250AB Algebraic Topology

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Contents

Introduction

In 250A, we will give an introduction to algebraic topology via de Rham cohomology of differentiable manifolds. Topics include the Poincarè Lemma, exact sequences, Mayer-Vietoris sequence, cohomology with compact supports, Poincaré duality, etc.

Then in 250B we will do singular homology and cohomology with integral coefficients, and prove more general things than can be done with just de Rham cohomology.

A guiding reference will be [BT82], but we will also use [Spi79, War83] for background material.

1 Lecture 1

1.1 Differentiable manifolds

Definition 1.1. A smooth manifold M^n is a second countable Hausdorff space which is locally Euclidean.

"Locally Euclidean" means that for each $p \in M$, there exist an open neighborhood U containing p and a smooth mapping $\phi : U \to \phi(U) \subset \mathbb{R}^n$ which is a homeomorphism onto its image. Furthermore, if two coordinate charts overlap, then the "overlap mapping"

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta) \tag{1.1}$$

is required to be a diffeomorphism. A collection of coordinate charts (U_α, ϕ_α) covering M is called an *atlas*. The collection of all possible coordinate charts on M compatible with some atlas is called a *differentiable structure*.

Definition 1.2. A mapping $\Psi : M \to N$ between smooth manifolds is a continuous mapping such that for each $p \in M$, there exists a coordinate system ϕ around p, and a coordinate system $\tilde{\phi}$ around $\Psi(p)$ such that

$$\tilde{\phi} \circ \Psi \circ \phi^{-1} \tag{1.2}$$

is smooth in some small neighborhood of $\phi(p)$.

It is easy to see that if $\Psi : M \to N$, and $\Phi : N \to M_1$, are smooth then $\Psi \circ \Psi : M \to M_1$ is smooth.

Definition 1.3. The category of smooth manifolds $\mathbf{Man}^\infty$ has objects as smooth manifolds and morphisms as smooth mappings, where composition of morphisms is just composition of mappings.

Composition of morphisms is obviously associative, i.e.,

$$(\Psi_1 \circ \Psi_2) \circ \Psi_3 = \Psi_1 \circ (\Psi_2 \circ \Psi_3) \tag{1.3}$$

and every manifolds has an identity morphism $id_X : X \to X$, which is obviously smooth, so this is indeed a category.

We could have just consider continuous mappings (instead of smooth mappings) in all the above definitions, to define the category of topological manifolds **Man** and continuous mappings. There is then a covariant functor between categories

$$F : \mathbf{Man}^{\infty} \to \mathbf{Man} \tag{1.4}$$

called a forgetful functor which simply maps $F(M) = M$ and $F(\Psi) = \Psi$.

Example 1.4. $\mathbb{R}, S^1, S^2, \mathbb{RP}^2, T^2, T^2 \# T^2, \ldots$ (see lecture note on Canvas page).

2 Lecture 2

2.1 Tangent vectors

Definition 2.1. A germ of a smooth function at p is an equivalence class $[f]_p$ where $f : U \to \mathbb{R}$ is a smooth function defined on a neighborhood of p and $f_1 \equiv f_2$ if there exists a neighborhood $U_3 \subset U_1 \cap U_2$ such $f_1 = f_2$ on U_3. The set of equivalence classes is denoted by $C^{\infty}(p)$.

Definition 2.2. A tangent vector at p, denoted by X_p is a linear derivation on germs of smooth functions around a point. That is,

$$X_p : C^{\infty}(p) \to \mathbb{R} \tag{2.1}$$

is linear over $\mathbb{R}$,

$$X_p(c_1[f_1]_p + c_2[f_2]_p) = c_1 X_p[f_1]_p + c_2 X_p[f_2]_p, \tag{2.2}$$

and

$$X_p([f]_p[g]_p) = X_p([f]_p)g(p) + f(p)X_p([g]_p). \tag{2.3}$$

The collection of all tangent vectors at p is denoted by T_pM.

Exercise 2.3. Show that T_pM is a vector space over $\mathbb{R}$ and

$$\dim(T_pM) = \dim(M) = n. \tag{2.4}$$

Definition 2.4. If $\Psi : M \to N$ is a smooth mapping between manifolds, then $\Psi_* : T_pM \to T_{\Psi(p)}N$ is the linear map defined by

$$(\Psi_* X_p)[f]_{\Psi(p)} = X_p([f \circ \Psi]_p) \tag{2.5}$$

We next give an alternate definition of a tangent vector. Let $p \in M$ and $c : (-\epsilon, \epsilon) \to M$ be a smooth curve with $c(0) = p$. Let $\phi : U \to \mathbb{R}^n$ be a coordinate system around p. Then

$$\phi \circ c : (-\epsilon, \epsilon) \to \mathbb{R}^n, \tag{2.6}$$

and we can consider

$$(\phi \circ c)'(0) = \frac{d}{dt}(\phi \circ c)|_{t=0} \in \mathbb{R}^n. \tag{2.7}$$

Given another smooth curve $\tilde{c} : (-\epsilon, \epsilon) \to M$ with $c(0) = p$, we say that $c \equiv c'$ if

$$(\phi \circ c)'(0) = (\phi \circ \tilde{c})'(0). \tag{2.8}$$

We will denote the equivalence class of a smooth curve at p by $[c]_p$.

Note that if ϕ_1 is another coordinate system around p, then

$$\begin{aligned}(\phi_1 \circ c)'(0) &= (\phi_1 \circ \phi^{-1} \circ \phi \circ c)'(0) = (\phi_1 \circ \phi^{-1})_*(\phi \circ c)'(0)\\ &= (\phi_1 \circ \phi^{-1})_*(\phi \circ \tilde{c})'(0) = (\phi_1 \circ \tilde{c})'(0),\end{aligned} \tag{2.9}$$

where $(\phi_1 \circ \phi^{-1})_*$ is the Jacobian matrix of partial derivatives (by the chain rule), so this notion is well-defined.

Definition 2.5. The tangent space $\tilde{T}_pM$ of M at p is the collection of all smooth curves $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$, modulo this equivalence relation.

Definition 2.6. If $\Psi : M \to N$ is a smooth mapping between manifolds, then $\Psi_{\tilde{*}} : \tilde{T}_pM \to \tilde{T}_{\Psi(p)}N$ is the linear map defined by

$$(\Psi_{\tilde{*}}[c]_p) = [\Psi \circ c]_{\Psi(p)}. \tag{2.10}$$

Exercise 2.7. Show that $\tilde{T}_pM$ is a vector space over $\mathbb{R}$ and

$$\dim(T_pM) = \dim(M) = n, \tag{2.11}$$

and that there is a natural isomorphism $\iota_p : \tilde{T}_pM \to T_pM$ given by

$$\iota_p([c]_p)[f]_p = \frac{d}{dt}(f \circ c)|_{t=0}. \tag{2.12}$$

Here "natural" means that if $\Psi : M \to N$ is a smooth mapping with $\Psi(p) = q$, then the diagram

$$\begin{array}{ccc} \tilde{T}_pM & \xrightarrow{\Psi_{\tilde{*}}} & \tilde{T}_{\Psi(p)}N \\ \Big\downarrow{\scriptstyle \iota_p} & & \Big\downarrow{\scriptstyle \iota_{\Psi(p)}} \\ T_pM & \xrightarrow{\Psi_*} & T_{\Psi(p)}N \end{array} \tag{2.13}$$

commutes.

Remark 2.8. Note that the first definition involving derivations on germs did not use any coordinate system, but the second definition did.

2.2 The tangent bundle

Definition 2.9. For a smooth manifold M, the tangent bundle $TM = \cup_{p\in M} T_pM$, and $\pi : TM \to M$ is defined by $\pi(X_p) = p$.

We next endow TM with a natural smooth manifold structure so that π is a smooth mapping. Given a coordinate system $\phi : U \to \mathbb{R}^n$, we write the coordinate functions as $x^i : U \to \mathbb{R}$ for $i = 1 \dots n$. Then ith coordinate vector field denoted by

$$\frac{\partial}{\partial x^i} \equiv \partial_i \tag{2.14}$$

is defined by

$$\partial_i[x^j]_p = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{2.15}$$

the Kronecker delta symbol. Equivalently, using the definition of the tangent space as equivalence classes of curves, we can define

$$(\partial_i)_p = [\phi^{-1}(x^1, \dots, t, \dots x^n)]_p, \tag{2.16}$$

where $\phi(p) = (x^1, \dots, x^n)$. Letting $\frac{\partial}{\partial t^i}$ denote the usual partial derivative operator in Euclidean space, clearly we have the relation

$$\phi_* \Big(\frac{\partial}{\partial x^i}\Big|_p \Big) = \frac{\partial}{\partial t^i}\Big|_{\phi(p)} \tag{2.17}$$

Also note that

$$\Big(\frac{\partial}{\partial x^i} f \Big)(p) = \frac{\partial (f \circ \phi)}{\partial t^i} (\phi(p)) \tag{2.18}$$

Given a coordinate system $\phi : U \to \mathbb{R}^n$, we define "inverse" local coordinates on TM

$$\Phi : \phi(U) \times \mathbb{R}^n \to TM \tag{2.19}$$

by

$$\Phi(x, v) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_{\phi^{-1}(x)}, \tag{2.20}$$

where $v = (v^1, \dots, v^n)$.

Let us consider coordinate systems (U_α, ϕ_α) and (U_β, ϕ_β) on M such that $U_\alpha \cap U_\beta \neq \emptyset$. Then we have

$$\Phi_\alpha(x, (v^1, \dots, v^n)) = \sum_{i=1}^n v^i \frac{\partial}{\partial x_\alpha^i}\Big|_{\phi_\alpha^{-1}(x)}, \tag{2.21}$$

and

$$\Phi_\beta(x, (\tilde{v}^1, \dots, \tilde{v}^n)) = \sum_{i=1}^n \tilde{v}^i \frac{\partial}{\partial x^i_\beta}\Big|_{\phi_\beta^{-1}(x)}. \tag{2.22}$$

We claim that

$$\frac{\partial}{\partial x^i_\beta} = \sum_{j=1}^n \frac{\partial x^j_\alpha}{\partial x^i_\beta} \frac{\partial}{\partial x^j_\alpha}. \tag{2.23}$$

The above formula is easily proved by plugging in the function x^j_α into each side, and using the defining property (2.15).

So then

$$\begin{aligned}
\Phi_\alpha^{-1} \circ \Phi_\beta(x, (\tilde{v}^1, \dots, \tilde{v}^n)) &= \Phi_\alpha^{-1}\Big(\sum_{i=1}^n \tilde{v}^i \frac{\partial}{\partial x^i_\beta}\Big|_{\phi_\beta^{-1}(x)} \Big) \\
&= \Phi_\alpha^{-1}\Big(\sum_{i=1}^n \tilde{v}^i \sum_{j=1}^n \frac{\partial x^j_\alpha}{\partial x^i_\beta} \frac{\partial}{\partial x^j_\alpha}\Big|_{\phi_\beta^{-1}(x)} \Big) \\
&= \Phi_\alpha^{-1}\left(\sum_{j=1}^n \left(\sum_{i=1}^n \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^i_\beta} \right) \frac{\partial}{\partial x^j_\alpha}\Big|_{\phi_\beta^{-1}(x)} \right) \\
&= \Phi_\alpha^{-1}\left(\sum_{j=1}^n \left(\sum_{i=1}^n \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^i_\beta} \right) \frac{\partial}{\partial x^j_\alpha}\Big|_{\phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1}(x)} \right).
\end{aligned} \tag{2.24}$$

Consequently,

$$\Phi_\alpha^{-1} \circ \Phi_\beta(x, (\tilde{v}^1, \dots, \tilde{v}^n)) = \Big(\phi_\alpha \circ \phi_\beta^{-1}(x), \sum_{i=1}^n \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^i_\beta} \Big) \tag{2.25}$$

The functions $\frac{\partial x^j_\alpha}{\partial x^i_\beta}$ are functions on M, and from (2.18) we have

$$\frac{\partial x^j_\alpha}{\partial x^i_\beta}(p) = \frac{\partial (x_\alpha \circ x_\beta^{-1})^j}{\partial t^i_\beta}(\phi_\beta(p)). \tag{2.26}$$

These functions are smooth since the overlap mappings are smooth. Furthermore, are *linear* isomorphisms in the second variable (invertibility follows from the chain rule, since $\phi_\alpha \circ \phi_\beta^{-1}$ is assumed to be a diffeomorphism). Therefore, the overlap mappings are smooth diffeomorphisms. So we have the following properties of TM:

- If M is a smooth manifold of dimension n, then TM is a smooth manifold of dimension $2n$,
- For any $p \in M$, $T_pM = \pi^{-1}(p)$ is an n-dimensional vector space.
- For any M, TM is noncompact.

(It is a vector bundle, which we will discuss in detail later, but it has special properties that any old vector bundle does not possess.)

3 Lecture 3

By the above, a smooth mapping $f : M \to N$ induces a mapping

$$f_* : TM \to TN, \tag{3.1}$$

called a "push-forward", which is linear on fibers and which makes the following diagram commutes

$$\begin{array}{ccc} TM & \xrightarrow{f_*} & TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{3.2}$$

It is easy to check that $f_* : TM \to TN$ is a smooth mapping where TM and TN are given the smooth manifold structures from the previous lecture.

Note the following important proposition.

Proposition 3.1 (The chain rule). *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)_* = h_* \circ f_* : TM \to TM' \tag{3.3}$$

Proof. For $f : M \to N$, choose a local coordinate ϕ on M and ψ on N such that the mapping

$$f_c = \psi \circ f \circ \phi^{-1} \tag{3.4}$$

is defined. Then

$$(f_c)_*\Big(\frac{\partial}{\partial t^i}\Big) = \sum_{k=1}^{m} \frac{\partial f_c^k}{\partial t^i} \frac{\partial}{\partial s^k}, \tag{3.5}$$

for $i = 1 \dots n$, where $n = \dim(M)$ and $m = \dim(N)$. Using this, the result is then reduced to the ordinary chain rule (details left to the reader). □

3.1 Review of theory of vector bundles

Definition 3.2. A smooth real vector bundle of rank k over a smooth manifold M^n is a topological space E together with a smooth projection

$$\pi : E \to M \tag{3.6}$$

such that

- For $p \in M$, $\pi^{-1}(p)$ is a vector space of dimension k over $\mathbb{R}$.
- There exists local trivializations, that is, there are smooth mappings

  $$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E \tag{3.7}$$

  which maps $p \times \mathbb{R}^k$ linearly onto the fiber $\pi^{-1}(p)$ for every $p \in U_\alpha$.

The transition functions of a bundle are defined as follows.

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \tag{3.8}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{3.9}$$

for $v \in \mathbb{R}^k$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{3.10}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satifsying (3.10), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$. (It turns out this bundle is uniquely defined up to bundle equivalence, which we will define below.) If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle.

Example 3.3. (The tangent bundle redux.) Given a coordinate system (U_α, x_α) on a smooth manifold M, let

$$\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_\alpha^i}\Big|_x. \tag{3.11}$$

On U_β, we have

$$\Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \sum_{i=1}^n \tilde{v}_i \frac{\partial}{\partial x_\beta^i}\Big|_x. \tag{3.12}$$

Recall that

$$\frac{\partial}{\partial x_\beta^i} = \sum_{j=1}^n \frac{\partial x_\alpha^j}{\partial x_\beta^i} \frac{\partial}{\partial x_\alpha^j}, \tag{3.13}$$

so then

$$\begin{aligned}
\Phi_\alpha^{-1} \circ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) &= \Phi_\alpha^{-1}\left(\sum_{i=1}^n \tilde{v}^i \frac{\partial}{\partial x_\beta^i} \right) \\
&= \Phi_\alpha^{-1}\left(\sum_{i=1}^n \tilde{v}^i \sum_{j=1}^n \frac{\partial x_\alpha^j}{\partial x_\beta^i} \frac{\partial}{\partial x_\alpha^j} \right) \\
&= \Phi_\alpha^{-1}\left(\sum_{j=1}^n \left(\sum_{i=1}^n \tilde{v}^i \frac{\partial x_\alpha^j}{\partial x_\beta^i} \right) \frac{\partial}{\partial x_\alpha^j} \right).
\end{aligned} \tag{3.14}$$

Consequently,

$$\Big(\varphi_{\alpha\beta}(x)(v^1, \ldots, v^n)\Big)^j = \sum_{i=1}^n \tilde{v}^i \frac{\partial x_\alpha^j}{\partial x_\beta^i} \tag{3.15}$$

3.2 Categories

A bundle mapping between vector bundles E_1 over M and E_2 over N is a mapping $F : E_1 \to E_2$ which maps fibers linearly to fibers and covers a smooth mapping between the base spaces. That is, the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N \end{array} \tag{3.16}$$

commutes.

Definition 3.4. The category **Vect** of smooth vector bundles over smooth manifolds is the collection of all vector bundle (of any rank) over smooth manifolds. The morphisms are the bundle mappings.

We therefore have a functor $F : \mathbf{Man}^\infty \to \mathbf{Vect}$ where **Vect** is the category of smooth vector bundles over smooth manifolds given by $M \to TM$ and $f : M \to N$ maps to $f_* : TM \to TN$. The mapping F satisfies $F(id_X) = Id_{TM}$ and by by Proposition 3.1, $F(f_1 \circ f_2) = F(f_1) \circ F(f_2)$, so this is a *covariant* functor.

Next, we define another category.

Definition 3.5. For a fixed smooth manifold M, the category $\mathbf{Vect(M)}$ is the collection of smooth vector bundles over M (of any rank). A morphism in this category is a mapping $F : E_1 \to E_2$ covering the identity mapping, that is, the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id_M} & M \end{array} \tag{3.17}$$

We say that bundles E_1 and E_2 over M are isomorphic if there exists an invertible bunble mapping between E_1 and E_2. If E is isomorphic to the trivial bundle over M, $\pi_M : M \times \mathbb{R}^k \to M$ defined by $\pi_M(p, v) = p$, the we say that E is trivial.

We next express the above in coordinates. Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions

$$f_\alpha : U_\alpha \to Hom(\mathbb{R}^{k_1}, \mathbb{R}^{k_2}) \tag{3.18}$$

defined by

$$f_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)). \tag{3.19}$$

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi^{E_2}_{\alpha\beta} f_\beta \varphi^{E_1}_{\beta\alpha}, \tag{3.20}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} f_\alpha = f_\beta \varphi^{E_1}_{\beta\alpha}. \tag{3.21}$$

Bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $f : E_1 \to E_2$. Obviously, this means $\text{rank}(E_1) = \text{rank}(E_2)$ and non-singularity of the local representatives, that is, $\det(f_\alpha) \neq 0$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions

$$f_\alpha : U_\alpha \to GL(k, \mathbb{R}) \tag{3.22}$$

such that

$$\varphi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{3.23}$$

Remark 3.6. In the above, we only defined morphisms in the category of vector bundle to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence. More on this later.

4 Lecture 4: Operations on bundles

4.1 Direct sums

If $V_1, \dots, V_k$ are vector spaces over $\mathbb{R}$, then the direct sum $V_1 \oplus \cdots \oplus V_k$ is the Cartesian product $V_1 \times \cdots \times V_k$ with the following vector space structure:

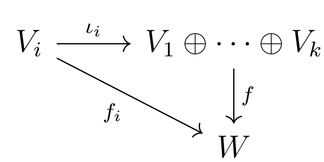
$$c(v_1, \dots, v_k) = (cv_1, \dots, cv_k) \tag{4.1}$$

$$(v_1, \dots, v_k) + (v_1', \dots, v_k') = (v_1 + v_1', \dots, v_k + v_k'), \tag{4.2}$$

for $c \in \mathbb{R}$. The space $V_1 \oplus \cdots \oplus V_k$ satisfies the following "universal" mapping property. For $1 \leq i \leq k$, let $\iota_i : V_i \to V_1 \oplus \cdots \oplus V_k$ be the inclusion mapping

$$\iota_i : v \mapsto (0, \dots, \overbrace{v}^{i}, \dots, 0). \tag{4.3}$$

Let W be any vector space, and $f_i : V_i \to W$ be linear mappings for $1 \leq i \leq k$. Then there is a *unique* linear map $f : V_1 \oplus \cdots \oplus V_k \to W$ which makes the following diagram



commute for $1 \leq i \leq k$.

Exercise 4.1. (i) Show that any vector space V with the above universal mapping property is isomorphic to the direct sum. (ii) Prove that

$$\dim_{\mathbb{R}}(V_1 \oplus \cdots \oplus V_k) = \sum_{i=1}^{k} \dim_{\mathbb{R}}(V_i). \tag{4.4}$$

(iii) Prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3). \tag{4.5}$$

Definition 4.2. Let $V_i, i \in \mathcal{I}$ be any collection of vector spaces. The Cartesian product $\Pi_{i\in\mathcal{I}}V_i$ is the collection of all functions

$$f : \mathcal{I} \to \cup_{i\in\mathcal{I}}V_i, \tag{4.6}$$

such that $f(i) \in V_i$ for all $i \in \mathcal{I}$. The direct product $\Pi_{i\in\mathcal{I}}V_i$ is the Cartesian product with the vector space structure

$$cf(i) = cf(i) \tag{4.7}$$

$$(f + g)(i) = f(i) + g(i). \tag{4.8}$$

The projection $\pi_i : \Pi_{i\in\mathcal{I}}V_i \to V_i$ is the mapping $\pi_i(f) = f(i)$. The above definition satisfies the following universal property. If V is any vector space and $\phi_i : V \to V_i$ are linear mappings for $i \in \mathcal{I}$, then there is a unique linear mapping $\phi : V \to \Pi_{i\in\mathcal{I}}V_i$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_i} & V_i \\ & \searrow^{\phi} & \uparrow{\scriptstyle \pi_i} \\ & & \Pi_{i\in\mathcal{I}}V_i \end{array}$$

commutes for each $i \in \mathcal{I}$. This property uniquely characterizes the direct product.

Definition 4.3. Let $V_i, i \in \mathcal{I}$ be any collection of vector spaces. The direct sum $\oplus_{i\in\mathcal{I}}V_i$ is the subspace of the direct product consisting of the functions f such that $f(i) \neq 0$ for only finitely many $i \in \mathcal{I}$.

Exercise 4.4. (i) Show that the direct sum satisfies the first universal property. (ii) If the index set is finite, then the direct product is isomorphic to the direct sum.

Definition 4.5. The direct sum of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is the vector bundle $\pi : E_1 \oplus E_2 \to M$ defined by $\pi^{-1}(p) = \pi_1^{-1}(p) \oplus \pi_2^{-1}(p)$. If $\Phi_1 : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^l \to \pi_1^{-1}(U)$ are local trivializations then

$$\Phi : U \times (\mathbb{R}^k \oplus \mathbb{R}^l) \to \pi^{-1}(U) \tag{4.9}$$

defined by

$$\Phi(x, (v_1, v_2)) = (\Phi_1(x.v_1), \Phi_2(x, v_2)) \tag{4.10}$$

is a local trivialization for $E_1 \oplus E_2$.

Note, the transition functions satisfy

$$\varphi_{\alpha\beta}^{E_1\oplus E_2} = \varphi_{\alpha\beta}^{E_1} \oplus \varphi_{\alpha\beta}^{E_2} \in GL(k + l, \mathbb{R}), \tag{4.11}$$

where this is the "block" matrix

$$\varphi_{\alpha\beta}^{E_1\oplus E_2}(x)(v, w) = \begin{pmatrix} \varphi_{\alpha\beta}^{E_1}(x)v & 0 \\ 0 & \varphi_{\alpha\beta}^{E_2}(x)w \end{pmatrix}. \tag{4.12}$$

4.2 Tensor products

Definition 4.6. If A is any set, then the free vector space over A is

$$\mathcal{F}(A) = \oplus_{a \in A} \mathbb{R}. \tag{4.13}$$

This can be thought of as the vector space with basis elements $a \in A$. That is, $\mathcal{F}(A)$ is the set of formal sums

$$\mathcal{F}(A) = \Big\{ \sum_{a \in A} c_a a \mid c_a \neq 0 \text{ for only finitely many } a \in A \Big\} \tag{4.14}$$

with vector space structure

$$c \sum_{a \in A} c_a a = \sum_{a \in A} (c c_a) a \tag{4.15}$$

$$\sum_{a \in A} c_a a + \sum_{a \in A} c'_a a = \sum_{a \in A} (c_a + c'_a) a. \tag{4.16}$$

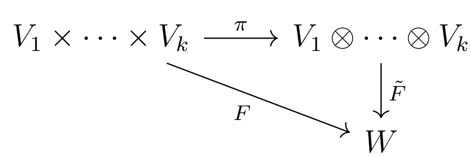
Definition 4.7. If $V_1, \dots, V_k$ are vector spaces over $\mathbb{R}$, then the tensor product $V_1 \otimes \cdots \otimes V_k$ is the free real vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$ modulo the subspace spanned by all elements of the form

$$(v_1, \dots, c v_i, \dots, v_k) - c(v_1, \dots, v_i, \dots, v_k) \tag{4.17}$$

$$(v_i, \dots, v_i + v'_i, \dots, v_k) - (v_i, \dots, v_i, \dots, v_k) - (v_i, v'_i, \dots, v_k), \tag{4.18}$$

for $c \in \mathbb{R}$.

The space $V_1 \otimes \cdots \otimes V_k$ satisfies the universal mapping property as follows. Let W be any vector space, and $F : V_1 \times \cdots V_k \to W$ be a multilinear mapping, i.e., F is linear when restricted to each factor, with the other variables held fixed. Then there is a unique *linear* map $\tilde{F} : V_1 \otimes \cdots \otimes V_k$ which makes the following diagram



commutative, where π is the projection to the quotient space, which we write as

$$\pi(v_1, \dots, v_k) = v_1 \otimes \cdots \otimes v_k. \tag{4.19}$$

We say that an element in $V_1 \otimes \cdots \otimes V_k$ of the form $v_1 \otimes \cdots \otimes v_k$ is *decomposable*. A general element of $V_1 \times \cdots \times V_k$ is not decomposable, but can always be written as a sum of decomposable elements.

Exercise 4.8. Prove that

$$\dim_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k) = \dim_{\mathbb{R}}(V_1) \cdots \dim_{\mathbb{R}}(V_k). \tag{4.20}$$

Also, prove that for 3 vector spaces V_1, V_2, V_3 we have

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3). \tag{4.21}$$

Definition 4.9. The tensor product of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is the vector bundle $\pi : E_1 \otimes E_2 \to M$ defined by $\pi^{-1}(p) = \pi_1^{-1}(p) \otimes \pi_2^{-1}(p)$. If $\Phi_1 : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^l \to \pi_1^{-1}(U)$ are local trivializations then consider

$$F : U \times (\mathbb{R}^k \times \mathbb{R}^l) \to \pi^{-1}(U) \tag{4.22}$$

defined by

$$F(x, (v_1, v_2)) = \Phi_1(x, v_1) \otimes \Phi_2(x, v_2). \tag{4.23}$$

This is clearly a multilinear mapping on each fiber, so by the universal property of tensor products, there is a unique induced mapping

$$\tilde{F} : U \times (\mathbb{R}^k \otimes \mathbb{R}^l) \to \pi^{-1}(U) \tag{4.24}$$

which, using an isomorphism $\mathbb{R}^k \otimes \mathbb{R}^l \cong \mathbb{R}^{kl}$, defines a local trivialization for $E_1 \otimes E_2$.

We could have equivalently defined the tensor product in terms of transition functions. To do this, note the following. If $\phi_1 \in GL(k, \mathbb{R})$ and $\phi_2 \in GL(l, \mathbb{R})$ then define

$$\phi_1 \times \phi_2 : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k \otimes \mathbb{R}^l \tag{4.25}$$

by

$$(\phi_1 \times \phi_2)(v_1, v_2) = \phi_1(v_1) \otimes \phi_2(v_2). \tag{4.26}$$

This is clearly a multilinear mapping, so by the universal property for tensor products, there is a unique induced mapping

$$\phi_1 \otimes \phi_2 : \mathbb{R}^k \otimes \mathbb{R}^l \to \mathbb{R}^k \otimes \mathbb{R}^l \tag{4.27}$$

Given transition functions for E_1

$$\phi^{E_1}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}), \tag{4.28}$$

and transition functions for E_2

$$\phi^{E_2}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(l, \mathbb{R}), \tag{4.29}$$

we define

$$\varphi^{E_1 \otimes E_2}_{\alpha\beta} = \varphi^{E_1}_{\alpha\beta} \otimes \varphi^{E_2}_{\alpha\beta} \in GL(kl, \mathbb{R}), \tag{4.30}$$

where we choose some isomorphism $\mathbb{R}^k \otimes \mathbb{R}^l \cong \mathbb{R}^{kl}$.

5 Lecture 5

5.1 Dual bundles

Definition 5.1. The dual of a vector space V is $V^* = Hom(V, \mathbb{R})$, which is the space of all linear mappings from V to $\mathbb{R}$.

Exercise 5.2. If V is finite-dimensional, show that $V^* \cong V$ and thus $\dim(V^*) = \dim(V)$.

Definition 5.3. The dual of a vector bundle $\pi : E \to M$ is the vector bundle $\Pi : E^* \to M$ defined by $\Pi^{-1}(p) = (\pi^{-1}(p))^*$. If $\Phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is a local trivialization then

$$\Phi^* : U \times (\mathbb{R}^k)^* \to \pi^{-1}(U) \tag{5.1}$$

defined by

$$\Phi^*(x, f)(v_p) = f(\pi_2 \circ \Phi^{-1}(v_p)) \tag{5.2}$$

is a local trivialization for E^*.

Exercise 5.4. Show that the transition functions of E^* are

$$\varphi^{E^*}_{\alpha\beta} = \left((\varphi^E_{\alpha\beta})^{-1}\right)^T = (\varphi^E_{\beta\alpha})^T. \tag{5.3}$$

5.2 Sections of bundles

Definition 5.5. Let $\pi : E \to M$ be a vector bundle. A section of a bundle is a smooth mapping $s : M \to E$ such that $\pi \circ s = id_M$. The space of sections is denoted by $\Gamma(E)$.

In other words, $s(x) \in E_x$, s maps x to a vector in the fiber over x. In terms of local trivializations we have the following. Let

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha) \tag{5.4}$$

be a local trivialization. Then

$$s_\alpha = \pi_2 \circ \Phi_\alpha^{-1} \circ s : U_\alpha \to \mathbb{R}^k \tag{5.5}$$

is called a local representative of s with respect to Φ_α. On U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\Phi_\beta : U_\beta \times \mathbb{R}^k \to \pi^{-1}(U_\beta). \tag{5.6}$$

Recall that the transition functions of a bundle are

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \tag{5.7}$$

defined by

$$\varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{5.8}$$

for $v \in \mathbb{R}^k$. Then for any $e_x \in \pi^{-1}(x)$, we have

$$\phi_{\alpha\beta}(x)(\pi_2 \circ \Phi_\beta^{-1}(e_x)) = \pi_2 \circ \Phi_\alpha^{-1}(e_x). \tag{5.9}$$

Choosing $e_x = s(x)$ we have

$$\phi_{\alpha\beta}(s)(\pi_2 \circ \Phi_\beta^{-1} \circ s(x)) = \pi_2 \circ \Phi_\alpha^{-1} \circ s(x), \tag{5.10}$$

or simply

$$\phi_{\alpha\beta} s_\beta = s_\alpha, \text{ on } U_\alpha \cap U_\beta, \tag{5.11}$$

which is the local transformation law for a section.

