Math 245B, Topics in Differential Geometry

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Winter Quarter 2020

Contents

Introduction

This quarter will be about complex manifolds and Kähler geometry. Following are just partial notes of some of the topics we will cover.

1 Lecture 1

1.1 Example of $\mathbb{R}^{2n} = \mathbb{C}^n$

Remark 1.1. For now we will denote $\sqrt{-1}$ by i. However, later we will not do this, because the letter i is sometimes used as an index.

We consider $\mathbb{R}^{2n}$ and denote the coordinates as $x^1, y^1, \ldots, x^n, y^n$. Letting $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, define complex one-forms

$$\begin{aligned} dz^j &= dx^j + idy^j, \\ d\bar{z}^j &= dx^j - idy^j, \end{aligned}$$

and complex tangent vectors

$$\begin{aligned} \partial/\partial z^j &= (1/2)\left(\partial/\partial x^j - i\partial/\partial y^j\right), \\ \partial/\partial \bar{z}^j &= (1/2)\left(\partial/\partial x^j + i\partial/\partial y^j\right). \end{aligned}$$

Note that

$$\begin{aligned} dz^j(\partial/\partial z^k) &= d\bar{z}^j(\partial/\partial \bar{z}^k) = \delta^{jk}, \\ dz^j(\partial/\partial \bar{z}^k) &= d\bar{z}^j(\partial/\partial z^k) = 0. \end{aligned}$$

The standard complex structure $J_0 : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ on $\mathbb{R}^{2n}$ is given by

$$J_0(\partial/\partial x^j) = \partial/\partial y^j, \quad J_0(\partial/\partial y^j) = -\partial/\partial x^j,$$

which in matrix form is written

$$J_0 = \text{diag}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{1.1}$$

Next, we complexify the tangent space $T \otimes \mathbb{C}$, and let

$$T^{(1,0)}(J_0) = \text{span}\{\partial/\partial z^j, j = 1 \ldots n\} = \{X - iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{1.2}$$

be the i-eigenspace and

$$T^{(0,1)}(J_0) = \text{span}\{\partial/\partial \bar{z}^j, j = 1 \ldots n\} = \{X + iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{1.3}$$

be the $-i$-eigenspace of J_0, so that

$$T \otimes \mathbb{C} = T^{(1,0)}(J_0) \oplus T^{(0,1)}(J_0). \tag{1.4}$$

The map J_0 also induces an endomorphism of 1-forms by

$$J_0(\omega)(v_1) = \omega(J_0 v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0(dx_j) = -dy_j, \quad J_0(dy_j) = +dx_j.$$

Then complexifying the cotangent space $T^* \otimes \mathbb{C}$, we have

$$\Lambda^{1,0}(J_0) = \text{span}\{dz^j, j = 1 \dots n\} = \{\alpha - iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.5}$$

is the i-eigenspace, and

$$\Lambda^{0,1}(J_0) = \text{span}\{d\bar{z}^j, j = 1 \dots n\} = \{\alpha + iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.6}$$

is the $-i$-eigenspace of J_0, and

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J_0) \oplus \Lambda^{0,1}(J_0). \tag{1.7}$$

We note that

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{1.8}$$

and similarly

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{1.9}$$

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{1.10}$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{1.11}$$

Note that we can characterize $\Lambda^{p,q}$ as those forms satisfying

$$\alpha(v_1, \dots, v_{p+q}) = 0, \tag{1.12}$$

if more than p if the v_j-s are in $T^{(1,0)}$ or if more than q of the v_j-s are in $T^{(0,1)}$.

Finally, we can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{1.13}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J is not a complex structure on the space $\Lambda^k_{\mathbb{C}}$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J-invariant.

1.2 Cauchy-Riemann equations

Let $f : \mathbb{C}^n \to \mathbb{C}^m$. Let the coordinates on $\mathbb{C}^n$ be given by

$$\{z^1, \dots z^n\} = \{x^1 + iy^1, \dots, x^n + iy^n\}, \tag{1.14}$$

and coordinates on $\mathbb{C}^m$ given by

$$\{w^1, \dots w^m\} = \{u^1 + iv^1, \dots, u^m + iv^m\} \tag{1.15}$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \text{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{1.16}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial u^1, \dots \partial/\partial u^m, \partial/\partial v^1, \dots \partial/\partial v^m\}. \tag{1.17}$$

Then the real Jacobian of

$$f = (f^1, \dots f^{2m}) = (u^1 \circ f, u^2 \circ f, \dots, v^{2m} \circ f). \tag{1.18}$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \dots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \dots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \dots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \tag{1.19}$$

Definition 1.2. A differentiable mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is pseudo-holomorphic if

$$f_* \circ J_{0,\mathbb{C}^n} = J_{0,\mathbb{C}^m} \circ f_*. \tag{1.20}$$

That is, the differential of f commutes with J_0.

We have the following characterization of pseudo-holomorphic maps.

Proposition 1.3. *A mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ is pseudo-holomorphic if and only if the Cauchy-Riemann equations are satisfied, that is, writing*

$$f(z^1, \dots z^m) = (f_1, \dots, f_n) = (u_1 + iv_1, \dots u_n + iv_n), \tag{1.21}$$

and $z^j = x^j + iy^j$, for each $j = 1 \dots n$, we have

$$\frac{\partial u_j}{\partial x^k} = \frac{\partial v_j}{\partial y^k} \qquad \frac{\partial u_j}{\partial y^k} = -\frac{\partial v_j}{\partial x^k}, \tag{1.22}$$

for each $k = 1 \dots m$, and these equations are equivalent to

$$\frac{\partial}{\partial \bar{z}^k} f_j = 0, \tag{1.23}$$

for each $j = 1 \dots n$ and each $k = 1 \dots m$

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{1.24}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{1.25}$$

which is exactly the Cauchy-Riemann equations. In the general case, rearrange the coordinates so that $(x^1, \ldots, x^m, y^1, \ldots, y^m)$ are the real coordinates on $\mathbb{R}^{2m}$ and $(u^1, \ldots, u^n, v^1, \ldots, v^n)$, such that the complex structure J_0 is given by

$$J_0(\mathbb{R}^{2m}) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{1.26}$$

and similarly for $J_0(\mathbb{R}^{2n})$. Then the computation in matrix form is entirely analogous to the case of $m = n = 1$.

Finally, we compute

$$\frac{\partial}{\partial \bar{z}^k} f_j = \frac{1}{2}\Big(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\Big)(u_j + iv_j) \tag{1.27}$$

$$= \frac{1}{2}\Big\{\frac{\partial}{\partial x^k}u_j - \frac{\partial}{\partial y^k}v_j + i\Big(\frac{\partial}{\partial x^k}v_j + \frac{\partial}{\partial y^k}u_j\Big)\Big\}, \tag{1.28}$$

the vanishing of which again yields the Cauchy-Riemann equations. □

From now on, if f is a mapping satisfying the Cauchy-Riemann equations, we will just say that f is *holomorphic*.

For any differentiable f, the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^m). \tag{1.29}$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \mathrm{span}\{\partial/\partial z^1, \ldots \partial/\partial z^n, \partial/\partial \overline{z}^1, \ldots \partial/\partial \overline{z}^n\}, \tag{1.30}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \mathrm{span}\{\partial/\partial w^1, \ldots \partial/\partial w^m, \partial/\partial \overline{w}^1, \ldots \partial/\partial \overline{w}^m\}. \tag{1.31}$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \cdots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \overline{z}^1} & \cdots & \frac{\partial f^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \cdots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \overline{z}^1} & \cdots & \frac{\partial f^m}{\partial \overline{z}^n} \\ \frac{\partial \overline{f}^1}{\partial z^1} & \cdots & \frac{\partial \overline{f}^1}{\partial z^n} & \frac{\partial \overline{f}^1}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \overline{f}^m}{\partial z^1} & \cdots & \frac{\partial \overline{f}^m}{\partial z^n} & \frac{\partial \overline{f}^m}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^m}{\partial \overline{z}^n}, \end{pmatrix} \tag{1.32}$$

where $(f^1, \dots f^m) = f$ now denotes the complex components of f. This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{1.33}$$

Notice that (1.32) is of the form

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{1.34}$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.35}$$

where $\mathbb{C}$-linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \tag{1.36}$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{1.37}$$

This corresponds to the embeddings

$$Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.38}$$

where the left $\mathbb{C}$-linear is with respect to J_0, via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{1.39}$$

Note that since the latter embedding is just a change of basis, if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB)\det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{1.40}$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(1,0)}. \tag{1.41}$$

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0, then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{1.42}$$

This corresponds to the embeddings

$$Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \tag{1.43}$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \tag{1.44}$$

We see that f is anti-holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(0,1)}. \tag{1.45}$$

Note that if f is antiholomorphic, then is it holomorphic with respect to the complex structure $-J_0$ on the domain (but still J_0 on the range).

Note that we can decompose $f_* = f_*^C + f_*^A$, where

$$f_*^C = \frac{1}{2}(f_* - Jf_*J) \tag{1.46}$$

$$f_*^A = \frac{1}{2}(f_* + Jf_*J), \tag{1.47}$$

and f_*^C is holomorphic, while f_*^A is anti-holomorphic. In block matrix form, this just says that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \tag{1.48}$$

2 Lecture 2

2.1 Complex Manifolds

Now we can define a complex manifold

Definition 2.1. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

A closely related notion is the following.

Definition 2.2. An *almost complex structure* is an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$.

If M is of real dimension n, and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^n, \tag{2.1}$$

which implies that n is even. Furthermore, by the discussion in the previous lecture, a complex manifold M is orientable and carries a natural orientation. It is moreover true that an almost complex manifold is orientable, we will see this later.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 2.3. *In any coordinate chart, define* $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ *by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{2.2}$$

Then $J_\alpha = J_\beta$ *on* $U_\alpha \cap U_\beta$ *and therefore gives a globally defined almost complex structure* $J : TM \to TM$ *satisfying* $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{2.3}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{2.4}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{2.5}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

Definition 2.4. An almost complex structure J is said to be *integrable* if J is induced from a collection of holomorphic coordinates on M.

