*) = 0$. In other words, any holomorphic line bundle on U is (equivalent to) a trivial bundle. This will not be true in general for domains in $\mathbb{C}^n$ for $n > 1$: as an example, consider a product of punctured discs, which has $H^2(\Delta^* \times \Delta^*; \mathbb{Z}) \simeq \mathbb{Z}$. This will also not be true for a *compact* Riemann surface Σ, which satisfies $H^2(\Sigma; \mathbb{Z}) = \mathbb{Z}$.

9 Lecture 9

9.1 Power series in several variables

We review some basic facts about power series in several variables. Some good references for this material are [FG02, Chapter 1], [JP08, Chapter 1], or [KP02, Chapter 2.1], We write a point $z = (z_1, \dots, z_n)$. The open polydisc with polyradius $r = (r_1, \dots, r_n)$ about a point $z_0 = (z_1^0, \dots, z_n^0)$ is the set

$$\Delta(z_0, r) = \{z \mid |z_j - z_j^0| < r_j,\ j = 1 \dots n\}. \tag{9.1}$$

We will let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ denote a multi-index, where $\mathbb{Z}_+$ denotes the non-negative integers. Define

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \tag{9.2}$$
$$|z|^\alpha = |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \tag{9.3}$$
$$\alpha! = \alpha_1! \cdots \alpha_n! \tag{9.4}$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{9.5}$$

Definition 9.1. The series $\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha$ converges at z if some rearrangement converges, that is, give some bijection $\phi : \mathbb{Z}_+ \to (\mathbb{Z}_+)^n$, the series

$$\sum_{j=0}^{\infty} a_{\phi(j)} (z - z_0)^{\phi(j)} \tag{9.6}$$

converges. The *domain of convergence* of the power series is the interior of the set of points of convergence.

In 1 variable we know that domains of convergence are discs. Regions of convergence in several variable can be more complicated.

Example 9.2. The domain of convergence of the series $\sum_{k=0}^{\infty} z^k w^k$ is $\{(z, w) \mid |zw| < 1\}$.

Example 9.3 (Boas)**.** The series $\sum_{n=1}^{\infty} z^n w^{n!}$ converges in the 3 sets

$$U_1 = \{(z, w) \mid |w| < 1\},\ U_2 = \{(0, w)\},\ U_3 = \{(z, w) \mid |z| < 1 \text{ and } |w| = 1\}. \tag{9.7}$$

Only U_1 is an open set; the sets U_2 and U_3 are 1 dimensional, and are not domains. The domain of convergence is U_1.

Lemma 9.4 (Abel)**.** *If $\sum_\alpha a_\alpha z^\alpha$ (centered at $z = 0$) converges at the point z' then it converges uniformly and absolutely for any point z of the form $z_j = \rho_j z_j'$ where $|\rho_j| < 1$. Furthermore, a point p belongs to the domain of convergence of the power series $\sum_\alpha a_\alpha z^\alpha$ if and only if there exists a neighborhood U of p, a constant C, and $r < 1$ such that $|a_\alpha z^\alpha| \leq C r^{|\alpha|}$ for all $z \in U$.*

Proof. Since the series converges at the point z', the terms must be bounded, so there exists a constant C so that $|a_\alpha||z'|^\alpha \leq C$. Let $\rho = \max\{|\rho_1|, \dots, |\rho_n|\} < 1$, and consider any point $z = (z_1, \dots, z_n)$ so that $|z_j| < \rho |z_j'|$. We then have

$$|a_\alpha||z|^\alpha \leq |a_\alpha| \rho^{|\alpha|} |z'|^\alpha \leq C \rho^{|\alpha|}. \tag{9.8}$$

So given an integer $N > 0$, we have

$$\begin{aligned}\sum_{|\alpha| \leq N} |a_\alpha||z|^\alpha &= \sum_{j=0}^{N} \sum_{|\alpha|=j} |a_\alpha||z|^\alpha \\ &\leq \sum_{j=0}^{N} \sum_{|\alpha|=j} C\rho^j\end{aligned} \tag{9.9}$$

How many multi-indices of length j are there? This is counting the number of non-negative integer solutions of

$$\alpha_1 + \cdots + \alpha_n = j. \tag{9.10}$$

To see this, let $\alpha_i' = \alpha_i + 1$, then we are interested in the number of positive integer solutions to

$$\alpha_1' + \cdots + \alpha_n' = j + n. \tag{9.11}$$

So we have a total of $j + n$ integers, dividing this up into n integers is the same as putting $n - 1$ partitions somewhere in the spaces between them, so the number is

$$\binom{j+n-1}{n-1}. \tag{9.12}$$

Continuing with the above calculation,

$$\begin{aligned}\sum_{|\alpha| \leq N} |a_\alpha||z|^\alpha &\leq C \sum_{j=0}^{N} \binom{j+n-1}{n-1} \rho^j \\ &= C \sum_{j=0}^{N} \frac{(j+n-1)!}{j!(n-1)!} \rho^j \\ &= \frac{C}{(n-1)!} \sum_{j=0}^{N} (j+n-1)(j+n-2)\cdots(j+1)\rho^j \\ &\leq C_n \sum_{j=0}^{N} j^n \rho^j.\end{aligned} \tag{9.13}$$

Applying the ratio test, we have

$$\lim_{j\to\infty} \frac{(j+1)^n \rho^{j+1}}{j^n \rho^j} = \lim_{j\to\infty} \left(\frac{j+1}{j}\right)^n \rho = \rho, \tag{9.14}$$

so the series converges provided $\rho < 1$.

If p belongs to the domain of convergence, then by definition the series converges in a neighborhood of p. Then by the first part it converges in some polydisc around the origin containing z, and we follow the first part of the proof.

□

Definition 9.5. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha \tag{9.15}$$

which converges absolutely and uniformly in a polydisc $\Delta(z_0, \hat{\epsilon})$ around z_0, for some positive polyradius $\hat{\epsilon}$.

9.2 Cauchy's formula in several complex variables

Basic reference are [GH78, Hör90, Nog16].

Definition 9.6. We say that f is holomorphic in U if it is $C^1(U)$ and satisfies the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial \overline{z}^j} = 0, \; j = 1 \cdots n. \tag{9.16}$$

Proposition 9.7. *Let U be an open set in $\mathbb{C}$. Then f is holomorphic in U if and only if f is complex analytic in U.*

Proof. Consider $n = 2$, the higher-dimensional case is similar. We assume that $U = \Delta(0, r_1) \times \Delta(0, r_2)$ is a polydisc, and $f \in C^1(\overline{U})$. If f is holomorphic in U, then for fixed z_1, the slice $f(z_1, z_2)$ is a 1-variable holomorphic function for $z_2 \in \Delta(0, r_2)$. This holds similarly for the other variable, so the Cauchy-Pompieu formula applied twice yields

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi i} \int_{|w_2| = r_2} \frac{f(z_1, w_2) dw}{w_2 - z_2} \\ &= \left(\frac{1}{2\pi i} \right)^2 \int_{|w_2| = r_2} \int_{|w_1| = r_1} \frac{f(w_1, w_2) dw}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned} \tag{9.17}$$

For any $(z_1^0, z_2^0) \in U$, we expand

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0 + z_1^0 - z_1} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \frac{1}{1 - \frac{z_1 - z_1^0}{w_1 - z_1^0}} \tag{9.18}$$

$$= \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0} \right)^k \tag{9.19}$$

$$= \frac{1}{(w_1 - z_1^0)(w_2 - z_2^0)} \sum_{l=0}^{\infty} \left(\frac{z_2 - z_2^0}{w_2 - z_2^0} \right)^l \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0} \right)^k \tag{9.20}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \tag{9.21}$$

We next show that we are justified in the last step. Let $(z_1^0, z_2^0) \in \Delta(0, r_1') \times \Delta(0, r_2')$ with $r_1' < r_1$ and $r_2' < r_2$. Then we have $|w_1 - z_1^0| > r_1 - r_1'$, and $|w_2 - z_2^0| > r_2 - r_2'$. For $|z_1 - z_1^0| < (r_1 - r_1')/2$ and $|z_2 - z_2^0| < (r_2 - r_2')/2$, we then have

$$|a_{kl}| = \left| \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1}(w_2 - z_2^0)^{l+1}} \right| \leq \frac{1}{(r_1 - r_1')(r_2 - r_2')} 2^{-k} 2^{-l}, \tag{9.22}$$

so the sum converges absolutely and uniformly in any smaller polydisc by Lemma 9.4. Interchanging the integration and summation in (9.17) then yields a power series expansion for f.

The converse is similar to the 1-variable case. If f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula (9.17). So then f does also by uniform convergence. Then we can differentiate under the integral to see that f is holomorphic. For more details, see [GH78, page 6]. $\square$

Remark 9.8. Note that the integral in (9.17) is just over a 2-dimensional torus contained in the boundary of the polydisc. The topological boundary of the polydisc is 3-dimensional, but it is not a manifold, it is $\partial(\Delta \times \Delta) = S^1 \times \Delta \cup \Delta \times S^1$, and these 2 sets intersect along the torus. The higher dimensional case is similar: the integral is over a real n-dimensional torus contained in the boundary of the polydisc.

Similar to the 1 variable case, we have the following corollaries.

Corollary 9.9. *If f holomorphic in Ω, then f is infinitely differentiable in Ω, and for any $z_0 \in \Omega$, f admits a power series expansion*

$$f = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha, \tag{9.23}$$

with

$$\begin{aligned} a_\alpha &= \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(z_0)}{\partial z^\alpha} \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{|w_n| = r_n} \cdots \int_{|w_1| = r_1} \frac{f(w_1, \cdots, w_n) dw_1 \cdots dw_n}{(w_1 - z_1^0)^{\alpha_1 + 1} \cdots (w_n - z_n^0)^{\alpha_n + 1}}, \end{aligned} \tag{9.24}$$

for any r such that the polydisc $\overline{\Delta(z_0, r)} \subset \Omega$.

Corollary 9.10 (The maximum principle)**.** *Let $\Omega \subset \mathbb{C}^n$ be connected, and $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$. Then $|f|$ does not assume its maximum at an interior point unless f is constant.*

Proof. We can assume that $\Omega = \Delta(0, r) = \Delta_1 \times \cdots \times \Delta_n$ is a polydisc centered at the origin, and that $|f|$ attains a maximum at the center point. The one variable function $z_1 \mapsto f(z_1, 0, \cdots, 0)$ is constant by the maximum principle in one variable, that is $f(z_1, 0, \cdots, 0) = f(0, \cdots, 0)$. Then, for any fixed z_1, the one variable function $z_2 \mapsto f(z_1, z_2, 0, \cdots, 0)$ is constant, so

$$f(z_1, z_2, 0, \cdots, 0) = f(z_1, 0, \cdots, 0) = f(0, \cdots, 0). \tag{9.25}$$

Iterating this argument shows that f is constant. $\square$

Corollary 9.11. *Let $K \subset \Omega$ be a compact subset. Then there exist constants $C_{|\alpha|}$, depending only upon K and Ω such that*

$$\sup_{z \in K} \left| \frac{\partial^\alpha f(z)}{\partial z^\alpha} \right| \leq C_{|\alpha|} \sup_{z \in \Omega} |f(z)|. \tag{9.26}$$

for all $u \in \mathcal{O}(\Omega)$.