Conversely, if a bundle $\pi : E \to M$ is given to us in terms of transition functions, then any collection of functions

$$s_\alpha : U_\alpha \to \mathbb{R}^k \tag{5.12}$$

satisfying (5.11) gives a well-defined smooth section $s : M \to E$.

5.3 Riemannian metrics on real vector bundles

If $\pi : E \to M$ is a real vector bundle, a Riemannian metric on E is a choice of smoothly varying positive definite symmetric inner product on each fiber. That is $g \in \Gamma(E^* \otimes E^*)$ satisfying

$$g(e_1, e_2) = g(e_2, e_1), \tag{5.13}$$

and

$$g(e, e) > 0 \text{ for } e \neq 0. \tag{5.14}$$

Proposition 5.6. *If E is any real vector bundle, then E admits a Riemannian metric.*

Proof. Let

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha) \tag{5.15}$$

be a local trivialization for U_α an open covering of M which is locally finite.. For $x \in U_\alpha$ and $e_1, e_2 \in E_x$, define

$$g_\alpha(e_1, e_2) = \langle \pi_2 \circ \Phi_\alpha^{-1}(e_1), \pi_2 \circ \Phi_\alpha^{-1}(e_2) \rangle, \tag{5.16}$$

where $\langle \cdot, \cdot, \rangle$ denotes the Euclidean inner product on $\mathbb{R}^k$. Next, let χ_α be a partition of unity subordinate to the cover U_α, that is

$$\text{supp}(\chi_\alpha) \subset U_\alpha, \quad 0 \leq \chi_\alpha \leq 1, \text{ and } \sum_\alpha \chi_\alpha = 1. \tag{5.17}$$

Define

$$g(e_1, e_2) = \sum_\alpha \phi_\alpha g_\alpha(e_1, e_2). \tag{5.18}$$

This is clearly symmetric since each g_α is symmetric. It is positive definite since it is a finite sum of positive terms at each point for any non-zero vector. □

Corollary 5.7. *For any real vector bundle E, $E^* \cong E$.*

Proof. Choose a Riemannian metric g on E. Then the mapping $\flat : E \to E^*$ defined by

$$\flat(e_1)(e_2) = g(e_1, e_2) \tag{5.19}$$

is an isomorphism on fibers, and covers the identity map. □

Definition 5.8. Given vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ over the same base space M, we say that E_1 is a subbundle of E_2, written $E_1 \subset E_2$ if each fiber $\pi_1^{-1}(x) \subset \pi_2^{-1}(x)$ is a vector subspace.

In bundle terms, existence of a Riemannian metric implies that there is always a non-zero section of $E^* \otimes E^*$, which says that

$$E^* \otimes E^* = A \oplus B \tag{5.20}$$

always admits a trivial 1-dimensional subbundle A. (This is because span$(g(x))$ defines a 1-dimensional subspace of every fiber, and the fact that any 1-dimensional bundle with a non-vanishing section must be a trivial bundle).

Of course, the metric gives a isomorphism

$$E^* \otimes E^* \cong E^* \otimes E \cong Hom(E, E), \tag{5.21}$$

and the latter bundle always admits the identity section. The latter choice is canonical, but the sub-bundle A is not.

Remark 5.9. We will soon see that there is an isomorphism of bundles

$$E \otimes E \cong \mathbb{R} \oplus S^2_0(E) \oplus \Lambda^2(E), \tag{5.22}$$

which you can think of as decomposing a matrix into a pure trace part, a symmetric traceless part, and a skew-symmetric part.

Definition 5.10. If $E_1 \subset E_2$ is a subbundle, then the quotient bundle E_2/E_1 is the vector bundle with fiber $\pi_2^{-1}(x)/\pi_1^{-1}(x)$ over x.

Exercise 5.11. Prove that the quotient bundle is a vector bundle. That is, find local trivializations for E_2/E_1.

Note the following corollary.

Corollary 5.12. *If $E_1 \subset E$ is a sub-bundle, then there exists a subbundle $E_2 \subset E$ such that*

$$E \cong E_1 \oplus E_2. \tag{5.23}$$

Furthermore, the quotient bundle $(E/E_1) \cong E_2$.

Proof. Choose a Riemannian metric g on E, and let $E_2 = (E_1)^\perp$. Use Gram-Schmidt to construct local trivializations for $(E_1)^\perp$ to show this is indeed a subbundle. The rest is just linear algebra. □

6 Lecture 6

6.1 Reduction of Structure group

Definition 6.1. If a bundle $\pi : E \to M$ is equivalent to a bundle which has transition functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to K$, where K is a subgroup of $GL(k, \mathbb{R})$, then we say that the structure group of E can be *reduced* to K.

Another way to state the results from the previous section is as follows.

Proposition 6.2. *We have the following.*

- *A bundle is trivial if and only if its structure group can be reduced to $\{Id\}$.*
- *The structure group of any real vector bundle $\pi : E \to M$ of rank k can be reduced to $O(k)$.*

Proof. The first case is obvious. For the second case, from above E admits a Riemannian metric. By Gram-Schmidt, for any point $x \in M$, there exists a neighborhood U_x and a local basis of sections $\{e_1, \ldots, e_k\}$ which are orthonormal at every point in U_x. Define local trivializations by

$$\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^{k} v^i e_i. \tag{6.1}$$

Then overlaps maps then necessarily satisfy

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k), \tag{6.2}$$

where $O(k)$ is the orthogonal group of $k \times k$ real matrices satisfying $AA^T = I_k$. □

6.2 Real line bundles

Note for a real 1-dimensional line bundle $\pi : L \to M$, we have that the structure group can be reduced to $O(1) = \{\pm 1\}$, Consider the set

$$\tilde{M} = \{v \in L \ | g(v, v) = 1\}. \tag{6.3}$$

Since there are exactly two unit norm vectors in any fiber, we have that $\pi : \tilde{M} \to M$ is a 2-fold covering space. So any real line bundle give an associated 2-fold covering space. Conversely, any 2-fold covering space gives a real line bundle, which is uniquely determined up to equivalence. To see this, note that a 2-fold covering space can be viewed as a fiber bundle with group $\mathbb{Z}_2$, and viewing $\mathbb{Z}_2 = \{\pm 1\} \subset GL(1, \mathbb{R})$, we naturally obtain an associated real line bundle.

Remark 6.3. Therefore real line bundles over M are in one-to-one correspondence with 2-fold covering spaces of M, up to equivalence. Using some covering space theory, the 2-fold coverings correspond to index 2 subgroups of $\pi_1(M)$, which is

$$Hom(\pi_1(M), \mathbb{Z}_2). \tag{6.4}$$

Later we will see that

$$Hom(\pi_1(M), \mathbb{Z}_2) \cong Hom(H_1(M), \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2), \tag{6.5}$$

the first cohomology group with $\mathbb{Z}_2$ coefficients.

Remark 6.4. Another way to understand this is through the following. After reduction the structure group to $\mathbb{Z}_2$, the transition functions of the bundle are given by

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2. \tag{6.6}$$

The condition on transition functions

$$\phi_{\alpha\gamma} = \phi_{\alpha\beta}\phi_{\beta\gamma} \tag{6.7}$$

says that $\phi_{\alpha\beta}$ form a Čech 1-cocycle, so

$$\phi_{\alpha\beta} \in \check{H}^1_{\mathfrak{U}}(M, \mathbb{Z}_2), \tag{6.8}$$

the first Čech cohomology group with coefficient in the constant sheaf $\mathbb{Z}_2$ with respect to the open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\mathcal{I}}$.

For $\phi_{\alpha\beta}$ to be a co-boundary, note that a 0-cocycle is a collection

$$f_\alpha : U_\alpha \to \mathbb{Z}_2 \tag{6.9}$$

and

$$(\delta f)_{\alpha\beta} = f_\beta \cdot f_\alpha^{-1} : U_\alpha \cap U_\beta \to \mathbb{Z}_2. \tag{6.10}$$

So for $\phi_{\alpha\beta}$ to be a co-boundary, we have f_α so that

$$\phi_{\alpha\beta} = f_\beta f_\alpha^{-1} \tag{6.11}$$

on $U_\alpha \cap U_\beta$ which is exactly the condition for the bundle to the equivalent to a trivial bundle.

For a sufficiently "good" open cover, it turns out that

$$\check{H}^1_{\mathfrak{U}}(M, \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2), \tag{6.12}$$

the ordinary first singular cohomology group with $\mathbb{Z}_2$ coefficients.

Example 6.5. (Tautological bundle on $\mathbb{RP}^n$) Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}), \ c \neq 0. \tag{6.13}$$

Define

$$\gamma_n^1 = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in [x]\} \tag{6.14}$$

We claim that γ_n^1 is a nontrivial 1-dimensional bundle over $\mathbb{RP}^n$. Assume by contradiction that it were the trivial bundle. Then there would exists a nowhere vanishing section $\sigma : \mathbb{RP}^n \to \gamma_n^1$. This is a mapping

$$\sigma : \mathbb{RP}^n \to \mathbb{RP}^n \times \mathbb{R}^{n+1} \tag{6.15}$$

of the form for $x \in S^n$,

$$\sigma([x]) = ([x], c(x) \cdot x) \tag{6.16}$$

For this to be well-defined, we require that $c(x) : S^n \to \mathbb{R}$ is a function satisfying $c(-x) = -c(x)$. Since c must take negative and positive values, by the intermediate value theorem, $c(x_0) = 0$ for some x_0, which is a contradiction.

For $n = 1$, we have that $\mathbb{RP}^1 \cong S^1$. There is the trivial bundle $S^1 \times \mathbb{R}$. We also know that there is the Mobius strip $S^1 \tilde{\times} \mathbb{R}$, which we can view as a line bundle over S^1.

Exercise 6.6. Show that γ_1^1 is isomorphic to the Mobius bundle.

7 Lecture 7

7.1 Exterior powers

Let V be a real vector space. The exterior algebra $\Lambda(V)$ is defined as

$$\Lambda(V) = \Big\{ \bigoplus_{k \geq 0} V^{\otimes^k} \Big\} / \mathcal{I} = \bigoplus_{k \geq 0} \Big\{ V^{\otimes^k} / \mathcal{I}_k \Big\} = \bigoplus_{k \geq 0} \Lambda^k V, \tag{7.1}$$

where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $v \otimes v \in V \otimes V$, and $\mathcal{I}_k = V^{\otimes^k} \cap \mathcal{I}$. The wedge product of $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$ is just the multiplication induced by the tensor product in this algebra, that is, lift v and w to $\tilde{v} \in V^{\otimes^p}$, and $\tilde{w} \in V^{\otimes^q}$, and define $v \wedge w = \pi(\tilde{v} \otimes \tilde{w})$, where $\pi : V^{\otimes^{p+q}} \to \Lambda^{p+q} V$ is the projection. This is easily seen to be well-defined. We say that an element in $\Lambda^k(V)$ of the form $v_1 \wedge \dots \wedge v_k$ is *decomposable*. A general element of $\Lambda^k(V)$ is not decomposable, but can always be written as a sum of decomposable elements.

The space $\Lambda^k(V)$ satisfies the universal mapping property as follows. Let W be any vector space, and let

$$F : \overbrace{V \times \dots \times V}^{k} \to W \tag{7.2}$$

be an alternating multilinear mapping. That is, F is multilinear and $F(v_1, \dots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

$$\begin{array}{ccc} \overbrace{V \times \cdots \times V}^{k} & \xrightarrow{\pi} & \Lambda^k(V) \\ & \searrow^{F} & \downarrow{\scriptstyle \tilde{F}} \\ & & W \end{array}$$

commutative, where π is the projection, which we denote as

$$\pi(v_1, \dots, v_k) = v_1 \wedge \cdots \wedge v_k. \tag{7.3}$$

Exercise 7.1. Prove the following properties of the wedge product.

- Bilinearity: $(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w$, and $(cv) \wedge w = c(v \wedge w)$ for $c \in \mathbb{R}$.
- If $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$, then $v \wedge w = (-1)^{pq} w \wedge v$.
- Associativity $(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3)$.

Exercise 7.2. If $\dim_{\mathbb{R}}(V) = n$, prove that $\Lambda^k(V) = \{0\}$ if $k > n$,

$$\dim(\Lambda^k(V)) = \binom{n}{k} \text{ if } 0 \leq k \leq n, \tag{7.4}$$

and

$$\dim(\Lambda(V)) = 2^n, \tag{7.5}$$

Definition 7.3. For a real vector bundle $\pi : E \to M$, we define $\Pi : \Lambda^p(E) \to M$ by $\Pi^{-1}(x) = \Lambda^p(\pi^{-1}(x))$. If $\Phi : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ is a local trivialization for E, then consider the mapping

$$F : U \times \overbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}^{p} \to \Pi^{-1}(U) \tag{7.6}$$

defined by

$$F(x, (v_1, \dots, v_p)) = \Phi(x, v_1) \wedge \cdots \wedge \Phi(x, v_k) \tag{7.7}$$

This is clearly an alternating multilinear mapping on fibers, so by the universal property, there is an unique induced mapping

$$\tilde{F} : U \times \Lambda^p(\mathbb{R}^k) \to \Pi^{-1}(U) \tag{7.8}$$

which is a local trivialization for $\Lambda^p(E)$.

We can equivalently define the pth exterior power in terms of transition functions. To do this, note that for any linear map $f : \mathbb{R}^k \to \mathbb{R}^k$, there is a naturally induced mapping

$$\Lambda^p f : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \tag{7.9}$$

define as follows. Define

$$F : \overbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}^{p} \to \Lambda^p(\mathbb{R}^k) \tag{7.10}$$

by

$$F(v_1, \ldots, v_p) = f(v_1) \wedge \cdots \wedge f(v_p) \tag{7.11}$$

This is clearly an alternating multilinear mapping, so by the universal property, there exists a unique mapping

$$\Lambda^p f = \tilde{F} : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k). \tag{7.12}$$

therefore for any vector bundle E, the pth exterior power $\Lambda^p(E)$ is defined to be the bundle with transition functions

$$\varphi_{\alpha\beta}^{\Lambda^p(E)} = \Lambda^p(\varphi_{\alpha\beta}^E). \tag{7.13}$$

Putting all of these together, we can define the following.

Definition 7.4. For a real vector bundle $\pi : E \to M$, define the exterior algebra bundle $\Lambda(E) = \oplus_{p=0}^k \Lambda^p(E)$.

Note in the above discussion, if we sum together all of the $\Lambda^p f$ mappings, we get an induced mapping between the exterior algebras

$$\Lambda(f) : \Lambda(\mathbb{R}^k) \to \Lambda(\mathbb{R}^k) \tag{7.14}$$

which satisfies

$$\Lambda(f)(\alpha \wedge \beta) = \Lambda(f)(\alpha) \wedge \Lambda(f)(\beta) \tag{7.15}$$

Therefore, the wedge product gives an algebra structure on each fiber of $\Lambda(E)$.

7.2 Orientability of real bundles

Note that if V is an n-dimensional vector space, then $\Lambda^n V$ is 1-dimensional. So if $L : V \to V$ is a linear transformation then $\Lambda^n L : \Lambda^n V \to \Lambda^n V$ is an endomorphism of a 1-dimensional vector space. Therefore $\Lambda^n(\omega) = c \cdot \omega$ for some scalar c. So we can make the following definition:

Definition 7.5. For a linear transformation $L : V \to V$, define $\det(L)$ to be the real number so that

$$\Lambda^n L(\omega) = \det(L) \cdot \omega. \tag{7.16}$$

Exercise 7.6. Show that this definition of determinant agrees with the usual linear algebra definition of determinant.

Proposition 7.7. *Let $\pi : E \to M$ be a real vector bundle of rank k. The following are equivalent.*

- *The line bundle $\Lambda^k(E)$ is trivial.*
- *$\Lambda^k(E)$ admits a nowhere zero section.*
- *The double cover $\tilde{M}$ corresponding to $\Lambda^k(E)$ is a trivial 2-fold covering space.*
- *The structure group of E can be reduced to*

$$GL_+(k, \mathbb{R}) \equiv \{A \in GL(k, \mathbb{R}) \mid \det(A) > 0\} \tag{7.17}$$

- *The structure group of E can be reduced to $SO(k)$*

Proof. The proof follows from the above discussion, with the following remarks. If $e_1, \dots, e_k$ is a local basis of sections, we say that $\{e_1, \dots, e_k\}$ is oriented if

$$e_1 \wedge \cdots \wedge e_k = f\omega, \tag{7.18}$$

with $f > 0$ and $\omega \in \Lambda^k(E)$ is the nowhere zero section. Restricting to local trivializations arising from oriented local bases of sections will give a reduction of structure group to $GL_+(k, \mathbb{R})$. □

Definition 7.8. We say that a real vector bundle $\pi : E \to M$ is *orientable* if any of the equivalent conditions in Proposition 7.7 are satisfied.

Remark 7.9. If we use the 2-fold covering notion, then we see that if $\pi_1(M) = \{e\}$ then every vector bundle over M is orientable. This is because any covering of a simply connected space is trivial. (Actually, we just need to assume that $H^1(M, \mathbb{Z}_2) = 0$.) Thus, every vector bundle over S^n is orientable for $n \geq 2$.

Example 7.10. Returning to $\mathbb{RP}^n$, since $\mathbb{RP}^n$ is double covered by S^n, we have $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$. Therefore there are exactly 2 real line bundles over $\mathbb{RP}^n$, the trivial bundle and the tautological line bundle. Note that if we put a Riemannian metric on the tautological bundle $\pi : \gamma_n^1 \to \mathbb{RP}^n$, then the total space of the unit sphere bundle is just S^n. But for the trivial bundle over $\mathbb{RP}^n$, the unit sphere bundle is just 2 copies of $\mathbb{RP}^n$.

8 Lecture 8

8.1 Induced mappings

Recall that if $L : V \to W$ is a linear mapping between vector spaces, then there is a mapping, $L^* : W^* \to V^*$ called the *transpose*, defined by the following. If $\omega \in W^*$, and $v \in V$, then

$$(L^*\omega)(v) = \omega(Lv). \tag{8.1}$$

This is called the transpose for the following reason. Let $dim(V) = n$, and $\dim(W) = m$. Let $e_1, \ldots, e_n$ be a basis of V and $f_1, \ldots, f_n$ be a basis of W. Let $e^1, \ldots, e^n$, and $f^1, \ldots, f^n$ denote the dual bases, that is

$$e^i(e_j) = \delta^i_j, \ 1 \leq i, j \leq n \tag{8.2}$$

$$f^i(f_j) = \delta^i_j, \ 1 \leq i, j \leq m. \tag{8.3}$$

We define the quantities L^j_i, $1 \leq i \leq n$, $1 \leq j \leq m$, by

$$Le_i = L^j_i f_j. \tag{8.4}$$

Note that if we write $v \in V$ as $v = v^i e_i$, and $w \in W$ as $w = w^i f_i$, then

$$Lv = L(v^i e_i) = v^i L(e_i) = (v^i L^j_i) f_j = \tag{8.5}$$

So the components of a vector transform like

$$\{v^i\} \mapsto \{L^j_i v^i\}, \tag{8.6}$$

which is the matrix corresponding to the transformation L.

We define the quantities $(L^*)^i_j$, $1 \leq i \leq m, \ 1 \leq j \leq n$, by

$$L^* f^i = (L^*)^i_j e^j \tag{8.7}$$

Plugging in the dual bases, we compute

$$(L^* f^i)(e_k) = (L^*)^i_j e^j(e_k) = (L^*)^i_j \delta^j_k = (L^*)^i_k. \tag{8.8}$$

However, by the definition of the transpose mapping, we have

$$(L^* f^i)(e_k) = f^i(Le_k) = f^i L^j_k f_j = L^j_k f^i(f_j) = L^j_k \delta^i_j = L^i_k \tag{8.9}$$

So if we write $\omega \in V^*$ as $\omega_i e^i$ and $\eta \in W^*$ as $\eta_j f^j$, the components of a dual vector transform like

$$\{\eta_j\} \mapsto \{L^i_j \eta_i\} \tag{8.10}$$

So the matrix corresponding to L^* in the dual basis is indeed the transpose matrix.

The mapping $L^* : W^* \to V^*$ induces a mapping

$$(L^*)^{\times p} : \overbrace{W^* \times \cdots \times W^*}^{p} \to (V^*)^{\otimes p} \tag{8.11}$$

by

$$(L^*)^{\times p}(\alpha^1, \ldots, \alpha^p) \equiv (L^* \alpha^1) \otimes \cdots \otimes (L^* \alpha^p). \tag{8.12}$$

This mapping is a multilinear mapping, so by the universal property of tensor products, this induces a unique mapping

$$(L^*)^{\otimes p} : (W^*)^{\otimes p} \to (V^*)^{\otimes p}. \tag{8.13}$$

By composing with the projection $\pi : (V^*)^{\otimes p} \to \Lambda^p(V^*)$, we obtain an alternating multilinear mapping

$$(L^*)^{\times p} : (W^*)^{\otimes p} \to \Lambda^p(V^*). \tag{8.14}$$

Now by the universal property of exterior products, this induces a mapping

$$\Lambda^p(L^*) : \Lambda^p(W^*) \to \Lambda^p(V^*). \tag{8.15}$$

Note that by taking the direct sum on all p-s, we obtain a mapping between the full exterior algebras

$$\Lambda(L^*) : \Lambda(W^*) \to \Lambda(V^*) \tag{8.16}$$

which is an algebra homomorphism, that is

$$\Lambda(L^*)(\alpha \wedge \beta) = (\Lambda(L^*)\alpha) \wedge (\Lambda(L^*)\beta). \tag{8.17}$$

8.2 Pull-back bundles

If M and N are smooth manifolds, and $\pi_N : E \to N$ is a vector bundle over N, then given a smooth mapping $f : M \to N$, define

$$f^*E = \{(p, v) \in M \times E \mid f(p) = \pi_N(v)\}. \tag{8.18}$$

Proposition 8.1. *The pullback f^*E is a vector bundle over M, with projection given by $\pi_1(p, v) = p$, and the fiber f^*E over $p \in M$ is identified with the fiber $E_{f(p)}$, i.e., the following diagram commutes*

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\pi_2} & E \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N. \end{array} \tag{8.19}$$

Proof. Let $\Phi : U \times \mathbb{R}^k \to \pi_N^{-1}(U)$ be a local trivialization for E. The set $f^{-1}(U)$ is open since f is continuous, and define

$$f^*\Phi : f^{-1}(U) \times \mathbb{R}^k \to \pi_1^{-1}(f^{-1}(U)) \tag{8.20}$$

by

$$f^*\Phi(x, v) = (x, \Phi(f(x), v)). \tag{8.21}$$

The reader can verfiy that this is a local trivialization for f^*E. $\square$

Next we note that sections can be pulled back to sections of the pullback bundle.

Definition 8.2. Let $f : M \to N$ be a smooth mapping between smooth manifolds, and $\pi : E \to N$ be a vector bundle over N. If $\sigma : N \to E$ is a section of E, then $(\sigma \circ f)(x) = (x, \sigma(f(x)))$ is a section of $\pi_1 : f^*E \to M$ and is called the pullback of σ under f.

The fact that this is a section of the pullback bundle is almost obvious, we just need to check that

$$\pi_1(\sigma \circ f)(x) = \pi_1(x, \sigma(f(x))) = x. \tag{8.22}$$

8.3 Push-forward of vector fields

Next, we restrict to tangent bundles. Let $f : M \to N$ be a smooth mapping between smooth manifolds. Then f^*TN is a vector bundle over M. Define

$$(f_*)_B : TM \to f^*TN \tag{8.23}$$

by

$$(f_*)_B(v_p) = (p, f_*v). \tag{8.24}$$

(the subscript B is short for "bundle mapping"). We have the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{(f_*)_B} & f^*TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{id} & M. \end{array} \tag{8.25}$$

Definition 8.3. If $X \in \Gamma(TM)$, then we can define $f_*X \in \Gamma(f^*TN)$, by

$$f_*X \equiv (f_*)_B \circ X. \tag{8.26}$$

In words: under smooth mappings, vector fields push-forward to sections of the pull-back bundle.

Remark 8.4. Note that for $f : M \to N$, although we can push-forward individual tangent vectors, in general there is *not* a mapping

$$f_* : \Gamma(TM) \to \Gamma(TN). \tag{8.27}$$

For example, f might not even be surjective. This is one reason we had to consider pull-back bundles in the above discussion.

8.4 Pull-back of differential forms

Noting that $(f^*(TN))^*$ is isomorphic to $f^*(T^*N)$, let us dualize the diagram (35.30) to obtain

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{id} & M. \end{array} \tag{8.28}$$

Note that if $\omega \in \Gamma(T^*N)$, then $\omega \circ f \in \Gamma(f^*(T^*N))$. Then, if $\omega \in \Gamma(T^*N)$, we can compose with the bundle mapping in (8.28) to define the following.

Definition 8.5. If $f : M \to N$ is a smooth mapping between smooth manifolds, then

$$f^*\omega \equiv f_B^* \circ (\omega \circ f) \in \Gamma(T^*M) \tag{8.29}$$

is call the pullback of ω under f.

More generally, we have the following.

Definition 8.6. A differential form is a section of $\Lambda^p(T^*M)$. That is, a differential form is a smooth mapping $\omega : M \to \Lambda^p(T^*M)$ such that $\pi \circ \omega = Id_M$, where $\pi : \Lambda^p(T^*M) \to M$ is the bundle projection map. We will write $\omega \in \Gamma(\Lambda^p(T^*M))$, or $\omega \in \Omega^p(M)$.

If $f : M \to N$ is a smooth mapping, then by the diagram (8.28) and the above discussion, we obtain induced mappings

$$\begin{array}{ccc} f^*(\Lambda^p(T^*N)) & \xrightarrow{\Lambda^p(f_B^*)} & \Lambda^p(T^*M) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{\quad id \quad} & M, \end{array} \tag{8.30}$$

which is linear on fibers, and therefore f_B^* is a smooth mapping.

Definition 8.7 (Pull-back of a differential form)**.** If $f : M \to N$ is a smooth mapping, and $\omega \in \Lambda^p(T^*N)$, then define $\omega \circ f \in \Gamma(f^*(\Lambda^p(T^*N))$ by $\omega \circ f(p) = (p, \omega_{f(p)})$. Then define

$$f^*\omega \equiv f_B^*(\omega \circ f) \in \Gamma(\Lambda^p(T^*M)). \tag{8.31}$$

For any manifold M, define

$$\Omega(M) = \Gamma(\Lambda(T^*M)) = \Gamma\Big(\bigoplus_{p\geq 0} \Lambda^p(T^*M)\Big) = \bigoplus_{p\geq 0} \Gamma(\Lambda^p(T^*M)) = \bigoplus_{p\geq 0} \Omega^p(M). \tag{8.32}$$

By taking the direct sum of the exterior powers, we obtain a mapping

$$f^* : \Omega(N) \to \Omega(M), \tag{8.33}$$

which by (8.17) satisfies

$$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta). \tag{8.34}$$

The proof of the following proposition is left as an exercise.

Proposition 8.8 (The chain rule)**.** *If $f : M \to N$, and $h : N \to M'$ are smooth maps, then*

$$(h \circ f)^* = f^* \circ h^* : \Omega(M') \to \Omega(M). \tag{8.35}$$

9 Lecture 9

9.1 The exterior derivative

Given a function $f \in C^\infty(M, \mathbb{R})$ we define $df \in \Omega^1(M)$ in two ways. First, viewing vector fields as derivations on smooth functions, we can define

$$df(X) \equiv X(f). \tag{9.1}$$

Alternatively, since $f : M \to \mathbb{R}$, we have $f_* : TM \to T\mathbb{R}$. But there is a natural identification $T_p\mathbb{R} \cong \mathbb{R}$ for any $p \in \mathbb{R}$, so we can view

$$f_* : TU \to \mathbb{R}, \tag{9.2}$$

which is naturally an element in $df \in \Omega^1(U)$.

Exercise 9.1. Verify that these two definitions agree.

For a coordinate system (U, x), and let $\frac{\partial}{\partial x^i}$ denote the coordinate vector field. Recall that viewing vector fields as derivations on germs of functions, this is characterized by

$$\frac{\partial}{\partial x^i}(x^j) = \delta_i^j. \tag{9.3}$$

We then define a local basis of 1-forms dx^i by

$$dx^i\Big(\frac{\partial}{\partial x^i}\Big) = \delta_j^i. \tag{9.4}$$

Note this is just the dual basis, but these are also $d(x^i)$ as defined above in (9.1).

An element $\alpha \in \Omega^p(U)$ can be written as

$$\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{9.5}$$

where the coefficients $\alpha_{i_1 \ldots i_p} : U \to \mathbb{R}$ are well-defined functions. Note these coefficients are only defined for strictly increasing sequences $i_1 < \cdots < i_p$.

We next define the exterior derivative operator [War83, Theorem 2.20].

Proposition 9.2. *There exists an exterior derivative operator*

$$d : \Omega^p(M) \to \Omega^{p+1}(M) \tag{9.6}$$

which is the unique linear mapping satisfying

- *For* $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.
- $d^2 = 0$.
- *If* $f \in C^\infty(M, \mathbb{R})$ *then* df *is the differential of* f *defined above.*

Proof. Note that the differential of a function is given locally by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \tag{9.7}$$

This is obviously well-defined and independent of the coordinate system. Given a p-form α, write α locally as in (9.5), and then define

$$\begin{aligned} d\alpha &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^n d\alpha_{i_1 \ldots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^n \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{9.8}$$

The first "anti-derivation" property is easily verified by computation. The second property holds on functions, because

$$d(df) = d\Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i\Big) = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0, \tag{9.9}$$

since the Hessian of a smooth function is symmetric.

For existence, we need to check that this definition is independent of the coordinate system. Let d' be the operator defined with respect to another coordinate system $x' : U \to \mathbb{R}^n$. Then

$$\begin{aligned} d'(\alpha) &= d'\Big(\sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\Big) \\ &= \sum_{|I|=p} (d'\alpha_{i_1 \dots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &\quad + \sum_{|I|=p} \alpha_{i_1 \dots i_p} \sum_k (-1)^{k-1} dx^{i_1} \wedge \cdots \wedge d'(dx^{i_k}) \wedge \cdots \wedge dx^{i_p} \\ &= \sum_{|I|=p} (d\alpha_{i_1 \dots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = d(\alpha), \end{aligned} \tag{9.10}$$

since d and d' agree on functions, and since $d'dx^i = d'd'x^i = 0$.

Then for any p-form α,

$$d(d\alpha) = d\Big(\sum_{|I|=p} (d\alpha_I) \wedge dx^I\Big) = \sum_{|I|=p} (d^2\alpha_I) \wedge dx^I - d\alpha_I \wedge d(dx^I) = 0. \tag{9.11}$$

Uniqueness is left as an exercise. □

An important fact is that d commutes with pull-back.