Remark 2.5. Let (M^2, g) be any oriented Riemannian surface. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure. In the case of Riemann surfaces, any such J is necessarily integrable, but this is not necessarily true in higher dimensions.

Let (M, J) be any almost complex manifold. Then we can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \tag{2.6}$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \tag{2.7}$$

is the i-eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \tag{2.8}$$

is the $-i$-eigenspace of J.

The map J also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{2.9}$$

where

$$\Lambda^{1,0} = \{\alpha - iJ\alpha, \alpha \in T_p^* M\} \tag{2.10}$$

is the i-eigenspace of J, and

$$\Lambda^{0,1} = \{\alpha + iJ\alpha, \alpha \in T_p^* M\} \tag{2.11}$$

is the $-i$-eigenspace of J.

If (M, J) is a complex manifold, then there exist coordinate systems around any point

$$(z^1, \dots, z^n) = (x^1 + iy^1, \dots, x^n + iy^n) \tag{2.12}$$

such that $T^{1,0}$ is spanned by

$$\frac{\partial}{\partial z^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big), \tag{2.13}$$

$T^{0,1}$ is spanned by

$$\frac{\partial}{\partial \bar{z}^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big), \tag{2.14}$$

$\Lambda^{1,0}$ is spanned by

$$dz^j \equiv dx^j + idy^j, \tag{2.15}$$

and $\Lambda^{0,1}$ is spanned by

$$d\bar{z}^j \equiv dx^j - idy^j, \tag{2.16}$$

for $j = 1 \dots n$.

Definition 2.6. A smooth mapping between $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is *pseudo-holomorphic* if

$$f_* \circ J_M = J_N \circ f_* \tag{2.17}$$

The next proposition follows from the above discussion on Cauchy-Riemann equations.

Proposition 2.7. *$f : M \to N$ is pseudo-holomorphic if and only if*

$$f_*(T^{1,0}(M)) \subset T^{1,0}(N), \tag{2.18}$$

if and only if

$$f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M). \tag{2.19}$$

If (M, J_M) and (N, J_N) are moreover complex manifolds, then f is pseudoholomorphic if and only if it is a holomorphic mapping in local holomorphic coordinate systems.

2.2 The Nijenhuis tensor

When is an almost complex structure integrable? To answer this question, we define the following tensor associated to an almost compelx structure.

Proposition 2.8. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} \tag{2.20}$$

is a tensor of type $(1,2)$ *and satisfies*

(i) $N(Y,X) = -N(X,Y)$.

(ii) $N(JX,JY) = -N(X,Y)$.

Proof. Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned}N(fX,Y) &= 2\{[J(fX),JY] - [fX,Y] - J[fX,JY] - J[J(fX),Y]\}\\ &= 2\{[fJX,JY] - [fX,Y] - J[fX,JY] - J[fJX,Y]\}\\ &= 2\{f[JX,JY] - (JY(f))JX - f[X,Y] + (Yf)X\\ &- J(f[X,JY] - (JY(f))X) - J(f[JX,Y] - (Yf)JX)\}\\ &= fN(X,Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X\}.\end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X,fY) = fN(X,Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y is obvious, and (ii) follows easily using $J^2 = -Id$. □

Notice that if M is of complex dimension 1, then there is a basis of the tangent space of the form $\{X, JX\}$, so

$$N(X,JX) = -N(JX,X) = -N(JX,J^2X) = N(JX,X), \tag{2.21}$$

which shows that the Nijenhuis tensor of any almost complex structure on a Riemann surface vanishes.

We have the following local formula for the Nijenhuis tensor.

Proposition 2.9. *In local coordinates, the Nijenhuis tensor is given by*

$$N^i_{jk} = 2\sum_{h=1}^{2n}(J^h_j\partial_h J^i_k - J^h_k\partial_h J^i_j - J^i_h\partial_j J^h_k + J^i_h\partial_k J^h_j) \tag{2.22}$$

Proof. We compute

$$\begin{aligned}&\frac{1}{2}N(\partial_j,\partial_k) = [J\partial_j, J\partial_k] - [\partial_j,\partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j,\partial_k]\\ &= [J^l_j\partial_l, J^m_k\partial_m] - [\partial_j,\partial_k] - J[\partial_j, J^l_k\partial_l] - J[J^l_j\partial_l,\partial_k]\\ &= I + II + III + IV.\end{aligned}$$

The first term is

$$
\begin{aligned}
I &= J^l_j \partial_l (J^m_k \partial_m) - J^m_k \partial_m (J^l_j \partial_l) \\
&= J^l_j (\partial_l J^m_k) \partial_m + J^l_j J^m_k \partial_l \partial_m - J^m_k (\partial_m J^l_j) \partial_l - J^m_k J^l_j \partial_m \partial_l \\
&= J^l_j (\partial_l J^m_k) \partial_m - J^m_k (\partial_m J^l_j) \partial_l .
\end{aligned}
$$

The second term is obviously zero. The third term is

$$III = -J(\partial_j (J^l_k) \partial_l) = -\partial_j (J^l_k) J^m_l \partial_m . \quad (2.23)$$

Finally, the fourth term is

$$III = \partial_k (J^l_j) J^m_l \partial_m . \quad (2.24)$$

Combining these, we are done. □

Next, we have

Theorem 2.10. *An real analytic almost complex structure J is integrable if and only if the Nijenhuis tensor vanishes.*

Proof. If J is integrable, then we can always find local coordinates so that $J = J_0$, and Proposition 2.9 shows that the Nijenhuis tensor vanishes. For the converse, the vanishing of the Nijenhuis tensor is the integrability condition for $T^{1,0}$ as a complex sub-distribution of $T \otimes \mathbb{C}$. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \quad (2.25)$$

But this is also a $(1, 0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \quad (2.26)$$

applying J, and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \quad (2.27)$$

which is exactly the vanishing of the Nijenhuis tensor. In the analytic case, the converse then follows using a complex version of the Frobenius Theorem, we will discuss later. □

Remark 2.11. The C^∞-case is more difficult, and is the main content of the Newlander-Nirenberg Theorem.

3 Lecture 3

3.1 The operators ∂ and $\overline{\partial}$

On any almost complex manifold (M, J), we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{3.1}$$

The real operator $d : \Lambda^k_{\mathbb{R}} \to \Lambda^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \Lambda^k_{\mathbb{C}} \to \Lambda^{k+1}_{\mathbb{C}} \tag{3.2}$$

by complexification. On a complex manifold, if α is a (p, q)-form, then locally we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{3.3}$$

where I and J are multi-indices of length p and q, respectively, and $\alpha_{I,J}$ are complex-valued functions. Using (1.33), we have the formula

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \Big) \wedge dz^I \wedge d\overline{z}^J. \tag{3.4}$$

Proposition 3.1. *For an almost complex structure J*

$$d(\Lambda^{p,q}) \subset \Lambda^{p+2,q-1} + \Lambda^{p+1,q} + \Lambda^{p,q+1} + \Lambda^{p-1,q+2}, \tag{3.5}$$

and J is integrable if and only if

$$d(\Lambda^{p,q}) \subset \Lambda^{p+1,q} + \Lambda^{p,q+1}. \tag{3.6}$$

(In a slight abuse of notation, by $\Lambda^{p,q}$, we mean the space of smooth sections of this bundle.)

Proof. Let $\alpha \in \Lambda^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) &= \sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{3.7}$$

This is easily seen to vanish if more than $p + 2$ of the X_j are of type $(1, 0)$ or if more than $q + 2$ are of type $(0, 1)$.

If J is integrable, then in a local complex coordinate system, (3.6) is easily seen to hold. For the converse we have the inclusions,

$$d(\Lambda^{1,0}) \subset \Lambda^{2,0} + \Lambda^{1,1} \text{ and } d(\Lambda^{0,1}) \subset \Lambda^{1,1} + \Lambda^{0,2}. \tag{3.8}$$

The formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{3.9}$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X,Y]$. So write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. Define $Z = [X,Y]$, then Z is also of type $(1,0)$, so

$$Z + iJZ = 0. \tag{3.10}$$

Writing this in terms of X' and Y' we see that

$$0 = 2(Z + iJZ) = -N(X',Y') - iJN(X',Y'), \tag{3.11}$$

which implies that $N \equiv 0$. □

Corollary 3.2. *On a complex manifold,* $d = \partial + \overline{\partial}$ *where* $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ *and* $\overline{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$*, and these operators satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{3.12}$$

Proof. These relations follow simply from $d^2 = 0$. □

Note that on a complex manifold, if α is a (p,q)-form written locally as

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{3.13}$$

then

$$\partial\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J. \tag{3.14}$$

$$\overline{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J. \tag{3.15}$$

Definition 3.3. *A form* $\alpha \in \Lambda^{p,0}$ *is holomorphic if* $\overline{\partial}\alpha = 0$.

It is easy to see that a $(p,0)$-form is holomorphic if and only if it can locally be written as

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{3.16}$$

where the α_I are holomorphic functions.

Definition 3.4. *The* (p,q) *Dolbeault cohomology group is*

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\{\alpha \in \Lambda^{p,q}(M) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M))}. \tag{3.17}$$

We will discuss these in more detail later, and just point out the following for now. If $f : M \to N$ is a holomorphic mapping between complex manifolds, then

$$f^*(\Lambda^{p,q}(N)) \subset \Lambda^{p,q}(M), \tag{3.18}$$

and

$$\overline{\partial} \circ f^* = f^* \circ \overline{\partial} \tag{3.19}$$

(because d commutes with f^*). Consequently, there is an induced mapping

$$f^* : H^{p,q}_{\overline{\partial}}(N) \to H^{p,q}_{\overline{\partial}}(M) \tag{3.20}$$

In particular, if f is a biholomorphism (one-to-one, onto, with holomorphic inverse), then the Dolbeault cohomologies of M and N are isomorphic.