Proof. Again, we just consider the case of 2 dimensions, the higher dimensional case is similar. Fix $(z_1^0, z_2^0) \in \Omega$, and let $\Delta(z^0, (r_1, r_2)) \subset \Omega$ be a polydisc. Then for $(z_1, z_2) \in \Delta(z^0, (r_1', r_2'))$ with $r_1' < r_1$ and $r_2' < r_2$, we have

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl}(z_1 - z_1^0)^k (z_2 - z_2^0)^l, \tag{9.27}$$

where

$$a_{kl} = \left(\frac{1}{2\pi i} \right)^2 \int_{|w_2|=r_2} \int_{|w_1|=r_1} \frac{f(w_1, w_2) dw}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \tag{9.28}$$

We then get *Cauchy's inequalities*

$$\left| \frac{\partial^{k+l} f}{\partial z_1^k \partial z_2^l}(z_1^0, z_2^0) \right| = k! l! |a_{kl}| \leq \frac{k! l! r_1 r_2}{(r_1 - r_1')^k (r_2 - r_2')^l} \sup_{w=(w_1,w_2),|w_1|=r_1,|w_2|=r_2} |f(w)|. \tag{9.29}$$

Since the distance of K to $\partial\Omega$ is positive, the claim follows. □

The following corollaries are proved exactly as before.

Corollary 9.12. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \to u$ converges uniformly to u in the C^0 norm as $n \to \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Corollary 9.13. *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

10 Lecture 10

10.1 Hartogs' Theorem

Consider the polydisc $\Delta(0, r) = \{z \in \mathbb{C}^n \mid |z_j| < r, j = 1, \ldots, n\}$, and for $0 < r' < r$, let $A(r', r) = \Delta(r) \setminus \overline{\Delta(r')}$.

Theorem 10.1 (Hartogs)**.** *Let $n > 1$, and $0 < r' < r$. If $f \in \mathcal{O}(\overline{A(r', r)})$, then there exists a holomorphic function $F \in \mathcal{O}(\Delta(r))$ with $F = f$ on $A(r', r)$.*

Proof. Write a point in $\mathbb{C}^n$ as (z, w), where $z \in \mathbb{C}^{n-1}$. We then have

$$\begin{aligned}\left(\Delta^{n-1}(0,r) \times \Delta^1(0,r)\right) \setminus \left(\Delta^{n-1}(0,r') \times \Delta^1(0,r')\right) \\ = A^{n-1}(r',r) \times \Delta^1(0,r) \cup \Delta^{n-1}(0,r) \times A^1(r',r).\end{aligned} \tag{10.1}$$

Then for each $z \in \Delta^{n-1}(0, r')$, the cross section of $A^n(r', r)$ is a 1-dimensional annulus $A^1(r', r)$. However if $z \in A^{n-1}(r', r)$, then the cross section is a disc $\Delta^1(0, r)$. Let's write down the expression

$$F(z, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{|w|=r} \frac{f(z,w)dw}{w - z_n} \tag{10.2}$$

For $r' < |z| < r$ and $|z_n| < r$ this equals to f, by Cauchy's Theorem. This expression clearly defines a holomorphic function in all of $\Delta(r)$. The function F agrees with f whenever $r' < |z| < r$. By the identity principle, $F = f$ in $A(r', r)$, so is the required extension of f. ☐

In particular, while non-removable isolated singularities exist when $n = 1$, isolated singularities are always removable if $n > 1$! We also have the following corollary.

Corollary 10.2. *Let $n > 1$. If $\Omega \subset \mathbb{C}^n$ is a domain and $f \in \mathcal{O}(\Omega)$ then the zero set of f, $Z_f \equiv \{z \in \Omega \mid f(z) = 0\}$ does not contain any isolated points.*

To prove the corollary, note that $1/f$ would be a holomorphic function with an isolated singularity, which doesn't exist.

The next result will show that zero sets of holomorphic functions are large, they are always complex $(n-1)$-dimensional in a very precise sense.

10.2 Weierstrass Preparation Theorem

For $n = 1$, we know that zeros of a non-constant holomorphic function are isolated. We want to investigate the structure of zero sets of holomorphic function in $\mathbb{C}^n$. So consider $f(z_1, \ldots, z_n) \in \mathcal{O}(U)$, where U is a neighborhood of the origin, and assume that $f(0, \ldots, 0) = 0$. Consider a complex line through the origin

$$L_z = \{c(z_1, \ldots, z_n) \; | c \in \mathbb{C}\}. \tag{10.3}$$

If we restrict $f|L$, we get a 1 variable holomorphic function. If this restricted function were to vanish identically for all complex lines through the origin, then f itself would be identically zero. So if f is not identically zero, there exists a least one line on which f is not identically zero. By a rotation, we may assume that this in the nth coordinate line. I.e., for fixed $(z_1, \ldots, z_{n-1})$, we look at the holomorphic function of one variable $w \mapsto f(z_1, \ldots, z_{n-1}, w)$. By assumption $f(0, \ldots, 0, w)$ is not identically zero, so there is a unique minimal d such that $f(0, \ldots, 0, w) = w^d h$, where h does not vanish at 0. Then there exists $r > 0$ and $\delta > 0$ such that $|f(0, \ldots, 0, w)| \geq \delta$ for $|w| = r$. By continuity, there exists $\epsilon > 0$ so that $|f(z_1, \ldots, z_{n-1}, w)| \geq \delta/2$ for $|w| = r$ and $|(z_1, \ldots, z_n)| < \epsilon$. We next recall the argument principle.

Corollary 10.3 (The argument principle)**.** *If $f \in \mathcal{M}(\overline{\Omega})$ with no poles on $\partial\Omega$, then*

$$\int_{\partial\Delta(z_0,r)} w^q \frac{f'(w)}{f(w)} dw = \sum_{j=1}^{d} m_j z_j^q \tag{10.4}$$

where w_j are the zeros and poles of f, with m_j is the order of f at w_j.

To simplify notation, let $z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$. First of all, we apply the argument principle to the function $w \mapsto f(z, w)$, with $q = 0$ to get that

$$\int_{|w|=r} \frac{\partial f/\partial w(z,w)}{f(z,w)} dw = \sum_j m_j, \tag{10.5}$$

where m_j is the multiplicity of f at the zeros of the function $w \mapsto (z, w)$ inside $|w| < r$ and $|z| < \epsilon$. This is equal to d for $z = 0$. Since it is a continuous integer valued function, the function $w \mapsto f(z, w)$ also has d zeroes, counted with multiplicity for $|w| < r$ and $|z| < \epsilon$. We let $b_1(z), \cdots b_d(z)$ denote the zeros of $f(z, w)$ for $z \in \mathbb{C}^{n-1}$ fixed. The problem is that these function cannot necessarily be chosen smoothly, because there is no canonical ordering of these zeros. We define

$$g(z, w) = (w - b_1(z)) \cdots (w - b_d(z)) \tag{10.6}$$

Obviously, the zero set of g is exactly the same as the zero set of f. However, this is not obviously holomorphic due to the ordering problem. However, we may write

$$g(z, w) = w^d - \sigma_1(b(z))w^{d-1} + \sigma_2(b(z))w^{d-2} + \cdots + (-1)^d \sigma_d(b(z)) \tag{10.7}$$

where the σ_i are the ith elementary symmetric function, i.e.,

$$\sigma_1(b_1, \dots, b_d) = b_1 + \cdots + b_d \tag{10.8}$$

$$\sigma_2(b_1, \dots, b_d) = \sum_{i \neq j} b_i b_j \tag{10.9}$$

$$\vdots \tag{10.10}$$

$$\sigma_d(b_1, \dots, b_d) = b_1 \cdots b_d. \tag{10.11}$$

We have the following lemma.

Lemma 10.4. *There exists polynomials P_j with rational coefficients so that the elementary symmetric function $\sigma_i = P_i(F_1, \cdots, F_i)$, where*

$$F_q(b) = b_1^q + \cdots + b_d^q. \tag{10.12}$$

for $1 \le i \le d$.

Proof. Obviously $\sigma_1 = P_1$. For σ_2, we have

$$\sigma_2 = \frac{1}{2}(F_1^2 - F_2). \tag{10.13}$$

For σ_3, we have

$$\sigma_3 = \frac{1}{6}F_1^3 - \frac{1}{2}F_2F_1 + \frac{1}{3}F_3. \tag{10.14}$$

Will leave the proof of the general case as an exercise (Hint: use a induction argument, or ask Newton). □

However, if we apply the argument principle again for $q > 0$, we will get

$$F_q(b(z)) = b_1(z)^q + \cdots + b_d(z)^q = \int_{|w|=r} \frac{w^q(\partial f/\partial w)(z,w)}{f(z,w)}dw \tag{10.15}$$

Thus we see that the power sums $F_q(z) = b_1(z)^q + \cdots + b_d(z)^q$ are analytic functions of z, for $|z| < \epsilon$. By the lemma, the elementary symmetric functions are analytic there, which proves that the function $g(z,w)$ defined above is analytic.

Now we consider the function

$$h(z,w) = \frac{f(z,w)}{g(z,w)}, \tag{10.16}$$

which is analytic away from the zero set of g. By construction, for each fixed z with $|z| < \epsilon$, $g(z,w)$ has exactly the same zeros as $f(z,w)$. So the 1 variable function $w \mapsto h(z,w)$ is bounded, and thus has a removable singularity at near any zero of $g(z,w)$, so h extends to a non-zero holomorphic function for $|w| < r$. But then the Cauchy integral formula

$$h(z,w) = \frac{1}{2\pi\sqrt{-1}} \int_{|u|=r} \frac{h(z,u)du}{u-w} \tag{10.17}$$

shows that h is continuously differentiable and holomorphic in all of the variables z_j. So we have proved

Theorem 10.5 (Weierstrass Preparation Theorem)**.** *Let $f(z,w)$ be an analytic function which vanishes at the origin, but does not vanish identically on the w-axis. Then there exists a unique Weierstrass polynomial of the form*

$$g(z,w) = w^d + a_1(z)w^{d-1} + \cdots + a_d(z), \tag{10.18}$$

with $a_j(z)$ analytic with $a_j(0) = 0$, such that $f = g \cdot h$ with $h(0) \neq 0$.

This gives us a very nice description of the zero set of holomorphic functions.

Corollary 10.6. *If f is holomorphic, then locally Z_f admits a projection to a $(n-1)$ dimension polydisc Δ^{n-1}, which represents Z_f as a d-fold cover of Δ^{n-1} , which is branched over the zero locus of a holomorphic function $\mathscr{D} \in \mathcal{O}(\Delta^{n-1})$.*

Proof. This was proved above, we just need to note that the set where $w \mapsto f(z,w)$ has multiple roots is given by the vanishing of the discriminant $\mathscr{D}$, which is a polynomial in the roots. □

We also can prove another removable singularity theorem which allows for a much larger singular set.