Proposition 9.3. *If $f : M \to N$ is a smooth mapping, and $\omega \in \Omega^p(N)$, then*

$$f^*(d_N \omega) = d_M(f^*\omega). \tag{9.12}$$

Proof. If ω is a 0-form, which is a function, then $f^*\omega = \omega \circ f$. So by above, we have

$$d(f^*\omega) = d(\omega \circ f) = (\omega \circ f)_*. \tag{9.13}$$

By the chain rule, we then have

$$d(f^*\omega) = \omega_* \circ f_*. \tag{9.14}$$

On the other hand, we have that

$$f^*(d\omega)(X) = d\omega(f_*(X)) = \omega_* \circ f_*(X). \tag{9.15}$$

So the claim is true on functions. Then if ω is a p-form, write

$$\omega = \sum_{|I|=p} \omega_I dx^I. \tag{9.16}$$

Since the pull-back operation is an algebra homomorphism, we have

$$f^*\omega = \sum_{|I|=p} (f^*\omega_I) f^* dx^I = \sum_{|I|=p} (\omega_I \circ f) d(x^I \circ f). \tag{9.17}$$

Then

$$d(f^*\omega) = \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f). \tag{9.18}$$

On the other hand, we have

$$d\omega = \sum_{|I|=p} (d\omega_I) \wedge dx^I, \tag{9.19}$$

so

$$\begin{aligned} f^*(d\omega) = \sum_{|I|=p} f^*(d\omega_I) \wedge f^* dx^I &= \sum_{|I|=p} d(f^*\omega_I) \wedge d(f^* x^I) \\ &= \sum_{|I|=p} d(\omega_I \circ f) \wedge d(x^I \circ f) = d(f^*\omega). \end{aligned} \tag{9.20}$$

□

9.2 de Rham cohomology

Let M be a smooth manifold. Since $d^2 = 0$, we have a "cochain" complex

$$\cdots \xrightarrow{d^{p-2}} \Omega^{p-1}(M) \xrightarrow{d^{p-1}} \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \xrightarrow{d^{p+1}} \cdots . \tag{9.21}$$

which terminates at $\Omega^n(M)$, where $n = \dim(M)$. Clearly we have that $Im(d^{p-1}) \subset Ker(d^p)$, so we can define the following vector spaces.

Definition 9.4. For $0 \leq p \leq n$, the pth de Rham cohomology group is

$$H^p_{dR}(M) = \frac{Ker\{d^p : \Omega^p(M) \to \Omega^{p+1}(M)\}}{Im\{d^{p-1} : \Omega^{p-1}(M) \to \Omega^p(M)\}}. \tag{9.22}$$

Note that

$$H^*_{dR}(M) \equiv \bigoplus_{p=0}^{n} H^p_{dR}(M) \tag{9.23}$$

has an algebra structure induced by the wedge product. To see this, for $[\alpha] \in H^p_{dR}(M)$ and $[\beta] \in H^q_{dR}(M)$, represented by $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, we have that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0, \tag{9.24}$$

so we define

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]. \tag{9.25}$$

To see that this is well-defined, we have

$$(\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d\gamma \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta), \tag{9.26}$$

since β is closed, so

$$[(\alpha + d\gamma) \wedge \beta] = [\alpha \wedge \beta]. \tag{9.27}$$

Well-definedness in the other factor is similar, or just use the skew-symmetry property of the wedge product. Therefore we have

$$\wedge : H^p_{dR}(M) \otimes H^q_{dR}(M) \to H^{p+q}_{dR}(M). \tag{9.28}$$

Note that from Proposition 9.2, we have

$$[\alpha] \wedge [\beta] = (-1)^{pq} [\beta] \wedge [\alpha]. \tag{9.29}$$

Next, let $f : X \to Y$ be a smooth mapping between smooth manifolds. As discussed before, we have a pullback operation on differential forms, $f^* : \Omega^*(Y) \to \Omega^*(X)$, which makes the following diagram commute

$$\begin{array}{ccc} \Omega^p(Y) & \xrightarrow{d^p_Y} & \Omega^{p+1}(Y) \\ \downarrow{\scriptstyle (f^*)^p} & & \downarrow{\scriptstyle (f^*)^{p+1}} \\ \Omega^p(X) & \xrightarrow{d^p_X} & \Omega^{p+1}(X). \end{array} \tag{9.30}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The de Rham cohomology algebra is a diffeomorphism invariant.

Corollary 9.5. *If $f : X \to Y$ then there are induced mappings*

$$(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X). \tag{9.31}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{9.32}$$

Consequently, if X and Y are diffeomorphic, then $H^p_{dR}(X) \cong H^p_{dR}(Y)$ for every $p \geq 0$, and moreover, the cohomology algebras are isomorphic $H^*_{dR}(X) \cong H^*_{dR}(Y)$.

Proof. We first note that any smooth mapping $f : X \to Y$ induces a well-defined mapping on cohomology $(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X)$ by the following. If $[\alpha] \in H^p_{dR}(Y)$ is represented by a form α, such that $d^p_Y \alpha = 0$, then we have

$$d^p_X (f^*)^p \alpha = (f^*)^{p+1} d^p_Y \alpha = (f^*)^{p+1} 0 = 0, \tag{9.33}$$

so we can define $f^*[\alpha] = [f^*\alpha]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$(f^*)^p(\alpha + d^{p-1}_Y \beta) = (f^*)^p \alpha + (f^*)^p d^{p-1}_Y \beta = (f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1} \beta, \tag{9.34}$$

so $[(f^*)^p(\alpha + d^p_Y \beta)] = [(f^*)^p \alpha + d^{p-1}_X (f^*)^{p-1} \beta] = [(f^*)^p \alpha]$.

If f is a diffeomorphism, then f^{-1} exists and is smooth, so we have

$$f \circ f^{-1} = id_Y, \quad f^{-1} \circ f = id_X, \tag{9.35}$$

and from Proposition 8.8, the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^*_{dR}(X)}, \quad (f^{-1})^* \circ f^* = id_{H^*_{dR}(Y)}, \tag{9.36}$$

Finally, since $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$, together these mappings form an algebra homomorphism on cohomology algebras, which will be an algebra isomorphism if X and Y are diffeomorphic. □

10 Lecture 10

Let us formalize some of the above notions.

10.1 Cochain complexes

A collection A^p of vector spaces for $p \geq 0$ and operators $\delta^p_A : A^p \to A^{p+1}$ for $p \geq 0$ satisfying $\delta^{p+1}_A \delta^p_A = 0$ is called a *cochain complex*.

$$\cdots \xrightarrow{\delta^{p-2}_A} A^{p-1} \xrightarrow{\delta^{p-1}_A} A^p \xrightarrow{\delta^p_A} A^{p+1} \xrightarrow{\delta^{p+1}_A} \cdots . \tag{10.1}$$

Definition 10.1. The pth cohomology of a chain complex is the vector space

$$H^p(A) = \frac{Ker\{\delta^p_A : A^p \to A^{p+1}\}}{Im\{\delta^{p-1}_A : A^{p-1} \to A^p\}} \tag{10.2}$$

Definition 10.2. A morphism $\alpha : A \to B$ of cochain complexes is a collection of mappings $\alpha^p : A^p \to B^p$ such that $\delta^p_B \alpha^p = \alpha^{p+1} \delta^p_A$ for $p \geq 0$. In other words, $\alpha : A \to B$ is a morphism if the following diagram commutes

$$\begin{array}{ccc} A^p & \xrightarrow{\delta^p_A} & A^{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ B^p & \xrightarrow{\delta^p_B} & B^{p+1}. \end{array} \tag{10.3}$$

Proposition 10.3. *Morphisms satisfy the following properties:*

- *Composition of morphisms: If $\alpha : A \to B$ and $\beta : B \to C$ are morphisms of chain complexes, then $\beta \circ \alpha : A \to C$ is a morphism.*
- *Associativity: If $\gamma : C \to D$ is another morphism, then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.*

Consequently, the collection of cochain complexes and morphisms of cochain complexes forms a category.

Proof. The diagram looks like

$$\begin{array}{ccc} A^p & \xrightarrow{d_A^p} & A^{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ B^p & \xrightarrow{d_B^p} & B^{p+1}. \\ \downarrow{\scriptstyle \beta^p} & & \downarrow{\scriptstyle \beta^{p+1}} \\ C^p & \xrightarrow{d_C^p} & C^{p+1}. \end{array} \tag{10.4}$$

We want to show that

$$\beta^{p+1} \circ \alpha^{p+1} \circ d_A^p = d_C^p \circ \beta^p \circ \alpha^p. \tag{10.5}$$

Using commutativity of the top square, the left hand side of (30.24) is

$$\beta^{p+1} \circ \alpha^{p+1} \circ d_A^p = \beta^{p+1} \circ d_B^p \circ \alpha^p. \tag{10.6}$$

Using commutativity of the bottom square, the right hand side of (30.24) is

$$d_C^p \circ \beta^p \circ \alpha^p = \beta^{p+1} \circ d_B^p \circ \alpha^p, \tag{10.7}$$

which proves (30.24).

Associativity is clear: $\gamma^p \circ (\beta^p \circ \alpha^p) = (\gamma^p \circ \beta^p) \circ \alpha^p$ holds for every $p \geq 0$ since composition of mappings is associative. □

Proposition 10.4. *A morphism of cochain complexes $\alpha : A \to B$ induces mappings $H^p\alpha : H^p(A) \to H^p(B)$. Furthermore, if $\beta : B \to C$ is another morphism of chain complexes, then*

$$H^p(\beta \circ \alpha) = H^p\beta \circ H^p\alpha. \tag{10.8}$$

Proof. Given $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$, we have

$$\delta_B^p \alpha^p a^p = \alpha^{p+1} \delta_A^p a^p = 0, \tag{10.9}$$

therefore we can define $(H^p\alpha^p)[a^p] = [\alpha^p a^p]$. To check that this is well-defined,

$$[\alpha^p(a^p + \delta_A^{p-1} a^{p-1})] = [\alpha^p a^p + \alpha^p \delta_A^{p-1} a^{p-1}] = [\alpha^p a^p + \delta_B^{p-1} \alpha^{p-1} a^{p-1}] = [\alpha^p a^p]. \tag{10.10}$$

Next,for $[a^p] \in H^p(A)$ represented by $a^p \in A^p$, we have

$$H^p(\beta \circ \alpha)[a^p] = [(\beta^p \circ \alpha^p)a^p] = [\beta^p(\alpha^p(a^p))] = H^p\beta^p[\alpha^p(a^p)] = H^p\beta(H^p\alpha[a^p]). \tag{10.11}$$

□

Definition 10.5. For $p \geq 0$, the pth cohomology functor H^p is the mapping between the category of chain complexes to the category of vector spaces (with morphisms being linear mappings) given by $A \mapsto H^p(A)$.

Proposition 10.6. *The functor H^p is a covariant functor.*

Proof. The functor H^p maps objects to objects, just by mapping the chain complex C to the vector space $H^p(C)$. Also for each morphism $\alpha : A \to B$ between chain complexes, we associate the morphism $H^p\alpha : H^p(A) \to H^p(B)$. The covariant property is (30.27). □

Definition 10.7. The pth de Rham cohomology functor is the functor from the category of smooth manifolds and smooth mappings to vector spaces the category of vector spaces and linear mappings given by $M \mapsto H^p_{dR}(M, \mathbb{R})$ and $f : X \to Y$ maps to $H^p f = (f^*)^p : H^p(Y) \to H^p(X)$.

Proposition 10.8. *The pth de Rham cohomology functor is a contravariant functor.*

Proof. This follows from $(g \circ f)^* = f^* \circ g^*$, and the fact that the composition of a covariant functor and a contravariant functor is a contravariant functor. □

10.2 Cochain homotopy between morphisms of cochain complexes

Definition 10.9. Let $f : A \to B$, and $g : A \to B$ be two morphisms of cochain complexes. We say that f is cochain homotopic to g if there exists mappings $S^p : A^p \to B^{p-1}$ such that

$$f^p - g^p = \delta_B^{p-1} S^p + S^{p+1} \delta_A^p. \tag{10.12}$$

Proposition 10.10. *If f is cochain homotopic to g then $H^p f = H^p g : H^p(A) \to H^p(B)$.*

Proof. Consider the mapping $H^p f - H^p g$, and take $[a^p] \in H^p(A)$ represented by $a^p \in A^p$ satisfying $\delta_A^p a^p = 0$. Then

$$\begin{aligned}(H^p f - H^p g)[a^p] &= (H^p(f - g))[a^p] = [(H^p(f - g))a^p] \\ &= [\delta_B^{p-1} S^p a^p + S^{p+1} \delta_A^p a^p = [\delta_B^{p-1} S^p a^p] = 0.\end{aligned} \tag{10.13}$$

□

10.3 Homotopy invariance of de Rham cohomology

Let M be a smooth manifold, possibly noncompact. Let $\Omega^p(M)$ denote the smooth p-forms on M. Recall that we have a cochain complex

$$\cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots, \tag{10.14}$$

and $H^p_{dR}(M)$ is defined to be the cohomology of this complex.

Let M be a differentiable n-manifold, and consider $N = M \times [0, 1]$. Let $\pi : N \to M$ be the projection $\pi(x, t) = x$. Also, let $\iota_t : M \to M \times [0, 1]$ be the inclusion $\iota_t(x) = (x, t)$.

Remark 10.11. The object $N = M \times [0,1]$ is a manifold with boundary. The reader can check that all the properties of differential forms proved above hold in the category of manifolds with boundary.

We next define a mapping

$$I^k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M) \tag{10.15}$$

by the following. Any k-form on N can be written as

$$\omega = h(x,t)\pi^*\phi_k + f(x,t)dt \wedge (\pi^*\phi_{k-1}), \tag{10.16}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega^0(M \times [0,1])$. Define

$$I^k(\omega) = \Big(\int_0^1 f(p,t)dt\Big)\phi_{k-1}. \tag{10.17}$$

Proposition 10.12. *For $\omega \in \Omega^k(N)$, we have*

$$(\iota_1)^*\omega - (\iota_0)^*\omega = d_M I^k\omega + I^{k+1}d_N\omega. \tag{10.18}$$

In other words, I^k is a cochain homotopy between $(\iota_0)^*$ and $(\iota_1)^*$.

Proof. Writing ω in the form (10.16), since $\iota_t^* dt = 0$, and $\pi \circ \iota_t = id_M$, the left hand side of (10.18) is

$$\begin{aligned}(\iota_1)^*\omega - (\iota_0)^*\omega &= (\iota_1)^*h(x,t)\pi^*\phi_k - (\iota_0)^*h(x,t)\pi^*\phi_k \\ &= (h(x,1) - h(x,0))\phi_k\end{aligned} \tag{10.19}$$

Next, assume that ω is just of the form

$$\omega = h(x,t)\pi^*\phi_k. \tag{10.20}$$

Then

$$d_N\omega = \Big(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt\Big) \wedge \pi^*\phi_k + h(x,t)\pi^*d_M\phi_k. \tag{10.21}$$

By defintion of I^*,

$$d_M I^k\omega = 0, \tag{10.22}$$

and

$$\begin{aligned}I^{k+1}d_N\omega &= I^{k+1}\Big\{\Big(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt\Big) \wedge \pi^*\phi_k + h(x,t)\pi^*d_M\phi_k\Big\} \\ &= I^{k+1}\Big\{\frac{\partial h}{\partial t}dt \wedge \pi^*\phi_k\Big\} = \Big(\int_0^1 \frac{\partial h}{\partial t}dt\Big)\phi_k = (h(x,1) - h(x,0))\phi_k.\end{aligned} \tag{10.23}$$

So the proposition holds for forms of this type.

Next, assume that ω is just of the form

$$\omega = f(x,t)dt \wedge (\pi^*\phi_{k-1}). \tag{10.24}$$

From (10.19) above, we have

$$(\iota_1)^*\omega - (\iota_0)^*\omega = 0. \tag{10.25}$$

Note that

$$\begin{aligned} d_N\omega &= \frac{\partial f}{\partial x}dx \wedge dt \wedge (\pi^*\phi_{k-1}) - f(x,t)dt \wedge \pi^*(d_M\phi_{k-1}) \\ &= -\frac{\partial f}{\partial x}dt \wedge \pi^*\Big(dx \wedge \phi_{k-1}\Big) - fdt \wedge \pi^*\Big(d_M\phi_{k-1}\Big). \end{aligned} \tag{10.26}$$

By defintion of I^k,

$$\begin{aligned} d_M I^k\omega &= d_M\Big\{\Big(\int_0^1 f(x,t)dt\Big)\phi_{k-1}\Big\} \\ &= \Big(\int_0^1 \frac{\partial f}{\partial x}dt\Big)dx \wedge \phi_{k-1} + \Big(\int_0^1 fdt\Big)d_M\phi_{k-1}. \end{aligned} \tag{10.27}$$

Next, by definition of I^{k+1} and (10.26), we have

$$I^{k+1}d_N\omega = -\Big(\int_0^1 \frac{\partial f}{\partial x}dt\Big)dx \wedge \phi_{k-1} - \Big(\int_0^1 fdt\Big)d_M\phi_{k-1}. \tag{10.28}$$

So on forms of this type, we have

$$d_M I^k\omega + I^{k+1}d_N\omega = 0. \tag{10.29}$$

So the proposition is true for forms of the second type. By linearity, the proposition holds for all forms, and we are done. □

Definition 10.13. Let X and Y be smooth manifolds. Smooth mappings $f, g : X \to Y$ are said to be smoothly homotopic if there exists a smooth mapping $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

Proposition 10.14. *Let X and Y be smooth manifolds. If $f, g : X \to Y$ are smoothly homotopic then*

$$f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X) \tag{10.30}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x,t)$, and note that

$$(\iota_t)^* : \Omega^*(X \times [0,1]) \to \Omega^*(X). \tag{10.31}$$

In Proposition 10.12, we constructed a cochain homotopy between ι_1^* and ι_0^*,

$$I^k : \Omega^k(X \times [0,1]) \to \Omega^{k-1}(X) \tag{10.32}$$

satisfying

$$(\iota_1)^* - (\iota_0)^* = I^{k+1} d_{X\times[0,1]} + d_X I^k. \tag{10.33}$$

By Proposition 10.10, we have that

$$(\iota_0)^* = (\iota_1)^* : H^k_{dR}(X \times [0,1]) \to H^k_{dR}(X). \tag{10.34}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, and H^k_{dR} is a contravariant functor, we have

$$f^* = (\iota_0)^* \circ F^*, \quad g^* = (\iota_1)^* \circ F^*, \tag{10.35}$$

therefore $f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X)$. □

11 Lecture 11

11.1 Homotopy type

Definition 11.1. Smooth manifolds X and Y have the same smooth homotopy type if there exist smooth mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is smoothly homotopic to Id_X and $f \circ g$ is smoothly homotopic to id_Y.

Corollary 11.2. *If X and Y have the same smooth homotopy type, then $H^*_{dR}(X) \cong H^*_{dR}(Y)$.*

Proof. From Proposition 32.2, we have

$$f^* \circ g^* = Id_{H^*_{dR}(X)} \tag{11.1}$$
$$g^* \circ f^* = Id_{H^*_{dR}(Y)}, \tag{11.2}$$

so f^* and g^* are isomorphisms. □

Some special cases of this are the following.

Definition 11.3. A smooth manifold X is smoothly contractible if X has the same smooth homotopy type as a point.

Corollary 11.4. *If X is smoothly contractible, then*

$$H^k_{dR}(X) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases}. \tag{11.3}$$

Proof. If $X = \{p\}$ is a single point. Then $\Omega^0(X)$ are functions $f : p \to \mathbb{R}$, so $\Omega^0(X) = \mathbb{R}$, and since $df = 0$, we have $H^0_{dR}(X) = \mathbb{R}$. There are no k-forms on X for $k > 0$, so the corollary follows. □

Example 11.5. A domain $A \subset \mathbb{R}^n$ is star-shaped if there exists a $p \in A$ such that for any $x \in A$, the line segment between p and x is contained in A. In this case, let $F : A\times[0,1] \to \mathbb{R}^n$ be the mapping $F(x,t) = (1-t)x + tp$. This shows that A is (smoothly) contractible, so A has the same de Rham cohomology groups as a point.

Definition 11.6. A subset (submanifold) $i : A \hookrightarrow X$ is a (smooth) deformation retraction of X if there exists a (smooth) mapping $r : X \to X$ such that

$$r \circ i = id_A, \tag{11.4}$$

and $i \circ r$ is (smoothly) homotopic to Id_X.

Corollary 11.7. *If A is a smooth deformation retraction of X then*

$$H^k_{dR}(A) \cong H^k_{dR}(X), \tag{11.5}$$

for all $k \geq 0$.

Example 11.8. Consider $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \subset \mathbb{R}^n$ given by $r(x) = x/|x|$. The mapping $F(x,t) = (1-t)x + t(x/|x|)$ is a smooth homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, so S^{n-1} is a smooth deformation retraction of $\mathbb{R}^n \setminus \{0\}$ and we therefore have

$$H^k_{dR}(S^{n-1}) = H^k_{dR}(\mathbb{R}^n \setminus \{0\}). \tag{11.6}$$

11.2 Exact sequences of cochain complexes

Definition 11.9. A sequence of vector spaces A, B, C, with linear mappings $\alpha : A \to B$, $\beta : B \to C$

$$0 \xrightarrow{0} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{0} 0 \tag{11.7}$$

is called *exact* if the kernel of each mapping is equal to the image of the previous mapping. That is $Ker(\alpha) = \{0\}$ if and only if α is injective. Next, $Ker(\beta) = Im(\alpha)$. Finally, $Im(\beta) = C$, if and only if β is surjective.

Let C_i be a co-complex of vector spaces for $i = 1, 2, 3$.

$$\cdots \xrightarrow{d_i^{p-2}} C_i^{p-1} \xrightarrow{d_i^{p-1}} C_i^p \xrightarrow{d_i^p} C_i^{p+1} \xrightarrow{d_i^{p+1}} \cdots. \tag{11.8}$$

with $d^2 = 0$. A morphism from C_i to C_j are mappings $\alpha^k : C_i^k \to C_j^k$ such that the following diagram commutes for every p

$$\begin{array}{ccc} C_i^p & \xrightarrow{d_i^p} & C_i^{p+1} \\ \downarrow{\scriptstyle \alpha^p} & & \downarrow{\scriptstyle \alpha^{p+1}} \\ C_j^p & \xrightarrow{d_j^p} & C_j^{p+1} \end{array} \tag{11.9}$$

For co-complexes C_1, C_2, C_3, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of co-complexes is exact if

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{11.10}$$

if the sequence

$$0 \xrightarrow{0} C_1^p \xrightarrow{\alpha^p} C_2^p \xrightarrow{\beta^p} C_3^p \xrightarrow{0} 0 \tag{11.11}$$

is exact for every p.

Lemma 11.10 (The zig-zag lemma for cochain complexes)**.** *If*

$$0 \xrightarrow{0} C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \xrightarrow{0} 0 \tag{11.12}$$

is a short exact sequence of co-complexes, then there exist connecting homomorphisms

$$\delta^p : H^p(C_3) \to H^{p+1}(C_1) \tag{11.13}$$

for every p such that the sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\delta^p} H^{p+1}(C_1) \longrightarrow \cdots \tag{11.14}$$

is exact.

Proof. We look at the huge commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1^{p-1} & \xrightarrow{\alpha^{p-1}} & C_2^{p-1} & \xrightarrow{\beta^{p-1}} & C_3^{p-1} & \longrightarrow & 0 \\
 & & \downarrow d_1^{p-1} & & \downarrow d_2^{p-1} & & \downarrow d_3^{p-1} & & \\
0 & \longrightarrow & C_1^{p} & \xrightarrow{\alpha^{p}} & C_2^{p} & \xrightarrow{\beta^{p}} & C_3^{p} & \longrightarrow & 0 \\
 & & \downarrow d_1^{p} & & \downarrow d_2^{p} & & \downarrow d_3^{p} & & \\
0 & \longrightarrow & C_1^{p+1} & \xrightarrow{\alpha^{p+1}} & C_2^{p+1} & \xrightarrow{\beta^{p+1}} & C_3^{p+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{11.15}$$

which has all horizontal rows exact.

To define the connecting homomorphism, take $c_3^p \in C_3^p$ with $d_3^p c_3^p = 0$. By exactness of the middle row, β_p is surjective, so $c_3^p = \beta^p(c_2^p)$ for some $c_2^p \in C_2^p$. Then since the diagram commutes, we have

$$\beta^{p+1} d_2^p c_2^p = d_3^p \beta^p c_2^p = d_3^p c_3^p = 0. \tag{11.16}$$

By exactness of the bottow row, we have $d_2^p c_2^p = \alpha^{p+1} c_1^{p+1}$ for some $c_1^{p+1} \in C_1^{p+1}$. Since C_1 is a co-complex, and by commutativity of the diagram, we have

$$0 = d_2^{p+1} d_2^p c_2^p = d_2^{p+1} \alpha^{p+1} c_1^{p+1} = \alpha^{p+2} d_1^{p+1} c_1^{p+1}, \tag{11.17}$$

which implies that $d_1^{p+1} c_1^{p+1} = 0$, since α^{p+2} is injective. So we define $\delta^p(c_3^p) = [c_1^{p+1}]$, the homology class of c_1^{p+1} in $H^{p+1}(C_1)$.

To prove this mapping is well-defined, assume that we started with $c_p^3 \in C_p^3$ which was of the form $c_p^3 = d_3^{p-1} c_3^{p-1}$. Then we can write $c_3^{p-1} = \beta^{p-1} c_2^{p-1}$, and the element $\tilde{c}_2^p = d_2^{p-1} c_2^{p-1}$ satisfies $\beta^p(\tilde{c}_2^p) = c_3^p$. But this element is exact, so the next step clearly gives zero. Independence of the choice of c_2^p is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing.

□

12 Lecture 12

12.1 Mayer-Vietoris for de Rham cohomology

Write $M = U \cup V$ as the union of two open sets in M. Then the following sequence is exact:

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{\beta^p} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{\alpha^p} \Omega^p(U \cap V) \longrightarrow 0 \tag{12.1}$$

where

$$\beta^p(\omega) = \big((i_{U \hookrightarrow M})^* \omega, (i_{V \hookrightarrow M})^* \omega\big). \tag{12.2}$$

and

$$\alpha^p(\omega_U, \omega_V) = (i_{U \cap V \hookrightarrow U})^* \omega_U - (i_{U \cap V \hookrightarrow V})^* \omega_V \tag{12.3}$$

To see this, β^p is obviously injective. For exactness at the middle step, obviously $\alpha^p \beta^p \omega = 0$. If $\beta^p(\omega_U, \omega_V) = 0$, then $\omega_U = \omega_V$ on $U \cap V$, so then (ω_U, ω_V) is a well-defined global form on M.

To show that α is onto, let $\omega \in \Omega^p(U \cap V)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \alpha(\phi_V \omega, -\phi_U \omega)$.

By the zig-zag lemma for cohomology, we obtain a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{12.4}$$

Let us review the definition of the mapping δ^p. Given a cohomology class $[\omega] \in H^p_{dR}(U \cap V)$, represented by $\omega \in \Omega^p(U \cap V)$ with $d\omega = 0$, we first write $\omega = \alpha^p(\phi_V \omega, -\phi_U \omega)$, then we apply the exterior derivative to get

$$(d(\phi_V \omega), -d(\phi_U \omega)) = (d\phi_V \wedge \omega, -d\phi_U \wedge \omega) \in \Omega^p(U) \oplus \Omega^p(V). \tag{12.5}$$

Note that on $U \cap V$, we have $(\phi_U + \phi_V)\omega = \omega$, so applying d to this equation, we have that $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$ on $U \cap V$, so together these define a global form

$$\delta^p \omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{12.6}$$

and we take the cohomology class of this form.

Remark 12.1. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

The following lemma will be useful.

Lemma 12.2. *If*

$$0 \longrightarrow V_1 \xrightarrow{\alpha} V_2 \longrightarrow \cdots \longrightarrow V_{k-1} \longrightarrow V_k \longrightarrow 0. \tag{12.7}$$

is exact, then

$$0 = \dim(V_1) - \dim(V_2) + \dim(V_3) + \cdots + (-1)^{k-1} \dim(V_k). \tag{12.8}$$

Proof. Induction. □

Example 12.3. S^n: Cover with 2 open sets U, V, with $U \cong \mathbb{R}^n \cong V$ and $U \cap V \cong S^{n-1}$, use the Mayer-Vietoris sequence and induction to get

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases} \tag{12.9}$$

First, consider the case of S^1.

$$\begin{array}{l} 0 \longrightarrow H^0_{dR}(S^1) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \\ \hookrightarrow H^1_{dR}(S^1) \xrightarrow{\beta^1} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^1} H^1_{dR}(U \cap V) \longrightarrow 0. \end{array} \tag{12.10}$$

But $U \cap V$ is contractible to 2 points, so this is

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta^0} \\ \hookrightarrow H^1_{dR}(S^1) \xrightarrow{\beta^1} 0. \end{array} \tag{12.11}$$

The lemma then says that $H^1_{dR}(S^1) \cong \mathbb{R}$.

Next, for $n > 1$, look at the beginning of the Mayer-Vietoris sequence

$$0 \longrightarrow H^0_{dR}(S^n) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots \tag{12.12}$$

But now $U \cap V$ is connected, so this is

$$0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \xrightarrow{\delta^0} \cdots . \tag{12.13}$$

Since β is injective, the kernel of α^0 is 1-dimensional. But α^0 has a 2-dimensional domain, so the image of α^0 is 1-dimensional, that is α^0 is surjective. So we can move to the next level and get

$$0 \longrightarrow H^1_{dR}(S^n) \xrightarrow{\beta^0} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^0} H^1_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots , \tag{12.14}$$

Since U and V are contractible, this says that $H^1_{dR}(S^n) = 0$ for $n \geq 2$.

Next, we look at the upper portion of the Mayer-Vietoris sequence

$$\begin{array}{l} \cdots \longrightarrow H^{n-2}_{dR}(U) \oplus H^{n-2}_{dR}(V) \xrightarrow{\alpha^{n-2}} H^{n-2}_{dR}(S^{n-1}) \xrightarrow{\delta^{n-2}} \\ \hookrightarrow H^{n-1}_{dR}(S^n) \xrightarrow{\beta^p} 0 \xrightarrow{\alpha^p} H^{n-1}_{dR}(S^{n-1}) \xrightarrow{\delta^{n-1}} \\ \hookrightarrow H^n_{dR}(S^n) \xrightarrow{\beta^{p+1}} 0. \end{array} \tag{12.15}$$

This yields

$$H^n_{dR}(S^n) \cong H^{n-1}_{dR}(S^{n-1}) \cong \mathbb{R}, \tag{12.16}$$

and

$$H^k_{dR}(S^n) \cong H^{k-1}_{dR}(S^{n-1}) = 0, \tag{12.17}$$

for $2 \leq k \leq n-1$, so this finishes the proof.

Example 12.4. Torus T^2. See lecture notes for details. The conclusion was that $1 - \dim(H^1_{dR}(T^2)) + \dim(H^2_{dR}(T^2)) = 0$.

13 Lecture 13

Definition 13.1. We say that a manifold M has a good cover U_i each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has the same de Rham cohomology as $\mathbb{R}^n$.

Corollary 13.2. *If M has a finite good cover, then the de Rham cohomology of M is finite-dimensional.*

Proof. Note that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{13.1}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{13.2}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

We prove the corollary using induction on the number of open sets in a finite good cover. To see this, let k be the number of sets in a good cover. For $k = 1$, we know the corollary is true. Assume the corollary is true up to k, and let $\{U_1, \ldots, U_{k+1}\}$ be a good cover of a manifold M. Let $U = U_1 \cup \cdots \cup U_k$, and let $V = U_{k+1}$. Then U and V have good covers with fewer that $k+1$ open sets, so their de Rham cohomology is finite-dimensional. Also, $U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}$ is a good cover of $U \cap V$, so the theorem is true for $U \cap V$ as well.