3.2 The operator d^c

For an integrable complex structure, we know that

$$d = \partial + \overline{\partial}, \tag{3.21}$$

and

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{3.22}$$

We can write these complex operators in the form

$$\overline{\partial} = \frac{1}{2}(d - id^c), \ \partial = \frac{1}{2}(d + id^c). \tag{3.23}$$

for a *real* operator $d^c : \Lambda^p \to \Lambda^{p+1}$ given by

$$d^c = i(\overline{\partial} - \partial), \tag{3.24}$$

which satisfies

$$d^2 = 0, \ dd^c + d^c d = 0, \ (d^c)^2 = 0. \tag{3.25}$$

We next have an alternative formula for d^c. Recall that $J : TM \to TM$ induces a dual mapping $J : T^*M \to T^*M$, and we extended to $J : \Lambda^r_{\mathbb{C}} \to \Lambda^r_{\mathbb{C}}$ by

$$J\alpha^{p,q} = i^{p-q}\alpha^{p,q}, \tag{3.26}$$

for a α a form of type (p,q). Notice that if $\alpha^r \in \Lambda^r_{\mathbb{C}}$, then

$$J^2\alpha^r = w \cdot \alpha^r, \text{ where } w \cdot \alpha^r = (-1)^r\alpha^r, \tag{3.27}$$

since

$$J^2\alpha^{p,q} = i^{2(p-q)}\alpha^{p,q} = (-1)^{p-q}\alpha^{p,q} = (1)^{p-q+2q}\alpha^{p,q} = (-1)^{p+q}\alpha^{p,q}. \tag{3.28}$$

Proposition 3.5. *For $\alpha \in \Lambda^r$, we have*

$$d^c \alpha = (-1)^{r+1} J d J \alpha. \tag{3.29}$$

Proof. For $\alpha \in \Lambda^{p,q}$, $p + q = r$, we compute

$$\begin{aligned} JdJ\alpha = i^{p-q} J d\alpha &= i^{p-q} J(\partial\alpha + \overline{\partial}\alpha) && (3.30)\\ &= i^{p-q}(i^{p+1-q}\partial\alpha + i^{p-q-1}\overline{\partial}\alpha) && (3.31)\\ &= i^{2(p-q)+1}\partial\alpha + i^{2(p-q)-1}\overline{\partial}\alpha && (3.32)\\ &= (-1)^{p+q}(i\partial\alpha - i\overline{\partial}\alpha) = (-1)^{r+1} d^c\alpha. && (3.33) \end{aligned}$$

□

We also note the formula

$$dd^c = 2i\partial\overline{\partial}. \tag{3.34}$$

4 Lecture 4

4.1 Automorphisms

If $J \in \Gamma(End(TM))$, recall the formula

$$(\mathcal{L}_X J)(Y) = \mathcal{L}_X(J(Y)) - J(\mathcal{L}_X Y) = [X, JY] - J([X, Y]). \tag{4.1}$$

Definition 4.1. An *infinitesimal automorphism* of a complex manifold is a real vector field X such that $\mathcal{L}_X J = 0$, where $\mathcal{L}$ denotes the Lie derivative operator.

It is straightforward to see that X is an infinitesimal automorphism if and only if its 1-parameter group of diffeomorphisms are holomorphic automorphisms, that is, $(\phi_s)_* \circ J = J \circ (\phi_s)_*$.

Proposition 4.2. *A vector field X is an infinitesimal automorphism if and only if*

$$J([X, Y]) = [X, JY], \tag{4.2}$$

for all vector fields Y.

Proof. We compute

$$[X, JY] = \mathcal{L}_X(JY) = \mathcal{L}_X(J)Y + J(\mathcal{L}_X Y) = \mathcal{L}_X(J)Y + J([X, Y]), \tag{4.3}$$

and the result follows. □

Proposition 4.3. *The set of infinitesimal automorphisms is a real Lie algebra under the Lie bracket. Furthermore, if $N \equiv 0$, then it is a complex Lie algebra, with complex structure given by J.*

Proof. First, recall the Jacobi identity for the Lie bracket:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{4.4}$$

Now let X and Y satisfy $\mathcal{L}_X J = 0$ and $\mathcal{L}_Y J = 0$. We need to show that $\mathcal{L}_{[X,Y]} J = 0$, so we compute

$$\begin{aligned} (\mathcal{L}_{[X,Y]} J)(Z) &= \mathcal{L}_{[X,Y]}(J(Z)) - J(\mathcal{L}_{[X,Y]} Z) \\ &= [[X, Y], J(Z)] - J([[X, Y], Z]). \end{aligned} \tag{4.5}$$

By the Jacobi identity,

$$\begin{aligned} (\mathcal{L}_{[X,Y]} J)(Z) &= -[[Y, J(Z)], X] - [[J(Z), X], Y] + J\big([[Y, Z], X] + [[Z, X], Y]\big) \\ &= -[J([Y, Z]), X] + [J([X, Z]), Y] + J\big([[Y, Z], X] + [[Z, X], Y]\big) \\ &= J\big([X, [Y, Z] - [Y, [X, Z]]) + [[Y, Z], X] + [[Z, X], Y]\big) = 0. \end{aligned} \tag{4.6}$$

For the second part, we need to show that if X is an infinitesimal automorphism, then JX is also. For this, we need to show that $\mathcal{L}_{JX} J = 0$, so we compute

$$\begin{aligned} (\mathcal{L}_{JX} J)(Z) &= \mathcal{L}_{JX}(JZ) - J(\mathcal{L}_{JX} Z) \\ &= [JX, JZ] - J([JX, Z]). \end{aligned} \tag{4.7}$$

From the definition of the Nijenhuis tensor,

$$N(X, Z) = 2\{[JX, JZ] - [X, Z] - J[X, JZ] - J[JX, Z]\} = 0, \tag{4.8}$$

so we have

$$\begin{aligned} (\mathcal{L}_{JX} J)(Z) &= [X, Z] + J([X, JZ]) \\ &= [X, Z] + J(J([X, Z])) = [X, Z] - [X, Z] = 0. \end{aligned} \tag{4.9}$$

Finally, Proposition 4.2 shows that the Lie bracket is complex linear in both arguments, so it is a complex Lie algebra. □

Next we assume that (M, J) is a complex manifold.

Definition 4.4. A *holomorphic vector field* on a complex manifold (M, J) is vector field $Z \in \Gamma(T^{1,0})$ which satisfies Zf is holomorphic for every locally defined holomorphic function f.

In complex coordinates, a holomorphic vector field can locally be written as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}, \tag{4.10}$$

where the Z^j are locally defined holomorphic functions. We extend the Lie bracket of real vector fields to complex vector fields by complex linearity.

Proposition 4.5. *If Z_1 and Z_2 are holomorphic vector fields, then $[Z_1, Z_2]$ is also a holomorphic vector field. Consequently, the space of holomorphic vector fields is a complex Lie algebra.*

Proof. If in local holomorphic coordinates,

$$Z_1 = \sum Z_1^j \frac{\partial}{\partial z^j},\ Z_2 = \sum Z_2^k \frac{\partial}{\partial z^k}, \tag{4.11}$$

with Z_1^j and Z_2^k holomorphic functions, then

$$\begin{aligned}[Z_1, Z_2] &= \sum_{j,k} Z_1^j \frac{\partial Z_2^k}{\partial z^j} \frac{\partial}{\partial z^k} - \sum_{j,k} Z_2^k \frac{\partial Z_1^j}{\partial z^k} \frac{\partial}{\partial z^j} \\ &= \sum_{j,k} \Big(Z_1^j \frac{\partial Z_2^k}{\partial z^j} - Z_2^j \frac{\partial Z_1^k}{\partial z^j} \Big) \frac{\partial}{\partial z^k}.\end{aligned} \tag{4.12}$$

Since $\partial\overline{\partial} = -\overline{\partial}\partial$, the coefficients are holomorphic functions. □

Proposition 4.6. *For $X \in \Gamma(TM)$, associate a vector field of type $(1,0)$ by mapping $X \mapsto X^{1,0} = \frac{1}{2}(X - iJX)$. This complex linear mappping maps the subspace of infinitesimal automorphisms maps isomorphically onto the space of holomorphic vector fields. Furthermore this mapping is an isomorphism of Lie algebras, that is, for infinitesimal automorphisms X and Y,*

$$[X, Y] = [X^{1,0}, Y^{1,0}]. \tag{4.13}$$

Proof. Choose a local holomorphic coordinate system $\{z^i\}$, and for real vector fields X' and Y', write

$$X = \frac{1}{2}(X' - iJX') = \sum X^j \frac{\partial}{\partial z^j}, \tag{4.14}$$

$$Y = \frac{1}{2}(Y' - iJY') = \sum Y^j \frac{\partial}{\partial z^j}. \tag{4.15}$$

We know that X' is an infinitesimal automorphism if and only if

$$J([X', Y']) = [X', JY'], \tag{4.16}$$

for all real vector fields Y'. This condition is equivalent to

$$\sum_j \overline{Y}^j \frac{\partial X^k}{\partial \overline{z}^j} = 0, \tag{4.17}$$

for each $k = 1 \dots n$, which is equivalent to X being a holomorphic vector field.

To see this, we rewrite (4.16) in terms of complex vector fields. We have

$$\begin{aligned} X' &= X + \overline{X} \quad JX' = i(X - \overline{X}) \\ Y' &= Y + \overline{Y} \quad JY' = i(Y - \overline{Y}) \end{aligned}$$

The left hand side of (4.16) is

$$\begin{aligned} J([X', Y']) &= J([X + \overline{X}, Y + \overline{Y}]) \\ &= J([X, Y] + [X, \overline{Y}] + [\overline{X}, Y] + [\overline{X}, \overline{Y}]). \end{aligned}$$

But from integrability, $[X, Y]$ is also of type $(1,0)$, and $[\overline{X}, \overline{Y}]$ is of type $(0,1)$. So we can write this as

$$J([X', Y']) = (i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y]). \tag{4.18}$$

Next, the right hand side of (4.16) is

$$[X + \overline{X}, i(Y - \overline{Y})] = i([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{4.19}$$

Then (4.18) equals (4.19) if and only if

$$J[X, \overline{Y}] + J[\overline{X}, Y] = -i[X, \overline{Y}] + i[\overline{X}, Y]. \tag{4.20}$$

This is equivalent to

$$J(Re([X, \overline{Y}])) = Im([X, \overline{Y}]). \tag{4.21}$$

This says that $[X, \overline{Y}]$ is a vector field of type $(0,1)$. We can write the Lie bracket as

$$\begin{aligned}
[X, \overline{Y}] &= \Big[\sum_j X^j \frac{\partial}{\partial z^j}, \sum_k \overline{Y}^k \frac{\partial}{\partial \overline{z}^k} \Big] \\
&= - \sum_j \overline{Y}^k \Big(\frac{\partial}{\partial \overline{z}^k} X^j\Big) \frac{\partial}{\partial z^j} + \sum_k X^j \Big(\frac{\partial}{\partial z^j} \overline{Y}^k\Big) \frac{\partial}{\partial \overline{z}^k},
\end{aligned}$$

and the vanishing of the $(1,0)$ component is exactly (4.17).