Theorem 10.7 (Riemann Extension Theorem)**.** *Let f be holomorphic in Ω, and let Z_f denote the zero set of f. Let g be holomorphic in $\Omega \setminus Z_f$ and bounded. Then g extends to a holomorphic function $G \in \mathcal{O}(\Omega)$.*

Proof. This is clearly local, so we can assume that f is a Weierstrass polynomial in a small polydisc. Recall above, we chose $r, \epsilon > 0$ so that for $|z| < \epsilon$ and $|w| = r$, f has no zeros, so g is holomorphic there. Also, for $|z| < \epsilon$, $w \mapsto f(z, w)$ only has finite many zeroes which are isolated. Since g is bounded, for $|z| < \epsilon$, the 1 variable function $w \mapsto g(z, w)$ has removable singularities. So we can extend to $G(z, w)$. By the Cauchy integral formula, we have

$$G(z, w) = \frac{1}{2\pi\sqrt{-1}} \int_{|u|=r} \frac{g(z, u)du}{u - w} \tag{10.19}$$

is continuous differentiable and holomorphic in the z variables, so is the required extension of g. □

11 Lecture 11

11.1 Complex differential forms

Using the coordinates

$$(z^1, \dots, z^n) = (x^1 + iy^1, \dots, x^n + iy^n), \tag{11.1}$$

recall the definitions

$$\frac{\partial}{\partial z^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big) \tag{11.2}$$

$$\frac{\partial}{\partial \bar{z}^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big), \tag{11.3}$$

which are differential operators on complex-valued functions. We define

$$T_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{\partial/\partial x^1, \dots, \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\} \tag{11.4}$$

which is a real $2n$-dimensional vector space, and define vector spaces

$$T^{1,0} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial z^j, j = 1 \dots n\} \tag{11.5}$$

$$T^{0,1} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial \bar{z}^j, j = 1 \dots n\} \tag{11.6}$$

$$T_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}, \tag{11.7}$$

which are complex vector spaces of complex dimension n, n, and $2n$, respectively. Note that $T_{\mathbb{C}} = T_{\mathbb{R}} \otimes \mathbb{C}$. Next define

$$\Lambda^1_{\mathbb{R}} = Hom(T_{\mathbb{R}}, \mathbb{R}), \tag{11.8}$$

to be the dual vector space to $T_{\mathbb{R}}$. We can write

$$\Lambda^1_{\mathbb{R}} = \text{span}\{dx^1, \dots, dx^n, dy^1, \dots, dy^n\}, \tag{11.9}$$

where the above is the dual basis to $\partial/\partial x^1, \dots, \partial/\partial y^n$. Next, define

$$\Lambda^{1,0} = Hom(T^{1,0}, \mathbb{C}) \tag{11.10}$$
$$\Lambda^{0,1} = Hom(T^{0,1}, \mathbb{C}) \tag{11.11}$$
$$\Lambda^1_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}. \tag{11.12}$$

Similarly, we have $\Lambda^1_{\mathbb{C}} = \Lambda^1_{\mathbb{R}} \otimes \mathbb{C}$. Then we can write

$$\Lambda^{1,0} = \text{span}\{dz^j, j = 1 \dots n\} \tag{11.13}$$
$$\Lambda^{0,1} = \text{span}\{d\bar{z}^j, j = 1 \dots n\}, \tag{11.14}$$

where dz^j is the dual basis to $\partial/\partial z^j$ and $d\overline{z}^j$ is the dual basis to $\partial/\partial z^j$, that is,

$$\begin{aligned} dz^j(\partial/\partial z^k) = d\bar{z}^j(\partial/\partial\bar{z}^k) = \delta^{jk}, \\ dz^j(\partial/\partial\bar{z}^k) = d\bar{z}^j(\partial/\partial z^k) = 0. \end{aligned} \tag{11.15}$$

Define $\Lambda^k_{\mathbb{C}}$ to be the vector space

$$\Lambda^k_{\mathbb{C}} = \text{span}\{dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \dots \wedge d\overline{z}^{j_q} \mid i_1 < \dots < i_p, j_1 < \dots < j_q, p + q = k\} \tag{11.16}$$

We can then extend this notation to all indices by

$$dz^{i_1} \wedge dz^{i_2} = -dz^{i_2} \wedge dz^{i_1} \tag{11.17}$$
$$d\overline{z}^{i_1} \wedge d\overline{z}^{i_2} = -d\overline{z}^{i_2} \wedge d\overline{z}^{i_1} \tag{11.18}$$
$$dz^i \wedge d\overline{z}^j = -d\overline{z}^j \wedge dz^i. \tag{11.19}$$

This extends to a product called the wedge product:

$$\wedge : \Lambda^{k_1}_{\mathbb{C}} \oplus \Lambda^{k_2}_{\mathbb{C}} \to \Lambda^{k_1+k_2}_{\mathbb{C}} \tag{11.20}$$

which is bilinear, associative, and satisfies

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \ \alpha \in \Lambda^p_{\mathbb{C}}, \ \beta \in \Lambda^q_{\mathbb{C}}. \tag{11.21}$$

Remark 11.1. Notice we can do a similar construction and define $\Lambda^k_{\mathbb{R}}$ which will satisfy $\Lambda^k_{\mathbb{C}} = \Lambda^k_{\mathbb{R}} \otimes \mathbb{C}$.

We define $\Lambda^{p,q} \subset \Lambda^{p+q}_{\mathbb{C}}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{11.22}$$

Noting that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}, \tag{11.23}$$

we have

$$\binom{2n}{k} = \sum_{p+q=k} \binom{n}{p} \cdot \binom{n}{q}. \tag{11.24}$$

Next, suppose we are given a domain $U \subset \mathbb{C}^n$. Let

$$C^\infty_{\mathbb{C}}(U) = \{f : U \to \mathbb{C} \mid f \text{ is smooth}\}. \tag{11.25}$$

$$\mathcal{E}^k_{\mathbb{C}}(U) = C^\infty_{\mathbb{C}}(U) \otimes \Lambda^k_{\mathbb{C}} \tag{11.26}$$

$$\mathcal{E}^{p,q}(U) = C^\infty_{\mathbb{C}}(U) \otimes \Lambda^{p,q}. \tag{11.27}$$

So we have that

$$\mathcal{E}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{E}^{p,q}. \tag{11.28}$$

If $\alpha \in \mathcal{E}^{p,q}(U)$, then we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{11.29}$$

where I and J are multi-indices of length p and q, respectively, and $\alpha_{I,J} : U \to \mathbb{C}$ are smooth complex-valued functions.

The wedge product extends to

$$\wedge : \mathcal{E}^{k_1}_{\mathbb{C}} \oplus \mathcal{E}^{k_2}_{\mathbb{C}} \to \mathcal{E}^{k_1+k_2}_{\mathbb{C}} \tag{11.30}$$

which is bilinear, associative, and satisfies

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \;\; \alpha \in \mathcal{E}^p_{\mathbb{C}}, \;\; \beta \in \mathcal{E}^q_{\mathbb{C}}. \tag{11.31}$$

Remark 11.2. We can do a similar construction and define $C^\infty_{\mathbb{R}}(U)$ and $\mathcal{E}^k_{\mathbb{R}}$, which will satisfy $\mathcal{E}^k_{\mathbb{C}} = \mathcal{E}^k_{\mathbb{R}} \otimes \mathbb{C}$.

11.2 The operators ∂ and $\overline{\partial}$ in $\mathbb{C}^n$

We first define the exterior derivative operator

$$d_{\mathbb{R}} : \mathcal{E}^k_{\mathbb{R}} \to \Omega^{k+1}_{\mathbb{R}} \tag{11.32}$$

by the following. For $f \in \mathcal{E}^k_{\mathbb{R}} = C^\infty_{\mathbb{R}}(U)$, define

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i + \sum_j \frac{\partial f}{\partial j^j} dy^j. \tag{11.33}$$

Then we extend to any differential forms of any degree by

$$d(f_{IJ} dx^I \wedge dy^J) = (df_{IJ}) \wedge dx^I \wedge dy^J. \tag{11.34}$$

Proposition 11.3. *The operator $d_{\mathbb{R}}$ satisfies*

$$d^2 = 0 \tag{11.35}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta. \tag{11.36}$$

Proof. We leave this as an exercise. □

By complexification, this operator extends to

$$d_{\mathbb{C}} : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{11.37}$$

The following is a key proposition.

Proposition 11.4. *We have*

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}. \tag{11.38}$$

Proof. We simply check on functions that the following formula holds

$$df = \sum_k \frac{\partial f}{\partial z^k} dz^k + \sum_k \frac{\partial f}{\partial \overline{z}^k} d\overline{z}^k. \tag{11.39}$$

If $\alpha \in \mathcal{E}^{p,q}(U)$, then

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{11.40}$$

so applying d to (11.40), we obtain

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \Big) \wedge dz^I \wedge d\overline{z}^J, \tag{11.41}$$

and we are done. □

We can therefore define operators

$$\partial : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{11.42}$$

$$\overline{\partial} : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{11.43}$$

by

$$\partial\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J \tag{11.44}$$

$$\overline{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J. \tag{11.45}$$

Using (17.1) and we have

$$\partial|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{11.46}$$

$$\overline{\partial}|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{11.47}$$

Corollary 11.5. *We have $d = \partial + \overline{\partial}$ for operators*

$$\partial : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q} \tag{11.48}$$
$$\overline{\partial} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}, \tag{11.49}$$

which satisfy

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{11.50}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{11.51}$$

If we plug in a form of type (p, q) the first term is of type $(p+2, q)$, the middle terms are of type $(p+1, q+1)$, and the last term is of type $(p, q+2)$. Since (17.1) is a direct sum, the claim follows. □

12 Lecture 12

12.1 De Rham cohomology

We can make the following definition since $d^2 = 0$.

Definition 12.1. Let $U \subset \mathbb{R}^n$ be a domain. For $0 \leq k \leq n$, the kth de Rham cohomology group is

$$H^k_{dR}(U) = \frac{\{\alpha \in \mathcal{E}^k(U) | d\alpha = 0\}}{d(\mathcal{E}^k(U))}, \tag{12.1}$$

Let $f : U \to V$ be a smooth mapping, where U and V are open subset of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then we can define the pullback of differential forms

$$f^* : \mathcal{E}^k(V) \to \mathcal{E}^k(U) \tag{12.2}$$

by

$$f^*\Big(\sum_{|I|=k} \alpha_I dy^I\Big) = \sum_{|I|=k} (\alpha_I \circ f) d(y^I \circ f) \tag{12.3}$$

Exercise 12.2. If $f : U \to V$ is smooth, then

$$d_U \circ f^* = f^* \circ d_V \tag{12.4}$$

and if $g : V \to W$ is smooth, then

$$(g \circ f)^* = f^* \circ g^*. \tag{12.5}$$

The de Rham cohomology groups enjoy the following functorality properties.

Proposition 12.3. *Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n, W \subset \mathbb{R}^l$ be domains. Let $f : U \to V$ be a smooth mapping. Then there are induced mappings*

$$f^* : H^k_{dR}(V) \to H^k_{dR}(U). \tag{12.6}$$

If $g : V \to W$ is smooth then so is $g \circ f : U \to W$ and

$$(g \circ f)^* = f^* \circ g^* : H^k_{dR}(W) \to H^k_{dR}(U). \tag{12.7}$$

In particular, if f is a diffeomorphism (one-to-one, onto, with smooth inverse), then the de Rham cohomologies of U and V are isomorphic.

Proof. We show that f^* induces a well-defined mapping on cohomology $f^* : H^k_{dR}(V) \to H^k_{dR}(U)$ by the following. If $[\alpha^k] \in H^k_{dR}(V)$ is represented by a form α^k, such that $d_V\alpha^k = 0$, then we have

$$d_U f^*\alpha^k = f^* d_V \alpha^k = f^*0 = 0, \tag{12.8}$$

so we can define $f^*[\alpha^k] = [f^*\alpha^k]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^k + d_V\beta^{k-1}) = f^*\alpha^k + f^*d_V\beta^{k-1} = f^*\alpha^k + d_U f^*\beta^{k-1}, \tag{12.9}$$

so we have

$$[f^*(\alpha^k + d_V\beta^{k-1})] = [f^*\alpha^k + d_U f^*\beta^{k-1}] = [f^*\alpha^k]. \tag{12.10}$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \tag{12.11}$$

holds on the level of forms. Finally, if f is a diffeomorphism, then f^{-1} exists and is smooth, so we have

$$f \circ f^{-1} = id_V, \quad f^{-1} \circ f = id_U, \tag{12.12}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^k_{dR}(U)}, \quad (f^{-1})^* \circ f^* = id_{H^k_{dR}(V)}, \tag{12.13}$$

□

12.2 Dolbeault cohomology

We can make the following definition since $\overline{\partial}^2 = 0$.