Now we look at the following portion of the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\alpha^{p-1}} H^{p-1}_{dR}(U \cap V) \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} \cdots \tag{13.3}$$

The above observation then implies that $H^p_{dR}(U \cup V)$ is finite-dimensional. □

Corollary 13.3. *If M is compact, then the de Rham cohomology of M is finite-dimensional.*

Proof. Using a Riemannian metric, there exists a covering of M by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map at any point, a geodesically convex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$, which we previously showed has the same de Rham cohomology as $\mathbb{R}^n$. It follows that every compact manifold admits a finite good cover. □

Exercise 13.4. (For those who do not like Riemannian geometry.) If M is compact and admits a triangulation, then show that M admits a finite good cover.

13.1 Mayer-Vietoris for cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega^p_c(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega^{p-1}_c(M) \xrightarrow{d} \Omega^p_c(M) \xrightarrow{d} \Omega^{p+1}_c(M) \xrightarrow{d} \cdots, \tag{13.4}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Write $M = U \cup V$ as the union of two open sets in M. Note that if $U_1 \subset U_2$ and $\omega \in \Omega^k_c(U_1)$ then ω extends to be a compactly supported form in U_2. Letting $\iota : U_1 \hookrightarrow U_2$ denote the inclusion mapping, we denote by $i_*\omega$ this extension map on forms. We claim that the following sequence is exact:

$$0 \longrightarrow \Omega^p_c(U \cap V) \xrightarrow{\tilde{\alpha}^p} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{\tilde{\beta}^p} \Omega^p_c(U \cup V) \longrightarrow 0 \tag{13.5}$$

where

$$\tilde{\alpha}^p(\omega_{U\cap V}) = \big((i_{U\cap V\hookrightarrow U})_*\omega_{U\cap V}, -(i_{U\cap V\hookrightarrow V})_*\omega_{U\cap V}\big) \tag{13.6}$$

and

$$\tilde{\beta}^p(\omega_U, \omega_V) = (i_{U\hookrightarrow M})_*\omega_U + (i_{V\hookrightarrow M})_*\omega_V. \tag{13.7}$$

To see this, $\tilde{\alpha}^p$ is obviously injective. For exactness at the middle step, obviously $\tilde{\beta}^p\tilde{\alpha}^p\omega = 0$. If $\tilde{\beta}^p(\omega_U, \omega_V) = 0$, then $\omega_U = -\omega_V$. This implies that the support of both forms is contained in $U \cap V$, and since they are equal there, take $\omega_{U\cap V} = \omega_U$, and then $(\omega_U, \omega_V) = \tilde{\alpha}^p(\omega_U)$.

To show that $\tilde{\beta}$ is onto, let $\omega \in \Omega^p_c(M)$. Let ϕ_U, ϕ_V be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$.

Consequently, from the ziz-zag Lemma, we obtain a long exact sequence

$$\cdots \xrightarrow{\tilde{\delta}^{p-1}} H^p_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} \cdots \tag{13.8}$$

Let us review the definition of the mapping $\tilde{\delta}^p$. Given a cohomology class $[\omega] \in H^p_{c,dR}(U\cup V)$, represented by $\omega \in \Omega^p_c(U \cup V)$ with $d\omega = 0$, we first write $\omega = \tilde{\beta}^p(\phi_U\omega, \phi_V\omega)$, then we apply the exterior derivative to get

$$(d(\phi_U\omega), d(\phi_V\omega)) = (d\phi_U \wedge \omega, d\phi_V \wedge \omega) \in \Omega^p_c(U) \oplus \Omega^p_c(V) \tag{13.9}$$

Either of these elements is supported in $U \cap V$ and then since $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$,

$$\tilde{\delta}^p\omega = [d\phi_U \wedge \omega] = [-d\phi_V \wedge \omega] \in H^{p+1}_{c,dR}(U \cap V). \tag{13.10}$$

Remark 13.5. This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

13.2 Integration of differential forms

Another important fact is that we can integrate top-dimensional differential forms on a compact manifold. But we need to recall orientability. First, an orientation on a n-dimensional vector space V is a choice of ordered basis $(v_1, \dots, v_n)$ with equivalence relation if 2 ordered bases are related by a change of basis matrix with positive determinant. There are exactly 2 such equivalence classes, and if M is a manifold, the oriented double cover of M denoted by $\tilde{M}$ is the double cover obtained by replacing a point p with the 2 orientations on T_pM.

Definition 13.6. A manifold M is orientable if any of the following equivalent conditions are satisfied.

- M admits an coordinate atlas (U_α, ϕ_α) such that the overlap maps are orientation-preserving $\phi_\alpha \circ \phi_\beta^{-1}$, that is, the Jacobian $(\phi_\alpha \circ \phi_\beta^{-1})_*$ has positive determinant.
- M admits a nowhere-zero n-form.
- The oriented double cover $\tilde{M} \to M$ is trivial, i.e., it has 2 components.

If M is orientable, the choice of one of the components of $\tilde{M}$ is called an *orientation* on M.

On an oriented n-dimensional manifold, the integral of $\omega \in \Omega^n(M)$ is defined as follows. Choose an oriented coordinate atlas (U_α, ϕ_α). First, assume that $\omega \in \Omega^n(M)$ has compact support in a single coordinate system U_α. Then

$$(\phi_\alpha)_*(\omega) = f dx^1 \wedge \cdots \wedge dx^n, \tag{13.11}$$

where $f : \phi_\alpha(U_\alpha) \to \mathbb{R}$ has compact support. Define

$$\int_M \omega \equiv \int_{\phi_\alpha(U_\alpha)} f dx^1 \dots dx^n. \tag{13.12}$$

By the change-of-variables formula for integrals, this definition is independent of coordinate system containing the support of ω.

Next, if M is compact, or if ω has compact support, let χ_α be a partition of unity subordinate to U_α, and define

$$\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega. \tag{13.13}$$

Since the sum is finite, this definition is independent of the choice of coordinate atlas and choice of partition of unity. To see this, let U_α and V_β be open covers with subordinate partitions of unity ρ_α, χ_β, respectively. Then

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \sum_\beta \chi_\beta \rho_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \omega = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta \omega, \tag{13.14}$$

since $\rho_\alpha \chi_\beta$ is suported in V_β. Therefore

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \sum_\alpha \rho_\alpha \chi_\beta \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega. \tag{13.15}$$

14 Lecture 14

14.1 Stokes' Theorem

Integration by parts on manifolds is the following.

Theorem 14.1 (Stokes' Theorem for manifolds with boundary)**.** *Let $(M, \partial M)$ be an oriented manifold with boundary of dimension n. If $\omega \in \Omega^{n-1}(M)$ has compact support, then*

$$\int_{\partial M} \omega = \int_M d\omega, \tag{14.1}$$

where the boundary has the orientation induced from the outer normal, i.e., if $v_i \in T_p(\partial M)$, then the ordered basis $(v_1, \dots v_{n-1})$ is oriented if $(v, v_1, \dots, v_{n-1})$ is positively oriented, for any outward pointing normal vector v.

Proof. A manifold with boundary, by definition, can be covered by usual coordinate charts in the interior together with coordinate charts (U_i, ϕ_i), where $\phi_i : U_i \to H^n$, where

$$H^n = \{(x^1, \dots x^n) \in \mathbb{R}^n \mid x^n \geq 0\}. \tag{14.2}$$

is the upper half space in $\mathbb{R}^n$, such that

$$\phi_i : U_i \cap \partial M \to \mathbb{R}^{n-1} \tag{14.3}$$

is a coordinate chart on ∂M viewed as an $(n-1)$-dimensional smooth manifold.

We first consider forms compactly supported in such a coordinate chart. Then just consider an $(n-1)$-form of the form

$$\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \tag{14.4}$$

Note that

$$d\omega = (-1)^{i-1} \partial_i f dx^1 \wedge \cdots \wedge dx^n \tag{14.5}$$

If $i < n$, then ω restricted to the boundary is zero, and

$$\int_{H^n} d\omega = (-1)^{i-1} \int_{H^n} \partial_i f dx^1 \cdots dx^n = 0, \tag{14.6}$$

by Fubini's Theorem and the fundamental theorem of calculus, since f has compact support. If $i = n$, then

$$\begin{aligned} \int_{H^n} d\omega &= (-1)^{n-1} \int_{H^n} \partial_n f dx^1 \cdots dx^n \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_n f dx^1 \cdots dx^n \\ &= (-1)^n \int_{\mathbb{R}^{n-1}} \omega(x^1, \dots, x^n, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{\partial H^n} \omega, \end{aligned} \tag{14.7}$$

since the outward normal is $-e_n$, so $\{-e_n, e_1, \ldots, e_{n-1}\}$ is oriented, which is equivalent to $(-1)^n$ times $\{e_1, \ldots, e_n\}$. In general ω is a sum of n-terms of the above type, so this proves Stokes' Theorem for $\omega \in \Omega^{n-1}(H^n)$ with compact support.

Next, we choose a partition of unity χ_i subordinate to the cover (U_i, ϕ_i), $\phi_i : U_i \to \mathbb{R}^n$, and write $\omega = \sum_i \chi_i \omega$. Let $\omega_i = \chi_i \omega$. Then for each i in the index set, we have

$$\begin{aligned} \int_M d\omega_i = \int_{U_i} d\omega_i = \int_{\phi_i^{-1}(U_i)} (\phi_i^{-1})^*(d\omega_i) = \int_{\phi_i^{-1}(U_i)} d(\phi_i^{-1})^*(\omega_i) \\ = \int_{H^n} d(\phi_i^{-1})^*(\omega_i) = \int_{\partial H^n} (\phi_i^{-1})^*(\omega_i) = \int_{\partial M} \omega_i, \end{aligned} \tag{14.8}$$

where the last equality holds since $\phi_i|_{\partial M}$ is a coordinate chart on ∂M as a $(n-1)$-dimensional manifold. Finally, we have

$$\int_M d\omega = \int_M d\Big(\sum_i \omega_i\Big) = \sum_i \int_M d\omega_i = \sum_i \int_{\partial M} \omega_i = \int_{\partial M} \sum_i \omega_i = \int_{\partial M} \omega. \tag{14.9}$$

□

14.2 Poincaré Lemma for cohomology with compact supports

Let M be a manifold, possibly noncompact. Let $\Omega_c^p(M)$ denote the smooth p-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega_c^{p-1}(M) \xrightarrow{d} \Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \cdots, \tag{14.10}$$

and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if M is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Theorem 14.2. *[Poincaré Lemma for compact supported cohomology] Let M be a differentiable n-manifold, then*

$$H^k_{c,dR}(M \times \mathbb{R}) \cong H^{k-1}_{c,dR}(M). \tag{14.11}$$

First, we define a mapping "integration over the fiber" by

$$\pi_* : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M) \tag{14.12}$$

by the following. Any k-form on $M \times \mathbb{R}$ can be written as

$$\omega = h(p,t)\pi^*\phi_k + f(p,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{14.13}$$

where $\phi_k \in \Omega^k(M)$ and $\phi_{k-1} \in \Omega^{k-1}(M)$, but $h, f \in \Omega_c^0(M \times \mathbb{R})$. Define

$$\pi_*(\omega) = \Big(\int_{-\infty}^{\infty} f(p,t)dt\Big)\phi_{k-1}, \tag{14.14}$$

noting that the integral is defined because ω is assumed to have compact support, and this form has compact support since f has compact support.

We claim that

$$d_M \circ \pi_* = \pi_* \circ d_{M\times\mathbb{R}}, \tag{14.15}$$

To see this, the left hand side of (14.15) is

$$\begin{aligned} d_M \circ \pi_*\omega &= d_M\Big(\Big(\int_{-\infty}^{\infty} f(p,t)dt\Big)\phi_{k-1}\Big) \\ &= \Big(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}dt\Big)dx \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(p,t)dt)\Big)d_M\phi_{k-1}. \end{aligned} \tag{14.16}$$

The right hand side of (14.15) is

$$\begin{aligned} \pi_* \circ d_N\omega &= \pi_*\Big(\frac{\partial h}{\partial t}dt \wedge \pi^*\phi_k + \frac{\partial f}{\partial x}dx \wedge \pi^*\phi_{k-1} \wedge dt + f(p,t)\pi^*(d_M\phi_{k-1}) \wedge dt\Big) \\ &= \pi_*\Big(\frac{\partial f}{\partial x}dx \wedge \pi^*\phi_{k-1} \wedge dt + f(p,t)\pi^*(d_M\phi_{k-1}) \wedge dt\Big) \\ &= \Big(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}dt\Big)dx \wedge \phi_{k-1} + \Big(\int_{-\infty}^{\infty} f(p,t)dt)\Big)d_M\phi_{k-1}, \end{aligned} \tag{14.17}$$

since the term involving h is zero because h has compact support, and using the fundamental theorem of calculus. Therefore π_* induces a mapping

$$\pi_* : H^k_{c,dR}(M \times \mathbb{R}) \to H^{k-1}_{c,dR}(M). \tag{14.18}$$

Next, we choose $e \in \Omega^1_c(\mathbb{R})$ with $\int_R e = 1$, and define

$$e_* : \Omega^k_c(M) \to \Omega^{k+1}_c(M \times \mathbb{R}) \tag{14.19}$$

by

$$e_*(\omega) = (\pi^*\omega) \wedge e. \tag{14.20}$$

It is not hard to see that

$$d_{M\times\mathbb{R}} \circ e_* = e_* \circ d_M. \tag{14.21}$$

To see this,

$$d_N \circ e_*(\omega) = d_N\pi^*\omega \wedge e = (d_N\pi^*\omega) \wedge e = \pi^*(d_M\omega) \wedge e = e_* \circ d_M(\omega). \tag{14.22}$$

Therefore e_* induces a mapping

$$e_* : H^k_{c,dR}(M) \to H^{k+1}_{c,dR}(M \times \mathbb{R}). \tag{14.23}$$

Let us write $e = \chi dt$, then

$$\pi_* \circ e_*(\omega) = \pi_*\Big(\chi(t)(\pi^*\omega) \wedge dt\Big) = \Big(\int_{-\infty}^{\infty} \chi(t)dt\Big)\omega = \omega \tag{14.24}$$

Therefore, we have $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$, so $\pi_* \circ e_* = 1$ on $H^k_{c,dR}(M)$.

Proposition 14.3. *We have $e_* \circ \pi_* = 1$ on $H^k_{c,dR}(M \times \mathbb{R})$. Consequently, π_* and e_* are isomorphisms on compactly supported cohomology.*

Proof. Again writing

$$\omega = h(p,t)\pi^*\phi_k + f(p,t)(\pi^*\phi_{k-1}) \wedge dt, \tag{14.25}$$

define a mapping

$$K : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R}) \tag{14.26}$$

by

$$K(\omega) = \pi^*\phi_{k-1}\Big(\int_{-\infty}^t f(x,s)ds - \Big(\int_{-\infty}^t e\Big) \int_{-\infty}^\infty f(x,s)ds\Big). \tag{14.27}$$

Note that the right hand side is indeed a $(k-1)$-form on $M \times \mathbb{R}$ with compact support, the ds-s are not 1-forms in this formula. We claim that if $\omega \in \Omega^k_c(M \times \mathbb{R})$ then

$$(1 - e_*\pi_*)\omega = (-1)^{k-1}(dK - Kd)\omega, \tag{14.28}$$

which can be separately verified for $\omega = h(p,t)\pi^*\phi_k$, and for forms of type $\omega = f(p,t)dt \wedge \pi^*\phi_{k-1}$.

For forms of the first type, we obviously have

$$(1 - e_*\pi_*)h(p,t)\pi^*\phi_k = h(p,t)\pi^*\phi_k. \tag{14.29}$$

On the other hand, since K is zero on forms of this type,

$$\begin{aligned}
(dK - Kd)(h(p,t)\pi^*\phi_k) &= -K\Big(\Big(\frac{\partial h}{\partial x}\Big)dx \wedge \pi^*\phi_k + \Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k + h(p,t)\pi^*d\phi_k\Big)\\
&= -K\Big(\Big(\frac{\partial h}{\partial t}\Big)dt \wedge \pi^*\phi_k\Big)\\
&= (-1)^{k-1}K\Big(\Big(\frac{\partial h}{\partial t}\Big)(\pi^*\phi_k) \wedge dt\Big)\\
&= (-1)^{k-1}\pi^*\phi_k\Big(\int_{-\infty}^t \frac{\partial h}{\partial t}ds - \Big(\int_{-\infty}^t e\Big)\int_{-\infty}^\infty \frac{\partial h}{\partial t}ds\Big)\\
&= (-1)^{k-1}(\pi^*\phi_k)h(p,t).
\end{aligned} \tag{14.30}$$

For forms of the second type, we have

$$\begin{aligned}
(1 - e_*\pi_*)f(p,t)\pi^*\phi_{k-1} \wedge dt &= f(p,t)\pi^*\phi_{k-1} \wedge dt - \Big(\int_{-\infty}^\infty f(p,t)dt\Big)(\pi^*\phi_{k-1}) \wedge e\\
&= \pi^*\phi_{k-1} \wedge \Big(f(p,t)dt - \Big(\int_{-\infty}^\infty f(p,t)dt\Big)e\Big)\\
&= (-1)^{k-1}\Big(f(p,t) - \Big(\int_{-\infty}^\infty f(p,t)dt\Big)\chi(t)\Big)\pi^*\phi_{k-1} \wedge dt
\end{aligned} \tag{14.31}$$

The verification that this is equal to $(-1)^{k-1}(dK - Kd)$ is left as an exercise.

Using Proposition 10.10, this formula then implies that $e_* \circ \pi_* = 1$ as a mapping on $H^k_{c,dR}(M \times \mathbb{R})$, and the proposition follows. □

Corollary 14.4. *We have*

$$H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{14.32}$$

and a generator for $H^n_{c,dR}(\mathbb{R}^n)$ *is given by any compactly supported* n*-form* μ *with* $\int_{\mathbb{R}^n} \mu = 1$*.*

Proof. We start with $M = \{p\}$ a single point. The above shows that

$$H^1_{c,dR}(\mathbb{R}) \cong H^0_{c,dR}(\{p\}) \cong \mathbb{R}. \tag{14.33}$$

Furthermore, the proof shows that a generator of the left hand side is $\chi(x^1)dx^1$. Next, we have

$$H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}, \tag{14.34}$$

and a generator of the left hand side is $\chi(x^1)dx^1 \wedge \chi(x^2)dx^2$. In general, a generator is given by will be

$$\chi(x^1) \cdots \chi(x^n) dx^1 \wedge \cdots \wedge dx^n. \tag{14.35}$$

Next, we use the fact that π_* is an isomorphism. The isomorphism

$$H^1_{c,dR}(\mathbb{R}) \cong H^0_{c,dR}(\{p\}) \cong \mathbb{R} \tag{14.36}$$

is given by

$$\phi_1 \mapsto \int_{\mathbb{R}} \phi_1 dx^1. \tag{14.37}$$

Then the isomorphism

$$H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}, \tag{14.38}$$

is given by

$$f(x^1, x^2) dx^1 \wedge dx^2 \mapsto \left(\int_{\mathbb{R}} f(x^1, x^2) dx^2 \right) dx^1 \tag{14.39}$$

Composing these isomorphisms and using Fubini's Theorem, we get

$$f(x^1, x^2) dx^1 \wedge dx^2 \mapsto \int_{\mathbb{R}^2} f(x^1, x^2) dx^1 \wedge dx^2. \tag{14.40}$$

In general, the isomorphism is given by

$$f(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n \mapsto \int_{\mathbb{R}^n} f(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n. \tag{14.41}$$

□

15 Lecture 15

Recall last time we showed that

$$H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \tag{15.1}$$

Remark 15.1. This shows that $H^*_{c,dR}(M)$ is not a homotopy invariant, since (14.32) is not the same as the cohomology of a point. But of course, $H^*_{c,dR}(M)$ *is* a diffeomorphism invariant.

If M is any oriented manifold of dimension n, then we have a pairing

$$\Omega^k(M) \times \Omega^{n-k}_c(M) \to \mathbb{R}, \tag{15.2}$$

given by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta. \tag{15.3}$$

By Stokes' Theorem, this mapping descends to cohomology, and since this mapping is bilinear, we obtain a pairing

$$PD : H^k_{dR}(M) \otimes H^{n-k}_c(M) \to \mathbb{R}. \tag{15.4}$$

In the case $M = \mathbb{R}^n$, note that $H^k_{c,dR}(\mathbb{R}^n) \cong H^{n-k}_{dR}(\mathbb{R}^n)$. Furthermore, we have an isomorphism

$$PD : H^k_{dR}(\mathbb{R}^n) \to (H^{n-k}_{c,dR}(\mathbb{R}^n))^* \tag{15.5}$$

given by $PD(\alpha)(\beta) = \int_{\mathbb{R}^n} \alpha \wedge \beta$. This is because we showed that an isomorphism of $H^n_{c,dR}(\mathbb{R}^n)$ and $\mathbb{R}$ is obtained from composing the isomorphisms

$$H^n_{c,dR}(\mathbb{R}^n) \xrightarrow{\pi^n_*} H^{n-1}_{c,dR}(\mathbb{R}^{n-1}) \xrightarrow{\pi^{n-1}_*} \cdots \xrightarrow{\pi^1_*} H^0_{c,dR}(\mathbb{R}^0), \tag{15.6}$$

where $\pi^k_* : H^k_{c,dR}(\mathbb{R}^k) \to H^{k-1}_{c,dR}(\mathbb{R}^{k-1})$ is the mapping induced by writing $\mathbb{R}^k = \mathbb{R}^{k-1} \times \mathbb{R}$ and using coordinates $(x^1, \ldots, x^{k-1}, t)$, writing $\omega \in \Omega^k_c(\mathbb{R}^k)$ as

$$\omega = f(x^1, \ldots, x^{k-1}, t) dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dt \tag{15.7}$$

and then

$$\pi^k_*(\omega) = \Big(\int_{-\infty}^{\infty} f(x^1, \ldots, x^{k-1}, t) dt \Big) dx^1 \wedge \cdots \wedge dx^{n-1}, \tag{15.8}$$

so the iterated map is

$$
\begin{aligned}
\pi^1_* \circ \cdots \circ \pi^n_*(\omega) &= \pi^1_* \circ \cdots \circ \pi^{n-1}_*\Big(\int_{-\infty}^{\infty} f(x^1,\dots,x^n)dx^n\Big)dx^1 \wedge \cdots \wedge dx^{n-1} \\
&= \pi^1_* \circ \cdots \circ \pi^{n-2}_*\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(x^1,\dots,x^{n-1},x^n)dx^n\Big)dx^{n-1}\Big)dx^1 \wedge \cdots \wedge dx^{n-2} \\
&= \int_{-\infty}^{\infty}\Big(\cdots\Big(\int_{-\infty}^{\infty} f(x^1,\dots,x^{n-1},x^n)dx^n\Big)dx^{n-1}\cdots\Big)dx^1 \\
&= \int_{\mathbb{R}^n} f(x^1,\dots,x^n)dx^1\cdots dx^n = \int_{\mathbb{R}^n}\omega
\end{aligned}
\tag{15.9}
$$

by Fubini's Theorem.

15.1 Star-shaped open sets in $\mathbb{R}^n$

Let U be a star-shaped open set in $\mathbb{R}^n$. Let us assume that the special point is the origin. Then we write

$$
U = \{tv \mid v \in S^{n-1}, 0 \le t < \rho(v)\} \tag{15.10}
$$

where $\rho : S^{n-1} \to \mathbb{R}_+$.

Lemma 15.2. *If ρ is smooth, then U is diffeomorphic to the unit ball in $\mathbb{R}^n$, and thus*

$$
H^k_{c,dR}(U) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \ne n \end{cases} \tag{15.11}
$$

Proof. Since U is an open set, clearly $B(0,\epsilon) \subset U$ for some $\epsilon > 0$, so $\rho(v) \ge \epsilon$. Choose a function $f : \mathbb{R} \to \mathbb{R}$ so that $0 \le f \le 1$,

$$
f(t) = \begin{cases} 0 & t \le \epsilon/2 \\ 1 & t \ge \epsilon \end{cases} \tag{15.12}
$$

and $f'(t) > 0$ for $0 < t \le \epsilon$. Define $h : B(0,1) \to U$ by

$$
h(tv) = \Big(t + (\rho(v) - \epsilon)f(t)\Big)v \tag{15.13}
$$

for $0 \le t \le 1$.

Then h is one-one and surjective. In a neighborhood of the origin, it is smooth with h_* invertible, because it is the identity map there. The radial derivative of h is $1 + (\rho(v) - \epsilon)f'(t) > 0$. Therefore h_* is invertible everywhere, so h is a diffeomorphism. □

Lemma 15.3. *In general, $\rho : S^{n-1} \to \mathbb{R}$ is upper semi-continuous. That is for any $v \in S^{n-1}$ and $\epsilon > 0$, there is a neighborhood $U_v \subset S^{n-1}$ of v such that $\rho(w) > \rho(v) - \epsilon$ for all $w \in U_v$.*

Proof. Choose a t so that $tv \in U$ with $\rho(v) - \epsilon < t$. Since U is an open set, there is a open ball $B(tv,\delta) \subset U$. Then there is a neighborhood $U_v \subset S^{n-1}$ of v such that $ty \in B(tv,\delta) \subset U$ for $y \in U_v$. This implies that $\rho(y) \ge t > \rho(v) - \epsilon$. □

The main result of the subsection is the following.

Corollary 15.4. *If U is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{c,dR}(U) \cong H^k_{c,dR}(\mathbb{R}^n)$ for all $0 \leq k \leq n$. Furthermore, an isomorphism of $H^n_{c,dR}(U)$ and $\mathbb{R}$ is given by integration.*

Proof. At the end of the proof we will show that given any compact subset $K \subset U$, there is a star-shaped open set V with $K \subset V \subset U$ and such that ρ_V is C^∞, and thus V is diffeomorphic to $\mathbb{R}^n$ by Lemma 15.2.

If $0 \leq k < n$, then $\omega = d\eta$, where $\eta \in \Omega^{k-1}_c(V)$, but we just view $\eta \in \Omega^{k-1}_c(U)$, which proves that $H^k_{c,dR}(U) = \{0\}$. For $k = n$, then we know that $\omega = d\eta + c\mu$, where μ is a compactly supported form in V with $\int_{\mathbb{R}^n} \mu = 1$ and $c \in \mathbb{R}$. Again we can view μ as a compactly supported form in U, which proves that $\dim(H^n_{c,dR}(U)) \leq 1$. If $\mu = d\gamma$ with $\gamma \in \Omega^{n-1}_c(U)$, then we can view $\gamma \in \Omega^{n-1}_c(V)$ for a nice V. Then Stokes' Theorem says that

$$\int_{\mathbb{R}^n} \mu = \int_{\mathbb{R}^n} d\gamma = \int_{\partial V} \gamma = 0, \tag{15.14}$$

a contradiction. Therefore $\dim(H^n_{c,dR}(U)) = 1$.

Finally, we will find the star-shaped domain V claimed above. For each $v \in S^{n-1}$, there is a number $t_v < \rho(v)$ so that all the points in K of the form uv must have $u < t_v$. That is, the segment $\{rv \,| t_v < r < \rho(v)\}$ does not hit K. This is because K, being compact, must stay at a positive distance from the boundary in the radial direction. Furthermore since K is closed, there is a neighborhood of $W_v \subset S^{n-1}$ of v such that the set $\{ry \,| t_v < r < \rho(v)\}$ does not hit K either. From Lemma 15.3, ρ is upper semicontinuous, so choosing $\epsilon = \rho(v) - t_v$, there is a neighborhood $U_v \subset S^{n-1}$ of v such that $\rho(w) > \rho(v) - (\rho(v) - t_v) = t_v$ for all $w \in U_v$. Therefore, by taking the intersection of these neighborhoods, there is a neighborhood of v, $W_v \subset S^{n-1}$ so that t_v works as t_y for all $y \in W_v$. Since S^{n-1} is compact, we can cover by finitely many $W_{v_1}, \ldots, W_{v_k}$. Let χ_i be a partition of unity subordinate to this cover. Define

$$\tilde{\rho} = t_{v_1}\chi_1 + \cdots t_{v_k}\chi_k. \tag{15.15}$$

Since $\chi_1 + \cdots + \chi_k = 1$, and each $t_{v_j} < \rho(x)$, we have $\tilde{\rho}(x) < \rho(x)$. Letting

$$V = \{tv \mid v \in S^{n-1},\ 0 \leq t < \tilde{\rho}(v)\} \tag{15.16}$$

then $K \subset V \subset U$, and $\rho_V = \tilde{\rho}$ is smooth.

□

Remark 15.5. It is actually true that U is diffeomorphic to $\mathbb{R}^n$, but this is more difficult to show, and we do not need such a strong result.

16 Lecture 16

16.1 Good covers

Next, let us modify our definition of good cover.

Definition 16.1. We say that a manifold M has a good cover U_i each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has the same de Rham cohomology as $\mathbb{R}^n$, the same compactly supported de Rham cohomology as $\mathbb{R}^n$.

Recall we proved ealier that if M has a finite good cover, then the de Rham cohomology is finite-dimensional. We next extend this to compactly supported cohomology.

Corollary 16.2. *If M has a finite good cover, then the compactly supported de Rham cohomology is finite-dimensional.*

Proof. Recall that if

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{16.1}$$

is exact at B, then

$$B \cong Ker(g) \oplus Im(g) \cong Im(f) \oplus Im(g). \tag{16.2}$$

Consequently, if A and C are both finite-dimensional, then B is also finite-dimensional.

We prove the corollary using induction on the number of open sets in a finite good cover. To see this, let k be the number of sets in a good cover. For $k = 1$, we know the corollary is true. Assume the corollary is true up to k, and let $\{U_1, \ldots, U_{k+1}\}$ be a good cover of a manifold M. Let $U = U_1 \cup \cdots \cup U_k$, and let $V = U_{k+1}$. Then U and V have good covers with fewer that $k+1$ open sets, so their compactly supported de Rham cohomology is finite-dimensional. Also, $U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}$ is a good cover of $U \cap V$, so the theorem is true for $U \cap V$ as well.

Now we look at the following portion of the compactly supported Mayer-Vietoris sequence

$$\cdots \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} H^{p+1}_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^{p+1}} \cdots \tag{16.3}$$

The above observation then implies that $H^p_{c,dR}(U \cup V)$ is finite-dimensional. ☐

Corollary 16.3. *If M is compact, then M admits a finite good cover.*

Proof. Using a Riemannian metric, there exists a covering of M by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map at any point, a geodesically convex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$. This is contractible, so from the Poincaré Lemma, it has the same de Rham cohomology as $\mathbb{R}^n$. Corollary 15.4 tells us that it also has the same compactly supported de Rham cohomology as $\mathbb{R}^n$, so we are done. ☐

17 Lecture 17

17.1 Poincaré Duality

Lemma 17.1 (The Five Lemma)**.** *Assume the diagram*

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{\alpha_1} & V_2 & \xrightarrow{\alpha_2} & V_3 & \xrightarrow{\alpha_3} & V_4 & \xrightarrow{\alpha_4} & V_5 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\ W_1 & \xrightarrow{\beta_1} & W_2 & \xrightarrow{\beta_2} & W_3 & \xrightarrow{\beta_3} & W_4 & \xrightarrow{\beta_4} & W_5 \end{array} \tag{17.1}$$

commutes, and has exact rows. If $\phi_1, \phi_2, \phi_4, \phi_5$ are isomorphisms, then ϕ_3 is also an isomorphism.