Finally, for infinitesimal automorphisms X and Y, we want to show that

$$[X, Y] \mapsto \frac{1}{4}[X - iJX, Y - iJY] = \frac{1}{4}\Big([X, Y] - [JX, JY] - i([JX, Y] + [X, JY])\Big). \tag{4.22}$$

Since X and Y are both infinitesimal automorphisms, we know that JX and JY are also. We then have

$$\begin{aligned}
&\frac{1}{4}\Big([X, Y] - [JX, JY] - i([JX, Y] + [X, JY])\Big) \\
&\qquad = \frac{1}{4}\Big([X, Y] - J([JX, Y]) - i(J([X, Y]) + J([X, Y])\Big) \\
&\qquad = \frac{1}{4}\Big([X, Y] - J^2([X, Y]) - 2iJ([X, Y])\Big) \\
&\qquad = \frac{1}{2}\Big([X, Y] - iJ([X, Y])\Big),
\end{aligned} \tag{4.23}$$

which is the indeed the image of the real Lie bracket $[X, Y]$. □

Proposition 4.7. *There is a first order differential operator*

$$\overline{\partial} : \Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}), \tag{4.24}$$

such that a vector field Z is holomorphic if and only if $\overline{\partial}(Z) = 0$.

Proof. Choose local holomorphic coordinates $\{z^j\}$, and write any section of Z of $T^{1,0}$, locally as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}. \tag{4.25}$$

Then define

$$\overline{\partial}(Z) = \sum_j (\overline{\partial} Z^j) \otimes \frac{\partial}{\partial z^j}. \tag{4.26}$$

This is in fact a well-defined global section of $\Lambda^{0,1} \otimes T^{1,0}$ since the transition functions of the bundle $T^{1,0}$ corresponding to a change of holomorphic coordinates are holomorphic.

To see this, if we have an overlapping coordinate system $\{w^j\}$ and

$$Z = \sum W^j \frac{\partial}{\partial w^j}. \tag{4.27}$$

Note that

$$\frac{\partial}{\partial z^j} = \frac{\partial w^k}{\partial z^j} \frac{\partial}{\partial w^j}, \tag{4.28}$$

which implies that

$$W^j = Z^p \frac{\partial w^j}{\partial z^p}. \tag{4.29}$$

We compute

$$\begin{aligned}
\overline{\partial}(Z) &= \sum \overline{\partial}(W^j) \otimes \frac{\partial}{\partial w^j} = \sum \overline{\partial}(Z^p \frac{\partial w^j}{\partial z^p}) \otimes \frac{\partial z^q}{\partial w^j} \frac{\partial}{\partial z^q} \\
&= \sum \frac{\partial w^j}{\partial z^p} \frac{\partial z^q}{\partial w^j} \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \delta_p^q \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \overline{\partial}(Z^j) \otimes \frac{\partial}{\partial z^j}.
\end{aligned}$$

□

5 Lecture 5

5.1 Integrability: power series method

We begin with a lemma.

Lemma 5.1. *Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear mapping satisfying $J^2 = -Id$. Then there exists an invertible matrix A such that $A^{-1} J A = J_{Euc}$.*

Proof. For $X \in \mathbb{R}^{2n}$, define

$$(a + ib)X = aX + bJX. \tag{5.1}$$

Then $\mathbb{R}^{2n}$ becomes an n-dimensional complex vector space. Let $X_1, \ldots, X_n$ be a complex basis. Then $X_1, JX_1, \ldots, X_n, JX_n$ is a basis of $\mathbb{R}^{2n}$ as a real vector space, and J is obviously standard in this basis. □

Remark 5.2. We will see below that we can even take A to be an orthogonal matrix.

We can assume that J is analytic around the origin in $\mathbb{C}^n$, then J has a convergent power series expansion

$$J = \sum_{j=0}^{\infty} J_j, \tag{5.2}$$

where J_j is a real polynomial which is homogeneous of degree j. By Lemma 5.1, we can assume that $J_0 = J_{Euc}$, after a linear change of coordinates.

For $k = 1, \ldots, n$, let us try and find a function $f : U \to \mathbb{C}$, where

$$f^k = \sum_{j=1}^{\infty} f_j^k, \tag{5.3}$$

where f_j^k is homogeneous of degree j, and which satisfies

$$\overline{\partial}_J f^k = 0, \;\; f_1^k = z^k. \tag{5.4}$$

Then by the inverse function theorem, $(f^1, \ldots, f^n)$ will form a holomorphic coordinate system in some possibly smaller neighborhood of the origin.

In the following, we will omit the superscript k. The equation we need to solve is

$$\begin{aligned} 0 = \overline{\partial}_J f &= \frac{1}{2}(df + iJdf) \\ &= \frac{1}{2}(df + i(J - J_0 + J_0)df) \\ &= \overline{\partial}_0 f + \frac{i}{2}(J - J_0)df. \end{aligned} \tag{5.5}$$

Writing this out term-by-term, we have the system

$$\begin{aligned} \overline{\partial}_0 f_1 &= 0 \\ \overline{\partial}_0 f_2 &= -\frac{i}{2} J_1 df_1 \\ \overline{\partial}_0 f_3 &= -\frac{i}{2}\Big(J_2 df_1 + J_1 df_2\Big), \end{aligned}$$

we see the general formula is

$$\overline{\partial}_0 f_l = -\frac{i}{2} \sum_{j+k=l} J_j df_k. \tag{5.6}$$

Proposition 5.3. *If f_j solves the above system for $j = 1, \ldots, p$, then the expression*

$$H_p = -\frac{i}{2} \sum_{j+k=p} J_j df_k \tag{5.7}$$

is a form of type $(0,1)$ with respect to J_0, and satisfies $\overline{\partial}_0 H_p = 0$.

Proof. The assumption implies that $f = \sum_{j=1}^{p} f_j$ satisfies

$$\overline{\partial}_J f = O(|z|^p) = H_p + O(|z|^{p+1}). \tag{5.8}$$

Since $\overline{\partial}_J f$ is of type $(0,1)$ with respect to J, we have

$$J\overline{\partial}_J f = -i\overline{\partial}_J f. \tag{5.9}$$

Expanding both sides of this equation yields

$$(J_0 + J_1 + \dots)(H_p + O(|z|^{p+1}) = -i(H_p + O(|z|^{p+1}), \tag{5.10}$$

and the leading term of this equation says that

$$J_0 H_p = -iH_p, \tag{5.11}$$

so H_p is of type $(0,1)$ with respect to J_0, as claimed.

For the next step, we use the assumption of integrability of J which implies that the operator $\overline{\partial}_J : \Lambda^{0,1}(J) \to \Lambda^{0,2}(J)$ defined by $\overline{\partial}_J \alpha = \Pi_{\Lambda^{0,2}(J)} d\alpha$ satisfies

$$\overline{\partial}_J \overline{\partial}_J f = 0, \tag{5.12}$$

for any function f.

Note that for $\alpha \in \Lambda^{0,1}(J)$, $J\alpha = -i\alpha$, so from Proposition 3.5, we have that

$$\overline{\partial}_J \alpha = \frac{1}{2}(d - id^c)\alpha = \frac{1}{2}(d\alpha - iJdJ\alpha) = \frac{1}{2}(d\alpha - Jd\alpha) \tag{5.13}$$

Expanding this, we obtain

$$\overline{\partial}_J \alpha = \frac{1}{2}(d\alpha - (J - J_0 + J_0)d\alpha) = \frac{1}{2}(d\alpha - J_0 d\alpha) - \frac{1}{2}(J - J_0)d\alpha. \tag{5.14}$$

Now we plug in $\alpha = \overline{\partial}_J f$, and by assumption

$$0 = \overline{\partial}_J \overline{\partial}_J f = \overline{\partial}_J (H_p + O(|z|^{p+1}) = \frac{1}{2}(dH_p - J_0 dH_p) + O(|z|^p). \tag{5.15}$$

But from the first part of the proof, H_p is of type $(0,1)$ with respect to J_0, so we conclude that

$$0 = \frac{1}{2}(dH_p - J_0 dH_p) = \overline{\partial}_0 H_p. \tag{5.16}$$

□

Proposition 5.4. *For each $1 \leq p < \infty$, there exists $f = \sum_{j=1}^{p} f_j$ satisfying $\overline{\partial}_J f = O(|z|^p)$.*

Proof. We prove this by induction. For $p = 1$, we have $f = z^k$, and then

$$\overline{\partial}_J z^k = \overline{\partial}_0 z^k + \frac{i}{2}(J - J_0)dz^k = 0 + O(|z|), \tag{5.17}$$

Assume that we have found a solution for $j = 1 \dots p$. Let $f = \sum_{j=1}^{p} f_j$, by the induction assumption, we have

$$\overline{\partial}_J f = H_p + O(|z|^{p+1}), \tag{5.18}$$

and by the above, we need to solve the equation

$$\overline{\partial}_0 f_{p+1} = H_p = -\frac{i}{2} \sum_{j+k=p} J_j df_j. \tag{5.19}$$

From Proposition 5.3, H_p is a form of type $(0,1)$ with respect to J_0, and satisfies $\overline{\partial}_0 H_p = 0$. We can therefore write

$$H_p = \alpha_{\bar{j}} d\overline{z}^j, \tag{5.20}$$

where

$$\frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^l} = \frac{\partial \alpha_{\bar{l}}}{\partial \overline{z}^j}, \; j, l = 1, \dots, n. \tag{5.21}$$

Define

$$f_{p+1} = \int_0^1 \sum_{j=1}^{n} \overline{z}^j \alpha_{\bar{j}}(z, t\bar{z}) dt. \tag{5.22}$$

Then we compute

$$\begin{aligned}
\frac{\partial f_{p+1}}{\partial \overline{z}^k} &= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^{n} \overline{z}^j \frac{\partial}{\partial \overline{z}^k}\Big(\alpha_{\bar{j}}(z, t\bar{z})\Big)\Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^{n} \overline{z}^j \frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^k}(z, t\bar{z}) t\Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^{n} \overline{z}^j \frac{\partial \alpha_{\bar{k}}}{\partial \overline{z}^j}(z, t\bar{z}) t\Big) \\
&= \int_0^1 \frac{d}{dt}\Big(t \alpha_{\bar{k}}(z, t\bar{z})\Big) dt = \alpha_{\bar{k}}(z, \bar{z}).
\end{aligned} \tag{5.23}$$

5.2 Convergence

Trying to prove convergence of the above procedure is a difficult exercise, which you can try to prove on your own. In the next lecture, we will see a nicer way to prove this which avoids such convergence issues.