Definition 12.4. Let $U \subset \mathbb{C}^n$ be a domain. For $0 \leq p, q \leq n$, the (p, q) Dolbeault cohomology group is

$$H^{p,q}_{\overline{\partial}}(U) = \frac{\{\alpha \in \mathcal{E}^{p,q}(U) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\mathcal{E}^{p,q-1}(U))}. \tag{12.14}$$

Next we define holomorphic mappings.

Definition 12.5. Let $f : U \to V$ be a C^1 mapping, where U and V are open subset of complex Euclidean spaces $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then f is *holomorphic* if writing $f = (f^1, \ldots, f^m)$, then $f^j : U \to \mathbb{C}$ is holomorphic for $j = 1, \ldots, m$.

Proposition 12.6. *If $f : U \to V$ is holomorphic, then f^* preserves the (p,q)-type decomposition differential forms, i.e.,*

$$f^* : \mathcal{E}^{p,q}(V) \to \mathcal{E}^{p,q}(U). \tag{12.15}$$

Proof. To see this, let $\alpha^{p,q} \in \mathcal{E}^{p,q}(V)$, of the form

$$\alpha^{p,q} = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz^I \wedge d\overline{z}^J, \tag{12.16}$$

Then by definition

$$f^*\alpha^{p,q} = \sum_{|I|=p,|J|=q} (\alpha_{IJ} \circ f) d(z^I \circ f) \wedge d(z^J \circ f). \tag{12.17}$$

But

$$d(z^j \circ f) = (\partial + \overline{\partial})(z^j \circ f) = \partial(z^j \circ f), \tag{12.18}$$

since $z^j \circ f$ is holomorphic, so this is a form of type $(1,0)$ on U. Similarly,

$$d(\overline{z}^j \circ f) = (\partial + \overline{\partial})(\overline{z^j \circ f}) = \overline{\partial}(\overline{z^j \circ f}), \tag{12.19}$$

since $\overline{z^j \circ f}$ is anti-holomorphic, so this is a form of type $(0,1)$ on U, and we are done. □

The Dolbeault cohomology groups enjoy the following functorality properties.

Proposition 12.7. *Let $U \subset \mathbb{C}^m, V \subset \mathbb{C}^n, W \subset \mathbb{C}^l$ be domains. Let $f : U \to V$ be holomorphic. Then there are induced mappings*

$$f^* : H^{p,q}(V) \to H^{p,q}(U). \tag{12.20}$$

If $g : V \to W$ is holomorphic, then so is $g \circ f : U \to W$ and

$$(g \circ f)^* = f^* \circ g^* : H^{p,q}(W) \to H^{p,q}(U). \tag{12.21}$$

In particular, if f is a biholomorphism (one-to-one, onto, with holomorphic inverse), then the Dolbeault cohomologies of U and V are isomorphic.

Proof. We know that the exterior derivative commutes with pullback,

$$d_U \circ f^* = f^* \circ d_V. \tag{12.22}$$

This is equivalent to

$$(\partial_U + \overline{\partial}_U) \circ f^* = f^* \circ (\partial_V + \overline{\partial}_V) \tag{12.23}$$

If we plug in $\alpha^{p,q} \in \Omega^{p,q}(V)$, we have 2 equations

$$\partial_U \circ f^* \alpha^{p,q} = f^* \circ \partial_V \alpha^{p,q} \tag{12.24}$$
$$\overline{\partial}_U \circ f^* \alpha^{p,q} = f^* \circ \overline{\partial}_V \alpha^{p,q} \tag{12.25}$$

The second equation implies that f^* induces a well-defined mapping on cohomology $f^* : H^{p,q}(V) \to H^{p,q}(U)$ by the following. If $[\alpha^{p,q}] \in H^{p,q}(V)$ is represented by a form $\alpha^{p,q}$, such that $\overline{\partial}_V \alpha^{p,q} = 0$, then we have

$$\overline{\partial}_U f^* \alpha^{p,q} = f^* \overline{\partial}_V \alpha^{p,q} = f^* 0 = 0, \tag{12.26}$$

so we can define $f^*[\alpha^{p,q}] = [f^*\alpha^{p,q}]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^{p,q} + \overline{\partial}_V \beta^{p,q-1}) = f^*\alpha^{p,q} + f^*\overline{\partial}_V \beta^{p,q-1} = f^*\alpha^{p,q} + \overline{\partial}_U f^* \beta^{p,q-1}, \tag{12.27}$$

so we have

$$[f^*(\alpha^{p,q} + \overline{\partial}_V \beta^{p,q-1})] = [f^*\alpha^{p,q} + \overline{\partial}_U f^* \beta^{p,q-1}] = [f^*\alpha^{p,q}]. \tag{12.28}$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \tag{12.29}$$

holds on the level of forms. Finally, if f is a biholomorphism, then f^{-1} exists and is holomorphic, so we have

$$f \circ f^{-1} = id_V, \quad f^{-1} \circ f = id_U, \tag{12.30}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^{p,q}(U)}, \quad (f^{-1})^* \circ f^* = id_{H^{p,q}(V)}, \tag{12.31}$$

□

Definition 12.8. A form $\alpha \in \mathcal{E}^{p,0}(U)$ is *holomorphic* if $\overline{\partial}\alpha = 0$, and we write that $\alpha \in \Omega^p(U)$.

Remark 12.9. We only talk about forms of type $(p, 0)$ being holomorphic, we never call a (p, q)-form holomorphic if $q > 0$. Also, we have (trivially)

$$\Omega^p(U) = H^{p,0}(U) = \{\alpha \in \mathcal{E}^{p,0}(U) \mid \overline{\partial}\alpha = 0\}. \tag{12.32}$$

Proposition 12.10. *A p-form $\alpha \in \mathcal{E}^{p,0}(U)$ is holomorphic if and only if it can be written as*

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{12.33}$$

where the $\alpha_I : U \to \mathbb{C}$ are holomorphic functions.

Proof. We have

$$\overline{\partial}\alpha = \sum_{|I|=p,k} \frac{\partial \alpha_I}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I. \tag{12.34}$$

So $\overline{\partial}\alpha = 0$ if and only if the α_I are holomorphic. □

Remark 12.11. So for $U \subset \mathbb{C}^n$ a domain, $\dim_{\mathbb{C}} H^{p,0}(U) = \infty$ is always infinite-dimensional for $0 \leq p \leq n$, in particular because any polynomial function in the z-variables is holomorphic.

Example 12.12. Let's review the case of a domain $U \subset \mathbb{C}$. First, $H^{0,0}_{\overline{\partial}}(U) = \mathcal{O}(U)$. Theorem 5.3 shows that $H^{0,1}_{\overline{\partial}}(U) = \{0\}$. The space $H^{1,0}_{\overline{\partial}}(U)$ consists of holomorphic 1-forms, but since $n = 1$, any holomorphic 1-form is of the form $f(z)dz$, where $f \in \mathcal{O}(U)$. So $H^{1,0}_{\overline{\partial}}(U) \cong \mathcal{O}(U)$. Finally,

$$H^{1,1}_{\overline{\partial}}(U) = \frac{\operatorname{Ker} \overline{\partial} : \Omega^{1,1} \to \Omega^{1,2}}{\operatorname{Image} \overline{\partial} : \Omega^{1,0} \to \Omega^{1,1}} = \frac{g dz \wedge d\overline{z}}{(\partial f / \partial \overline{z}) d\overline{z} \wedge dz} = \{0\}, \tag{12.35}$$

which also follows from Theorem 5.3.

13 Lecture 13

13.1 Jacobians

Let $f : \mathbb{C}^n \to \mathbb{C}^m$. Let the coordinates on $\mathbb{C}^n$ be given by

$$\{z^1, \dots z^n\} = \{x^1 + iy^1, \dots, x^n + iy^n\}, \tag{13.1}$$

and coordinates on $\mathbb{C}^m$ given by

$$\{w^1, \dots w^m\} = \{u^1 + iv^1, \dots, u^m + iv^m\} \tag{13.2}$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \operatorname{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{13.3}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \operatorname{span}\{\partial/\partial u^1, \dots \partial/\partial u^m, \partial/\partial v^1, \dots \partial/\partial v^m\}. \tag{13.4}$$

Then the real Jacobian of

$$f = (f^1, \dots f^{2m}) = (u^1 \circ f, u^2 \circ f, \dots, v^{2m} \circ f). \tag{13.5}$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \dots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \dots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \dots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \tag{13.6}$$

We define

$$J_{0,\mathbb{C}^n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{13.7}$$

Note that $J_{0,\mathbb{C}^n}^2 = -Id$. We have the following characterization of holomorphic mappings.

Proposition 13.1. *A C^1 mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if and only if*

$$f_* \circ J_{0,\mathbb{C}^n} = J_{0,\mathbb{C}^m} \circ f_*. \tag{13.8}$$

That is, the differential of f commutes with J_0.

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{13.9}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{13.10}$$

which is exactly the Cauchy-Riemann equations. The argument in general is similar, and is left as an exercise. □

For any differentiable f, the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^m). \tag{13.11}$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \text{span}\{\partial/\partial z^1, \dots \partial/\partial z^n, \partial/\partial \overline{z}^1, \dots \partial/\partial \overline{z}^n\}, \tag{13.12}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial w^1, \dots \partial/\partial w^m, \partial/\partial \overline{w}^1, \dots \partial/\partial \overline{w}^m\}. \tag{13.13}$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \cdots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \overline{z}^1} & \cdots & \frac{\partial f^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \cdots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \overline{z}^1} & \cdots & \frac{\partial f^m}{\partial \overline{z}^n} \\ \frac{\partial \overline{f}^1}{\partial z^1} & \cdots & \frac{\partial \overline{f}^1}{\partial z^n} & \frac{\partial \overline{f}^1}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \overline{f}^m}{\partial z^1} & \cdots & \frac{\partial \overline{f}^m}{\partial z^n} & \frac{\partial \overline{f}^m}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^m}{\partial \overline{z}^n}, \end{pmatrix} \tag{13.14}$$

where $(f^1, \dots, f^m) = f$ now denotes the complex components of f. This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{13.15}$$

Notice that (13.14) is of the form

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{13.16}$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{13.17}$$

where $\mathbb{C}$-linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \tag{13.18}$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{13.19}$$

This corresponds to the embeddings

$$Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{13.20}$$

where the left $\mathbb{C}$-linear is with respect to J_0, via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{13.21}$$

Note that since the latter embedding is just a change of basis, if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB) \det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{13.22}$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{1,0}) \subset T^{1,0}. \tag{13.23}$$

13.2 Holomorphic inverse function theorem

Proposition 13.2 (Holomorphic inverse function theorem)**.** *Let $f : U \to V$ be holomorphic where U and V are open subsets of $\mathbb{C}^n$. Let $z_0 \in U$ satisfy*

$$\det\left(\frac{\partial f^j}{\partial z^k}\right)(z_0) \neq 0. \tag{13.24}$$

Then there exists neighborhoods U' of z_0 and V' of $f(z_0)$ such that $f : U' \to V'$ is a biholomorphism.