Proof. Injectivity of ϕ_3: If $\phi_3(v_3) = 0$, then $\beta_3(\phi_3(v_3) = 0 = \phi_4\alpha_3(v_3)$. Since ϕ_4 is injective, $\alpha_3(v_3) = 0$. By exactness, $v_3 = \alpha_2(v_2)$. Then $\phi_3\alpha_2(v_2) = 0 = \beta_2\phi_2(v_2)$. By exactness, $\phi_2(v_2) = \beta_1(w_1)$. By surjectivity of ϕ_1, $w_1 = \phi_1(v_1)$. Then

$$\phi_2(v_2) = \beta_1\phi_1(v_1) = \phi_2\alpha_1(v_1), \tag{17.2}$$

but since ϕ_2 is injective, this implies that $v_2 = \alpha_1(v_1)$. Finally, $v_2 = \alpha_2(v_2) = \alpha_2\alpha_1(v_1) = 0$, by exactness.

The proof of surjectivity is similar, and left to the reader. □

Lemma 17.2. *If the sequence*

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \tag{17.3}$$

is exact at W_2, then the dual sequence

$$W_3^* \xrightarrow{\beta^*} W_2^* \xrightarrow{\alpha^*} W_1^* \tag{17.4}$$

is exact at W_2^*.

Proof. First, if $w_3^* \in W_3^*$, and $w_1 \in W_1$, then

$$\alpha^*(\beta^* w_3^*)(w_1) = (\beta^* w_3^*)(\alpha(w_1)) = w_3^*(\beta\alpha(w_1)) = 0, \tag{17.5}$$

since $\beta \circ \alpha = 1$ by assumption. This proves that $Im(\beta^*) \subset Ker(\alpha^*)$. For the other direction, if $w_2^* \in Ker(\alpha^*)$, then for all $w_1 \in W_1$, $\alpha^*(w_2^*)(w_1) = w_2^*(\alpha(w_1))$. So the element $0 = w_2^* \circ \alpha \in W_1^*$. We want to find $w_3^* \in W_3^*$ such that $w_2^* = \beta^* w_3^*$. For all $w_2 \in W_2$, this is $w_2^*(w_2) = w_3^*\beta w_2$, which is just $w_2^* = w_3^* \circ \beta$. So if $w_3 \in W_3$ is of the form $\beta(w_2)$ then define

$$w_3^*(w_3) \equiv w_2^*(w_2). \tag{17.6}$$

If $w_3 = \beta(w_2')$, then $\beta(w_2 - w_2') = 0$, so $w_2 - w_2' = \alpha(w_1)$. Then

$$w_2^*(w_2 - w_2') = w_2^*(\alpha(w_1)) = (w_2^*\alpha)(w_1) = 0. \tag{17.7}$$

So we have defined w_3^* on the subspace $Im(\beta) \subset W_3$. To extend to a linear mapping on all of W_3, just take any subspace so that $W_3 = Im(\beta) \oplus W$, and define w_3^* to vanish on W. Then the condition $w_2^* = w_3^* \circ \beta$ is obviously satisifed. □

Theorem 17.3. *If M^n is orientable and has a finite good cover, then*

$$PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{17.8}$$

is an isomorphism for all $0 \leq k \leq n$.

Proof. Let $m = n - k$, and consider the diagram

$$
\begin{array}{ccccccccc}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_{dR}(U \cap V) & \xrightarrow{\delta^{k-1}} & H^{k}_{dR}(U \cup V) & \xrightarrow{\beta^{k}} & H^{k}_{dR}(U) \oplus H^{k}_{dR}(V) & \xrightarrow{\alpha^{k}} & H^{k}_{dR}(U \cap V) \\
\downarrow{\scriptstyle PD\oplus PD} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD\oplus PD} & & \downarrow{\scriptstyle PD} \\
\big(H^{m+1}_{c,dR}(U) \oplus H^{m+1}_{d,dR}(V)\big)^* & \xrightarrow{(\tilde{\alpha}^{m+1})^*} & H^{m+1}_{c,dR}(U \cap V)^* & \xrightarrow{(\tilde{\delta}^{m})^*} & H^{m}_{c,dR}(U \cup V)^* & \xrightarrow{(\tilde{\beta}^{m})^*} & \big(H^{m}_{c,dR}(U) \oplus H^{m}_{c,dR}(V)\big)^* & \xrightarrow{(\tilde{\alpha}^{m})^*} & H^{m}_{c,dR}(U \cap V)^*
\end{array}
\tag{17.9}
$$

The top horizontal row is exact since it is the usual Mayer-Vietoris sequence. The bottom horizontal row is exact since is the dual exact sequence of the Mayer-Vietoris sequence with compact support. We next claim that this diagram commutes up to sign, so by changing some of the vertical maps to their negatives if necessary, we obtain a commutative diagram.

Consider the square

$$
\begin{array}{ccc}
H^{k-1}_{dR}(U \cap V) & \xrightarrow{\delta^{k-1}} & H^{k}_{dR}(U \cup V) \\
\downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} \\
H^{m+1}_{c,dR}(U \cap V)^* & \xrightarrow{(\tilde{\delta}^{m})^*} & H^{m}_{c,dR}(U \cup V)^*
\end{array}
\tag{17.10}
$$

For the mapping

$$PD \circ \delta^{k-1} : H^{k-1}_{dR}(U \cap V) \to H^{m}_{c,dR}(U \cup V)^* \tag{17.11}$$

let's take an element $[\omega] \in H^{k-1}_{dR}(U \cap V)$, and an element $[\tau] \in H^{m}_{c,dR}(U \cup V)$. Then

$$(PD \circ \delta^{k-1}[\omega])[\tau] = PD(\delta^{k-1}[\omega])[\tau] = \int_M (\delta^{k-1}\omega) \wedge \tau. \tag{17.12}$$

Recall from our discussion of the Mayer-Vietoris sequence that

$$\delta^{k-1}\omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \tag{17.13}$$

This form is supported in $U \cap V$, so we have

$$(PD \circ \delta^{k-1}[\omega])[\tau] = \int_{U\cap V} (\delta^{k-1}\omega) \wedge \tau = \int_{U\cap V} (-d\phi_U \wedge \omega) \wedge \tau. \tag{17.14}$$

Next, we look at the mapping

$$(\tilde{\delta}^m)^* \circ PD : H^{k-1}_{dR}(U \cap V) \to H^{m}_{c,dR}(U \cup V)^*. \tag{17.15}$$

We then have

$$\big((\tilde{\delta}^m)^* \circ PD[\omega]\big)[\tau] = PD[\omega](\tilde{\delta}^m[\tau]) = \int_{U\cap V} \omega \wedge \tilde{\delta}^m \tau. \tag{17.16}$$

Recall from our discussion of the compactly supported Mayer-Vietoris sequence that

$$\tilde{\delta}^m \tau = [d\phi_U \wedge \tau] = [-d\phi_V \wedge \tau] \in H^{m+1}_{c,dR}(U \cap V). \tag{17.17}$$

So we have

$$\big((\tilde{\delta}^m)^* \circ PD[\omega]\big)[\tau] = \int_{U\cap V} \omega \wedge d\phi_U \wedge \tau = (-1)^{k-1} \int_{U\cap V} d\phi_U \wedge \omega \wedge \tau. \tag{17.18}$$

So we see that

$$(\tilde{\delta}^m)^* \circ PD = (-1)^k PD \circ \delta^{k-1} \tag{17.19}$$

Next, we look at the square

$$\begin{array}{ccc} H^k_{dR}(U\cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U) \oplus H^k_{dR}(V) \\ \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD\oplus PD} \\ H^m_{c,dR}(U\cup V)^* & \xrightarrow{(\tilde{\beta}^m)^*} & \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^* \end{array} \tag{17.20}$$

Next, a lemma

Lemma 17.4. *Let $f : A \to B \oplus C$ be a linear map between finite dimensional vector spaces. Write $f = (f_B, f_C)$, where $f : A \to B$ and $f_C : A \to C$. Then $f^* : (B \oplus C)^* \to A^*$ is given by*

$$f^*(b^*, c^*)(a) = b^*(f_B(a)) + c^*(f_C(a)), \tag{17.21}$$

where we used the isomorphism $(B \oplus C)^* \cong B^* \oplus C^*$.

Proof. We leave the proof of this as an exercise. □

For the mapping

$$(PD \oplus PD) \circ \beta^k : H^k_{dR}(U \cup V) \to \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^*, \tag{17.22}$$

choose $[\omega] \in H^k_{dR}(U \cup V)$, $[\tau_1] \in H^m_{c,dR}(U)$ and $[\tau_2] \in H^m_{c,dR}(V)$, and we have

$$\begin{aligned} \big((PD \oplus PD) \circ \beta^k[\omega]\big)([\tau_1],[\tau_2]) &= \big(PD_U \circ \beta^k_U[\omega]\big)([\tau_1]) + PD_V \circ \beta^k_V[\omega]\big)([\tau_2]) \\ &= \int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2. \end{aligned} \tag{17.23}$$

Next, we look at the mapping

$$(\tilde{\beta}^m)^* \circ PD : H^k_{dR}(U \cup V) \to \big(H^m_{c,dR}(U) \oplus H^m_{c,dR}(V)\big)^*, \tag{17.24}$$

for which we have

$$\begin{aligned} \Big((\tilde{\beta}^m)^* \circ PD[\omega]\Big)([\tau_1],[\tau_2]) &= PD[\omega](\tilde{\beta}^m(([\tau_1],[\tau_2])) = PD[\omega](\tau_1 + \tau_2) \\ &= \int_M \omega \wedge (\tau_1 + \tau_2) = \int_M \omega \wedge \tau_1 + \int_M \omega \wedge \tau_2 \\ &= \int_U \omega \wedge \tau_1 + \int_V \omega \wedge \tau_2, \end{aligned} \tag{17.25}$$

since τ_1 has compact support on U and τ_2 has compact support on V. So we have that

$$(PD \oplus PD) \circ \beta^k = (\tilde{\beta}^m)^* \circ PD. \tag{17.26}$$

We leave the remaining α square(s) as an exercise.

By the five lemma, if the outer 4 vertical maps are isomorphisms, then so is the central vertical map. The proof is completed by induction on the number of open sets in the good cover, since we know it is true for $\mathbb{R}^n$ from the previous lecture. □

Corollary 17.5. *If M^n is a connected and orientable n-manifold with a finite good cover, then $H^n_{c,dR}(M) \cong \mathbb{R}$. If M is compact, then $H^n_{dR}(M) \cong \mathbb{R}$. If M is noncompact, then $H^n_{dR}(M) = \{0\}$.*

Corollary 17.6. *If M^n is a connected and orientable n-manifold with a finite good cover then $H^k_{dR}(M)$ and $H^{n-k}_{c,dR}(M)$ have the same dimension. If M is moreover compact, then $H^k_{dR}(M)$ and $H^{n-k}_{dR}(M)$ have the same dimension.*

Corollary 17.7. *If M^n is a compact oriented odd-dimensional manifold, then the Euler characteristic $\chi(M) = 0$.*

Remark 17.8. Poincaré duality is also true for singular homology with $\mathbb{Z}$ coefficients on a orienable manifold. If M is not orientable, then it is still true for $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Next topics in the Spring: Poincare duality on nonorientable manifolds, Kunneth formula, Thom isomorphism, singular homology and cohomology, de Rham's Theorem.

18 Lecture 18

18.1 Degree of a smooth mapping

Definition 18.1. A mapping $f : X \to Y$ between topological spaces is proper if the inverse image of any compact set is compact.

Exercise 18.2. Let X and Y be metric spaces. For a sequence of points $x_i \in X$, we say that $lim_{i\to\infty} x_i = \infty$ if given any compact subset $K \subset X$, then there exists an integer N so that $x_i \in X \setminus K$ for $i > N$. Show that $f : X \to Y$ is proper iff for any sequence $x_i \in X$ such that $lim_{i\to\infty} x_i = \infty$, then $lim_{i\to\infty} f(x_i) = \infty$.

Exercise 18.3. If Y is a manifold and $f : X \to Y$ is proper and continuous, then f is a closed mapping, that is, f maps closed sets to closed sets.

Let $f : M \to N$ be a proper smooth mapping between n-dimensional connected and oriented smooth manifolds. Since f is proper, $f^* : \Omega^n_c(N) \to \Omega^n_c(M)$, and therefore there is an induced mapping $f^* : H^n_{c,dR}(N) \to \Omega^n_{c,dR}(M)$. From Poincaré duality, we know that $H^n_{c,dR}(M) \cong H^0_{dR}(M) \cong \mathbb{R}$, and similarly for N. Recall that the Poincaré duality isomorphism is given by $[\omega] \mapsto \int_M \omega$. Therefore we can make the following definition.

Definition 18.4. The degree of f is the real number $\deg(f)$ so that

$$\int_M f^*\omega = \deg(f)\int_N \omega \tag{18.1}$$

for all $\omega \in \Omega^n_c(N)$.

Proposition 18.5. *If $f : M \to N$ and $g : N \to \tilde{M}$ are both proper then $g \circ f : N \to \tilde{M}$ is proper and*

$$\deg(g \circ f) = \deg(g) \circ \deg(f) \tag{18.2}$$

Proof. The composition of proper maps is obviously proper. Given $\omega \in \Omega^n_c(\tilde{M})$ then

$$\int_M (g\circ f)^*\omega = \int_M f^*g^*\omega = \deg(f)\int_N g^*\omega = \deg(f)\deg(g)\int_{\tilde{M}}\omega. \tag{18.3}$$

□

Proposition 18.6. *Let $f : M \to N$ be a diffeomorphism. Then $\deg(f) = 1$ is f is orientation preserving, and $\deg(f) = -1$ if f is orientation reversing.*

Proof. This follows from the change of variables formula, the definition of the integral is clearly invariant under diffeomorphisms, but only up to sign. □

Proposition 18.7. *If M and N are compact, and $f : M \to N$ is smoothly homotopic to $g : M \to N$, then $\deg(f) = \deg(g)$.*

Proof. If M and N are compact, we know that $H^n_{c,dR}(M) = H^n_{dR}(M)$ and $H^n_{c,dR}(N) = H^n_{dR}(N)$. By the Poincaré Lemma, $f^* = g^* : H^n_{c,dR}(N) \to H^n_{c,dR}(M)$. □

Remark 18.8. This is not true in the noncompact case. The functions z and z^2 as mappings from $\mathbb{C}$ to itself are properly homotopic, yet have different degrees.

Proposition 18.9. *If $f : M \to N$ is proper and not surjective, then $\deg(f) = 0$.*

Proof. If f is not surjective, then there exists $q \in N$ which is not in the image of f. Furthermore, since f is a closed mapping, there exists a neighborhood U of q which contains no points in the image of f. Let χ be an n-form supported in U with $\int_U \chi = 1$. But $f^*\chi \equiv 0$, so $\int_M f^*\chi = 0$, and thus $\deg(f) = 0$. □

Proposition 18.10. *If $f : M \to N$ is proper, then $\deg(f) \in \mathbb{Z}$. Furthermore, if $q \in N$ be a regular value of f, and let $f^{-1}(q) = \{p_1, \dots, p_k\}$. Then*

$$\deg(f) = \sum_i \mathrm{sgn}(f_*|_{p_i}), \tag{18.4}$$

where

$$\mathrm{sgn}(f_*|_{p_i}) = \det(f_*|_{p_i})/|\det(f_*|_{p_i})|, \tag{18.5}$$

Proof. By Sard's Theorem, there exists a regular value q. We can choose a neighborhood U of q so that $f^{-1}(U) = U_1 \coprod \cdots \coprod U_k$ and such that $f : U_i \to U$ is a diffeomorphism. Let $\chi \in \Omega^n_c(U)$ satisfy $\int_U \chi = 1$. Then $f^*\chi$ is supported in $U_1 \cup \cdots \cup U_k$, and we have

$$\int_M f^*\chi = \sum_{i=1}^k \int_{U_i} f^*\chi = \sum_{i=1}^k \mathrm{sgn}(f_*|_{p_i})\int_U \chi = \sum_{i=1}^k \mathrm{sgn}(f_*|_{p_i}). \tag{18.6}$$

□

18.2 Applications of degree

Corollary 18.11. *There does not exist any non-zero vector field on S^n for n even.*

Proof. Let $A : S^n \to S^n$ be the antipodal map. We first claim that $\deg(A) = -1$ for n even. Clearly A is a diffeomorphism, so we just need to check if it is orientation preserving or not. The standard orientation of S^n is given by

$$\omega = \left(x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \cdots \wedge dx^{n+1}). \tag{18.7}$$

Clearly

$$A^*(\omega) = (-1)^{n+1}\omega. \tag{18.8}$$

So if n is even, we have $\deg(A) = -1$.

But if X is a non-zero vector field on S^n, let $\gamma_p(t)$ be the portion of the great circle such that $\gamma_p(0) = p$, $\gamma_p(1) = -p$ and such that $\gamma_p'(0)$ points in the direction of X_p. Then $H(p,t) = \gamma_p(t)$ is a homotopy between Id and A. This is a contradiction since $\deg(Id) = 1$. □

Remark 18.12. Odd-dimensional spheres always have a non-zero vector field:

$$X = (-x_2, x_1, -x_4, x_3, \cdots) \tag{18.9}$$

We also have the following:

Proposition 18.13. *If $f : S^n \to S^n$ is smooth and $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.*

Proof. If no fixed point, then for $p \in S^n$, the line segment from $f(p)$ to $-p$ does not hit the origin. Then we can define

$$H(p,t) = \frac{(1-t)f(p) - tp}{|(1-t)f(p) - tp|}, \tag{18.10}$$

which is a homotopy between f and the antipodal map. □

Next, we have

Proposition 18.14. *Let M be a smooth n-dimensional oriented compact manifold with connected boundary ∂M. Let N be a compact connected oriented $(n-1)$ manifold. Let $g : \partial M \to N$ be a smooth mapping which extends to a smooth mapping $G : M \to N$. Then $\deg(g) = 0$.*

Proof. Let ω be any smooth $(n-1)$ form on N such that $\int_N \omega = 1$. Obviously $d_N\omega = 0$ since N is of dimension $n-1$. Then by Stokes' Theorem

$$\deg(g) = \int_{\partial M} g^*\omega = \int_{\partial M} G^*\omega = \int_M d_M(G^*\omega) = \int_M G^*(d_N\omega) = 0. \tag{18.11}$$

□

Corollary 18.15 (Brouwer fixed point theorem)**.** *If* $f : \overline{B^n} \to \overline{B^n}$ *is smooth, then* f *has a fixed point.*

Proof. Assume by contradiction that f has no fixed point. Then define $G : \overline{B^n} \to S^{n-1}$ by

$$G(x) = \frac{x - F(x)}{|x - F(x)|}. \tag{18.12}$$

Letting $g = G|_{S^{n-1}}$, by the previous proposition, we have $\deg(g) = 0$.

However, define $H : S^{n-1} \times [0,1] \to S^{n-1}$ by

$$H(x,t) = \frac{x - tF(x)}{|x - tF(x)|}. \tag{18.13}$$

Clearly, the denominator never vanishes, so H is a smooth homotopy between g and the identity map. Since $\deg(Id) = 1$, this is a contradiction. $\square$

Theorem 18.16 (Fundamental theorem of algebra)**.** *Any nonconstant polynomial in* $\mathbb{C}$ *has a zero.*

Proof. Let $P(x) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Recall that $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$ where $(z^1, z^2) \sim \lambda(z^1, z^2)$ for $\lambda \neq 0$. It is easy to see that we can extend $P(x) : S^2 \to S^2$, as a holomorphic map (it is a meromorphic function on $\mathbb{C}$ with a single pole at infinity of order n). Since P is holomorphic, it is orientation preserving. Since P is a polynomial, the set of critical values is a finite set. Since P is non-constant, P attains some value $q \in S^2$ which is not critical. Since P is orientation preserving at all regular points, the degree must then be non-zero (in fact, the degree is n). By the above proposition, this implies that P is surjective. $\square$

19 Lecture 19

19.1 Real projective spaces

Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{RP}^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$(v_1, \dots v_{n+1}) \sim (cv_1, \dots, cv_{n+1}),\ c \neq 0. \tag{19.1}$$

Since every line through the origin hits the unit sphere in exactly two points, we can desribe $\mathbb{RP}^n$ as a quotient space. That is,

$$\mathbb{RP}^n = S^n / \mathbb{Z}_2, \tag{19.2}$$

where $\mathbb{Z}_2$ acts by $p \mapsto A(p) = -p$. Let $\pi : S^n \to \mathbb{RP}^n$ denote the projection mapping.

Proposition 19.1. $\mathbb{RP}^n$ *is orientable if* n *is odd, and non-orientable if* n *is even.*

Proof. In the last lecture, we saw that

$$\omega = \left(x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \cdots \wedge dx^{n+1}) \tag{19.3}$$

is a nowhere-vanishing n-form on $S^n \subset \mathbb{R}^{n+1}$. Clearly,

$$A^*\omega = (-1)^{n+1}\omega. \tag{19.4}$$

So if n is odd, ω is invariant under the above $\mathbb{Z}_2$ action and thus descends to be nowhere-zero n-form on $\mathbb{RP}^n$.

In n is even, then $A^*\omega = -\omega$. This says that A is orientation-reversing. If $\mathbb{RP}^n$ were orientable, then it would have a non-zero n-form $\omega \in \Omega^n(\mathbb{RP}^n)$, and the pull back form $\pi^*\omega$ would be a non-zero n-form on S^n which is invariant under A:

$$A^*\pi^*\omega = (\pi \circ A)^*\omega = \pi^*\omega, \tag{19.5}$$

since $\pi \circ A = \pi$. This says that A is orientation-preserving, which is a contradiction. □

We next compute the de Rham cohomology of $\mathbb{RP}^n$.

Theorem 19.2. *We have*

$$H^k_{dR}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k < n \\ \mathbb{R} & k = n \text{ odd} \\ 0 & k = n \text{ even} \end{cases} \tag{19.6}$$

Proof. Since $A : S^n \to S^n$ satisfies $A^2 = Id_{S^n}$. For each $0 \leq k \leq n$, we have that

$$\Omega^k(S^n) = \Omega^k_+(S^n) \oplus \Omega^k_-(S^n) \tag{19.7}$$

where

$$\Omega^k_\pm(S^n) = \{\omega \in \Omega^k(S^n) \mid A^*\omega = \pm\omega\}, \tag{19.8}$$

because we can write

$$\omega = \frac{1}{2}(\omega + A^*\omega) + \frac{1}{2}(\omega - A^*\omega) \tag{19.9}$$

We claim that

$$\pi^* : \Omega^k(\mathbb{RP}^n) \to \Omega^k_+(S^n) \subset \Omega^k(S^n), \tag{19.10}$$

and is an isomorphism. Just as above $\pi \circ A = \pi$ implies that $A^*\pi^*\omega = \pi^*\omega$, so clearly the image of the pull-back lies in the space of invariant forms. Next, we need to show that if $\omega \in \Omega^k_+(S^n)$, then ω is the pull-back of a form $\alpha \in \Omega^k(\mathbb{RP}^n)$. That is, if $A^*\omega = \omega$, then for $p \in S^n$, and $X_1, \ldots, X_k \in T_pS^n$,

$$\omega_p(X_1, \ldots, X_k) = (\pi^*\alpha)_p(X_1, \ldots, X_k) = \alpha_{\pi(p)}(\pi_*X_1, \ldots, \pi_*X_k). \tag{19.11}$$

Let us use this equation to define $\alpha_{\pi(p)}$. We need to prove this is well-defined. Given $[p] \in \mathbb{RP}^n$, there are exactly 2 preimages p and $A(p) = -p$. The mappings $(\pi_*)_p : T_pS^n \to T_{[p]}\mathbb{RP}^n$, $(\pi_*)_{A(p)} : T_{A(p)}S^n \to T_{[p]}\mathbb{RP}^n$, and $(A_*)_p : T_pS^n \to T_{A(p)}S^n$ are isomorphisms. Given $[p] \in \mathbb{RP}^n$ and $Y_1, \dots Y_k \in T_{[p]}\mathbb{RP}^n$, there are exactly 2 choices:

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_p((\pi_*)_p^{-1}Y_1, \dots, (\pi_*)_p^{-1}Y_k), \tag{19.12}$$

or

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A(p)}((\pi_*)_{A(p)}^{-1}Y_1, \dots, (\pi_*)_{A(p)}^{-1}Y_k). \tag{19.13}$$

Since $\pi \circ A = \pi$, we have

$$(\pi_*)_{A(p)}(A_*)_p = (\pi_*)_p. \tag{19.14}$$

Since all of the mappings are isomorphisms, this implies that

$$(\pi_*)_{A(p)}^{-1} = (A_*)_p(\pi_*)_p^{-1}, \tag{19.15}$$

so (19.13) can be rewritten as

$$\alpha_{[p]}(Y_1, \dots, Y_k) = \omega_{A(p)}((A_*)_p(\pi_*)_p^{-1}Y_1, \dots, (A_*)_p(\pi_*)_p^{-1}Y_k). \tag{19.16}$$

The condition that ω is invariant under A, $A^*\omega = \omega$ says that

$$(A^*\omega)_p(X_1, \dots, X_k) = \omega_{A(p)}(A_*X_1, \dots, A_*X_k) \tag{19.17}$$

Choosing $X_i = (\pi_*)_p^{-1}Y_i$, we see that (19.12) = (19.13), therefore α is well-defined.

We next note that if $A^*\omega = \omega$ then

$$A^*d\omega = dA^*\omega = d\omega, \tag{19.18}$$

so the exterior derivative maps

$$d : \Omega_+^k(S^n) \to \Omega_+^{k+1}(S^n). \tag{19.19}$$

We therefore have the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(\mathbb{RP}^n) & \xrightarrow{d} & \Omega^k(\mathbb{RP}^n) & \xrightarrow{d} & \Omega^{k+1}(\mathbb{RP}^n) & \xrightarrow{d} & \cdots \\ & & \downarrow{\pi^*} & & \downarrow{\pi^*} & & \downarrow{\pi^*} & & \\ \cdots & \xrightarrow{d} & \Omega_+^{k-1}(S^n) & \xrightarrow{d} & \Omega_+^k(S^n) & \xrightarrow{d} & \Omega_+^{k+1}(S^n) & \xrightarrow{d} & \cdots \end{array} \tag{19.20}$$

Since π^* is an isomorphism, we have

$$H_{dR}^k(\mathbb{RP}^n) = H^k(\Omega_+^*(S^n)) \tag{19.21}$$

We next note that if $A^*\omega = -\omega$ then

$$A^*d\omega = dA^*\omega = -d\omega, \tag{19.22}$$

so the exterior derivative maps

$$d : \Omega^k_-(S^n) \to \Omega^{k+1}_-(S^n). \tag{19.23}$$

We can also decompose the de Rham complex on S^n by

$$\Omega^{k-1}_+(S^n) \oplus \Omega^{k-1}_-(S^n) \xrightarrow{d\oplus d} \Omega^k_+(S^n) \oplus \Omega^k_-(S^n) \xrightarrow{d\oplus d} \Omega^{k+1}_+(S^n) \oplus \Omega^{k+1}_-(S^n). \tag{19.24}$$

This implies that

$$H^k_{dR}(S^n) = H^k(\Omega^*_+(S^n)) \oplus H^k(\Omega^*_-(S^n)). \tag{19.25}$$

Next, note that since A is a diffeomorphism satisfying $A^2 = Id_{S^n}$, we have that A induces a mapping on cohomology

$$A^* : H^k_{dR}(S^n) \to H^k_{dR}(S^n), \tag{19.26}$$

which also satisfies $(A^*)^2 = Id_{H^k(S^n)}$. Consequently, we can decompose

$$H^k_{dR}(S^n) = H^k_+(S^n) \oplus H^k_-(S^n), \tag{19.27}$$

where $H^k_+(S^n), H^k_-(S^n)$ are the invariant and anti-invariant cohomology classes, respectively. Equivalently, these are the $+1$ and -1 eigenspaces of A^*. We next claim that

$$H^k(\Omega^*_\pm(S^n)) = H^k_\pm(S^n). \tag{19.28}$$

This follows because we have two decompositions

$$\begin{aligned} H^k_{dR}(S^n) &= H^k(\Omega^*_+(S^n)) \oplus H^k(\Omega^*_-(S^n)) \\ &= H^k_+(S^n) \oplus H^k_-(S^n), \end{aligned} \tag{19.29}$$

the first factors are the $+1$ eigenspace, and the second factors are the -1 eigenspace, so they must be equal.

To finish the proof, we clearly have $H^k_{dR}(\mathbb{RP}^n) = \{0\}$ for $0 < k < n$. For $k = n$, we know that $A^* = (-1)^{n+1}$ acting on $H^n_{dR}(S^n)$, so we have that $H^n_{dR}(\mathbb{RP}^n) = \mathbb{R}$ if n is odd, and $H^n_{dR}(\mathbb{RP}^n) = \{0\}$ if n is even.

□

20 Lecture 20

20.1 Finite group quotients

Let M be a smooth manifold, and Γ be a finite group. An left action of Γ on M is a smooth mapping

$$A : \Gamma \times M \to M \tag{20.1}$$

satisfying

$$A(g_1 g_2, p) = A(g_1, A(g_2, p)). \tag{20.2}$$

For each $g \in \Gamma$, then mapping $A_g : M \to M$ is a diffeomorphism since it has inverse $A_{g^{-1}}$. Also, $A(e, p) = p$ for all $p \in M$, where e is the identity element of Γ.

Definition 20.1. The action A is *free* if $A(g, p) = p$ for some $p \in M$ implies that $g = e$.

We define the quotient space M/Γ as the set of equivalence classes $[p]$ where the equivalence relation is $p_1 \sim p_2$ if there exist $g \in \Gamma$ such that $A(g, p_1) = p_2$.