□

6 Lecture 6

6.1 Integrability: holomorphic Frobenius method

Let (M^{2n}, J) be an almost complex manifold. Choose local coordinates $(x^1, \ldots, x^{2n})$ near a point $p \in M$. The complex vector fields

$$X_j = \frac{\partial}{\partial x^k} - iJ\frac{\partial}{\partial x^k} \tag{6.1}$$

for $j = 1 \ldots 2n$ are sections of $T^{1,0}(U)$, where U is a small neighborhood of p. Without loss of generality, we can assume that $\operatorname{span}\{X_j, j = 1, \ldots n\} = T^{1,0}(U)$.

We want to find complex valued functions $\tilde{x}^j = \tilde{x}^j(x^1, \ldots, x^{2n}), j = 1, \ldots n$, so that

$$\frac{\partial}{\partial \tilde{x}^j} = \sum_{k=1}^{n} a_j^k X_k, \tag{6.2}$$

where $a_j^k : U \to \mathbb{C}$, and such that $\tilde{x}^j, j = 1, \ldots n$ form a local coordinate system around p.

The assumption that J is integrable, implies that

$$[X_k, X_l] = \sum_{j=1}^{n} b_{kl}^j X_j, \tag{6.3}$$

where $b_{kl}^j : U \to \mathbb{C}$.

We can write

$$X_k = \sum_{j=1}^{2n} f_k^j(x) \frac{\partial}{\partial x^j}, \tag{6.4}$$

where $f_k^j : U \to \mathbb{C}$, and taking a conjugate, we have

$$\overline{X}_k = \sum_{j=1}^{2n} g_k^j(x) \frac{\partial}{\partial x^j}, \tag{6.5}$$

where $g_k^j = \overline{f}_k^j$.

Let us assume that p is the origin in $\mathbb{R}^{2n}$. If we assume everything is real analytic, then we can expand these equations in a power series around 0, and replacing the x^j with complex coordinates z^j, we can define the holomorphic vector fields

$$\Xi_k = \sum_{j=1}^{2n} f_k^j(z) \frac{\partial}{\partial z^j} \tag{6.6}$$

$$\tilde{\Xi}_k = \sum_{j=1}^{2n} g_k^j(z) \frac{\partial}{\partial z^j}. \tag{6.7}$$

in some neighborhood of the origin $\tilde{U} \subset \mathbb{C}^{2n}$. Let $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$ be the totally real subspace, we have that

$$\Xi_k|_{\mathbb{R}^{2n}} = X_k, \ \tilde{\Xi}_k|_{\mathbb{R}^{2n}} = \overline{X}_k \tag{6.8}$$

Notice that the collection $\Xi_k, \tilde{\Xi}_k, k = 1, \dots n$ are linearly independent, in some possibly smaller neighborhood.

The equation (6.3) implies that

$$[\Xi_k, \Xi_l] = \sum_{j=1}^{n} \beta_{kl}^j \Xi_j, \tag{6.9}$$

where $\beta_{kl}^j : \tilde{U} \to \mathbb{C}$, and $\beta_{kl}^j|_{\mathbb{R}^{2n}} = b_{kl}^j$.

This says that

$$\mathcal{D} = \operatorname{span}\{\Xi_k, k = 1, \dots, n\} \subset T^{1,0}(\tilde{U}) \subset T^{1,0}(\mathbb{C}^{2n}) \tag{6.10}$$

is an integrable distribution, which is *holomorphic*, i.e., it is locally spanned by holomorphic vector fields, which is equivalent to saying that $\mathcal{D}$ is a complex rank n holomorphic subbundle of the complex rank $2n$ holomorphic bundle $T^{1,0}(\tilde{U})$.

By the holomorphic version of the Frobenius theorem, there exists a holomorphic mapping

$$\tilde{F} : \tilde{U} \to \mathbb{C}^n \tag{6.11}$$

such that the fibers of F are leaves of the foliation defined by $\mathcal{D}$. That is,

$$\mathcal{D} = Ker\{\tilde{F}_* : T^{1,0}(\tilde{U}) \to T^{1,0}\mathbb{C}^n\}, \tag{6.12}$$

and then $\tilde{F}|_{\mathbb{R}^{2n}} : U \to \mathbb{C}^n$ is the desired holomorphic coordinate system in a small neighborhood of the origin.

To prove the holomorphic Frobenius Theorem, let $D = Re(\mathcal{D})$, which is a real rank $2n$ subbundle of the real rank $4n$ bundle $T(\tilde{U})$. We claim that D is integrable. This follows from the formula

$$[Re(X), Re(Y)] = \frac{1}{4} Re[X, Y], \tag{6.13}$$

which holds for *holomorphic* vector fields, as proved above.

By the real Frobenius Theorem, there exists a mapping $F : \tilde{U} \to \mathbb{R}^{2n}$, denoted by $F = (f_1, \dots, f_{2n})$, such that fibers of F are leaves of the foliation defined by D. That is, $D = Ker\{F_* : T\tilde{U} \to T\mathbb{R}^{2n}\}$. One can then show that

$$\tilde{F} = (f_1 + if_2, \dots, f_{2n-1} + if_{2n}) : \tilde{U} \to \mathbb{C}^n \tag{6.14}$$

is holomorphic.

7 Lecture 7

7.1 Hermitian metrics

We next consider (M, J, g) where g is a Riemannian metric, and we assume that g and J are compatible. That is,

$$g(X, Y) = g(JX, JY). \tag{7.1}$$

The metric g is called an almost-Hermitian metric. If J is also integrable, then g is called Hermitian. We extend g by complex linearity to a symmetric inner product on $T \otimes \mathbb{C}$. The following will be useful later.

Proposition 7.1. *There exist elements $\{X_1, \dots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{7.2}$$

is an ONB for $\mathbb{R}^{2n}$ with respect to g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{7.3}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of $span\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. □

To a Hermitian metric $(\mathbb{R}^{2n}, J, g)$ we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{7.4}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{7.5}$$

Since

$$\omega(JX, JY) = \omega(X, Y), \tag{7.6}$$

this form is a real form of type $(1, 1)$, and is called the *Kähler form* or *fundamental 2-form*.

In Euclidean space, this form is

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{7.7}$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{7.8}$$

Note that this defines an orientation on $\mathbb{C}^n$, which we will refer to as the natural orientation. Note also that

$$\omega^n = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{7.9}$$

Corollary 7.2. *Any almost complex manifold (M, J) is orientable.*

Proof. Given J, there always exists an almost-Hermitian metric h with respect to J. To see this, let g be any Riemannian metric, and let

$$h(X, Y) = g(X, Y) + g(JX, JY). \tag{7.10}$$

Let $\omega = h(JX, Y)$ be the Kähler form, then $\omega^n \in \Lambda^{n,n}_{\mathbb{R}} \cong \Lambda^{2n}_{\mathbb{R}}$ is a nowhere vanishing n-form. It is nowhere-vanishing since at any point, we can assume we are Euclidean by Proposition 7.1. □

The following proposition gives a fundamental relation between the covariant derivative of J, the exterior derivative of ω and the Nijenhuis tensor.

Proposition 7.3. *Let (M, g, J) be an almost Hermitian manifold. Then*

$$2g((\nabla_X J)Y, Z) = -d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2} g(N(Y, Z), JX). \tag{7.11}$$

Proof. The covariant derivative of an endomorphism is given by

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y) \tag{7.12}$$

so we have

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) - g(J(\nabla_X Y), Z). \tag{7.13}$$

Since g is J-invariant, and $J^2 = -Id$, it follows that

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) + g(\nabla_X Y, JZ). \tag{7.14}$$

The Riemannian connection is defined by

$$\begin{aligned} \langle \nabla_X Y, Z \rangle = \frac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & -\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \Big). \end{aligned} \tag{7.15}$$

Now apply formula (7.15) to both terms on the right hand side of (7.14) to obtain

$$\begin{aligned} 2g((\nabla_X J)Y, Z) = {} & Xg(JY, Z) + JYg(X, Z) - Zg(X, JY) \\ & - g(JY, [X, Z]) - g(Z, [JY, X]) + g(X, [Z, JY]) \\ & + Xg(Y, JZ) + Yg(JZ, X) - JZg(X, Y) \\ & - g(Y, [X, JZ]) - g(JZ, [Y, X]) + g(X, [JZ, Y]). \end{aligned} \tag{7.16}$$

Next, using (3.7), we compute

$$\begin{aligned} d\omega(X, JY, JZ) &= X\omega(JY, JZ) - JY\omega(X, JZ) + JZ\omega(X, JY) \\ &\quad - \omega([X, JY], JZ) + \omega([X, JZ], JY) - \omega([JY, JZ], X]) \\ &= Xg(J^2Y, JZ) - JYg(JX, JZ) + JZg(JX, JY) \\ &\quad - g(J[X, JY], JZ) + g(J[X, JZ], JY) - g(J[JY, JZ], X]) \\ &= -Xg(Y, JZ) - JYg(X, Z) + JZg(X, Y) \\ &\quad - g([X, JY], Z) + g([X, JZ], Y) + g([JY, JZ], JX]). \end{aligned} \tag{7.17}$$