Proof. By the above, since f is holomorphic, we have

$$\det(\mathcal{J}_{\mathbb{R},z_0}) = \left| \det \Big(\frac{\partial f^j}{\partial z^k} \Big)(z_0) \right|^2 \neq 0. \tag{13.25}$$

By the (real) smooth inverse function theorem, there exists a smooth inverse $f^{-1} : V' \to U'$, for some neighborhoods. We need to check that f^{-1} is holomorphic. For this, we differentiate the equation $z = f^{-1}(f(z))$ using the complex chain rule to get

$$0 = \frac{\partial}{\partial \overline{z}^j} f^{-1} \circ f(z) = \frac{\partial f^{-1}}{\partial w^k} \frac{\partial f^k}{\partial \overline{z}^j} + \frac{\partial f^{-1}}{\partial \overline{w}^k} \frac{\partial \overline{f}^k}{\partial \overline{z}^j} = \frac{\partial f^{-1}}{\partial \overline{w}^k} \overline{\Big(\frac{\partial f^k}{\partial z^j} \Big)} \tag{13.26}$$

The latter matrix has non-zero determinant in a neighborhood of z_0, from which we conclude that f^{-1} is holomorphic. □

Corollary 13.3. *Let $f : U \to \mathbb{C}$ be holomorphic, and $w \in \mathbb{C}$ be a complex regular value. That is,*

$$\nabla_{\mathbb{C}} f = \Big(\frac{\partial f}{\partial z^1}, \dots, \frac{\partial f}{\partial z^n} \Big)(z) \neq 0 \tag{13.27}$$

for all $z \in f^{-1}(w)$. Then $f^{-1}(w)$ is a complex submanifold of U of complex codimension 1. That is, near any point $z \in f^{-1}(w)$, there exists a neighborhood U' of z and a holomorphic mapping $\Psi : V' \to U'$, where V' is a neighbohood of the origin in $\mathbb{C}^n$, such that

$$f \circ \Psi = z_n \tag{13.28}$$

and therefore $(f \circ \Psi)^{-1}(w)$ is locally a hyperplane.

Proof. This is an application of the inverse function theorem, the proof is left as an exercise. □

Exercise 13.4. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be given by

$$f(z_1, z_2) = z_1^2 + z_2^3 + z_1 z_2. \tag{13.29}$$

Determine the set of $w \in \mathbb{C}$ such that $f^{-1}(w)$ is a submanifold.

Exercise 13.5. Generalize Corollary 13.3 to the case of holomorphic $f : U \to \mathbb{C}^m$ for $m > 1$.

13.3 Anti-holomorphic mappings

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0, then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{13.30}$$

This corresponds to the embeddings

$$Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \tag{13.31}$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \tag{13.32}$$

We see that f is anti-holomorphic if and only if

$$f_*(T^{1,0}) \subset T^{0,1}. \tag{13.33}$$

For an arbitary mapping, not necessarily holomorphic or anti-holomorphic, we can decompose $f_* = f_*^C + f_*^A$, where

$$f_*^C = \frac{1}{2}\left(f_* - J_0 f_* J_0\right) \tag{13.34}$$

$$f_*^A = \frac{1}{2}\left(f_* + J_0 f_* J_0\right), \tag{13.35}$$

and f_*^C is holomorphic, while f_*^A is anti-holomorphic. In block matrix form, this just says that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \tag{13.36}$$

13.4 Almost complex structures

Recall that

$$T_{\mathbb{R}}(\mathbb{C}^n) = \text{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{13.37}$$

Above, we defined

$$J_0 : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^n) \tag{13.38}$$

by

$$J_{0,\mathbb{C}^n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{13.39}$$

which satisfies $J_{0,\mathbb{C}^n}^2 = -Id$. This mapping is called an *almost complex structure*. Using this mapping, we have some nice descriptions of the various spaces we previously introduced. Upon complexification, we have the decomposition.

$$T_{\mathbb{R}}(\mathbb{C}^n) \otimes \mathbb{C} = T_{\mathbb{C}}(\mathbb{C}^n) = T^{1,0} \oplus T^{0,1}. \tag{13.40}$$

We extend J_0 to a mapping

$$J_0 : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^n) \tag{13.41}$$

by complex linearity. Using J_0, we can describe the above decomposition as follows:

Proposition 13.6. *We have*

$$T^{1,0} = span\{\partial/\partial z^j, j = 1 \dots n\} = \{X - iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{13.42}$$

is the i-eigenspace of J_0 and

$$T^{0,1} = span\{\partial/\partial \bar{z}^j, j = 1 \dots n\} = \{X + iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{13.43}$$

is the $-i$-eigenspace of J_0.

Proof. We leave as an easy exercise. □

Next, recall that

$$\Lambda^1_{\mathbb{R}} = Hom(T_{\mathbb{R}}, \mathbb{R}) = \text{span}\{dx^1, \dots, dx^n, dy^1, \dots, dy^n\}, \tag{13.44}$$

is the dual vector space to $T_{\mathbb{R}}$. Upon complexification, we have a decomposition

$$\Lambda^1_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}. \tag{13.45}$$

The map J_0 also induces an endomorphism of 1-forms

$$J_0^T : \Lambda^1_{\mathbb{R}} \to \Lambda^1_{\mathbb{R}} \tag{13.46}$$

by

$$J_0^T(\omega)(v_1) = \omega(J_0 v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0^T(dx_j) = -dy_j, \quad J_0^T(dy_j) = +dx_j,$$

that is

$$J^T_{0,\mathbb{C}^n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{13.47}$$

which satisfies $(J_0^T)^2 = -Id$. We extend J_0^T to a mapping

$$J_0^T : \Lambda^1_{\mathbb{C}} \to \Lambda^1_{\mathbb{C}} \tag{13.48}$$

by complex linearity. Using J_0^T, we can describe the above decomposition as follows.

Proposition 13.7. *We have*

$$\Lambda^{1,0} = Hom(T^{1,0}, \mathbb{C}) = span\{dz^j, j = 1 \dots n\} = \{\alpha - iJ_0^T\alpha, \alpha \in \Lambda^1_{\mathbb{R}}(\mathbb{R}^{2n})\} \tag{13.49}$$

is the i-eigenspace of J_0^T, and

$$\Lambda^{0,1} = Hom(T^{0,1}, \mathbb{C}) = span\{d\bar{z}^j, j = 1 \dots n\} = \{\alpha + iJ_0^T\alpha, \alpha \in \Lambda^1_{\mathbb{R}}(\mathbb{R}^{2n})\} \tag{13.50}$$

is the $-i$-eigenspace of J_0^T.

Proof. This is left as an exercise. □

We also defined

$$\Lambda^k_{\mathbb{C}} = \Lambda^k_{\mathbb{R}} \otimes \mathbb{C} \tag{13.51}$$

and proved that there is a decomposition

$$\Lambda^k_{\mathbb{C}} = \Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{13.52}$$

with

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{13.53}$$

Finally, we can define

$$J_0^T : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C} \tag{13.54}$$

by letting

$$J_0^T(\alpha) = i^{p-q}\alpha, \tag{13.55}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J_0^T is not an almost complex structure on the space $\Lambda^k_{\mathbb{C}}$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J-invariant.

14 Lecture 14

14.1 The $\overline{\partial}$-equation for $(0, 1)$-forms and Hartogs' Theorem

A reference for this section is [HL84, Section 1.2]. For $n \geq 2$, and $g \in \mathcal{E}^{0,1}(U)$, the equation $\overline{\partial} f = g$ is not always solvable. This follows from (17.14): applying $\overline{\partial}$ yields a compatibility condition $\overline{\partial} g = 0$. The following is in sharp contrast to the case $n = 1$.

Proposition 14.1. *Let $n \geq 2$ and $g \in \mathcal{E}^{0,1}_0(\mathbb{C}^n)$ (compact support) have C^∞ regularity and satisfy $\overline{\partial} g = 0$. Then there exists a smooth $f \in C^\infty_0(\mathbb{C}^n)$ (also having compact support) with $\overline{\partial} f = g$. Furthermore, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus supp(g)$.*

Proof. We write $g = \sum_{j=1}^n g_j d\overline{z}^j$. Define

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(w, z_2, \ldots, z_n)}{w - z_1} dw \wedge d\overline{w}. \tag{14.1}$$

The integral is defined since g_1 has compact support. Make the change of variable $\xi = w - z_1$, and we can write f as

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\xi + z_1, z_2, \ldots, z_n)}{\xi} d\xi \wedge d\bar{\xi}. \tag{14.2}$$

This shows that we can differentiate under the integral sign to conclude that f has C^∞ regularity. Furthermore,

$$\begin{aligned}\frac{\partial f(z_1,\dots,z_n)}{\partial \overline{z}^1} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(\xi+z_1,z_2,\dots,z_n)}{\partial \overline{z}^1}\frac{1}{\xi} d\xi\wedge d\bar{\xi} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(\xi+z_1,z_2,\dots,z_n)}{\partial \bar{\xi}}\frac{1}{\xi} d\xi\wedge d\bar{\xi} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(w,z_2,\dots,z_n)}{\partial \overline{w}}\frac{1}{w-z_1} dw\wedge d\overline{w} = g_1(z_1,\dots,z_n),\end{aligned} \tag{14.3}$$

by the Cauchy-Pompieu formula applied to a large ball containing the support of g. The condition that $\overline{\partial} g = 0$ means that

$$0 = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial g_j}{\partial \overline{z}_k} d\overline{z}^k \wedge d\overline{z}^j, \tag{14.4}$$

so

$$\frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_j} \tag{14.5}$$

for all $1 \leq j,k \leq n$. Then differentiating (14.2) for $j \geq 2$, we obtain

$$\begin{aligned}\frac{\partial f(z_1,\dots,z_n)}{\partial \overline{z}_j} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_1(\xi+z_1,z_2,\dots,z_n)}{\partial \overline{z}^j}\frac{1}{\xi} d\xi\wedge d\bar{\xi} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_j(\xi+z_1,z_2,\dots,z_n)}{\partial \overline{z}^1}\frac{1}{\xi} d\xi\wedge d\bar{\xi} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g_j(w,z_2,\dots,z_n)}{\partial \overline{w}}\frac{1}{w-z_1} dw\wedge d\overline{w} = g_j(z_1,\dots,z_n).\end{aligned} \tag{14.6}$$

So the equation $\overline{\partial} f = g$ is satisfied everywhere. Finally, since g has compact support, it follows that f is holomorphic on the complement of a large ball $B_r(0)$ containing the support of g. But (14.1) shows that f vanishes when $\max\{|z_2|,\dots,|z_n|\} > r$ for some r sufficiently large. Therefore f is a holomorphic function on $\mathbb{C}^n \setminus \overline{B_r(0)}$ which vanishes on the open subset $V = \{\max\{|z_2|,\dots,|z_n|\} > r\}$. By unique continuation, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus \mathrm{supp}(g)$. □

Theorem 14.2 (Hartogs)**.** *Let $n \geq 2$, U a domain, and $K \subset U$ a compact subset of U such that $U \setminus K$ is connected. Then if $u \in \mathcal{O}(U \setminus K)$, there exists $\tilde{u} \in \mathcal{O}(U)$ with $\tilde{u}|_{U\setminus K} = u$.*

Proof. Let $0 \leq \chi \in C_0^\infty(U)$ and $\chi \equiv 1$ on K. Define $g = \overline{\partial}(\chi \cdot u) = \overline{\partial}((\chi - 1)\cdot u)$. Since

$$\overline{\partial}(\chi u) = u\overline{\partial}(\chi) + \chi\overline{\partial}(u) = u\overline{\partial}(\chi) + 0, \tag{14.7}$$

we see that g extends smoothly to U, that is, $g \in \mathcal{E}_0^{0,1}(U)$, and $\overline{\partial} g = 0$. By Proposition 14.1, there exists $f \in C_0^\infty(\mathbb{C}^n)$ with $\overline{\partial} f = g$. So then we let $\tilde{u} = (1-\chi)u + f$. This satisfies

$$\overline{\partial}\tilde{f} = -g + \overline{\partial}(f) = 0, \tag{14.8}$$

so $\tilde{u} \in \mathcal{O}(U)$. Let V denote the unbounded component of the complement of the support of χ. Since $\text{supp}(g) \subset \text{supp}(\chi)$, from Proposition 14.1, we have that $f \equiv 0$ in V, so $\tilde{u} = u$ in $U \cap V$. But since $U \setminus K$ is connected and $V \cap (U \setminus K) \neq \emptyset$, we have $\tilde{u} = u$ in $U \setminus K$ from unique continuation. □

Example 14.3. This gives another proof that point singularities are removable for $n \geq 2$. Also, polydics are removable: if u is holomorphic on $\Delta \setminus \overline{\Delta'}$, where $\overline{\Delta'} \subset \Delta$ then u extends to a holomorphic function on Δ. We also see that the same is true for balls $\overline{B_{r_1}(0)} \subset B_{r_2}(0)$ with $r_1 < r_2$.