Proposition 20.2. *If the action is free, then the quotient space M/Γ is a manifold. Furthermore, $\pi : M \to M/\Gamma$ is a covering space of order $|\Gamma|$ with deck transformation group Γ.*

Proof. We will leave this as an exercise. □

Definition 20.3. The space of invariant k-forms

$$\Omega^k_+(M) = \{\omega \in \Omega^k(M) \mid A_g^*\omega = \omega \text{ for all } g \in \Gamma\}. \tag{20.3}$$

Proposition 20.4. *The mapping $\pi^* : \Omega^k(M/\Gamma) \to \Omega^k_+(M)$ is an isomorphism.*

Proof. For each $g \in \Gamma$, we have $\pi \circ A_g = \pi$ which implies that $A^*\pi^*\omega = \pi^*\omega$, so clearly the image of the pull-back lies in the space of invariant forms. Next, we need to show that if $\omega \in \Omega^k_+(M)$, then ω is the pull-back of a form $\alpha \in \Omega^k(M/\Gamma)$. That is, if $A_g^*\omega = \omega$ for all $g \in \Gamma$, then for $p \in M$, and $X_1, \ldots, X_k \in T_pM$,

$$\omega_p(X_1, \ldots, X_k) = (\pi^*\alpha)_p(X_1, \ldots, X_k) = \alpha_{\pi(p)}(\pi_* X_1, \ldots, \pi_* X_k). \tag{20.4}$$

Let p be any preimage of $[p]$ under the projection π. The mapping $(\pi_*)_p : T_pM \to T_{[p]}(M/\Gamma)$ is an isomorphism. Given $Y_1, \ldots Y_k \in T_{[p]}(M/\Gamma)$, we define

$$\alpha_{[p]}(Y_1, \ldots, Y_k) = \omega_p((\pi_*)_p^{-1}Y_1, \ldots, (\pi_*)_p^{-1}Y_k). \tag{20.5}$$

We need to show this is well-defined. Let $\tilde{p}$ be any other preimage. Then there exists $g \in \Gamma$ such that $\tilde{p} = A_g p$. Using $A_g p$ instead of p in the definition yields

$$\alpha_{[p]}(Y_1, \ldots, Y_k) = \omega_{A_g(p)}((\pi_*)_{A_g(p)}^{-1}Y_1, \ldots, (\pi_*)_{A_g(p)}^{-1}Y_k). \tag{20.6}$$

Since $\pi \circ A_g = \pi$, we have

$$(\pi_*)_{A_g(p)}((A_g)_*)_p = (\pi_*)_p. \tag{20.7}$$

Since all of these mappings are isomorphisms, this implies that

$$(\pi_*)_{A_g(p)}^{-1} = ((A_g)_*)_p(\pi_*)_p^{-1}, \tag{20.8}$$

so (20.6) can be rewritten as

$$\alpha_{[p]}(Y_1, \ldots, Y_k) = \omega_{A_g(p)}(((A_g)_*)_p(\pi_*)_p^{-1}Y_1, \ldots, (A_*)_p(\pi_*)_p^{-1}Y_k). \tag{20.9}$$

The condition that ω is invariant under A_g, $A_g^*\omega = \omega$ says that

$$\omega_p(X_1, \ldots, X_k) = (A_g^*\omega)_p(X_1, \ldots, X_k) = \omega_{A_g(p)}((A_g)_*X_1, \ldots, (A_g)_*X_k). \tag{20.10}$$

Choosing $X_i = (\pi_*)_p^{-1}Y_i$, we see that (20.5) = (20.6), therefore α is well-defined. □

Proposition 20.5. *We have*

$$H^k_{dR}(M/\Gamma) = H^k(\Omega^*_+(M)) \tag{20.11}$$

Proof. If $A^*\omega = \omega$ then

$$A^* d\omega = dA^*\omega = d\omega, \tag{20.12}$$

so the exterior derivative maps

$$d : \Omega^k_+(M) \to \Omega^{k+1}_+(M). \tag{20.13}$$

We therefore have the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \Omega^{k-1}(M/\Gamma) & \xrightarrow{d} & \Omega^k(M/\Gamma) & \xrightarrow{d} & \Omega^{k+1}(M/\Gamma) & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle\pi^*} & & \downarrow{\scriptstyle\pi^*} & & \downarrow{\scriptstyle\pi^*} & & \\
\cdots & \xrightarrow{d} & \Omega^{k-1}_+(M) & \xrightarrow{d} & \Omega^k_+(M) & \xrightarrow{d} & \Omega^{k+1}_+(M) & \xrightarrow{d} & \cdots
\end{array} \tag{20.14}$$

Since π^* is an isomorphism, this finishes the proof. □

Next, we have the following

Proposition 20.6. *The induced mapping*

$$\pi^* : H^k_{dR}(M/\Gamma) \to H^k_{dR}(M) \tag{20.15}$$

is injective.

Proof. We have that $\Omega^*_+(M) \subset \Omega^*(M)$ is a subcocomplex. This induces a mapping

$$H^k(\Omega^*_+(M)) \to H^k_{dR}(M) \tag{20.16}$$

by the following. Take an equivalence class $[\omega] \in H^k(\Omega^*_+(M))$ represented by $\omega \in \Omega^k_+(M)$, and map this to the cohomology class $[\omega] \in H^k_{dR}(M)$. This is well-defined, since if $\omega = d\alpha$ where $\alpha \in \Omega^{k-1}_+(M)$ then obviously $[\omega] = 0$ in $H^k_{dR}(M)$ also.

By the previous proposition, we just need to show that the mapping (20.16) is an injection. For this, we need to show that if $\omega \in \Omega^k_+(M)$ satisfies $\omega = d\alpha$ for $\alpha \in \Omega^{k-1}(M)$, then $\omega = d\beta$, where $\beta \in \Omega^{k-1}_+(M)$. For this, simply define

$$\beta = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} A^*_g \alpha. \tag{20.17}$$

For any $g' \in G$, this satisfies

$$A^*_{g'}\beta = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} A^*_{g'} A^*_g \alpha = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} A^*_{gg'} \alpha = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} A^*_g \alpha = \beta \tag{20.18}$$

so $\beta \in \Omega^{k-1}_+(M)$. Then

$$d\beta = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} dA^*_g \alpha = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} A^*_g d\alpha = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} d\alpha = d\alpha. \tag{20.19}$$

□

Definition 20.7. The kth Betti number of M is

$$b^k(M) = \dim(H^k_{dR}(M)). \tag{20.20}$$

We can phrase the above result as follows.

Theorem 20.8. *If $\pi : \tilde{M} \to M$ is a covering space then*

$$b^k(\tilde{M}) \geq b^k(M). \tag{20.21}$$

Definition 20.9. The Euler characteristic of M is

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b^k(M). \tag{20.22}$$

The following theorem is more difficult to prove (with the right machinery however, it is easy).

Theorem 20.10. *If $\pi : \tilde{M} \to M$ is a covering space with deck transformation group Γ then*

$$\chi(\tilde{M}) = |\Gamma| \cdot \chi(M). \tag{20.23}$$

We will prove this a bit later.

20.2 Compactly supported cohomology

If M is noncompact, the mapping $\pi : M \to M/\Gamma$ is proper. Therefore we have

$$\pi^* : \Omega^k_c(M/\Gamma) \to \Omega^k_{c,+}(M). \tag{20.24}$$

The above arguments holds verbatim for compactly supported cohomology, so we have:

Proposition 20.11. *The mapping $\pi^* : \Omega^k_c(M/\Gamma) \to \Omega^k_{c,+}(M)$ is an isomorphism, and*

$$H^k_{c,dR}(M/\Gamma) = H^k(\Omega^*_{c,+}(M)). \tag{20.25}$$

Furthermore, the induced mapping

$$\pi^* : H^k_{c,dR}(M/\Gamma) \to H^k_{c,dR}(M) \tag{20.26}$$

is injective.

21 Lecture 21

21.1 Top cohomology of nonorientable manifolds

Theorem 21.1. *If M is a smooth manifold of dimension n which is non-orientable and connected then*

$$H^n_{dR}(M) = \{0\} \quad \text{and} \quad H^n_{c,dR}(M) = \{0\}. \tag{21.1}$$

Proof. Recall the construction of the orientable double cover $\pi : \tilde{M} \to M$: the bundle $\Lambda^n(M)$ is a real line bundle. Endow this bundle with a Riemannian metric, and then $\tilde{M}$ is the unit sphere bundle. Since M is non-orientable, $\tilde{M}$ is connected. The mapping $A : \omega_p \mapsto -\omega_p$ is clearly a free $\mathbb{Z}_2$-action on $\tilde{M}$, and $M = \tilde{M}/\mathbb{Z}_2$.

We claim that $\tilde{M}$ is orientable. Given $p \in M$, there are precisely 2 preimages $\tilde{p}$ and $A\tilde{p}$ under π. The point $\tilde{p} = \omega_p$ is, by definition, a non-zero n-form on T_pM, so determines an orientation on T_pM. The mapping $\pi_* : T_{\tilde{p}}\tilde{M} \to T_pM$ is an isomorphism, so we give $T_{\tilde{p}}\tilde{M}$ the induced orientation. Similarly, we give $T_{A\tilde{p}}$ the induced orientation. This clearly gives a smooth orientation on $\tilde{M}$, called the *tautological* orientation. Note that the mapping A is orientation-reversing (otherwise, the quotient space would also be orientable).

For compactly supported cohomology, we have

$$\pi^* : H^k_{c,dR}(M/\Gamma) \to H^k_{c,dR}(M) \tag{21.2}$$

is injective. Given $\omega \in \Omega^n_c(M)$, we have $\tilde{\omega} = \pi^*\omega$ satisfies $A^*\tilde{\omega} = \tilde{\omega}$. But

$$\int_{\tilde{M}} \tilde{\omega} = -\int_{\tilde{M}} A^*\tilde{\omega} = -\int_{\tilde{M}} A^*\tilde{\omega}, \tag{21.3}$$

since A is orientation-reversing. Consequently,

$$\int_{\tilde{M}} \tilde{\omega} = 0. \tag{21.4}$$

By Poincaré duality, this implies that $[\tilde{\omega}] = 0 \in H^n_{c,dR}(\tilde{M})$. But since π^* is injective, this implies that $[\omega] = 0 \in H^n_{c,dR}(M)$.

For the regular de Rham cohomology, we only need to consider the case that M is noncompact. Then $\tilde{M}$ is also noncompact. By Poincaré duality, we have

$$H^n_{dR}(\tilde{M}) \cong \left(H^0_{c,dR}(\tilde{M})\right)^* = \{0\}. \tag{21.5}$$

But since

$$\pi^* : H^k_{dR}(M) \to H^k_{dR}(\tilde{M}) \tag{21.6}$$

is injective, this implies that $H^n_{dR}(M) = \{0\}$. □

21.2 Summary

Let us summarize our results on the top-dimensional cohomology of any connected smooth n-dimensional manifold.

Theorem 21.2. *Let M be a connected smooth n-dimensional manifold. Then*

$$H^n_{dR}(M) = \begin{cases} \mathbb{R} & M \textit{ compact and orientable} \\ 0 & M \textit{ otherwise} \end{cases} \tag{21.7}$$

and

$$H^n_{c,dR}(M) = \begin{cases} \mathbb{R} & M \textit{ orientable} \\ 0 & M \textit{ otherwise} \end{cases}. \tag{21.8}$$

Proof. If M is orientable, Poincaré duality says

$$H^n_{dR}(M) \cong (H^0_{c,dR}(M))^* \tag{21.9}$$

If M is compact, the right hand side is 1-dimensional. If M is non-compact, the right hand side is 0-dimensional. In the orientable case, we also have

$$H^n_{c,dR}(M) \cong (H^0_{dR}(M))^*, \tag{21.10}$$

and the right hand side is always 1-dimensional, if M is either compact or non-compact.

The non-orientable cases follow from Theorem 21.1.

□

21.3 Compact surfaces

By "surface" we will mean a 2-dimensional smooth manifold. We will compute the de Rham cohomology of any compact surface. First, we need to define the connected sum operation.

Given surfaces M_1, M_2 choose points $p_i \in M_i$, $i = 1, 2$. Let U_i be a neighborhood of p_1 which is diffeomorphic to $B_0(2)$, a ball centered at the origin of radius 2 in $\mathbb{R}^2 = \mathbb{C}$. Let $\Psi_i : B(0,2) \to M_i$ be a diffeomorphism such $\Psi_i(B_0(2)) = U_i$ and $\Psi_i(0) = p_i$. Let $V_i = M_i \setminus \Psi_i(B_0(1/2))$.

Definition 21.3. We define $M_1 \# M_2 = V_1 \coprod V_2 / \sim$ where the equivalence relation is $z \sim w$ if $z \in \Psi_1(B_0(2) \setminus B_0(1/2))$ and $w \in \Psi_2(B_0(2) \setminus B_0(1/2))$ satisfy

$$\Psi_1^{-1}(z) \sim (\Psi_2^{-1}(w))^{-1}, \tag{21.11}$$

where the right hand side means the inverse as a complex number.

Exercise 21.4. Prove the following properties of the connected sum:

(i) Show that $M_1 \# M_2$ is a surface.

(ii) Show that if M_1 and M_2 are orientable, then so is $M_1 \# M_2$. (Hint: the map $z \mapsto z^{-1}$ is orientation-preserving).

(iii) If either M_1 or M_2 is non-orientable, then $M_1 \# M_2$ is non-orientable.

Theorem 21.5 (Classification of compact surfaces)**.** *If M is an orientable compact surface then M is diffeomorphic to*

$$S^2 \quad or \quad \overbrace{T^2 \# \cdots \# T^2}^{k} \equiv k \# T^2, \tag{21.12}$$

for some $k \in \mathbb{Z}_+$. If M is a non-orientable compact surface then M is diffeomorphic

$$\overbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}^{k} \equiv k \# \mathbb{RP}^2 \tag{21.13}$$

for some $k \in \mathbb{Z}_+$.

We will not prove this, but will determine the de Rham cohomology groups of each of these cases. This will show that these examples are pairwise non-diffeomorphic.

Theorem 21.6. *We have*

$$H^k_{dR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & k = 1 \end{cases}, \tag{21.14}$$

and $\chi(S^2) = 2$. *If* $M = g\#T^2$, *then*

$$H^k_{dR}(M) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^{2g} & k = 1 \end{cases}, \tag{21.15}$$

and $\chi(M) = 2 - 2g$. *If* $M = (g+1)\#\mathbb{RP}^2$ *then*

$$H^k_{dR}(M) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^{g} & k = 1 \\ 0 & k = 2 \end{cases}, \tag{21.16}$$

and $\chi(M) = 1 - g$.

The remainder of this section will be occupied with the proof. We already know this is true for S^2 and $\mathbb{RP}^2$. For T^2, we can cover by open sets U, V, such that U and V retract to S^1 and $U \cap V$ retracts to 2 copies of S^1. The Mayer-Vietoris sequence gives

$$\begin{array}{l} 0 \longrightarrow H^0_{dR}(T^2) \longrightarrow H^0_{dR}(S^1) \oplus H^0_{dR}(S^1) \longrightarrow H^0_{dR}(S^1 \coprod S^1) \longrightarrow \\ \longrightarrow H^1_{dR}(T^2) \longrightarrow H^1_{dR}(S^1) \oplus H^1_{dR}(S^1) \longrightarrow H^1_{dR}(S^1 \coprod S^1) \longrightarrow \\ \longrightarrow H^2_{dR}(T^2) \longrightarrow 0. \end{array} \tag{21.17}$$

This is

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \\ \longrightarrow H^1_{dR}(T^2) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \\ \longrightarrow \mathbb{R} \longrightarrow 0. \end{array} \tag{21.18}$$

Recall the lemma:

Lemma 21.7. *If*

$$0 \longrightarrow V_1 \xrightarrow{\alpha} V_2 \longrightarrow \cdots \longrightarrow V_{k-1} \longrightarrow V_k \longrightarrow 0. \tag{21.19}$$

is exact, then

$$0 = \dim(V_1) - \dim(V_2) + \dim(V_3) + \cdots + (-1)^{k-1} \dim(V_k). \tag{21.20}$$

Applying the lemma yields that $H^1_{dR}(T^2) = \mathbb{R} \oplus \mathbb{R}$ and $\chi(T^2) = 0$.

22 Lecture 22

22.1 Compact surfaces

We continue the proof of Theorem 21.6. Next, we have

Proposition 22.1. *The Euler characteristic of a connect sum of surfaces is given by*

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2. \tag{22.1}$$

Proof. Given any decomposition of $M = U \cup V$ into the union of 2 open sets, we have the Mayer-Vietoris sequence,

$$\begin{array}{l} 0 \longrightarrow H^0(M) \longrightarrow H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(U \cap V) \\ \hookrightarrow H^1_{dR}(M) \longrightarrow H^1_{dR}(U) \oplus H^1_{dR}(V) \longrightarrow H^1_{dR}(U \cap V) \\ \hookrightarrow H^2_{dR}(M) \longrightarrow H^2_{dR}(U) \oplus H^2_{dR}(V) \longrightarrow H^1_{dR}(U \cap V) \longrightarrow 0. \end{array} \tag{22.2}$$

Lemma 21.7 implies that

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V). \tag{22.3}$$

Given a surface M and $p \in M$, cover M by open sets $U = M \setminus \{p\}$, V a small ball around p, and $U \cap V \sim S^1$. Applying (24.3),

$$\chi(M \setminus \{p\}) = \chi(M) - 1. \tag{22.4}$$

Next, we cover $M_1 \# M_2$ by $U \sim M_1 \setminus \{p_1\}$, $V \sim M_2 \setminus \{p_2\}$ and such that $U \cap V$ retracts onto S^1. Applying (24.3) with (22.4) yields

$$\chi(M_1 \# M_2) = \chi(M_1) - 1 + \chi(M_2) - 1 - \chi(S^1) = \chi(M_1) + \chi(M_2) - 2. \tag{22.5}$$

□

In the orientable case, the proposition implies that

$$\chi\left(g \# T^2\right) = 2 - 2g. \tag{22.6}$$

In the nonorientable case, the proposition implies that

$$\chi\big((g+1)\#\mathbb{RP}^2\big) = 1 - g. \tag{22.7}$$

and the dimension of the middle de Rham homology group follows since we know the dimension of the top de Rham cohomology group.

Proposition 22.2. *We have the miscellaneous facts about surfaces:*

- *The orientable double cover of $(g+1)\#\mathbb{RP}^2$ is $g\#T^2$.*
- *The Klein bottle is diffeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2$.*
- *We have $T^2\#\mathbb{RP}^2$ is diffeomorphic to $3\#\mathbb{RP}^2$.*
- *Any compact nonorientable surface is diffeomorphic to a compact orientable surface connect sum with an $\mathbb{RP}^2$ or Klein bottle.*

We leave the proof as an exercise.

22.2 Triangulations

Define the standard p-simplex to be

$$\Delta^p = \Big\{(t_0,\dots,t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\Big\}. \tag{22.8}$$

For $0 \leq i \leq p$, the ith face of Δ^p is the $(p-1)$-simplex

$$\Delta^p_i : \Delta^{p-1} \to \Delta^p \tag{22.9}$$

defined by

$$(t_0,\dots,t_{p-1}) \mapsto (t_0,\dots,t_{i-1},0,t_i,\dots t_{p-1}). \tag{22.10}$$

More generally, for $k < p-1$, a k-face of Δ^p is a simplex obtained from Δ_p obtained by setting $p-k$ of the coordinates equal to 0.

Definition 22.3. If M is a smooth compact n-dimensional manifold, a triangulation of M is collection of diffeomorphisms

$$c^n_i : \Delta^n \to M \tag{22.11}$$

for $i = 1 \dots N$ whose images cover M and such that if

$$c^n_i(\Delta^p) \cap c^n_j(\Delta^p) \neq \emptyset, \tag{22.12}$$

for $i \neq j$, then the intersection is exactly a k-face of both simplices for $0 < k \leq p-1$.

We will refer to image of c^n_i as an n-simplex of the triangulation, and the image of any k-face of a simplex will be called a k-simplex of the triangulation. Let α_k be the number of k-simplices in a triangulation.

Theorem 22.4. *If a smooth compact surface M admits a triangulation, then*

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2 = V - E + F. \tag{22.13}$$

Proof. Clearly, we can assume that M is connected. Let U be the union of small balls around the barycenters (center of mass) of the 2-simplices p_i. Let $V_1 = M \setminus \cup_{i=1}^{\alpha_2}\{p_i\}$. Then $U \cap V$ is homotopic to the disjoint union of α_2 copies of S^1.

Applying the Mayer-Vietoris sequence to U and V yields that

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^{\alpha_2} \oplus H^0_{dR}(V_1) \longrightarrow \mathbb{R}^{\alpha_2} \\ \hookrightarrow H^1_{dR}(M) \longrightarrow H^1_{dR}(V_1) \longrightarrow \mathbb{R}^{\alpha_2} \\ \hookrightarrow H^2_{dR}(M) \longrightarrow H^2_{dR}(V_1) \longrightarrow 0. \end{array} \tag{22.14}$$

Lemma 21.7 implies that

$$0 = 1 - \alpha_2 - b^0(V_1) + \alpha_2 - b^1(M) + b^1(V_1) - \alpha_2 + b^2(M) - b^2(V_1), \tag{22.15}$$

or

$$\chi(M) = \chi(V_1) + \alpha_2. \tag{22.16}$$

We next apply the Mayer-Vietoris sequence on V_1, with a new U and V. For this, if 2 2-simplices intersect along a 1-face, then we can connect the barycenters by a curve which intersects the 1-face in the barycenter of the 1-face.

Then let U be the disjoint union of sets diffeomorphic to balls which are slight "fattenings" of slight shrinkings of the curves (so that they are disjoint near the endpoints). Let V_0 be the complement in V_1 of the union of the curves joining the barycenters. Then $V_1 = U \cup V_0$, V_1 is the union of α_1 balls, and the set V_0 deformation retracts onto the set of 0-faces. Also, the intersection consists of $2\alpha_1$ sets diffeomorphic to balls, since each curve cuts the fattenings into 2 pieces.

$$\begin{array}{l} 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^{\alpha_1} \oplus \mathbb{R}^{\alpha_0} \longrightarrow \mathbb{R}^{2\alpha_1} \\ \hookrightarrow H^1_{dR}(V_1) \longrightarrow 0 \end{array} \tag{22.17}$$

Lemma 21.7 implies that

$$\chi(V_1) = \alpha_0 - \alpha_1. \tag{22.18}$$

Combining with (22.16), we have

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2. \tag{22.19}$$

□

Remark 22.5. In higher dimensions, it is true that

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \alpha_k, \tag{22.20}$$

but we leave this as an exercise. (The above proof basically extends to the higer-dimensional case; see [Spi79, Chapter 11].)

Corollary 22.6. *If a finite group Γ acts freely on a compact manifold M, then*

$$\chi(M) = |\Gamma| \cdot \chi(M/\Gamma). \tag{22.21}$$

Proof. If M/Γ has a triangulation with α_k k-simplices, then we can pull-back the triangulation to M, which is a triangulation with $\tilde{\alpha}_k = |\Gamma|\alpha_k$ k-simplices. □

23 Lecture 23

23.1 Complex projective space

Complex projective spaces is defined to be the space of lines through the origin in $\mathbb{C}^{n+1}$. This is equivalent to $\mathbb{C}^{n+1}/\sim$, where $\sim$ is the equivalence relation

$$(z^0, \dots, z^n) \sim (w^0, \dots, w^n) \tag{23.1}$$

if there exists $\lambda \in \mathbb{C}^*$ so that $z^j = \lambda w^j$ for $j = 1 \dots n$. The equivalence class of $(z^0, \dots, z^n)$ will be denoted by $[z^0 : \dots : z^n]$. Letting $U_j = \{[z^0 : \dots : z^n] | z^j \neq 0\}$, $\mathbb{CP}^n$ is covered by $(n+1)$ coordinate charts $\phi_j : U_j \to \mathbb{C}^n$ defined by

$$\phi_j : [z^0 : \dots : z^n] \mapsto \left(\frac{z^0}{z_j}, \dots, \frac{z^{j-1}}{z_j}, \frac{z^{j+1}}{z_j}, \dots, \frac{z^n}{z_j} \right), \tag{23.2}$$

with inverse given by

$$\phi_j^{-1} : (w^1, \dots, w^n) \mapsto [w^1 : \dots w^{j-1} : 1 : w^j : \dots : w^n]. \tag{23.3}$$

The overlap maps are holomorphic, which gives $\mathbb{CP}^n$ the structure of a complex manifold, and is therefore orientable.

Theorem 23.1. *The de Rham cohomology of $\mathbb{CP}^n$ is given by*

$$H^k_{dR}(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \tag{23.4}$$

Proof. We note that $\mathbb{CP}^n = U \cup V$, where U is diffeomorphic to $\mathbb{C}^n$, V is a tubular neighborhood of $\mathbb{CP}^{n-1}$ and $U \cap V$ deformation retracts onto S^{2n-1}. The Mayer-Vietoris sequence gives

$$
\begin{array}{l}
0 \longrightarrow H^0(\mathbb{CP}^n) \longrightarrow H^0_{dR}(\mathbb{C}^2) \oplus H^0_{dR}(\mathbb{CP}^{n-1}) \longrightarrow H^0_{dR}(S^{2n-1}) \\
\hookrightarrow H^1_{dR}(\mathbb{CP}^n) \longrightarrow H^1_{dR}(\mathbb{CP}^{n-1}) \longrightarrow H^1_{dR}(S^{2n-1}) \\
\hookrightarrow H^2_{dR}(\mathbb{CP}^n) \longrightarrow H^2_{dR}(\mathbb{CP}^{n-1}) \longrightarrow H^2_{dR}(S^{2n-1}) \longrightarrow \cdots
\end{array}
\tag{23.5}
$$

The theorem follows by induction since we know the de Rham cohomology of S^{2n-1} is non-zero only in degrees 0 and $2n-1$. $\square$

Theorem 23.2. *The de Rham cohomology ring of* $\mathbb{CP}^n$ *is*

$$H^*_{dR}(\mathbb{CP}^n) \cong \mathbb{R}[\omega]/(\omega^{n+1}), \tag{23.6}$$

where u *is an element of degree* 2.

Proof. Proos is by induction. Assume that the statement is true for $n-1$, so that

$$H^*_{dR}(\mathbb{CP}^{n-1}) \cong \mathbb{R}[\alpha]/(\alpha^n), \tag{23.7}$$

for $\alpha \in H^2_{dR}(\mathbb{CP}^{n-1})$. The Mayer-Vietoris sequence shows that the inclusion $i : \mathbb{CP}^{n-1} \to \mathbb{CP}^n$ induces an isomorphism

$$i^* : H^k(\mathbb{CP}^n) \cong H^k(\mathbb{CP}^{n-1}) \tag{23.8}$$

for $k < 2n$, so there is a element $\omega \in H^2(\mathbb{CP}^n)$ so that $i^*\omega = \alpha$. Then $i^*\omega^k = \alpha^k$ is not zero for $k < n$.

Next, we use the isomorphism from Poincaré duality

$$H^{2n-2}_{dR}(\mathbb{CP}^n) \cong \left(H^2_{dR}(\mathbb{CP}^n)\right)^* \tag{23.9}$$

with isomorphism given by the paring

$$([\omega^{n-1}], [\omega]) \mapsto \int_{\mathbb{CP}^n} \omega^n, \tag{23.10}$$

which implies that ω^n is not zero in $H^{2n}_{dR}(\mathbb{CP}^n)$. $\square$

Remark 23.3. Later, we will give a more geometric way to understand this using intersection theory.

23.2 Branched covers

Definition 23.4. Let M and N be compact surfaces. We say that $f : M \to N$ is a degree d branched covering if there exist $S = \{p_1, \dots p_k\} \in N$ such that f is is a d-fold covering space away from S, $f^{-1}(p_i)$ is finite, and near and $q \in f^{-1}(p_i)$, f is diffeomorphically conjugate to $z \mapsto z^{d_q}$, for some integer d_q.

Note that the sum of the branching degrees $d_{i,j}$ for $q_{i,j} \in f^{-1}(p_i)$ must satisfy $\sum_j d_{i,j} = d$.

Remark 23.5. Assuming that M and N are compact Riemann surfaces. Then any nonconstant holomorphic mapping $f : M \to N$ is a branched covering. This follows because the set of critical values must be finite; in local holomorphic coordinates, a critical point satisfies $\frac{\partial}{\partial z} f = 0$, since f is nonconstant, and this equation can only have finitely many zeroes. Away from the critical values, f must be a covering space, with the number of sheets given by the degree of f, which we know is the multiplicity of $f_* : H^2_{dR}(M) \to H^2_{dR}(N)$.

Theorem 23.6 (Riemann-Hurwitz formula)**.** *If $f : M \to N$ is a degree d branched covering, then*

$$\chi(M) = d \cdot \chi(N) - \sum_{p \in M} (d_p - 1), \tag{23.11}$$

where d_p is the local branching degree at p.

Proof. Consider a triangulation of N which has vertices at all of the critical values of f. This lifts to a triangulation of M which has d times the number of faces, and d times the number of edges. Then number of vertices is d times the number of vertices which are not at branching points. At a branching point, the number of vertices is reduced by d_p. □

23.3 Hypersurfaces in $\mathbb{CP}^n$

Let f_d be a homogeneous degree d polynomial in the variable $(z_0, \dots z_n)$, that is

$$f_d = \sum_{|I|=d} a_I z^I \tag{23.12}$$

where I is a multi-index of length $n+1$. Although f_d is not a well-defined function from $\mathbb{CP}^n$ to $\mathbb{C}$, the subset

$$V_f = \{p \in \mathbb{CP}^n : f_d(p) = 0\} \tag{23.13}$$

is a well-defined subset of $\mathbb{CP}^n$.

Also, for each $i = 0 \dots n$, $\frac{\partial}{\partial z^i} f_d$ is a homogeneous degree $d-1$ polynomial. We have the following

Proposition 23.7. *If f_d and $\frac{\partial}{\partial z^i} f_d$ for $i = 0 \dots n$ have no common zeroes, then V_f is a submanifold.*

Proof. This follows from the implicit function theorem, details are left as an exercise. □

We will now restrict to $\mathbb{CP}^2$, and consider the hypersurface defined by

$$f_d = z_0^d + z_1^d + z_2^d. \tag{23.14}$$

We claim this is a submanifold: we have

$$\frac{\partial}{\partial z^i} f_d = d z_i^{d-1}. \tag{23.15}$$

The set where $\frac{\partial}{\partial z^i} f_d = 0$ is the subset $\{[z_0, z_1, z_2] \in \mathbb{CP}^n \mid z_i = 0\}$. These 3 subsets have no common zero, therefore V_f is a smooth Riemann surface.

Proposition 23.8. *The surface V_{f_d} is compact orientable and has genus*

$$g = \frac{(d-1)(d-2)}{2}. \tag{23.16}$$

Proof. The subset V is closed, and since $\mathbb{CP}^2$ is compact, V is compact. We know V has a complex structure, and since holomorphic maps are orientation-preserving, V is orientable. To find the genus, let $\pi : \mathbb{CP}^2 \setminus [1,0,0] \to \mathbb{CP}^1$ by $[z_0, z_1, z_2] \mapsto [z_1, z_2]$. Since the point $[1,0,0]$ is not on $V = V_{f_d}$, the restriction of π to V gives a holomorphic mapping $\pi : V \to \mathbb{CP}^1$. Since f is a degree d polynomial, this mapping has degree d. The branch points are the subset

$$\{[z_1, z_2] \in \mathbb{CP}^1 \mid z_1^d + z_2^d = 0\} = \{[1, \zeta_d]\}, \tag{23.17}$$

where ζ_d is a dth root of -1. There are exactly d of these, so there are d branch points of order d. The Riemann-Hurwitz formula then gives

$$\chi(V) = d\chi(\mathbb{CP}^1) - d(d-1) = -d^2 + 3d. \tag{23.18}$$

But we know that $\chi(V) = 2 - 2g$, and solving for g yields (23.16). □

The first few values of this, starting with $d = 1$ are

$$0, 0, 1, 3, 5, 10, 15, 21... \tag{23.19}$$

The degree 1 case is a line, so is obviously $\mathbb{CP}^1$. The degree 2 case is also an S^2. The degree 3 case is a torus; this is called an elliptic curve.