The next term is

$$\begin{aligned} d\omega(X, Y, Z) &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &\quad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X]) \\ &= Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) \\ &\quad - g(J[X, Y], Z) + g(J[X, Z], Y) - g(J[Y, Z], X]). \end{aligned} \tag{7.18}$$

The last term is

$$\begin{aligned} &\frac{1}{2}g(N(Y, Z), JX) \\ &\quad = g([JY, JZ], JX) - g([Y, Z], JX) - g(J[Y, JZ], JX) - g(J[JY, Z], JX) \\ &\quad = g([JY, JZ], JX) - g([Y, Z], JX) - g([Y, JZ], X) - g([JY, Z], X) \end{aligned} \tag{7.19}$$

We then obtain the right hand side of (7.11) is

$$\begin{aligned} &- d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX) \\ &= Xg(Y, JZ) + JYg(X, Z) - JZg(X, Y) \\ &\quad + g([X, JY], Z) - g([X, JZ], Y) - g([JY, JZ], JX]) \\ &+ Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) \\ &\quad - g(J[X, Y], Z) + g(J[X, Z], Y) - g(J[Y, Z], X]) \\ &+ g([JY, JZ], JX) - g([Y, Z], JX) - g([Y, JZ], X) - g([JY, Z], X). \end{aligned} \tag{7.20}$$

The first two terms of the last line cancel out with terms on the previous lines, so this simplifies to

$$\begin{aligned} &- d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX) \\ &= Xg(Y, JZ) + JYg(X, Z) - JZg(X, Y) + g([X, JY], Z) - g([X, JZ], Y) \\ &+ Xg(JY, Z) - Yg(JX, Z) + Zg(JX, Y) - g(J[X, Y], Z) + g(J[X, Z], Y) \\ &- g([Y, JZ], X) - g([JY, Z], X), \end{aligned} \tag{7.21}$$

and each of these 12 terms appears exactly once in (7.16). □

Corollary 7.4. *If (M, g, J) is Hermitian, then $d\omega = 0$ if and only if J is parallel.*

Proof. Since $N = 0$, this follows immediately from (7.11). □

Corollary 7.5. *If (M, g, J) is almost Hermitian, $\nabla J = 0$ implies that $d\omega = 0$ and $N = 0$.*

Proof. If J is parallel, then ω is also. The corollary follows from the fact that the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ can be written in terms of covariant differentiation.

$$d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{7.22}$$

which follows immediately from (1.33) using normal coordinates around a point. This shows that a parallel form is closed, so the corollary then follows from (7.11). □

Definition 7.6. An almost Hermitian manifold (M, g, J) is *Kähler* if J is integrable and $d\omega = 0$, or equivalently, if $\nabla J = 0$,

Note that if (M, g, J) is Kähler, then ω is a parallel $(1, 1)$-form.

Proposition 7.7. *There are the following equivalences:*

- *M^{2n} is almost complex if and only if the structure group of the principal frame bundle can be reduced from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$.*
- *(M^{2n}, g, J) is almost Hermitian if and only if the structure group of the bundle of orthonormal frames can be reduced from $O(2n)$ to $\mathrm{U}(n)$.*
- *(M^{2n}, g, J) is Kähler if and only if the holonomy group is contained in $\mathrm{U}(n)$.*

8 Lecture 8

8.1 Complex tensor notation

Choosing any real basis of the form $\{X_1, JX_1, \ldots, X_n, JX_n\}$, let us abbreviate

$$Z_\alpha = \frac{1}{2}\Big(X_\alpha - iJX_\alpha\Big) \tag{8.1}$$

$$Z_{\overline{\alpha}} = \frac{1}{2}\Big(X_\alpha + iJX_\alpha\Big), \tag{8.2}$$

and define

$$g_{\alpha\beta} = g(Z_\alpha, Z_\beta) \tag{8.3}$$

$$g_{\overline{\alpha}\overline{\beta}} = g(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) \tag{8.4}$$

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) \tag{8.5}$$

$$g_{\overline{\alpha}\beta} = g(Z_{\overline{\alpha}}, Z_\beta). \tag{8.6}$$

Notice that

$$\begin{aligned} g_{\alpha\beta} = g(Z_\alpha, Z_\beta) &= \frac{1}{4} g(X_\alpha - iJX_\alpha, X_\beta - iJX_\beta) \\ &= \frac{1}{4}\Big(g(X_\alpha, X_\beta) - g(JX_\alpha, JX_\beta) - i(g(X_\alpha, JX_\beta) + g(JX_\alpha, X_\beta))\Big) \\ &= 0, \end{aligned}$$

since g is J-invariant, and $J^2 = -Id$. Similarly,

$$g_{\overline{\alpha}\overline{\beta}} = 0, \tag{8.7}$$

Also, from symmetry of g, we have

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) = g(Z_{\overline{\beta}}, Z_\alpha) = g_{\overline{\beta}\alpha}. \tag{8.8}$$

However, applying conjugation, since g is real we have

$$\overline{g_{\alpha\overline{\beta}}} = \overline{g(Z_\alpha, Z_{\overline{\beta}})} = g(Z_{\overline{\alpha}}, Z_\beta) = g(Z_\beta, Z_{\overline{\alpha}}) = g_{\beta\overline{\alpha}}, \tag{8.9}$$

which says that $g_{\alpha\overline{\beta}}$ is a Hermitian matrix.

We repeat the above for the fundamental 2-form ω, and define

$$\omega_{\alpha\beta} = \omega(Z_\alpha, Z_\beta) = i g_{\alpha\beta} = 0 \tag{8.10}$$

$$\omega_{\overline{\alpha}\overline{\beta}} = \omega(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = -i g_{\overline{\alpha}\overline{\beta}} = 0 \tag{8.11}$$

$$\omega_{\alpha\overline{\beta}} = \omega(Z_\alpha, Z_{\overline{\beta}}) = i g_{\alpha\overline{\beta}} \tag{8.12}$$

$$\omega_{\overline{\alpha}\beta} = \omega(Z_{\overline{\alpha}}, Z_\beta) = -i g_{\overline{\alpha}\beta}. \tag{8.13}$$

The first 2 equations are just a restatement that ω is of type $(1,1)$. Also, note that

$$\omega_{\alpha\overline{\beta}} = i g_{\alpha\overline{\beta}}, \tag{8.14}$$

defines a skew-Hermitian matrix.

On a complex manifold, the fundamental 2-form in holomorphic coordinates takes the form

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta. \tag{8.15}$$

Remark 8.1. Note that for the Euclidean metric, we have $g_{\alpha\overline{\beta}} = \frac{1}{2}\delta_{\alpha\beta}$, so

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{8.16}$$

Proposition 8.2. *(M, g, J) is Kähler if and only if in any local holomorphic coordinate system,*

$$\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} = \frac{\partial g_{k\overline{\beta}}}{\partial z^\alpha}, \tag{8.17}$$

Proof. If (M, g, J) is Kähler, then

$$
\begin{aligned}
0 = d\omega &= i \sum_{\alpha,\beta=1}^{n} (dg_{\alpha\overline{\beta}}) \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta=1}^{n} (\partial g_{\alpha\overline{\beta}} + \overline{\partial} g_{\alpha\overline{\beta}}) \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta=1}^{n} \Big\{ \sum_{k} \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{k}} dz^{k} \Big) + \sum_{k} \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^{k}} d\overline{z}^{k} \Big) \Big\} \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{k}} dz^{k} \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} + i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^{k}} d\overline{z}^{k} \wedge dz^{\alpha} \wedge d\overline{z}^{\beta}.
\end{aligned}
\tag{8.18}
$$

However, the first term is a form of type $(2, 1)$, and the second term is a form of type $(1, 2)$ so both sums must vanish, which is equivalent to (8.17). The converse follows by reversing the above calculation. □

We also see that the Kähler condition on a Hermitian manifold is equivalent to $\overline{\partial}\omega = 0$, which is also equivalent to $\partial\omega = 0$, since ω is real.

8.2 Existence of local Kähler potential

First, a special case of the $\overline{\partial}$-Poincaré lemma.

Lemma 8.3. *If α is a smooth $(0, 1)$-form in a closed ball $\overline{B} \subset \mathbb{C}^n$ satisfying $\overline{\partial}\alpha = 0$, then there exists $f : B \to \mathbb{C}$ such that $\alpha = \overline{\partial} f$.*

Proof. Write $\alpha = \sum_{j=1}^{n} \alpha_{\overline{j}} d\overline{z}^{j}$. Then

$$
0 = \overline{\partial}\alpha = \sum_{j,k=1}^{n} \frac{\partial \alpha_{\overline{j}}}{\partial \overline{z}^{k}} d\overline{z}^{k} \wedge d\overline{z}^{j}. \tag{8.19}
$$

This implies that

$$
\frac{\partial \alpha_{\overline{j}}}{\partial \overline{z}^{k}} = \frac{\partial \alpha_{\overline{k}}}{\partial \overline{z}^{j}} \tag{8.20}
$$

for all $1 \leq j, k \leq n$.

We want to find f such that $\partial f = \alpha$, which in components is

$$
\frac{\partial f}{\partial \overline{z}^{k}} = \alpha_{\overline{k}} \tag{8.21}
$$

for all $1 \leq k \leq n$.