14.2 Dolbeault cohomology of a polydisc

Some references for this section are [GH78, Section 0.2] or [Nog16, Section 3.6].

Proposition 14.4. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), and $\omega \in \mathcal{E}^{p,q}(U)$ satisfies $\overline{\partial}\omega = 0$ for $q \geq 1$, then given any polyradius $s < r$, there exists $\eta \in \mathcal{E}^{p,q-1}(\Delta(r))$ with $\overline{\partial}\eta = \omega$ satisfied in $\Delta(s)$.*

Proof. Step 1: reduce to case of $\mathcal{E}^{0,q}$. If $\omega \in \mathcal{E}^{p,q}(U)$,

$$\omega = \sum_{|I|=p,|J|=q} \omega_{IJ} dz^I \wedge d\overline{z}^J. \tag{14.9}$$

Define

$$\omega_I = \sum_{|J|=q} \omega_{IJ} d\overline{z}^J. \tag{14.10}$$

Then $\omega_I \in \mathcal{E}^{0,q}$, and $\overline{\partial}\omega_I = 0$. If $\omega_I = \overline{\partial}\eta_I$, then

$$\overline{\partial}(dz^I \wedge \eta_I) = (-1)^p dz^I \wedge \overline{\partial}\eta_I = (-1)^p dz^I \wedge \omega_I, \tag{14.11}$$

so

$$\overline{\partial}(\sum_{|I|=p} dz^I \wedge \eta_I) = (-1)^p \sum_{|I|=p} dz^I \wedge \omega_I = (-1)^p \sum_{|I|=p} \sum_{|J|=q} \omega_{IJ} dz^I \wedge d\overline{z}^J, \tag{14.12}$$

and we are done with Step 1.

Step 2. Given $s < r$, if $\omega \in \mathcal{E}^{0,q}(\Delta(r))$ and $\overline{\partial}\omega = 0$ in $\Delta(r)$, then there exists $\eta \in \mathcal{E}^{0,q-1}(\Delta(r))$ with $\overline{\partial}\eta = \omega$ satisfied in $\Delta(s)$. Choose cutoff functions $0 \leq \chi_j(t) \leq 1$ so that

$$\chi_i(t) = \begin{cases} 1 & t \leq s_j \\ 0 & t \geq r_j \end{cases}. \tag{14.13}$$

We begin with $q = 1$. Note that $\omega \in \mathcal{E}^{0,1}(\Delta(r))$, but it does not have compact support, so we proceed differently than in the proof of Proposition 14.1. Write

$$\omega = \sum_k \omega_k d\overline{z}^k, \tag{14.14}$$

and define

$$\eta_1(z^1,\dots,z^n) = \frac{1}{2\pi i}\int_{|w^j|\le r_j} \frac{\chi_1(w_1)\omega_j(w^1,z^2,\dots,z^n)}{w^1-z^1} dw^1\wedge d\overline{w}^1. \tag{14.15}$$

Then $\partial\eta_1/\partial\overline{z}^1 = \chi_1\omega_1$, and we have

$$\overline{\partial}\eta_1 = \sum_l \frac{\partial\eta_1}{\partial\overline{z}^l} d\overline{z}^l = \chi_1\omega_1 d\overline{z}^1 + \sum_{j>1}\frac{\partial\eta_1}{\partial\overline{z}^j}d\overline{z}^j. \tag{14.16}$$

That is, we have solved the $d\overline{z}^1$-term, modulo terms involving $d\overline{z}^j$ for $j>1$ (we have not even used the fact that $\overline{\partial}\omega = 0$ yet!) Next, we consider the case

$$\omega = \sum_{k>1}\omega_k d\overline{z}^k, \tag{14.17}$$

Since $\overline{\partial}\omega = 0$, this tells us that $\partial\omega_2/\partial\overline{z}^1 = 0$. Next, we define

$$\eta_2(z^1,\dots,z^n) = \frac{1}{2\pi i}\int_{|w^2|\le r_2} \frac{\chi_2(w_2)\omega_2(z^1,w^2,z^3,\dots,z^n)}{w^2-z^2} dw^2\wedge d\overline{w}^2. \tag{14.18}$$

Then $\partial\eta_2/\partial\overline{z}^2 = \chi_2\omega_2$ and $\partial\eta_2/\partial\overline{z}^1 = 0$, so we have

$$\overline{\partial}\eta_2 = \sum_l \frac{\partial\eta_2}{\partial\overline{z}^j} d\overline{z}^j = \chi_2\omega_2 d\overline{z}^2 + \sum_{j>2}\frac{\partial\eta_2}{\partial\overline{z}^j}d\overline{z}^j. \tag{14.19}$$

Assume that we can solve all the terms involving $d\overline{z}^k$ for $k\le l$, and

$$\omega = \sum_{k>l}\omega_k d\overline{z}^k. \tag{14.20}$$

Since $\overline{\partial}\omega = 0$, this tells us that $\partial\omega_{l+1}/\partial\overline{z}^j = 0$ for $j\le l$. Then we define

$$\eta_{l+1}(z^1,\dots,z^n) = \frac{1}{2\pi i}\int_{|w^{l+1}|\le r_{l+1}} \frac{\chi_{l+1}(w_{l+1})\omega_{l+1}(z^1,\dots,w^{l+1},\dots,z^n)}{w^{l+1}-z^{l+1}} dw^{l+1}\wedge d\overline{w}^{l+1}. \tag{14.21}$$

Then $\partial\eta_{l+1}/\partial\overline{z}^{l+1} = \chi_{l+1}\omega_{l+1}$ and $\partial\eta_{l+1}/\partial\overline{z}^j = 0$ for $j\le l$, so we have

$$\overline{\partial}\eta_{l+1} = \sum_{j>l} \frac{\partial\eta_{l+1}}{\partial\overline{z}^j} d\overline{z}^j = \chi_{l+1}\omega_{l+1} d\overline{z}^{l+1} + \sum_{j>l+1}\frac{\partial\eta_{l+1}}{\partial\overline{z}^j}d\overline{z}^j. \tag{14.22}$$

By induction, we are done with the case of $q=1$.

Next, consider the case of $q=2$. Then

$$\omega = \sum_{1\le k<l}\omega_{kl} d\overline{z}^k\wedge d\overline{z}^l = \sum_{1<l}\omega_{1l}d\overline{z}^1\wedge d\overline{z}^l + \sum_{1<k<l}\omega_{kl}d\overline{z}^k\wedge d\overline{z}^l. \tag{14.23}$$

Define $\eta = \sum_{1<k} \eta_{1k} d\overline{z}^k$, where

$$\eta_{1k}(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|w^1| \leq r_1} \frac{\chi_1(w^1)\omega_{1k}(w^1, z^2, \ldots, z^n)}{w^1 - z^1} dw^1 \wedge d\overline{w}^1. \tag{14.24}$$

Then η_{1k} solves $\partial\eta_{1k}/\partial\overline{z}^1 = \chi_1\omega_{1k}$. So then

$$\overline{\partial}\eta = \sum_{1<k} \frac{\partial\eta_{1k}}{\partial\overline{z}^l} d\overline{z}^l \wedge d\overline{z}^k = \sum_{1<k} \chi_1\omega_{1k} d\overline{z}^1 \wedge d\overline{z}^k + R \tag{14.25}$$

where R doesn't include any $d\overline{z}^1$-s. So we have solved the terms in ω involving $d\overline{z}^1$-s. We next assume that ω is of the form

$$\omega = \sum_{1<k<l} \omega_{kl} d\overline{z}^k \wedge d\overline{z}^l = \sum_{2<l} \omega_{2l} d\overline{z}^2 \wedge d\overline{z}^l + \sum_{2<k<l} \omega_{kl} d\overline{z}^k \wedge d\overline{z}^l. \tag{14.26}$$

Let $\eta = \sum_{2<k} \eta_{2k} d\overline{z}^k$ where

$$\eta_{2k} = \frac{1}{2\pi i} \int_{|w^2| \leq r_2} \frac{\chi_2(w^2)\omega_{2k}(z^1, w^2, z^3, \ldots z^n)}{w^2 - z^2} dw^2 \wedge d\overline{w}^2. \tag{14.27}$$

Then $\partial\eta_{2k}/\partial\overline{z}^2 = \chi_2\omega_{2k}$. Furthermore, since $\overline{\partial}\omega = 0$, $\partial\eta_{2k}/\partial\overline{z}^1 = 0$. So then

$$\overline{\partial}\eta = \sum_{2<k} \overline{\partial}(\eta_{2k} d\overline{z}^k) = \sum_{2<k} \sum_{2\leq l} \frac{\partial\eta_{2k}}{\partial\overline{z}^l} d\overline{z}^l \wedge d\overline{z}^k = \sum_{2<k} \chi_2\omega_{2k} d\overline{z}^2 \wedge d\overline{z}^k + R, \tag{14.28}$$

where R only has terms $d\overline{z}^k \wedge d\overline{z}^l$ for $k, l \geq 3$. So we have solved as the term in ω having $d\overline{z}^1$-s or $d\overline{z}^2$-s. By a similar induction argument as in the $q = 1$ case, we can solve all terms in this manner. The case of $q > 2$ is similar, and details left as an exercise. □

We next upgrade this to have no shrinkage.

Theorem 14.5. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), then $H^{p,q}_{\overline{\partial}}(U) = \{0\}$ for $q \geq 1$.*

Proof. Choose a monotone increasing sequence of polyradii $r_1 < r_2 < \ldots$ with $\lim_{j\to\infty} r_j = r$. Given $\omega \in \mathcal{E}^{0,q}(\Delta(r)$, by Step 2, we can find $\eta_j \in \mathcal{E}^{0,q-1}(\Delta(r))$ with $\overline{\partial}\eta_j = \omega$ on $\Delta(r_j)$. We do not know that the sequence η_j will converge. However, $\overline{\partial}(\eta_{j+1} - \eta_j) = 0$ in $\Delta(r_j)$. If $q \geq 2$, then by Step 2, we can find $\beta_{j+1} \in \mathcal{E}^{0,q-2}(\Delta(r_j))$ solving $\overline{\partial}(\beta_{j+1}) = \eta_{j+1} - \eta_j$ in $\Delta(r_{j-1})$. We then consider the sequence $\eta'_{j+1} = \eta_{j+1} - \overline{\partial}(\beta_{j+1})$. Then $\eta'_{j+1} \in \mathcal{E}^{0,q-2}(\Delta(r_j)$ and

$$\overline{\partial}(\eta'_{j+1}) = \overline{\partial}\eta_{j+1} - \overline{\partial}^2(\beta_{j+1}) = \omega \tag{14.29}$$

in $\Delta(r_{j-1})$, and this new sequence now obviously converges to a solution $\eta \in \mathcal{E}^{0,q}(\Delta(r)$ with $\overline{\partial}\eta = \omega$ in $\Delta(r)$.