Remark 23.9. Actually, the result holds for any smooth degree d hypersurface in $\mathbb{CP}^2$, not just the degree d Fermat hypersurface. To see this, one can show that any smooth degree d hypersurface can be connected to the degree d Fermat hypersurface through smooth hypersurfaces. This is because the subset of the projective space of coefficients of non-smooth hypersurfaces is a lower dimensional set which cannot disconnect. So we can connect any degree d hypersurface to the Fermat one by a curve of coefficients $\gamma(t)$. One can then use the implicit function theorem to show that all of the hypersurfaces in this family must be diffeomorphic (aka Ehresmann's Theorem).

24 Lecture 24

We will use the following lemma.

Lemma 24.1. *If $N^k \subset M^n$ is a compact submanifold, and n is even, then*

$$\chi(M) = \chi(N) + \chi(M \setminus N). \tag{24.1}$$

Proof. Given any decomposition of $M = U \cup V$ into the union of 2 open sets, we have the Mayer-Vietoris sequence,

$$\begin{array}{l} 0 \longrightarrow H^0(M) \longrightarrow H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(U \cap V) \\ \hookrightarrow H^1_{dR}(M) \longrightarrow H^1_{dR}(U) \oplus H^1_{dR}(V) \longrightarrow H^1_{dR}(U \cap V) \\ \hookrightarrow \cdots \qquad \cdots \qquad \cdots \\ \hookrightarrow H^{2n}_{dR}(M) \longrightarrow H^{2n}_{dR}(U) \oplus H^{2n}_{dR}(V) \longrightarrow H^{2n}_{dR}(U \cap V) \longrightarrow 0. \end{array} \tag{24.2}$$

Lemma 21.7 implies that

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V). \tag{24.3}$$

Now, we choose $U = M \setminus N$, $V = D_\epsilon(N)$ a tubular neighborhood of N, which deformation retracts onto N. Then $U \cap V$ is a punctured disc bundle over N, and deformation retracts onto $\partial V = S_\epsilon(N)$, which is a sphere bundle over N. Since n is assumed to be even, $S_\epsilon(N)$ is a smooth odd-dimensional manifold (for ϵ sufficiently small). If $S_\epsilon(N)$ is orientable, then by Corollary 17.7, $\chi(S_\epsilon(N)) = 0$. If $S_\epsilon(N)$ is non-orientable, then the orientable double cover $\tilde{S}$ satisfies $\chi(\tilde{S}) = 0$. Then by Corollary 22.6

$$0 = \chi(\tilde{S}) = 2\chi(S_\epsilon(N)). \tag{24.4}$$

So in either case we have

$$\chi(M) = \chi(M \setminus N) + \chi(N). \tag{24.5}$$

□

Remark 24.2. The Lemma is also true in odd dimensions, but we will need an extra tool to prove this.

24.1 Degree d hypersurface in $\mathbb{CP}^n$

We now consider $V_f \subset \mathbb{CP}^n$, where

$$f_d = \sum_{i=0}^{n} z_i^d. \tag{24.6}$$

We have $\frac{\partial f_d}{\partial z_i} = d z_i^{d-1}$. The zero set of this is

$$Z_i = \{[z_0, \dots z_n] \in \mathbb{CP}^n \mid z_i = 0\} \cong \mathbb{CP}^{n-1}. \tag{24.7}$$

Clearly $Z_0 \cap \dots \cap Z_n = \emptyset$, so by Proposition 23.7, V_f is a smooth submanifold. We will also call V_f as V_d^n.

Consider the projection

$$\pi : \mathbb{CP}^n \setminus [1, 0, \dots, 0] \to \mathbb{CP}^{n-1} \tag{24.8}$$

given by $\pi[z_0, \dots, z_n] = [z_1, \dots, z_n]$. Since $[1, 0, \dots, 0] \notin V_f$, we can consider

$$\pi : V_d^n \to \mathbb{CP}^{n-1} \tag{24.9}$$

Given $[z_1, \dots, z_n] \in \mathbb{CP}^{n-1}$, the equation

$$z_0^d = -\sum_{i=1}^n z_i^d \tag{24.10}$$

will have exactly d distinct root, unless

$$\sum_{i=1}^n z_i^d = 0, \tag{24.11}$$

in which case there is a single solution $z_0 = 0$.

We say that $\pi : V_d^n \to \mathbb{CP}^{n-1}$ is d-fold cover of $\mathbb{CP}^{n-1}$, brached along a degree d hypersurface $V_d^{n-1} \subset \mathbb{CP}^{n-1}$.

This means that

$$\pi : V_d^n \setminus \pi^{-1}(V_d^{n-1}) \to \mathbb{CP}^{n-1} \setminus V_d^{n-1} \tag{24.12}$$

is a d-fold covering space. By Corollary 22.6, we have

$$\chi\big(V_d^n \setminus \pi^{-1}(V_d^{n-1})\big) = d\big(\chi(\mathbb{CP}^{n-1} \setminus V_d^{n-1})\big) \tag{24.13}$$

Using Lemma 24.1, we have

$$\chi\big(V_d^n\big) - \chi\big(\pi^{-1}(V_d^{n-1})\big) = d\big(\chi(\mathbb{CP}^{n-1}) - \chi(V_d^{n-1})\big). \tag{24.14}$$

But $\pi : \pi^{-1}(V_d^{n-1}) \to V_d^{n-1}$ is a diffeomorphism, and using Theorem 23.1, we have the recursive formula

$$\chi(V_d^n) = nd + (1-d)\chi(V_d^{n-1}). \tag{24.15}$$

Note that V_d^1 consists of d distinct points in $\mathbb{CP}^1$, so

$$\chi(V_d^2) = 2d + (1-d)d = -d^2 + 3d = d(3-d) \tag{24.16}$$

which we derived last time. For $n = 3$, we have

$$\chi(V_d^3) = 3d + (1-d)d(3-d) = d(d^2 - 4d + 6). \tag{24.17}$$

24.2 The quadric hypersurface in $\mathbb{CP}^3$

We let $f = z_0^2 + z_1^2 + z_2^2 + z_3^3$. Then from (24.17) we know that $V_f = V_2^3$ satisfies $\chi(V_2^3) = 4$. Let's see what else we can say about this case.

Make the change of variables

$$z_0 = w_0 + iw_1,\ z_1 = iw_0 + w_1,\ z_2 = iw_2 - w_3,\ z_3 = -w_2 + iw_3, \tag{24.18}$$

Then V_2^3 is equivalent to the hypersurface in $\mathbb{CP}^2$ defined by

$$\{[w_0, w_1, w_2, w_3] \mid w_0 w_1 = w_2 w_3\}. \tag{24.19}$$

We define a mapping $\sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ by

$$\sigma([z_0, z_1], [z_2, z_3]) = [z_0 z_2, z_1 z_3, z_0 z_3, z_1 z_2] \tag{24.20}$$

Then obviously $\sigma(\mathbb{CP}^1 \times \mathbb{CP}^1) \subset V_2^3$.

Proposition 24.3. *The mapping σ is a diffeomorphism from $S^2 \times S^2$ to V_2^3.*

Proof. Consider the point $[1, 0, 0, 0] \in V_2^3$. The preimage under σ is defined by the equations

$$z_0 z_2 = 1,\ z_1 z_3 = 0,\ z_0 z_3 = 0,\ z_1 z_2 = 0 \tag{24.21}$$

Clearly this implies that $z_1 = 0$ and $z_3 = 0$ so there is a unique preimage point

$$\sigma^{-1}([1, 0, 0, 0]) = ([1, 0], [1, 0]). \tag{24.22}$$

Near this point, we can write the mapping as

$$([1, z], [1, w]) \mapsto [1, zw, w, z] \tag{24.23}$$

So locally, the mapping is from $\mathbb{C}^2 \to \mathbb{C}^3$

$$(z, w) \mapsto [zw, w, z], \tag{24.24}$$

This is a graph, so the differential is an isomorphism on the tangent space to the origin. Consequently, we have $\deg(\sigma) = 1$, so σ is surjective by Proposition 18.9. The action of the automorphism group is transitive, so the above calculation shows that the differential is an isomorphism at any point, so there are no critical points, and the degree 1 statement implies that every point has a single preimage since the mapping is holomorphic. $\square$

Corollary 24.4. *The de Rham cohomology groups of $S^2 \times S^2 = V_2^3$ are*

$$H^k(S^2 \times S^2) = \begin{cases} \mathbb{R} & k = 0, 4 \\ \mathbb{R}^2 & k = 2 \\ 0 & k = 1, 3 \end{cases}. \tag{24.25}$$

Proof. Since S^2 is simply connected, $S^2 \times S^2$ is also. To see this, let $\gamma : S^1 \to S^2 \times S^2$ be a closed loop. Then $\gamma_i = \pi_i \gamma : S^1 \to S^2$, and since S^2 is simply connected, there exists homotopies $H_i : [0,1] \times S^1 \to S^2$ such that $H_i(0,t) = \gamma_i$, and $H_i(1,t) = p_i$, where p_i are any 2 points in S^2. Then $H = (H_1, H_2) : [0,1] \times S^1 \to S^2 \times S^2$ is a homotopy between γ and a constant mapping.

We claim this implies that $H^1_{dR}(S^2 \times S^2) = 0$. To see this, given any $\alpha \in \Omega^1(S^2 \times S^2)$ with $d\alpha = 0$, fix any point $(p_0, q_0) \in S^2 \times S^2$ and define

$$f(p,q) = \int_\gamma \alpha = \int_{[0,1]} \gamma^* \alpha, \tag{24.26}$$

where γ is any smooth path from (p_0, q_0) to (p,q). This is well defined because if γ_1 and γ_2 are 2 such paths, then $(-\gamma_2) * \gamma_1$ is a closed path, and therefore bounds a 2-disc D. By Stokes' Theorem

$$\int_{(-\gamma_2)*\gamma_1} \alpha = \int_D d\alpha = 0. \tag{24.27}$$

We therefore have $b^1 = 0$. By Poincaré duality $b^3 = 0$. By the above computations, we know that $\chi(S^2 \times S^2) = 4$, which implies that $b_2 = 2$. □

Exercise 24.5. Improve the above argument to show that if $\pi_1(M)$ is finite, then $H^1_{dR}(M) = 0$.

Remark 24.6. This will also follow from the Kunneth formula.

It follows from the Lefschetz hyperplane theorem that V^3_d is simply connected. We will probably not have time to prove this, but this implies the following.

Proposition 24.7. *The de Rham cohomology of V^3_d is given by*

$$H^k(V^3_d) = \begin{cases} \mathbb{R} & k = 0,4 \\ \mathbb{R}^{d^3 - 4d^2 + 6d - 2} & k = 2 \\ 0 & k = 1,3 \end{cases}. \tag{24.28}$$

24.3 The non-orientable case

Later, we will show that $\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$. But today, we will show that $\mathbb{RP}^2$ can be embedded in $\mathbb{R}^4$. We define a mapping from $\mathbb{R}^3$ to $\mathbb{R}^6$ by

$$\phi : (x,y,z) \mapsto (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz) \tag{24.29}$$

When restricted to $S^2 \subset \mathbb{R}^3$, this mapping is invariant under the antipodal map, so we get a mapping

$$\phi : \mathbb{RP}^2 \to \mathbb{R}^6 \tag{24.30}$$

which is easily checked to be an embedding. The image of ϕ lies in the subset of $\mathbb{R}^6$:

$$V = \Big\{(x_1,\ldots,x_6) \in \mathbb{R}^6 \mid x_1 + x_2 + x_3 = 1, \sum_{i=1}^{6} x_i^2 = 1\Big\} \tag{24.31}$$

This is $S^5 \subset \mathbb{R}^6$ intersected with a hyperplane, so is diffeomorphic to S^4. So we have a mapping of

$$\phi : \mathbb{RP}^2 \to S^4. \tag{24.32}$$

Since $S^4 \setminus \{p\}$ is diffeomorphic to $\mathbb{R}^4$ under stereographic projection, we find the claimed embedding. This is called the Veronsese $\mathbb{RP}^2$ in S^4.

Note we also have that $\mathbb{RP}^2$ embeds into $\mathbb{CP}^2$ as the submanifold with all real coordinates. There is an involution of $\mathbb{CP}^2$ given by

$$C : [z_0, z_1, z_2] \mapsto [\overline{z}_0, \overline{z}_1, \overline{z}_2]. \tag{24.33}$$

The fixed point set of this involution is $\mathbb{RP}^2$. so we have that $\mathbb{CP}^2$ is a double cover of $\mathbb{CP}^2/\mathbb{Z}_2$, branched over an $\mathbb{RP}^2$. The above higher dimensional branched covering formula says that

$$\chi(\mathbb{CP}^2) = 2\chi(\mathbb{CP}^2/\mathbb{Z}_2) - (2-1)\chi(\mathbb{RP}^2), \tag{24.34}$$

which implies that $\chi(\mathbb{CP}^2/\mathbb{Z}_2) = 2$. It turns out that $\mathbb{CP}^2/\mathbb{Z}_2$ is diffeomorphic to S^4, and the branch locus is the above embedded $\mathbb{RP}^2$! We will leave this as a challenging exercise.

25 Lecture 25

25.1 Long exact sequence in cohomology with compact support

We will let $N \subset M$ be a compact submanifold of a smooth manifold M, which is not necessarily compact. Then $M \setminus N$ is a manifold, and we can consider

$$\Omega^p_c(M \setminus N) \xrightarrow{\ e\ } \Omega^p_c(M) \xrightarrow{\ i^*\ } \Omega^p(N), \tag{25.1}$$

where e is the extension map, and $i : N \to M$ is the inclusion mapping. We clearly have that $i^* \circ e = 0$, or $Im(e) \subset Ker(i^*)$. However, the opposite inclusion is not true. This is because in $\omega \in \Omega^p_c(M)$ restricts to zero on N, then it is not necessarily 0 in a neighborhood of N.

To fix this problem, we instead consider the germs of forms on N (similar to how we defined germs of functions in the beginning of the course). Let $\omega_i \in \Omega^p(U_i)$, where U_i is a neighborhood of N in M. We say that $\omega_1 \sim \omega_2$ if there is a neighborhood U_3 of N such that $\omega_1|_{U_3} = \omega_2|_{U_3}$. We call the set of such equivalence classes by $\mathcal{G}^p(N)$. We can extend $d : \mathcal{G}^p(N) \to \mathcal{G}^{p+1}(N)$ just by taking d of a representative form. If 2 forms agree in a neighborhood of N, then their exterior derivatives will also agree in the same neighborhood of N. The pull-back mapping $i^* : \Omega^p_c(M) \to \mathcal{G}^p(N)$ is defined just by pulling back a form to a neighborhood of U, and then projection to a germ.

Proposition 25.1. *The sequence of co-complexes*

$$0 \longrightarrow \Omega^p_c(M \setminus N) \xrightarrow{\ e\ } \Omega^p_c(M) \xrightarrow{\ i^*\ } \mathcal{G}^p(N) \longrightarrow 0 \tag{25.2}$$

is exact.

Proof. Obviously, e is injective. We have that $i^* \circ e = 0$. To see this, if $\omega \in \Omega^p_c(M \setminus N)$, then ω is zero in a neighborhood of U. Then obviously the germ of ω is zero. This implies that $Im(e) \subset Ker(i^*)$. For the reverse inclusion, let $\omega \in \Omega^p_c(M)$ such that $i^*\omega = 0$. This means that there is a neighborhood U of N such that $\omega|_U = 0$. Then $\omega' = \omega_{M \setminus N}$ has compact support, and $\omega = e\omega'$.

To finish, we need to show that the mapping i^* is surjective. Given a germ $[\omega] \in \mathcal{G}^p(N)$, it is represented by a form ω defined on a neighborhood U of N. Let $U' \subset U$ be a smaller neighborhood with $\overline{U'} \subset U$. Let χ be a cutoff function so that $\chi = 1$ on U' and 0 on $M \setminus U$. Then $\omega' = \chi\omega \in \Omega^p_c(M)$ and $i^*\omega' = \omega$. □

Lemma 25.2. *The cohomology of $H^k(\mathcal{G}^*(N)) \cong H^k_{dR}(N)$ for $k \geq 0$.*

Proof. First, we note there is a morphism of cochain complexes from $\mathcal{G}^*(N)$ to $\Omega^p(N)$ given by $\iota^* : \mathcal{G}^p(N) \to \Omega^p(N)$ such that the following diagram commutes for every p

$$\begin{array}{ccc} \mathcal{G}^p(N) & \xrightarrow{d_{\mathcal{G}}} & \mathcal{G}^{p+1}(N) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \Omega^p(N) & \xrightarrow{d_N} & \Omega^{p+1}(N), \end{array} \tag{25.3}$$

where $d_{\mathcal{G}}$ is the extension to germs of the exterior derivative, and ι^* is just restriction of any representative of a germ to N. This induces a mapping

$$\iota^* : H^p(\mathcal{G}^*(N)) \to H^p_{dR}(N), \tag{25.4}$$

which we claim is an isomorphism.

Let U be a tubular neighborhood of N, which we identify with a small disc bundle in the normal bundle of N, and we have $\pi : U \to N$. We define $\pi^* : \Omega^p(N) \to \mathcal{G}^p(N)$ given taking the germ of the pull-back under π. The following diagram commutes

$$\begin{array}{ccc} \mathcal{G}^p(N) & \xrightarrow{d_{\mathcal{G}}} & \mathcal{G}^{p+1}(N) \\ {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow \\ \Omega^p(N) & \xrightarrow{d_N} & \Omega^{p+1}(N), \end{array} \tag{25.5}$$

Recall that $\pi : U \to N$ is a homotopy equivalence which is moreover a retraction. Note that $\pi \circ \iota = Id_N$ implies that

$$\iota^* \circ \pi^* = Id_{H^p_{dR}(N)}. \tag{25.6}$$

Consequently, ι^* is surjective. For the other direction we have that $\iota \circ \pi$ is homotopic to Id_U. Let $H : U \times [0,1] \to U$ be a homotopy so that $H(p,1) = \iota \circ \pi$ and $H(p,0) = Id_U$. Recall the homotopy operator

$$I^k : \Omega^*(U \times [0,1]) \to \Omega^{k-1}(U) \tag{25.7}$$

which was given by integration over the fiber. From Proposition 10.12, this satisfies for $\omega \in \Omega^k(U \times [0,1])$),

$$(\iota_1)^*\omega - (\iota_0)^*\omega = d_U I^k \omega + I^{k+1} d_{U\times[0,1]}\omega. \tag{25.8}$$

where $\iota_t(p) = (p,t)$. We have $H \circ \iota_0 = \iota \circ \pi$, and $H \circ \iota_1 = Id_U$. So given $\omega \in \Omega^k(U)$, we can substitute $H^*\omega$ in (25.8) to get

$$(\iota \circ \pi)^*\omega - \omega = d_U I^k H^*\omega + I^{k+1} d_{U\times[0,1]} H^*\omega. \tag{25.9}$$

Consider the mapping $\pi^*\iota^* : \mathcal{G}^p(N) \to \mathcal{G}^p(N)$. Given $[\omega] \in \mathcal{G}^p(N)$ such that $d_{\mathcal{G}}[\omega] = 0$, then take a representative $\omega_U \in \Omega^p(N)$, such that $d\omega_U = 0$ for some open set U with $N \subset U$. There exists a sequence $U_1 \supset U_2 \supset \cdots$ of tubular neighborhoods of N so that the intersection is N. So without loss of generality, we can assume that U is a small tubular neighborhood as above. The above formula then shows that

$$\pi^* \circ \iota^*\omega_U = \omega_U + d_U I^k H^*\omega_U, \tag{25.10}$$

which implies that

$$\pi^* \circ \iota^*[\omega] = [\omega] \in H^p(\mathcal{G}^*(N)). \tag{25.11}$$

Consequently, ι^* is injective, and we are done.

□

From the proposition, the zig-zag lemma yields an exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{c,dR}(M \setminus N) \xrightarrow{e} H^p_{c,dR}(M) \xrightarrow{i^*} H^p_{dR}(N) \xrightarrow{\delta^p} H^{p+1}_{c,dR}(M \setminus N) \longrightarrow \cdots, \tag{25.12}$$

which we refer to as the long exact sequence in cohomology with compact support.

Example 25.3. Let $M = \mathbb{CP}^n$ and $N = \mathbb{CP}^{n-1}$. Then $M \setminus N \cong \mathbb{C}^n$, so this sequence gives a very easy inductive proof of the cohomology of complex projective space.

25.2 Generalized Jordan curve theorem

Theorem 25.4. *Let N be a compact connected proper submanifold of an connected orientable M^{n+1} satisfying $b^1(M) = 0$. Then the number of components of $N \setminus M$ is equal to* $\dim H^n(N) + 1$.

Proof. The above exact sequence yields

$$\begin{array}{ccccc} H^n_{c,dR}(M \setminus N) & \longrightarrow & H^n_{c,dR}(M) & \longrightarrow & H^n_{dR}(N) \\ \hookrightarrow H^{n+1}_{c,dR}(M \setminus N) & \longrightarrow & H^{n+1}_{c,dR}(M) & \longrightarrow & H^{n+1}_{dR}(N). \end{array} \tag{25.13}$$

We use Poincaré duality to get

$$H^n_{c,dR}(M) \cong (H^1_{dR}(M))^* \cong \{0\}, \tag{25.14}$$

by the assumption that $b^1(M) = 0$. Since M is orientable and connected, we know that

$$H^{n+1}_{c,dR}(M) \cong \mathbb{R}. \tag{25.15}$$

Since N is a proper submanifold, it has dimension less than or equal to n, so

$$H^{n+1}_{c,dR}(N) \cong \{0\}. \tag{25.16}$$

Therefore, we have the short exact sequence

$$0 \longrightarrow H^n_{dR}(N) \longrightarrow H^{n+1}_{c,dR}(M \setminus N) \longrightarrow \mathbb{R} \longrightarrow 0. \tag{25.17}$$

Using Poincaré duality again,

$$H^{n+1}_{c,dR}(M \setminus N) \cong H^0_{dR}(M \setminus N). \tag{25.18}$$

Finally, Lemma 21.7 yields that

$$\#\text{ of components of } M \setminus N = \dim(H^0_{dR}(M \setminus N)) = 1 + \dim(H^n_{dR}(N)). \tag{25.19}$$

□

Remark 25.5. We have some remarks:

- The example of $\{p\} \times S^1 \subset S^1 \times S^1$ shows that the assumption on $b^1(M) = 0$ is necessary.
- Also, the example of $S^1 \cong \mathbb{RP}^1 \subset \mathbb{RP}^2$ shows that the orientability of M is also necessary, since $\mathbb{RP}^2 \setminus \mathbb{RP}^{n-1}$ is a 2-disc, and is connected.
- However, consider $\mathbb{RP}^2 \subset \mathbb{RP}^3$. The assumptions of the theorem are satisfied, and the number of components of the complement is 1.

Corollary 25.6. *Let M and N be as above, If N is codimension 2 or higher, then $M \setminus N$ is connected.*

Proof. This is obvious since $H^n_{dR}(N)$ is necessarily zero. □

Remark 25.7. A few remarks:

- An example from last time was $\mathbb{RP}^2 \subset S^4$, the complement $S^4 \setminus \mathbb{RP}^2$ is connected. But also, we had $\pi : \mathbb{CP}^2 \to S^4$, since $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ must also be connected this implies that $\pi : \mathbb{CP}^2 \setminus \mathbb{RP}^2 \to S^4 \setminus \mathbb{RP}^2$ must be a nontrivial double cover. It turns out that $\pi_1(S^4 \setminus \mathbb{RP}^2) = \mathbb{Z}_2$ and this is the universal cover. To see this, it is not hard to show that $S^4 \setminus \mathbb{RP}^2$ is a rank 2 real vector bundle over $\mathbb{RP}^2$, and thus is homotopy equivalent to $\mathbb{RP}^2$, so $\pi_1(S^4 \setminus \mathbb{RP}^2) \cong \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.

- Also, for $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$, the complement is a disc, and is connected. This holds for $V_d \subset \mathbb{CP}^n$, the complement is connected (actually true for any subvariety).

Theorem 25.8. *If $M = \mathbb{R}^{n+1}$ and N is compact, codimension 1, and connected, then N must be orientable, and the number of components of $\mathbb{R}^{n+1} \setminus M$ is exactly 2.*

Proof. If N is orientable, then $H^n_{dR}(M) \cong \mathbb{R}$, so this follows from the above. We will next show that even if N is not orientable, then the number of components of $\mathbb{R}^{n+1} \setminus M$ is greater than or equal to 2, so the only possibility is that $H^n_{dR}(M) \cong \mathbb{R}$ and therefore M must be orientable. We will just give an outline of the proof.

Given $p \in \mathbb{R}^{n+1} \setminus N$, let S_p be a small sphere around p, and let $\pi_p : N \to S_p$ be the projection onto S_p, just using the obvious retraction of $\mathbb{R}^{n+1} \setminus \{p\}$ onto S_p.

Then we can define the *winding number* of N around p to be

$$\deg_2(\pi_p) = \begin{cases} 1 & \#\{f^{-1}(q)\} \text{ is odd} \\ 0 & \#\{f^{-1}(q)\} \text{ is even} \end{cases} \tag{25.20}$$

for any regular value $q \in S_p$.

Exercise 25.9. Prove the following properties of the mod 2 degree.

- Show this is well-defined, independent of the choice of regular value (hint: use that the boundary of any 1-manifold with boundary must always have an even number of points).

- Show this is constant on components of $\mathbb{R}^{n+1} \setminus N$ (prove homotopy invariance).

- If p_1 and p_2 are such that the straight line segment between p_1 and p_2 hits N transversally at a single point, then $\deg_2(\pi_{p_1}) = 1 - \deg_2(\pi_{p_2})$. (Hint: we can assume that N is locally just the hyperplane $\{(x_1, \ldots, x_n, 0)\}$, and p_1 and p_2 are the points $\{(0, \ldots, 0, \pm\epsilon)\}$.

The exercise implies that the number of component of $\mathbb{R}^{n+1} \setminus N$ is at least 2. Thus the only possiblity is that $\dim(H^n_{dR}(N)) = 1$ and therefore N is orientable. □

Corollary 25.10. *A compact non-orientable surface $k\#\mathbb{RP}^2$ cannot be embedded into $\mathbb{R}^3$.*

26 Lecture 26

26.1 Euler characteristic

Proposition 26.1. *Let $N \subset M^n$ be a compact embedded submanifold of an orientable smooth manifold M.*

$$\chi(M) = \chi(M \setminus N) + (-1)^n \chi(N) \tag{26.1}$$

Proof. We use the exact sequence of a pair for cohomology with compact supports:

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{c,dR}(M \setminus N) \xrightarrow{e} H^p_{c,dR}(M) \xrightarrow{i^*} H^p_{dR}(N) \xrightarrow{\delta^p} H^{p+1}_{c,dR}(M \setminus N) \longrightarrow \cdots, \tag{26.2}$$

Using Lemma 21.7, and Poincaré duality (since M is orientable), we have

$$(-1)^n \chi(M \setminus N) - (-1)^n \chi(M) + \chi(N) = 0. \tag{26.3}$$

□

Remark 26.2. If n is even, this of course agree with Proposition 24.1. In case n is odd, this says that

$$\chi(M) = \chi(M \setminus N) - \chi(N) \tag{26.4}$$

But the proof of Proposition 24.1 showed that

$$\chi(M) = \chi(M \setminus N) + \chi(N) - \chi(S_\epsilon(N)) \tag{26.5}$$

where $S_\epsilon(N)$ is the unit sphere bundle of the normal bundle of N. Combining these two results says that in this case we must have

$$\chi(S_\epsilon(N)) = 2\chi(N). \tag{26.6}$$

As an illustration of this, consider $S^5 \subset \mathbb{R}^6$ and $S^2 = S^5 \cap \{(x_1, x_2, x_3, 0, 0, 0)\}$. Then $\chi(S_\epsilon(N)) = 2 \cdot \chi(S^2) = 4$. So $S_\epsilon(N)$ is a 4-manifold with Euler characteristic 4. This is not surprising because the normal bundle is trivial, so $D_\epsilon(N) = S^2 \times D^3$, so $S_\epsilon(N) = \partial D_\epsilon(N) = S^2 \times S^2$.

26.2 Manifolds with boundary

Proposition 26.3. *If M is a manifold with boundary which has compact boundary, then there is an exact sequence*

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{c,dR}(M \setminus \partial M) \xrightarrow{e} H^p_{c,dR}(M) \xrightarrow{i^*} H^p_{dR}(\partial M) \xrightarrow{\delta^p} H^{p+1}_{c,dR}(M \setminus \partial M) \longrightarrow \cdots. \tag{26.7}$$

Proof. The proof is almost exactly the same as the proof of (25.12) above, with $N = \partial M$ using tubular neighborhoods of ∂M in M. □

Example 26.4. For example, this can be used to give another computation of $H^k_{c,dR}(\mathbb{R}^n)$ by applying to the pair $(M, \partial M) = (D^n, S^{n-1})$, where D^n is the closed disc.

We can also use this to compute the following.

Proposition 26.5. *Let $m, n \in \mathbb{Z}_+$, then*

$$H^k_{dR}(S^n \times S^m) = \begin{cases} \mathbb{R} & k = 0, m+n \\ \mathbb{R} & k = m \text{ or } n \text{ if } m \neq n \\ \mathbb{R}^2 & k = m \text{ if } m = n \\ 0 & \text{otherwise} \end{cases} \tag{26.8}$$

Proof. Let $M = D^{m+1} \times S^n$. Then $S^m \times S^n = \partial M$, and $M \setminus \partial M = B^{m+1} \times S^n$, where B^{m+1} is an open disc. By Poincaré duality, we have

$$H^p_{c,dR}(B^{m+1} \times S^n) \cong (H^{m+1+n-p}_{dR}(B^{m+1} \times S^n))^* \cong (H^{m+1+n-p}_{dR}(S^n))^*, \tag{26.9}$$

where the last isomorphism follows since de Rham cohomology is a homotopy invariant. This in nonzero only if $p = m + n + 1$ or $p = m + 1$.

Also M is compact, so $H^p_{c,dR}(M) = H^p_{dR}(M)$. But M is is a deformation retract of $B^{m+1} \times S^n$, so

$$H^p_{c,dR}(M) \cong H^p_{dR}(B^{m+1} \times S^n) \cong H^p_{dR}(S^n). \tag{26.10}$$

The result then follow from this and the exact sequence (26.7). □

Remark 26.6. It follows from this that $\chi(S^m \times S^n) = \chi(S^m) \times \chi(S^n)$. Actually, it is true in general that $\chi(M \times N) = \chi(M) \times \chi(N)$, this will follows from the Künneth formula.

26.3 Relative cohomology

We will assume that $N \subset M$ is a compact submanifold of a compact manifold M. There are two ways to define the *relative cohomology* $H^p_{dR}(M, N)$.

Definition 26.7 (Godbillon)**.** First, define $\Omega^p(M, N) \subset \Omega^p(M)$ of forms which vanish on N. Then obviously $d : \Omega^p(M, N) \to \Omega^{p+1}(M, N)$, and we can define

$$H^p_{dR}(M, N) = H^p(\Omega^*(M, N)). \tag{26.11}$$

There is a short exact sequence

$$0 \longrightarrow \Omega^p(M, N) \longrightarrow \Omega^p(M) \xrightarrow{\iota^*} \Omega^p(N) \longrightarrow 0. \tag{26.12}$$

The zig-zag lemma shows that there is a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(M, N) \longrightarrow H^p_{dR}(M) \xrightarrow{i^*} H^p_{dR}(N) \xrightarrow{\delta^p} H^{p+1}_{dR}(M, N) \longrightarrow \cdots, \tag{26.13}$$

Compare this with the long exact sequence,

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{c,dR}(M\setminus N) \xrightarrow{e} H^p_{dR}(M) \xrightarrow{i^*} H^p_{dR}(N) \xrightarrow{\delta^p} H^{p+1}_{c,dR}(M\setminus N) \longrightarrow \cdots. \tag{26.14}$$

The five lemma shows that we have an isomorphism

$$H^p_{dR}(M,N) \cong H^p_c(M\setminus N). \tag{26.15}$$

There is another definition of relative cohomology.