Recall from one complex variable that if $B \subset \mathbb{C}$, and $g : \overline{B} \to \mathbb{C}$ is smooth, then there exists $f : B \to \mathbb{C}$ such that $\frac{\partial}{\partial \overline{z}} f = g$. The solution can be written explicitly as

$$
f(z) = \frac{1}{2\pi i} \int_{B} g(w) \frac{dw \wedge d\overline{w}}{w - z}. \tag{8.22}
$$

So we define

$$f(z^1, \dots z^n) = \frac{1}{2\pi i} \int_B \alpha_{\overline{1}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1}. \tag{8.23}$$

By the above remark, we have $\partial_{\overline{1}} f = \alpha_{\overline{1}}$. Next, for $k > 1$,

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}^k}(z^1, \dots z^n) &= \frac{1}{2\pi i} \int_B \frac{\partial}{\partial \overline{z}^k} \alpha_{\overline{1}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1} \\ &= \frac{1}{2\pi i} \int_B \frac{\partial}{\partial \overline{z}^1} \alpha_{\overline{k}}(w, z^2, \dots z^n) \frac{dw \wedge d\overline{w}}{w - z^1} \\ &= \alpha_{\overline{k}}(z^1, \dots, z^n), \end{aligned} \tag{8.24}$$

and we are done. □

We will prove the following very special property of Kähler metrics.

Proposition 8.4. *If (M, g, J) is Kähler then for each $p \in M$, there exists an open neighborhood U of p and a function $u : U \to \mathbb{R}$ such that $\omega = i\partial\overline{\partial}u$.*

Proof. Choose local homorphic coordinates z^j around p. Then in a ball B in these coordinates, since ω is a real closed 2-form, from the usual Poincaré lemma, there exists a real 1-form α such that $\omega = d\alpha$ in B. Next, write $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where $\alpha^{1,0}$ is a 1-form of type $(1, 0)$, and $\alpha^{0,1}$ is a 1-form of type $(0, 1)$. Since α is real, $\overline{\alpha^{1,0}} = \alpha^{0,1}$. Next,

$$\begin{aligned} \omega = d\alpha &= \partial\alpha + \overline{\partial}\alpha \\ &= \partial\alpha^{1,0} + \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0} + \overline{\partial}\alpha^{0,1} \end{aligned} \tag{8.25}$$

The first and last terms on the right hand side are forms of type $(2, 0)$ and $(0, 2)$, respectively. Since ω is of type $(1, 1)$, we must have $\overline{\partial}\alpha^{0,1} = 0$. Since we are in a ball in $\mathbb{C}^n$, the $\overline{\partial}$−Poincaré Lemma 8.3 says that there exists a function $f : B \to \mathbb{C}$ such that $\alpha^{0,1} = \overline{\partial} f$ in B. Substituting this into (8.25), we obtain

$$\omega = \partial\overline{\partial} f + \overline{\partial}\partial\overline{f} = i\partial\overline{\partial}(2Im(f)). \tag{8.26}$$

□

Proposition 8.5. *(M, g, J) is Kähler if and only if for each $p \in M$, there exists a holomorphic coordinate system around p such that*

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} (\delta_{jk} + O(|z|^2)_{jk}) dz^j \wedge d\overline{z}^k, \tag{8.27}$$

as $|z| \to 0$.

Proof. If this is true then $d\omega(p) = 0$ for any point p, so $d\omega \equiv 0$. Conversely, we can assume that $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$. From Proposition 8.4, we can find $u : B \to \mathbb{R}$ so that

$$u = c_0 + Re(c_{1j}z^j) + Re(c_{2ij}z^iz^j + c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3), \tag{8.28}$$

and $\omega = i\partial\overline{\partial}u$. But the first terms on the left hand side are in the kernel of the $\partial\overline{\partial}$-operator, so by subtracting these terms, we can assume that

$$u = Re(c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3). \tag{8.29}$$

Then since $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$, we have that

$$u = \frac{1}{2}|z|^2 + Re\{a_{jkl}z^jz^kz^l + b_{jkl}\overline{z}^jz^kz^l)\} + O(|z|^4). \tag{8.30}$$

Consider the coordinate change

$$z^k = w^k + \sum c_{klm}w^lw^m. \tag{8.31}$$

This will eliminate the b_{jkl} terms in the expansion of u, and the remaining cubic terms are annihilated by the $\partial\overline{\partial}$-operator, so by subtracting those terms, we can arrange that

$$u = \frac{1}{2}|w|^2 + O(|w|^4), \tag{8.32}$$

and (8.27) follows. □

9 Lecture 9

9.1 L^2 adjoints

For the real operator $d : \Lambda^p \to \Lambda^{p+1}$, the formal L^2-adjoint d^* is defined by

$$\int_M \langle d^*\alpha, \beta\rangle dV = \int_M \langle \alpha, d\beta\rangle dV, \tag{9.1}$$

where $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$, and where $\langle\cdot,\cdot\rangle = g(\cdot,\cdot)$, and dV is the oriented Riemannian volume element.

The Riemannian inner product on forms extends by complex linearity to an inner product on complex valued forms. For α and β be sections of $\Lambda^k_{\mathbb{C}}$, we define the Hermitian inner product of α and β to be

$$(\alpha, \beta) = g(\alpha, \overline{\beta}). \tag{9.2}$$

The formula (9.1) holds for complex valued forms. Replacing β with $\overline{\beta}$, we have

$$\int_M \langle d^*\alpha, \overline{\beta}\rangle dV = \int_M \langle \alpha, d\overline{\beta}\rangle dV. \tag{9.3}$$

But since d is a real operator, $d\overline{\beta} = \overline{d\beta}$, so we can write this as

$$\int_M (d^*\alpha, \beta) dV = \int_M (\alpha, d\beta) dV. \tag{9.4}$$

That is, d^* is the L^2 adjoint of d with respect to the Hermitian inner product.

We next want to compute the formal L^2 adjoints of other operators. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{p,q+1}), \tag{9.5}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p,q+1}) \xrightarrow{\overline{\partial}^*} \Gamma(\Lambda^{p,q}), \tag{9.6}$$

is defined as follows. For $\alpha \in \Gamma(\Lambda^{p,q+1})$ and $\beta \in \Gamma(\Lambda^{p,q})$, we have

$$\int_M (\alpha, \overline{\partial}\beta) dV = \int_M (\overline{\partial}^* \alpha, \beta) dV, \tag{9.7}$$

where dV denotes the Riemannian volume element. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\partial} \Gamma(\Lambda^{p+1,q}), \tag{9.8}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p+1,q}) \xrightarrow{\partial^*} \Gamma(\Lambda^{p,q}), \tag{9.9}$$

is defined similarly.

The Hodge Laplacian is $\Delta_H : \Lambda^p \to \Lambda^p$ defined by

$$\Delta_H = d^*d + dd^*. \tag{9.10}$$

We also have the following Laplacians on (p,q)-forms

$$\Delta_\partial : \Lambda^{p,q} \to \Lambda^{p,q} \tag{9.11}$$
$$\Delta_{\overline{\partial}} : \Lambda^{p,q} \to \Lambda^{p,q}. \tag{9.12}$$

are defined by

$$\Delta_\partial = \partial^*\partial + \partial\partial^* \tag{9.13}$$
$$\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*. \tag{9.14}$$

Remark 9.1. By definition, Δ_∂ and $\Delta_{\overline{\partial}}$ preserve the type, but we do not know whether Δ_H maps $\Lambda^{p,q}$ to $\Lambda^{p,q}$ i.e., there is no obvious reason why it should preserve the type.

9.2 Hodge star operator

For a real oriented Riemannian manifold of dimension n, the Hodge star operator is a mapping

$$* : \Lambda^p \to \Lambda^{n-p} \tag{9.15}$$

defined by

$$\alpha \wedge *\beta = g_{\Lambda^p}(\alpha, \beta) dV_g, \tag{9.16}$$

for $\alpha, \beta \in \Lambda^p$, where dV_g is the oriented Riemannian volume element. Note that

$$*^2 = (-1)^{p(n-p)} Id_{\Lambda^p}. \tag{9.17}$$

The Hodge star operator yields an explicit formula for d^*.

Proposition 9.2. *On a Riemannian manifold (M, g), for $\alpha \in \Omega^p(M)$, we have*

$$d^*\alpha = (-1)^{n(p+1)+1} * d * \omega. \tag{9.18}$$

Proof. For $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$, we compute

$$\begin{aligned}
\int_M \langle \alpha, d\beta \rangle dV &= \int_M d\beta \wedge *\alpha \\
&= \int_M \Big(d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big) \\
&= \int_M (-1)^{p+(n-p+1)(p-1)} \beta \wedge * * d * \alpha \\
&= \int_M \langle \beta, (-1)^{n(p+1)+1} * d * \alpha \rangle dV \\
&= \int_M \langle \beta, d^*\alpha \rangle dV.
\end{aligned} \tag{9.19}$$

□

If M is a complex manifold of complex dimension $m = n/2$, and g is a Hermitian metric, then the Hodge star extends to the complexification

$$* : \Lambda^p \otimes \mathbb{C} \to \Lambda^{2m-p} \otimes \mathbb{C}. \tag{9.20}$$

Proposition 9.3. *We have*

$$* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}. \tag{9.21}$$

Proof. This is easily seen to hold on $\mathbb{C}^n$, therefore it holds any any point of a Hermitian manifold (it is not a differential operator). □

Therefore the operator

$$\bar{*}: \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \tag{9.22}$$

defined by

$$\bar{*}\alpha = \overline{*\alpha} \tag{9.23}$$

is a $\mathbb{C}$-antilinear mapping and satisfies

$$\alpha \wedge \bar{*}\beta = g_{\Lambda^p}(\alpha, \overline{\beta}) dV_g. \tag{9.24}$$

for $\alpha, \beta \in \Lambda^p \otimes \mathbb{C}$.