If $q = 1$, then we prove exactly like we did in the case of $n = 1$, by approximating the difference $\eta_{j+1} - \eta_j$ by a polynomial P_{j+1} to obtain a sequence so that

$$\sup_{z\in K} |\eta_{j+1}(z) - \eta_j(z)| < 2^{-j}, \tag{14.30}$$

and we obtain a sequence converging on compact subsets to a solution. □

Remark 14.6. Using Laurent series instead of polynomials, a similar proof works to prove that Theorem 14.5 also holds for products $\Delta^*(r_1) \times \cdots \times \Delta^*(r_k) \times \Delta(r_{k+1}) \times \cdots \Delta(r_{k+l})$, that is, we can allow punctured 1-dimensional disks. With a lot more work, one can also show that Theorem 14.5 holds for $\Omega_1 \times \cdots \times \Omega_n$ with $\Omega_j \subset \mathbb{C}$ are domains. Note the result is NOT true for a punctured polydisc $\Delta(0, r) \setminus \{0\}$ for $n \geq 2$, but we cannot prove that yet.

Remark 14.7. Theorem 14.5 also holds for a ball $B(0, r) \subset \mathbb{C}^n$. However, this is difficult to prove directly. One could use the Bochner-Martinelli kernel instead of the Cauchy kernel to prove Proposition 14.4. Then one would also need to prove that the $B(0, r)$ is a Runge domain, that is, $\mathcal{O}(B(0, r))$ can be approximated by holomorphic polynomials uniformly on compact subsets. However, it seems actually easier to prove this more generally for any pseudoconvex domain (using Hörmander's L^2 methods), and then show that $B(0, r)$ is pseudoconvex.

15 Lecture 15

15.1 Almost complex manifolds

Definition 15.1. An *almost complex manifold* is a real manifold with an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$.

The following lemma shows that we can always take J to be standard at any *point*.

Lemma 15.2. *Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear mapping satisfying $J^2 = -Id$. Then there exists an invertible matrix A such that $A^{-1}JA = J_{Euc}$.*

Proof. For $X \in \mathbb{R}^{2n}$, define

$$(a + ib)X = aX + bJX. \tag{15.1}$$

Then $\mathbb{R}^{2n}$ becomes an n-dimensional complex vector space. Let $X_1, \ldots, X_n$ be a complex basis. Then $X_1, JX_1, \ldots, X_n, JX_n$ is a basis of $\mathbb{R}^{2n}$ as a real vector space, and J is obviously standard in this basis. □

Remark 15.3. The Newlander-Nirenberg Theorem deals with the following question: when can we make J standard in a *neighborhood* of a point? As we will see shortly, this cannot possibly be true for an arbitrary almost complex structure; there is an *integrability condition* which must be satisfied.

All of the linear algebra we discussed above in $\mathbb{C}^n$ can be done on an almost complex manifold (M, J). We can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \tag{15.2}$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \tag{15.3}$$

is the i-eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \tag{15.4}$$

is the $-i$-eigenspace of J.

The map J also induces an endomorphism of 1-forms by

$$J^T(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{15.5}$$

where

$$\Lambda^{1,0} = \{\alpha - iJ^T\alpha, \alpha \in T_p^*M\} \tag{15.6}$$

is the i-eigenspace of J^T, and

$$\Lambda^{0,1} = \{\alpha + iJ^T\alpha, \alpha \in T_p^*M\} \tag{15.7}$$

is the $-i$-eigenspace of J^T.

Next, we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} \tag{15.8}$$

decomposes as a direct sum.

Remark 15.4. This gives a necessary topological obstruction for existence of an almost complex structure: the bundle of complex k-forms must decompose into to a direct sum of subbundles as in (15.8).

We can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{15.9}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$. Note we can also extend J to k-forms by

$$J\alpha(X_1, \ldots, X_k) = \alpha(JX_1, \ldots, JX_k). \tag{15.10}$$

Exercise 15.5. Check that these two definitions of J on k-forms agree.

Definition 15.6. A triple (M, J, g) where J is an almost complex structure, and g is a Riemannian metric is *almost Hermitian* if

$$g(X, Y) = g(JX, JY) \tag{15.11}$$

for all $X, Y \in TM$. We also say that g is *compatible* with J.

Proposition 15.7. *Given a linear J with $J^2 = -Id$ on $\mathbb{R}^{2n}$, and a positive definite inner product g on $\mathbb{R}^{2n}$ which is compatible with J, there exist elements $\{X_1, \dots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{15.12}$$

is an ONB for $\mathbb{R}^{2n}$ with respect to g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{15.13}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of $\text{span}\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. □

Definition 15.8. To an almost Hermitian structure (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y) \tag{15.14}$$

called the *Kähler form* or *fundamental 2-form*.

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{15.15}$$

Furthermore, since

$$\omega(JX, JY) = \omega(X, Y), \tag{15.16}$$

this form is a real form of type $(1, 1)$. That is, $\omega \in \Gamma(\Lambda^{1,1}_{\mathbb{R}})$, where $\Lambda^{1,1}_{\mathbb{R}} \subset \Lambda^{1,1}$ is the real subspace of elements satisfying $\overline{\omega} = \omega$.

In Euclidean space $(\mathbb{R}^{2n}, J_0, g_{Euc})$, the fundamental 2-form is

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{15.17}$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \dots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n \tag{15.18}$$

Note that this defines an orientation on $\mathbb{C}^n$, which we will refer to as the natural orientation. Note also that

$$\omega_{Euc}^n = n! \cdot dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n. \tag{15.19}$$

Proposition 15.9. *If* (M, J) *is almost complex, then* $\dim(M)$ *is even and* M *is orientable.*

Proof. If M is of real dimension m, and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^m, \tag{15.20}$$

which implies that m is even. We will henceforth write $m = 2n$. Next, let g be any Riemannian metric on M. Then define

$$h(X, Y) = g(X, Y) + g(JX, JY). \tag{15.21}$$

Then $h(JX, JY) = h(X, Y)$ is J-invariant, so (M, J, h) is almost Hermitian. We then consider the fundamental 2-form

$$\omega(X, Y) = h(JX, Y). \tag{15.22}$$

This is a form of type $(1, 1)$, so $\omega^n \in \Lambda^{n,n}_{\mathbb{R}} \cong \Lambda^{2n}_{\mathbb{R}}$ is a top degree $2n$-form. It is nowhere-vanishing since at any point $x \in M$ by Proposition 15.7 we can assume that both $J_x = J_{Euc}$ and $g_x = g_{Euc}$, so $\omega^n(x) \neq 0$ by (15.19). Therefore, ω gives a globally defined orientation on M. □

Example 15.10. For example, $\mathbb{RP}^n$ does not admit any almost complex structure, since it is non-orientable for n even.

Definition 15.11. A smooth mapping between $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is *pseudo-holomorphic* if

$$f_* \circ J_M = J_N \circ f_* \tag{15.23}$$

We have a useful characterization of pseudo-holomorphic mappings.

Proposition 15.12. *A mapping* $f : M \to N$ *between almost complex manifolds* (M, J_M) *and* (N, J_N) *is pseudo-holomorphic if and only if*

$$f_*(T^{1,0}(M)) \subset T^{1,0}(N), \tag{15.24}$$

if and only if

$$f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M). \tag{15.25}$$

16 Lecture 16

16.1 Complex manifolds

We next define a complex manifold.

Definition 16.1. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

Example 16.2. Since holomorphic mappings are orientation-preserving by (13.22), any complex manifold is necessarily orientable. For example, $\mathbb{RP}^n$ does not admit any complex structure. Note that we knew from Example 15.10 above that there is no almost complex structure.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 16.3. *In any coordinate chart, define* $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ *by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{16.1}$$

Then $J_\alpha = J_\beta$ *on* $U_\alpha \cap U_\beta$ *and therefore gives a globally defined almost complex structure* $J : TM \to TM$ *satisfying* $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{16.2}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{16.3}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{16.4}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

The next proposition follows from the above discussion on Cauchy-Riemann equations.

Proposition 16.4. *If* (M, J_M) *and* (N, J_N) *are complex manifolds, then* $f : M \to N$ *is pseudo-holomorphic if and only if is a holomorphic mapping in local holomorphic coordinate systems.*

Definition 16.5. An almost complex structure J is said to be a *complex structure* if J is induced from a collection of holomorphic coordinates on M.

Proposition 16.6. *An almost complex structure* J *is a complex structure if and only if for any* $x \in M$*, there is a neighborhood* U *of* x *and a pseudo-holomorphic mapping* $\phi : (U, J) \to (\mathbb{C}^n, J_0)$ *which has non-vanishing Jacobian at* x*. Equivalently, there exist* n *pseudo-holomorphic functions* $f^j : U \to \mathbb{C}, j = 1 \dots n$*, with linearly independent differentials at* x*.*

Proof. By the inverse function theorem, ϕ gives a coordinate system in a possible smaller neighborhood of of x. The overlap mappings are pseudo-holomorphic mappings with respect to J_0, so they satisfy the Cauchy-Riemann equations, and are therefore holomorphic. The components of ϕ are functions $f^j, j = 1 \dots n$ with linearly independent differentials, and conversely, $\phi = (f^1, \dots, f^n)$ is a local coordinate system. □

Proposition 16.7. *A real 2-dimensional manifold admits an almost complex structure if and only if it is oriented.*

Proof. We have already proved the forward direction. Let M^2 be any oriented surface, and choose any Riemannian metric g on M. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric to identify $\Lambda^1 \cong TM$, we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure. □

Remark 16.8. In this case, any such J is necessarily a complex structure. This is equivalent to the problem of existence of isothermal coordinates, we will prove this soon.

16.2 The Nijenhuis tensor

When does an almost complex structure arise from a true complex structure? To answer this question, we define the following tensor associated to an almost complex structure.

Proposition 16.9. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X,Y) = 2\{[JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]\} \tag{16.5}$$

is in $\Gamma(T^*M \otimes T^*M \otimes TM)$ *and satisfies*

$$N(Y,X) = -N(X,Y), \tag{16.6}$$
$$N(JX, JY) = -N(X,Y), \tag{16.7}$$
$$N(X, JY) = N(JX, Y) = -J(N(X,Y)). \tag{16.8}$$

Proof. Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned}
N(fX,Y) &= 2\{[J(fX), JY] - [fX,Y] - J[fX, JY] - J[J(fX), Y]\} \\
&= 2\{[fJX, JY] - [fX,Y] - J[fX, JY] - J[fJX, Y]\} \\
&= 2\{f[JX, JY] - (JY(f))JX - f[X,Y] + (Yf)X \\
&- J(f[X, JY] - (JY(f))X) - J(f[JX,Y] - (Yf)JX)\} \\
&= fN(X,Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X\}.
\end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X, fY) = fN(X,Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y (16.6) is obvious, and (16.7) follows easily using $J^2 = -Id$. For (16.8)

$$N(X, JY) = -N(JX, J^2Y) = N(JX, Y), \tag{16.9}$$

and

$$
\begin{aligned}
N(X, JY) &= 2\{[JX, J^2Y] - [X, JY] - J[X, J^2Y] - J[JX, JY]\} \\
&= 2\{-[JX, Y] - [X, JY] + J[X, Y] - J[JX, JY]\} \\
&= 2J\{J[JX, Y] + J[X, JY] + [X, Y] - [JX, JY]\} \\
&= -2J\{N(X, Y)\}.
\end{aligned} \tag{16.10}
$$

□

Proposition 16.10. *For a C^1 almost complex structure J,*

$$
N_J \in \Gamma\Big(\big\{\big(\Lambda^{2,0} \otimes T^{0,1}\big) \oplus \big(\Lambda^{0,2} \otimes T^{1,0}\big)\big\}_{\mathbb{R}}\Big). \tag{16.11}
$$

Consequently, if $\dim(M) = 2n$, then the Nijenhuis tensor has $n^2(n-1)$ independent real components. In particular, if $n = 1$, then $N_J \equiv 0$.