Proposition 9.4. *The L^2-adjoints of $d, \overline{\partial}, \overline{\partial}^*$ are given by*

$$d^* = -\bar{*}\ d\ \bar{*} \tag{9.25}$$
$$\partial^* = -\bar{*}\ \partial\ \bar{*} \tag{9.26}$$
$$\overline{\partial}^* = -\bar{*}\ \overline{\partial}\ \bar{*}, \tag{9.27}$$

Proof. The dimension of an almost complex manifold is even, so know that $d^* = -*d*$. Taking a conjugate of this equation yields the first formula. Apply the first formula to $d = \partial + \overline{\partial}$, we have

$$\partial^* + \overline{\partial}^* = d^* = -\bar{*}\ d\ \bar{*} = -\bar{*}\ \partial\ \bar{*} - \bar{*}\ \overline{\partial}\ \bar{*} \tag{9.28}$$

Considering the degrees of the operators on the right hand side yields the last 2 formulas. □

Corollary 9.5. *On a Hermitian manifold, we have*

$$\Delta_{\overline{\partial}}\bar{*} = \bar{*}\Delta_{\overline{\partial}} \tag{9.29}$$

Proof. We compute on $\Lambda^k_{\mathbb{C}}$,

$$\Delta_{\overline{\partial}}\bar{*} = (\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)\bar{*} = (-\bar{*}\overline{\partial}\bar{*}\overline{\partial} - \overline{\partial}\bar{*}\overline{\partial}\bar{*})\bar{*} = -\bar{*}\overline{\partial}\bar{*}\overline{\partial}\bar{*} + (-1)^{k+1}\overline{\partial}\bar{*}\overline{\partial} \tag{9.30}$$

On the other hand,

$$\bar{*}\Delta_{\overline{\partial}} = \bar{*}(-\bar{*}\overline{\partial}\bar{*}\overline{\partial} - \overline{\partial}\bar{*}\overline{\partial}\bar{*}) = (-1)^{k+1}\overline{\partial}\bar{*}\overline{\partial} - \bar{*}\overline{\partial}\bar{*}\overline{\partial}\bar{*}. \tag{9.31}$$

□

10 Lecture 10

10.1 Serre duality

Letting

$$\mathbb{H}^{p,q}(M,g) = \{\alpha \in \Lambda^{p,q} | \Delta_{\overline{\partial}}\alpha = 0\}, \tag{10.1}$$

Hodge theory tells us that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathbb{H}^{p,q}(M,g), \tag{10.2}$$

is finite-dimensional, and that

$$\Lambda^{p,q} = \mathbb{H}^{p,q}(M,g) \oplus Im(\Delta_{\overline{\partial}}) \tag{10.3}$$

$$= \mathbb{H}^{p,q}(M,g) \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^*), \tag{10.4}$$

with this being an orthogonal direct sum in L^2.

Corollary 10.1. *Let (M,J) be a compact complex manifold of complex dimension n. Then*

$$H^{p,q}_{\overline{\partial}}(M) \cong (H^{n-p,n-q}_{\overline{\partial}}(M))^*, \tag{10.5}$$

and therefore

$$b^{p,q}(M) = b^{n-p,n-q}(M) \tag{10.6}$$

Proof. From Corollary 9.5, the mapping $\overline{*}$ preserves the space of harmonic forms, and is invertible. The result then follows from Hodge theory. The dual appears since the operator $\overline{*}$ is $\mathbb{C}$-antilinear. □

10.2 The Laplacian on a Kähler manifold

Let L denote the mapping

$$L : \Lambda^{p,q} \to \Lambda^{p+1,q+1} \tag{10.7}$$

given by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form. Define

$$\Lambda \equiv L^* : \Lambda^{p,q} \to \Lambda^{p-1,q-1}. \tag{10.8}$$

Proposition 10.2. *If $(M, J.g)$ is Kähler then*

$$[\Lambda, \partial] = i\overline{\partial}^*, \ [\Lambda, \overline{\partial}] = -i\partial^*, \ [\Lambda, d] = -(d^c)^* \tag{10.9}$$

$$[L, \partial^*] = i\overline{\partial}, \ [L, \overline{\partial}^*] = -i\partial, \ [L, d^*] = -d^c. \tag{10.10}$$

Proof. Note that the second identity is the conjugate of the first. Therefore, if the first identity is true,

$$[\Lambda, d] = [\Lambda, \partial + \overline{\partial}] = [\Lambda, \partial] + [\Lambda, \overline{\partial}] = i\overline{\partial}^* - i\partial^* = (-i(\overline{\partial} - \partial))^* = -(d^c)^*, \quad (10.11)$$

then the third identity follows. The last three identities are just the adjoints of the first three.

So to prove all of these identities, we only need to prove the first. To prove the first identity, one proves this for $\mathbb{C}^n$ with the standard Kähler form. The proof is a 2 page calculation, and is left as an exercise. Then for an arbitrary Kähler manifold, the identity follows by using Kähler normal coordinates at any point, and the fact that the identity only depends on the metric and its first derivatives at the point. □

On a Kähler manifold, we have the following very special occurrence.

Proposition 10.3. *For $\alpha \in \Gamma(\Lambda^{p,q})$, if (M, J, g) is Kähler, then*

$$\Delta_H \alpha = 2\Delta_\partial \alpha = 2\Delta_{\overline{\partial}} \alpha. \quad (10.12)$$

Proof. We first show that

$$\Delta_H = \Delta_\partial + \Delta_{\overline{\partial}}. \quad (10.13)$$

To see this

$$\begin{aligned} \Delta_H = dd^* + d^*d = &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= \partial\partial^* + \partial^*\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} + \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial} \\ &= \Delta_\partial + \Delta_{\overline{\partial}} + \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial}. \end{aligned} \quad (10.14)$$

Using Proposition 10.2,

$$\begin{aligned} i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) &= \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ &= \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial \\ &= \partial\Lambda\partial - \partial\Lambda\partial = 0. \end{aligned} \quad (10.15)$$

The sum of the last two terms in (10.14) also vanishes, just by taking the conjugate of the above computation, and (10.13) follows.

To finish the proof, we show that

$$\Delta_\partial = \Delta_{\overline{\partial}} \quad (10.16)$$

To see this, we again use Proposition 10.2, to compute

$$\begin{aligned} i\Delta_\partial = i\partial\partial^* + i\partial^*\partial &= \partial(-[\Lambda, \overline{\partial}]) - [\Lambda, \overline{\partial}]\partial \\ &= \partial\overline{\partial}\Lambda - \partial\Lambda\overline{\partial} - \Lambda\overline{\partial}\partial + \overline{\partial}\Lambda\partial. \end{aligned} \quad (10.17)$$

Also, we compute

$$
\begin{aligned}
i\Delta_{\overline{\partial}} = i\overline{\partial}\overline{\partial}^* + i\overline{\partial}^*\overline{\partial} &= \overline{\partial}([\Lambda,\partial]) + [\Lambda,\partial]\overline{\partial} \\
&= \overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial} \qquad (10.18)\\
&= \overline{\partial}\Lambda\partial + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial},
\end{aligned}
$$

from which (10.16) follows.

□

Using Hodge theory, we get the following structure on the cohomology of a Kähler manifold.

Proposition 10.4. *If* (M, J, g) *is a compact Kähler manifold, then*

$$H^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M), \tag{10.19}$$

and

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{q,p}_{\overline{\partial}}(M)^*. \tag{10.20}$$

Consequently,

$$b^k(M) = \sum_{p+q=k} b^{p,q}(M) \tag{10.21}$$

$$b^{p,q}(M) = b^{q,p}(M). \tag{10.22}$$

Proof. This follows because if a harmonic k-form is decomposed as

$$\phi = \phi^{p,0} + \phi^{p-1,1} + \cdots + \phi^{1,p-1} + \phi^{0,p}, \tag{10.23}$$

then

$$0 = \Delta_H\phi = 2\Delta_{\overline{\partial}}\phi^{p,0} + 2\Delta_{\overline{\partial}}\phi^{p-1,1} + \cdots + 2\Delta_{\overline{\partial}}\phi^{1,p-1} + 2\Delta_{\overline{\partial}}\phi^{0,p}, \tag{10.24}$$

therefore

$$\Delta_{\overline{\partial}}\phi^{p-k,k} = 0, \tag{10.25}$$

for $k = 0 \ldots p$.

Next,

$$\overline{\Delta_{\overline{\partial}}\phi} = \Delta_{\partial}\overline{\phi}, \tag{10.26}$$

so conjugation sends harmonic forms to harmonic forms. □

This yields a topologicial obstruction for a complex manifold to admit a Kähler metric:

Corollary 10.5. *If (M, J, g) is a compact Kähler manifold, then the odd Betti numbers of M are even.*

Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$

$$(z_1, z_2) \to 2^k(z_1, z_2). \tag{10.27}$$

This is a free and properly discontinuous action, so the quotient $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ is a manifold, which is called a primary Hopf surface. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$, which has $b^1 = 1$, therefore it does not admit any Kähler metric.

10.3 Lefschetz decomposition

We will not prove this completely here, but just motivate by the following brief discussion.

Proposition 10.6. *On a Kähler manifold, we have*

$$[L, \Delta_H] = 0, \ [\Lambda, \Delta_H] = 0. \tag{10.28}$$

Proof. Since Δ_H is self-adjoint, these identities are equivalent. Next, we have

$$[L, d] = 0. \tag{10.29}$$

To see this, for any α,

$$d(L\alpha) = d(\omega \wedge \alpha) = \omega \wedge d\alpha = L(d\alpha), \tag{10.30}$$

since the Kähler form ω is closed. By taking adjoints, we have

$$[\Lambda, d^*] = 0. \tag{10.31}$$

Then we use Proposition 10.2 to compute

$$\begin{aligned} \Lambda\Delta_H &= \Lambda dd^* + \Lambda d^* d \\ &= d\Lambda d^* - (d^c)^* d^* + d^* \Lambda d \\ &= dd^* \Lambda - (d^c)^* d^* + d^*(d\Lambda - (d^c)^*) \\ &= \Delta_H \Lambda - (d^c d + dd^c)^*. \end{aligned} \tag{10.32}$$

But the operators d and d^c anti-commute, so we are done. □

This proposition implies that the operators L and Λ map harmonic forms to harmonic forms. This yields an extra decomposition on cohomology called the Lefschetz decomposition, which we do not have time to discuss further here.

10.4 The Hodge diamond

The following picture is called the Hodge diamond:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & & \vdots & & & & \\
h^{n,0} & & \cdots & & \vdots & & \cdots & & h^{0,n}. \\
 & & & & \vdots & & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}
\tag{10.33}
$$

Reflection about the center vertical is conjugation. Reflection about the center horizontal is Hodge star. The composition of these two operations, or rotation by π, is Serre duality.

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