Proof. If we complexify, just using (16.6), we have

$$
\begin{aligned}
N_J &\in \Gamma((\Lambda^2 \otimes TM) \otimes \mathbb{C})) \\
&= \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}\big) \otimes \big(T^{1,0} \oplus T^{0,1}\big)\Big).
\end{aligned} \tag{16.12}
$$

But (16.7) says that the $\Lambda^{1,1}$ component vanishes. So we have

$$
N_J \in \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2}\big) \otimes \big(T^{1,0} \oplus T^{0,1}\big)\Big). \tag{16.13}
$$

Using (16.8), for $X', Y' \in \Gamma(TM)$, we have

$$
\begin{aligned}
&N_J(X' - iJX', Y' - iJY') \\
&\quad = N_J(X', Y') - N_J(JX', JY') - iN_J(JX', Y') - iN_J(X', JY') \\
&\quad = N_J(X', Y') + N_J(X', Y') + iJN_J(X', Y') + iJN_J(X', Y') \\
&\quad = 2N_J(X', Y') + 2iJN_J(X', Y'),
\end{aligned} \tag{16.14}
$$

which lies in $T^{0,1}$. This shows that the $\Lambda^{2,0} \otimes T^{1,0}$ component vanishes, so the $\Lambda^{0,2} \otimes T^{0,1}$ component also vanishes, and (16.11) follows since N_J is a real tensor. □

We have the following local formula for the Nijenhuis tensor.

Proposition 16.11. *In local coordinates, the Nijenhuis tensor is given by*

$$
N^i_{jk} = 2\sum_{h=1}^{2n}(J^h_j \partial_h J^i_k - J^h_k \partial_h J^i_j - J^i_h \partial_j J^h_k + J^i_h \partial_k J^h_j) \tag{16.15}
$$

Proof. We compute

$$
\begin{aligned}
&\frac{1}{2}N(\partial_j, \partial_k) = [J\partial_j, J\partial_k] - [\partial_j, \partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j, \partial_k] \\
&= [J^l_j \partial_l, J^m_k \partial_m] - [\partial_j, \partial_k] - J[\partial_j, J^l_k \partial_l] - J[J^l_j \partial_l, \partial_k] \\
&= I + II + III + IV.
\end{aligned}
$$

The first term is

$$
\begin{aligned}
I &= J_j^l \partial_l (J_k^m \partial_m) - J_k^m \partial_m (J_j^l \partial_l) \\
&= J_j^l (\partial_l J_k^m) \partial_m + J_j^l J_k^m \partial_l \partial_m - J_k^m (\partial_m J_j^l) \partial_l - J_k^m J_j^l \partial_m \partial_l \\
&= J_j^l (\partial_l J_k^m) \partial_m - J_k^m (\partial_m J_j^l) \partial_l .
\end{aligned}
$$

The second term is obviously zero. The third term is

$$
III = -J(\partial_j (J_k^l) \partial_l) = -\partial_j (J_k^l) J_l^m \partial_m . \tag{16.16}
$$

Finally, the fourth term is

$$
III = \partial_k (J_j^l) J_l^m \partial_m . \tag{16.17}
$$

Combining these, we are done. □

Definition 16.12. If J is an almost complex structure of class C^1 satisfying $N_J \equiv 0$, then we say that J is *integrable.*

Corollary 16.13. *If (M, J) arises from a complex structure, then J is integrable.*

Proof. In local holomorphic coordinates $J = J_0$ is a constant tensor, and $N(J) = 0$ follows from Proposition 16.11. □

Next, we have an alternative characterization of the vanishing of the Nijenhuis tensor.

Proposition 16.14. *For an almost complex structure J the Nijenhius tensor $N(J) = 0$ if and only if for any 2 vector fields $X, Y \in \Gamma(T^{1,0})$, their Lie bracket $[X, Y] \in \Gamma(T^{1,0})$.*

Proof. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. The commutator is

$$
[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \tag{16.18}
$$

But this is also a $(1, 0)$ vector field if and only if

$$
[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \tag{16.19}
$$

applying J, and moving everything to the left hand side, this says that

$$
[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \tag{16.20}
$$

which is exactly the vanishing of the Nijenhuis tensor. □

17 Lecture 17

17.1 The operators ∂ and $\overline{\partial}$ for an integrable almost complex structure

Recall that on any almost complex manifold (M, J), we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{17.1}$$

We define $\mathcal{E}^k, \mathcal{E}^k_{\mathbb{C}}, \mathcal{E}^{p,q}$ to be the space of smooth sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. The real operator $d : \mathcal{E}^k_{\mathbb{R}} \to \mathcal{E}^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{17.2}$$

by complexification.

Proposition 17.1. *For a C^1 almost complex structure J*

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+2,q-1} \oplus \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1} \oplus \mathcal{E}^{p-1,q+2}, \tag{17.3}$$

and $N_J = 0$ if and only if

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}. \tag{17.4}$$

if and only if

$$d(\mathcal{E}^{1,0}) \subset \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1} \tag{17.5}$$

if and only if

$$d(\mathcal{E}^{0,1}) \subset \mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2} \tag{17.6}$$

Proof. Let $\alpha \in \mathcal{E}^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) &= \sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{17.7}$$

This is easily seen to vanish if more than $p + 2$ of the X_j are of type $(1, 0)$ or if more than $q + 2$ are of type $(0, 1)$, and (17.3) follows.

Next, assume that (17.6) is satisfied. Let $\alpha \in \mathcal{E}^{0,1}$, then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{17.8}$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X, Y]$. Proposition 16.14 implies that $N(J) \equiv 0$. Conversely, if $N(J) \equiv 0$, then we can reverse the steps in this argument to obtain (17.6). Equation (17.5) is just the conjugate of (17.6).

Recall that if $\alpha \in \mathcal{E}^k$ and $\beta \in \mathcal{E}^l$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{17.9}$$

The formula (17.4) then follows from this. $\square$

If $N_J = 0$, we can therefore define operators

$$\partial : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{17.10}$$

$$\overline{\partial} : \mathcal{E}^k_{\mathbb{C}} \to \mathcal{E}^{k+1}_{\mathbb{C}} \tag{17.11}$$

using (17.1) and

$$\partial|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{17.12}$$

$$\overline{\partial}|_{\mathcal{E}^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{17.13}$$

Corollary 17.2. *For a C^1 almost complex structure J with $N_J = 0$, $d = \partial + \overline{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{17.14}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{17.15}$$

If we plug in a form of type (p, q) the first term is of type $(p+2, q)$, the middle terms are of type $(p+1, q+1)$, and the last term is of type $(p, q+2)$. Since (17.1) is a direct sum, the claim follows. □

Remark 17.3. If we assume that (M, J) arises from a complex structure, then we can just define these operators in local holomorphic coordinates like we did in $\mathbb{C}^n$ (and the prove that they define global operators). The point of the above is that we only assumed integrability, and did not use any local holomorphic coordinates.

17.2 Real form of the equations

Recall that for $n = 1$, any almost complex structure J satisfies $N_J = 0$, so there is no integrability condition. Let's look at various forms of the equations.

We just look in an open set in real coordinates (x, y), and then we have

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}. \tag{17.16}$$

The only condition is

$$-I = J^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} \tag{17.17}$$

If we assume that J is not too far from J_0, then $b \sim -1$ and $c \sim 1$, so we must have

$$a + d = 0, \quad a^2 + bc = -1. \tag{17.18}$$

Note that since $b \sim -1$, we can solve $c = -(1 + a^2)/b$, but we won't need to do this now. So we just consider

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix}. \tag{17.19}$$

We want to find a pseudo-holomorphic mapping

$$\phi : (U, J) \to (\mathbb{C}, J_0) \tag{17.20}$$

which has non-vanishing Jacobian at 0. So we want to solve

$$\phi_* \circ J = J_0 \circ \phi_* \tag{17.21}$$

If we write

$$\phi(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}, \tag{17.22}$$

then the pseudoholomorphic condition is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{17.23}$$

which yields the 4 equations

$$\begin{matrix} au_x + cu_y = -v_x & bu_x - au_y = -v_y \\ av_x + cv_y = u_x & bv_x - av_y = u_y \end{matrix}. \tag{17.24}$$

This looks like 4 first-order equations for 2 unknown functions, so one wouldn't expect a solution. However, the first two equations imply the second two:

$$av_x + cv_y = a(-au_x - cu_y) + c(-bu_x + au_y) = (-a^2 - bc)u_x = u_x, \tag{17.25}$$

and

$$bv_x - av_y = b(-au_x - cu_y) + a(bu_x - au_y) = (-bc - a^2)u_y = u_y, \tag{17.26}$$

using the condition that $a^2 + bc = -1$.

Example 17.4. Let's now do an example. Consider

$$J = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix}. \tag{17.27}$$

We have

$$J^2 = \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} \begin{pmatrix} 2x & -1 \\ 1 + 4x^2 & -2x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17.28}$$

so this is indeed an almost complex structure.

From (17.24), the pseudoholomorphic equations are

$$2xu_x + (1 + 4x^2)u_y = -v_x \tag{17.29}$$

$$-u_x - 2xu_y = -v_y. \tag{17.30}$$

If a sufficiently smooth solution exists, then we have $v_{xy} = v_{yx}$, which yields

$$(2xu_x + (1 + 4x^2)u_y)_y = -(u_x + 2xu_y)_x \tag{17.31}$$

This can be rewritten as

$$u_{xx} + 4xu_{xy} + (1 + 4x^2)u_{yy} + 2u_y = 0. \tag{17.32}$$

Remark 17.5. This is a nice equation, because it looks like

$$\Delta_0 u + \text{lower order terms}. \tag{17.33}$$

We will return to this viewpoint later.

For now, we just notice that, by inspection, $u = x$ is obviously a solution. We then return to the pseudoholomorphic equations, and find that

$$v_x = -2x, \quad v_y = 1, \tag{17.34}$$

so we can choose $v = -x^2 + y$. So our solution is $\phi = (u, v) = (x, y - x^2)$. The Jacobian at the origin is clearly non-degenerate, so we have found a holomorphic coordinate system. Note that the mapping $\phi : \mathbb{R}^2 \to \mathbb{C}$ is defined everywhere. It is injective: if we have $(x_1, y_1 - x_1^2) = (x_2, y_2 - x_2^2)$ then the first component says that $x_1 = x_2$ and the second component then implies that $y_1 = y_2$. It is also surjective: given any $(u, v) \in \mathbb{C}$, we let $x_2 = u$, and then we need to solve $y - u^2 = v$, which obviously has a solution $y = -u^2 + v$. Thus we have found that

$$\phi : (\mathbb{R}^2, J) \to (\mathbb{C}, J_0) \tag{17.35}$$

is a global biholomorphism! Note that any function of the form $f(x, y) = h(x + i(y - x^2))$, where h is a holomorphic function with respect to J_0, is then holomorphic for J, for example

$$f(x, y) = e^x(\cos(y - x^2) + i \sin(y - x^2)). \tag{17.36}$$

18 Lecture 18

Course to be continued in the Winter quarter.

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