Definition 26.8 (BottTu)**.** Define $\Omega^p_{BT}(M,N) = \Omega^p(M)\oplus\Omega^{p-1}(N)$, and a differential $d : \Omega^p_{BT}(M,N) \to \Omega^{p+1}_{BT}(M,N)$ by

$$d(\omega,\theta) = (d\omega, \iota^*\omega - d\theta), \tag{26.16}$$

where $\iota : N \to M$ is the inclusion. Define $H^p_{BT}(M,N)$ to be the cohomology of this complex.

This is a short exact sequence of co-complexes (which commutes up to sign), so there is a short exact sequence

$$0 \longrightarrow \Omega^{p-1}(N) \xrightarrow{\alpha} \Omega^p_{BT}(M,N) \xrightarrow{\beta} \Omega^p(M) \longrightarrow 0, \tag{26.17}$$

where $\alpha(\theta) = (0,\theta)$ and $\beta(\omega,\theta) = \omega$. The zig-zag lemma shows that there is a long exact sequence

$$\cdots \xrightarrow{\alpha} H^p_{BT}(M,N) \xrightarrow{\beta} H^p_{dR}(M) \xrightarrow{\delta} H^p_{dR}(N) \xrightarrow{\alpha} H^{p+1}_{BT}(M,N) \longrightarrow \cdots. \tag{26.18}$$

Comparing with the above exact sequence (26.14), we again have an isomorphism

$$H^p_{BT}(M,N) \cong H^p_c(M\setminus N). \tag{26.19}$$

so both cohomologies are the same.

Exercise 26.9. Give a direct proof that $H^p_{dR}(M,N) \cong H^p_{BT}(M,N)$. Here is an outline, due to Johannes Ebert on mathoverflow. For either theory, consider the sequence of complexes

$$0 \longrightarrow \Omega^p(M,N) \xrightarrow{\alpha} \Omega^p(U,N)\oplus\Omega^p(M\setminus N) \xrightarrow{\beta} \Omega^p(U\setminus N) \longrightarrow 0, \tag{26.20}$$

where U is a tubular neighborhood of N. Define a morphism from Godbillon to Bott-Tu, $\Theta : \Omega^p(M,N) \to \Omega^p_{BT}(M,N)$ by $\omega \mapsto (\omega, 0)$. Prove that both theories agree if $N = \emptyset$. Also show that if $\iota : N \to M$ is a homotopy equivalence, then $H^p(M,N) = \{0\}$ in either theory. The result then follows from the five lemma.

26.4 Duality on manifolds with boundary

This subsection is really just a remark. We have the following.

Theorem 26.10 (Poincaré-Lefscetz duality)**.** *If M is compact orientable manifold with boundary, then we have*

$$H^p_{dR}(M, \partial M) \cong (H^{n-p}_{dR}(M))^*. \tag{26.21}$$

Proof. From the above, we have that

$$H^p_{dR}(M, \partial M) \cong H^p_{c,dR}(M \setminus \partial M). \tag{26.22}$$

Since M is orientable, we can use Poincaré duality on the interior of M to obtain

$$H^p_{c,dR}(M \setminus \partial M) \cong (H^{n-p}_{dR}(M \setminus \partial M))^*. \tag{26.23}$$

But $M \setminus \partial M$ is homotopy equivalent to M, so

$$H^{n-p}_{dR}(M \setminus \partial M) \cong H^{n-p}_{dR}(M), \tag{26.24}$$

and the result follows by combining the above isomorphisms. □

27 Lecture 27

27.1 Künneth formula

Let M and N be smooth manifolds. Let $\pi : M \times N \to M$ denote the projection onto the first factor, and $\rho : M \times N \to N$ be projection onto the second factor. There is a mapping from

$$K : \Omega^p(M) \times \Omega^q(N) \to \Omega^{p+q}(M \times N) \tag{27.1}$$

given by $K : (\omega, \phi) \mapsto \pi^*\omega \wedge \rho^*\phi$. Since K is bilinear, there is an induced mapping

$$K : \Omega^p(M) \otimes \Omega^q(N) \to \Omega^{p+q}(M \times N) \tag{27.2}$$

Note that if $d\omega = 0$ and $d\phi = 0$ then

$$\begin{aligned} K((\omega + d\alpha), \phi) &= \pi^*(\omega + d\alpha) \wedge \rho^*\phi \\ &= \pi^*\omega \wedge \rho^*\phi + d\pi^*\alpha \wedge \rho^*\phi \\ &= \pi^*\omega \wedge \rho^*\phi + d(\pi^*\alpha \wedge \rho^*\phi). \end{aligned} \tag{27.3}$$

Consequently, there is an induced mapping

$$K : H^p_{dR}(M) \otimes H^q_{dR}(N) \to H^{p+q}(M \times N). \tag{27.4}$$

By taking direct sums, we obtain a mapping

$$\psi : \bigoplus_{p+q=k} H^p(M) \otimes H^q(N) \to H^k(M \times N). \tag{27.5}$$

The next theorem says that Ψ is an isomorphism for each k.

Theorem 27.1 (Künneth formula)**.** *For any $k \in \mathbb{Z}, k \geq 0$, we have*

$$H^k_{dR}(M \times N) \cong \bigoplus_{p+q=k} H^p_{dR}(M) \otimes H^q_{dR}(N). \tag{27.6}$$

Proof. If $M = U \cup V$, then consider the Mayer-Vietoris sequence on M:

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{dR}(U \cup V) \xrightarrow{\beta^p} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{\alpha^p} H^p_{dR}(U \cap V) \xrightarrow{\delta^p} \cdots \tag{27.7}$$

In the category of vector spaces, tensor products preserve exact sequences, so we have an exact sequence

$$\cdots \longrightarrow H^p(U \cup V) \otimes H^{k-p}(N) \longrightarrow (H^p(U) \otimes H^{k-p}(N)) \oplus (H^p(V) \otimes H^{k-p}(N)) \longrightarrow H^p(U \cap V) \otimes H^{k-p}(N) \longrightarrow \cdots \tag{27.8}$$

Next, take the direct sum on p from 0 to k, and we have a long exact sequence. Consider the following diagram.

$$\begin{array}{ccccc} \oplus_{p=0}^k H^p(U \cup V) \otimes H^{k-p}(N) & \longrightarrow & \oplus_{p=0}^k (H^p(U) \otimes H^{k-p}(N)) \oplus (H^p(V) \otimes H^{k-p}(N)) & \longrightarrow & \oplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(N) \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\ H^k((U \cup V) \times N) & \longrightarrow & H^k(U \times N) \oplus H^k(V \times N) & \longrightarrow & H^k((U \cap V) \times N), \end{array} \tag{27.9}$$

where the lower row is the Mayer-Vietoris sequence with respect to the open cover $\{U \times N, V \times N\}$ of $M \times N$. This is straightforward to check commutativity. If we continue the diagram to the right, we see the following square

$$\begin{array}{ccc} \oplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(N) & \xrightarrow{\delta_1} & \oplus_{p=0}^k H^{p+1}(U \cup V) \otimes H^{k-p}(N) \\ \downarrow \psi & & \downarrow \psi \\ H^k((U \cap V) \times F) & \xrightarrow{\delta_2} & H^{k+1}((U \cup V) \times N). \end{array} \tag{27.10}$$

If $\omega \otimes \phi \in H^p(U \cap V) \otimes H^{k-p}(N)$. Then

$$\psi\delta_1(\omega \otimes \phi) = \psi((\delta\omega) \otimes \phi) = \pi^*(\delta\omega) \wedge \rho^*\phi \tag{27.11}$$
$$\delta_2\psi(\omega \otimes \phi) = \delta_2(\pi^*\omega \wedge \rho^*\phi). \tag{27.12}$$

Recall the definition of δ: if ρ_U, ρ_V is a partition of unity with respect to the covering $\{U, V\}$ of M, then

$$\delta(\omega) = \begin{cases} d(\rho_V \omega) & \text{in } U \\ -d(\rho_U \omega) & \text{in } V. \end{cases} \tag{27.13}$$

Note also that $\pi^*\rho_U, \pi^*\rho_V$ is a partition of unity with respect to the open covering $\{U \times N, V \times N\}$ of $M \times N$. Therefore

$$\delta_2(\gamma) = \begin{cases} d(\pi^*\rho_V \gamma) & \text{in } U \times N \\ -d(\pi^*\rho_U \gamma) & \text{in } V \times N. \end{cases} \tag{27.14}$$

On the set $U \times N$, we have

$$\begin{aligned}\psi\delta_1(\omega \otimes \phi) &= \pi^*(d(\rho_V \omega)) \wedge \rho^*\phi \\ &= d(\pi^*\rho_V) \wedge \pi^*\omega \wedge \rho^*\phi,\end{aligned} \tag{27.15}$$

and

$$\begin{aligned}\delta_2\psi(\omega \otimes \phi) &= \delta_2(\pi^*\omega \wedge \rho^*\phi) \\ &= d\big(\pi^*\rho_V(\pi^*\omega \wedge \rho^*\phi)\big) \\ &= d(\pi^*\rho_V) \wedge \pi^*\omega \wedge \rho^*\phi.\end{aligned} \tag{27.16}$$

If $M = \mathbb{R}^n$, this is the Poincaré Lemma, see Proposition 10.12. The result then follows by the five lemma and the usual argument of induction on the number of sets in a good cover of M. $\square$

Corollary 27.2. *Let*

$$T^n = \overbrace{S^1 \times \cdots \times S^1}^{n}, \tag{27.17}$$

then

$$\dim(H^k(T^n)) = \binom{n}{k} \tag{27.18}$$

Let $m, n \in \mathbb{Z}_+$*, then*

$$H^k_{dR}(S^n \times S^m) = \begin{cases} \mathbb{R} & k = 0, m+n \\ \mathbb{R} & k = m \textit{ or } n \textit{ if } m \neq n \\ \mathbb{R}^2 & k = m \textit{ if } m = n \\ 0 & \textit{otherwise} \end{cases} \tag{27.19}$$

Theorem 27.3 (Künneth formula for cohomology with compact support)**.** *Let M and N be orientable. Then for any $k \in \mathbb{Z}, k \geq 0$, we have*

$$H^k_{c,dR}(M \times N) \cong \bigoplus_{p+q=k} H^p_{c,dR}(M) \otimes H^q_{c,dR}(N). \tag{27.20}$$

Proof. If M and N are orientable, then $M \times N$ is orientable. The result then follows from the Künneth formula for ordinary de Rham cohomology, and Poincaré duality. $\square$

Exercise 27.4. Show the above result is true without any orientability assumption. (Hint: use the Mayer-Vietoris sequence for compactly supported cohomology, and imitate the above proof of Künneth for ordinary de Rham cohomology.)

27.2 The Thom isomorphism

This section is straight from Bott-Tu [BT82]

Theorem 27.5. *Let $\pi : E \to M$ be a rank n orientable vector bundle over a compact orientable manifold M^m. Then*

$$H^k_c(E) \cong H^{k-n}(M). \tag{27.21}$$

Proof. First, it is straightforward to see that the total space of an orientable vector bundle over an orientable manifold is an orientable manifold, just by taking a covering U_α of M which is an oriented atlas of M, and which has transition functions in $GL_+(n, \mathbb{R})$. Then

$$H^k_{c,dR}(E) \cong (H^{m+n-k}_{dR}(E))^* \cong (H^{m+n-k}_{dR}(M))^* \cong H^{k-n}_{dR}(M). \tag{27.22}$$

The first isomorphism is from Poincaré duality on E. The second is from homotopy invariance of de Rham cohomology. The third is from Poincaré duality on M. □

We would like to understand this isomorphism more explicitly, as well as generalize it to allow M to be noncompact. Consider the complex of forms $\Omega^*_{cv}(E)$ which have compact support in the fiber direction. Define "integration over the fiber"

$$\pi_* : \Omega^k_{cv}(E) \to \Omega^{k-n}(M) \tag{27.23}$$

as follows. Take an oriented local trivialization of E, $\Phi_\alpha : U_\alpha \times \mathbb{R}^n \to E$. Use coordinates $(x_1, \ldots, x_m)$ on U_α and $(t_1, \ldots, t_m)$ on $\mathbb{R}^n$. Then if $\omega \in \Omega^k_{cv}(U_\alpha \times \mathbb{R}^n)$, we have

$$\omega = \pi^* \phi f(x,t) dt^I, \tag{27.24}$$

where $|I| < n$ or

$$\omega = \pi^* \phi f(x,t) dt^1 \wedge \cdots \wedge dt^n. \tag{27.25}$$

We define $\pi_* \omega = 0$ in the first case and

$$\pi_* \omega = \Big(\int_{\mathbb{R}^n} f(x,t) dt \Big) \phi \tag{27.26}$$

in the second case.

Exercise 27.6. Show that this definition is independent of the local trivialization.

The exercise then implies that π_* is well-defined.

Proposition 27.7. *We have $\pi_* \circ d_E = d_M \circ \pi_*$.*

Proof. If ω is of the first type, then $\pi_* \omega = 0$. If $|I| < n - 1$, then the left hand side also vanishes. If $|I| = n - 1$, then for example consider the case that

$$\omega = \pi^* \phi f(x,t) dt_1 \wedge \cdots dt_{n-1}. \tag{27.27}$$

Then

$$\pi_* d\omega = \pm 1 \Big(\int \frac{\partial}{\partial t_n} f(x,t) dt \Big) \phi = 0 \tag{27.28}$$

since f has compact support in the t-direction. We leave the case of forms of the second type as an exercise. □

As a corollary, we see that π_* induces a mapping

$$\pi_* : H^k_{cv}(E) \to H^{k-n}_{dR}(M). \tag{27.29}$$

We will show that this mapping is an isomorphism.

Proposition 27.8. *Let $\pi : E \to M$ be as above, $\tau \in \Omega^k(M)$ and $\omega \in \Omega^l_{cv}(E)$. Then*

$$\pi_*(\pi^*\tau \wedge \omega) = \tau \wedge \pi_*\omega, \tag{27.30}$$

and if $k = m + n - l$, then

$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge \pi_*\omega. \tag{27.31}$$

Proof. If ω is of the first type, then it is easy to see both sides are zero. If ω is of the second type, then locally we have

$$\omega = \pi^*\phi f(x,t) dt^1 \wedge \cdots \wedge dt^n, \tag{27.32}$$

and

$$\begin{aligned} \pi_*(\pi^*\tau \wedge \omega) &= \pi_*\Big(\pi^*(\tau \wedge \phi) f(x,t) dt^1 \wedge \cdots \wedge dt^n\Big) \\ &= \tau \wedge \phi \Big(\int_{\mathbb{R}^n} f(x,t) dt \Big) \\ &= \tau \wedge \pi_*\omega. \end{aligned} \tag{27.33}$$

The second follows from this upon integration. □

Theorem 27.9 (Thom isomorphism)**.** *The mapping*

$$\pi_* : H^k_{cv}(E) \to H^{k-n}_{dR}(M). \tag{27.34}$$

is an isomorphism.

Proof. Let $\{U, V\}$ be a covering of M, and let ρ_U, ρ_V be a partition of unity subordinate to $\{U, V\}$. Then $\pi^*\rho_U, \pi^*\rho_V$ is a partition of unity subordinate to $\{\pi^{-1}(U), \pi^{-1}(V)\}$. We have an exact sequence

$$0 \longrightarrow \Omega^p_{cv}(\pi^{-1}(U \cup V)) \xrightarrow{\beta^p} \Omega^p_{cv}(\pi^{-1}(U)) \oplus \Omega^p(\pi^{-1}(V)) \xrightarrow{\alpha^p} \Omega^p_{cv}(\pi^{-1}(U \cap V)) \longrightarrow 0, \tag{27.35}$$

which yields an exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{cv}(\pi^{-1}(U \cup V)) \xrightarrow{\beta^p} H^p_{cv}(\pi^{-1}(U)) \oplus H^p_{cv}(\pi^{-1}(V)) \xrightarrow{\alpha^p} H^p_{cv}(\pi^{-1}(U \cap V)) \xrightarrow{\delta^p} \cdots. \tag{27.36}$$

Consider the diagram

$$\begin{array}{ccccc} H^p_{cv}(\pi^{-1}(U \cup V)) & \xrightarrow{\beta^p} & H^p_{cv}(\pi^{-1}(U)) \oplus H^p_{cv}(\pi^{-1}(V)) & \xrightarrow{\alpha^p} & H^p_{cv}(\pi^{-1}(U \cap V)) \\ \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ H^{p-n}_{dR}(U \cup V) & \xrightarrow{\beta^p} & H^{p-n}_{dR}(U) \oplus H^{p-n}_{dR}(V) & \xrightarrow{\alpha^p} & H^{p-n}_{dR}(U \cap V). \end{array} \tag{27.37}$$

This is easily verified to be commutative. It we extend the diagram to the right, we have the square

$$\begin{array}{ccc} H^p_{cv}(\pi^{-1}(U\cap V)) & \xrightarrow{\delta_1} & H^{p+1}_{cv}(\pi^{-1}(U\cup V)) \\ \downarrow{\pi_*} & & \downarrow{\pi_*} \\ H^{p-n}_{dR}(U\cap V) & \xrightarrow{\delta_2} & H^{p+1-n}_{dR}(U\cap V). \end{array} \tag{27.38}$$

Using Proposition 27.8, we have on $\pi^{-1}(U)$, for $\omega \in H^p_{cv}(\pi^{-1}(U\cap V)$,

$$\begin{aligned} \pi_*\delta_1\omega &= \pi_*(d((\pi^*\rho_V)\omega)) \\ &= \pi_*(\pi^*(d\rho_V)\wedge\omega) \\ &= (d\rho_V)\wedge\pi_*\omega = \delta_2\pi_*\omega. \end{aligned} \tag{27.39}$$

The same argument shows that the diagram is commutative if we extend to the left.

If $U \cong \mathbb{R}^m$, then we have that

$$\pi_* : H^k_{c,dR}(U\times\mathbb{R}^n) \to H^{k-n}_{dR}(U). \tag{27.40}$$

The proof is the same as the Poincaré Lemma for compact supports, see Theorem 14.2. The result then follows from the five lemma and induction on the number of open sets in a good cover of M. □

Remark 27.10. It is true that $H^p_{cv}(E) \cong H^p_{BT}(E, E_0)$, where E_0 is the complement of the zero-section, we will return to this later.

28 Lecture 28

28.1 Thom class

In the previous lecture, we showed that

$$\pi_* : H^k_{cv}(E) \to H^{k-n}_{dR}(M). \tag{28.1}$$

is an isomorphism for $k \geq n$.

Definition 28.1. The *Thom class* of an oriented vector bundle $\pi : E \to M$ is the class $\Phi = (\pi_*)^{-1}(1) \in H^n_{cv}(E)$ where $1 \in H^0_{dR}(M)$.

Note that by Proposition 27.8, we have

$$\pi_*(\pi^*\omega\wedge\Phi) = \omega\wedge\pi_*\Phi = \omega, \tag{28.2}$$

and since π_* is invertible, this implies that

$$\pi_*^{-1}(\omega) = \pi^*\omega\wedge\Phi. \tag{28.3}$$

That is, the inverse of π_* for any degree is given by wedging with the Thom class.

Proposition 28.2. *The Thom class of a rank n oriented vector bundle is the unique cohomology class in $H^n_{cv}(E)$ which restricts to a generator of $H^n_c(F) \cong \mathbb{R}$ for each fiber F.*

Proof. Since $\pi_* \Phi = 1$, we have that Φ_F is a compactly supported n-form with integral equal to 1. For the converse, assume that $\Phi' \in H^n_{cv}(E)$ is any class with this property. Then by Proposition 27.8, we have

$$\pi_*((\pi^*\omega) \wedge \Phi') = \omega \wedge \pi_* \Phi' = \omega, \tag{28.4}$$

so Φ' must be the Thom class since the inverse to π_* is unique. □

28.2 Poincaré dual of a submanifold

Let $\iota : \Sigma^k \hookrightarrow M^n$ be a compact oriented submanifold of a smooth oriented manifold M^n. We define a functional

$$F_c : H^k_c(M) \to \mathbb{R} \tag{28.5}$$

by

$$F_c(\omega) = \int_\Sigma i^*\omega. \tag{28.6}$$

By Stokes' Theorem, this is well-defined. By Poincaré duality, we have $H^{n-k}(M) \cong (H^k_c(M))^*$, with isomorphism given by $PD(\alpha)(\beta) \mapsto \int_M \beta \wedge \alpha$. Consequently, there is a cohomology class $\eta_\Sigma \in H^{n-k}(M)$ so that for all $[\omega] \in H^k_c(M)$,

$$\int_\Sigma i^*\omega = \int_M \omega \wedge \eta_\Sigma. \tag{28.7}$$

Note that since Σ is compact, we can also define

$$F : H^k(M) \to \mathbb{R} \tag{28.8}$$

by

$$F(\omega) = \int_\Sigma i^*\omega. \tag{28.9}$$

Again, by Poincaré duality, there is a cohomology class $\eta'_\Sigma \in H^{n-k}_c(M)$ such that for all $[\omega] \in H^k(M)$,

$$\int_\Sigma i^*\omega = \int_M \omega \wedge \eta'_\Sigma. \tag{28.10}$$

Definition 28.3. If $\Sigma^k \subset M^n$ is a closed oriented submanifold of an oriented manifold M^n, then η_Σ is called the closed Poincaré dual of Σ in M, and η'_Σ is called the the compact Poincaré dual of $\Sigma \in M$.

If M is also compact, then $\eta_\Sigma = \eta'_\Sigma$, but in general they could be different: just consider $\{0^n\} \subset \mathbb{R}^n$. The closed Poincaré dual lives in $H^n(\mathbb{R}^n)$, which is trivial. But the compact Poincaré dual lives in $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$. This has the property that for any $f \in C^\infty(\mathbb{R}^n)$ with $df = 0$,

$$f(0) = \int_{\mathbb{R}^n} f \eta'_{0^n}. \tag{28.11}$$

Of course, f must be constant so we can choose η'_{0^n} to be any n-form which has integral equal to 1. Note that we can choose the support to be in an arbitrarily small neighborhood of 0^n. However, in the following, we will just consider the closed Poincaré dual.

We have the exact sequence of vector bundles

$$0 \longrightarrow T_\Sigma \xrightarrow{\iota_*} TM|_\Sigma \longrightarrow N_\Sigma \longrightarrow 0, \tag{28.12}$$

where N_Σ is defined to be the quotient bundle $TM|_\Sigma / T_\Sigma$. Since every short exact sequence of vector spaces splits, we have the isomorphism

$$T_M|_\Sigma = N_\Sigma \oplus T_\Sigma, \tag{28.13}$$

and we give the normal bundle the orientation which makes the direct sum orientation agree with that of M.

As we mentioned before, there exists a tubular neighborhood U of Σ which is diffeomorphic to the normal bundle N_Σ. Let $\Phi_\Sigma \in H^{n-k}_{cv}(U)$ denote the Thom class of this bundle. Let $j : U \hookrightarrow M$ be the inclusion mapping. Then since Φ_Σ has compact support in the fiber direction, we can consider $j_*\Phi_\Sigma \in H^{n-k}(M)$.

Proposition 28.4. *The image of the Thom class of the normal bundle of Σ under j_* represents the Poincaré dual of Σ, that is*

$$j_*\Phi_\Sigma = \eta_\Sigma \in H^{n-k}(M). \tag{28.14}$$

Proof. Call the normal bundle projection $\pi : U \to \Sigma$, and let $i : \Sigma \hookrightarrow U$ denote the inclusion of Σ as the zero section. Then $\pi \circ i = Id_\Sigma$ and $i \circ \pi$ is homotopic to Id_U. Consequently, if $\omega \in \Omega^k_c(M)$, we have

$$\omega = \pi^* i^* \omega + d\tau, \tag{28.15}$$

where $\tau \in \Omega^{k-1}(U)$. Since $j_*\Phi$ is supported in U, we have

$$\int_M \omega \wedge j_*\Phi = \int_U \omega \wedge \Phi = \int_U (\pi^* i^* \omega + d\tau) \wedge \Phi = \int_U (\pi^* i^* \omega) \wedge \Phi. \tag{28.16}$$

The last equality holds using Stokes' theorem

$$\int_U d\tau \wedge \Phi = \int_U d(\tau \wedge \Phi) = \int_{\partial U} \tau \wedge \Phi = 0, \tag{28.17}$$

since Φ is closed and vanishes on ∂U. By the integral formula (27.31), we then have

$$\int_M \omega \wedge j_*\Phi = \int_\Sigma i^*\omega \wedge \pi_*\Phi = \int_\Sigma i^*\omega, \tag{28.18}$$

since $\pi_*\Phi = 1$ by the defining property of Φ. ∎

We also have the following.

Proposition 28.5. *The Thom class of an oriented vector bundle $\pi : E \to M$ over an oriented manifold M is the Poincaré dual of the zero section.*

Proof. Let $\iota : M \hookrightarrow E$ be the zero section. We have the exact sequence

$$0 \longrightarrow T_M \xrightarrow{\iota_*} TE|_{\iota(M)} \longrightarrow N_{\iota(M)} \longrightarrow 0. \tag{28.19}$$

But the normal bundle of $\iota(M)$ is clearly isomorphic to the bundle E, so the result follows from Proposition 28.4. □

28.3 Intersection Theory

Let Σ^k and Σ^{n-k} be compact oriented submanifolds of oriented M of complementary dimension, which intersect transversally. That is, if $p \in \Sigma^k \cap \Sigma^{n-k}$ then

$$TM|_p = T\Sigma^k|_p \oplus T\Sigma^{n-k}|_p. \tag{28.20}$$

This implies that

$$TM|_p = N_{\Sigma^k \cap \Sigma^{n-k}} = N\Sigma^k|_p \oplus N\Sigma^{n-k}|_p. \tag{28.21}$$

Corollary 28.6. *If $\eta_1 \in H^k(M)$ and $\eta_2 \in H^{n-k}(M)$. If η_1 is Poincaré dual to a compact submanifold Σ^{n-k} and η_2 is Poincaré dual to a compact submanifold Σ^k, and Σ^{n-k} and Σ^k intersect transversally, then*

$$PD(\eta_1)(\eta_2) = \int_M \eta_2 \wedge \eta_1 \tag{28.22}$$

is the oriented intersection number of points in $\Sigma^{n-k} \cap \Sigma^k$.

Proof. To see this, we can just take coordinates near p which flatten out both submanifolds. That is $\Sigma^{n-k} \subset \mathbb{R}^{n-k} \times 0^k$, and $\Sigma^k \subset 0^{n-k} \times \mathbb{R}^k$. Then use the defining property of the Thom classes and Proposition 28.4. □

Remark 28.7. This gives a geometric picture of the cohomology ring of $\mathbb{CP}^n$. Let $[\omega] \in H^2(\mathbb{CP}^2)$ be dual to a linear $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Since any linear $\mathbb{CP}^k$ intersects $\mathbb{CP}^{n-k}$ in a single point with positive intersection (since these are complex submanifolds), we see that $[\omega^k] \in H^{2k}(\mathbb{CP}^2)$ is the Poincaré dual of a linear subspace $\mathbb{CP}^{n-k} \subset \mathbb{CP}^n$.

28.4 Euler class

Let $\pi : E \to M$ be an oriented vector bundle of rank n over a compact oriented manifold M.

Definition 28.8. The Euler class $e(E) \in H^n(M)$ is $\sigma^*\Phi_E$, where $\sigma : M \to E$ is any section.

Since any 2 sections are homotopic, this is well-defined, independent of the choice of section. If the rank of the bundle $n = \dim(M)$, then we can define

$$\chi(E) = \int_M e(E). \tag{28.23}$$

Proposition 28.9. *If $\pi : E \to M$ is an oriented vector bundle of rank n over a compact oriented manifold M of dimension n, then $\chi(E)$ is the oriented intersection number of a section which is transverse to the zero section with the zero section.*

Proof. Let $\iota : M \hookrightarrow E$ be the zero section. Then

$$\begin{aligned}\int_M e(E) &= \int_M \iota^*\Phi = \int_M \iota^*\Phi \wedge \pi_*\Phi \\ &= \int_E \pi^*\iota^*\Phi \wedge \Phi = \int_E \Phi_E \wedge \Phi_E = \int_E \eta_{\iota(M)} \wedge \eta_{\iota(M)}.\end{aligned} \tag{28.24}$$

□

Corollary 28.10. *If there exists a non-zero section of $\sigma : M \to E$, then $\chi(E) = 0$.*

Now we restrict to $E = TM$.

Proposition 28.11. *Let M be compact and oriented. Then*

$$\int_M e(TM) = \chi(M). \tag{28.25}$$

Proof. Choose X to be a vector field which is transverse to the zero section. By the corollary, $\chi(TM)$ counts the signed number of zeroes of X. To finish, take a triangulation of M. Then there exists a vector field on M which has a single zero at the barycenter of each i-face, with positive intersection on the even faces, and negative intersection on the odd faces. We know from our previous work that this is the Euler characteristic. □

29 Lecture 29

29.1 Gysin sequence

We let $\pi_E : E \to M$ be a rank k oriented vector bundle over the smooth compact manifold M. Choose a Riemannian metric g on E, and consider

$$D_\epsilon(E) = \{v \in E \mid g(v,v) \leq \epsilon^2\} \tag{29.1}$$
$$B_\epsilon(E) = \{v \in E \mid g(v,v) < \epsilon^2\} \tag{29.2}$$
$$S_\epsilon(E) = \{v \in E \mid g(v,v) = \epsilon^2\}, \tag{29.3}$$

which are the closed disc bundle, the open disc bundle, and the sphere bundle, respectively. Note the $D_\epsilon(E)$ is a manifold with boundary $S_\epsilon(E)$, and with interior $B_\epsilon(E)$. Note that $\pi_S : S_\epsilon(E) \to M$ is a fiber bundle with fiber S^{k-1}.

Since M is compact, $D_\epsilon(E)$ is also compact, so the long exact sequence for manifolds with boundary is

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{c,dR}(B_\epsilon(E)) \xrightarrow{e} H^p_{dR}(D_\epsilon(E)) \xrightarrow{i^*} H^p_{dR}(S_\epsilon(E)) \xrightarrow{\delta^p} H^{p+1}_{c,dR}(B_\epsilon(E)) \longrightarrow \cdots . \tag{29.4}$$

Note that $B_\epsilon(E)$ is diffeomorphic to E, and E deformation retracts onto $D_\epsilon(E)$, so we obtain the following exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p_{c,dR}(E) \xrightarrow{e} H^p_{dR}(E) \xrightarrow{i^*} H^p_{dR}(S_\epsilon(E)) \xrightarrow{\delta^p} H^{p+1}_{c,dR}(E) \longrightarrow \cdots ,$$

where the inclusion mapping is now $i : S_\epsilon(E) \hookrightarrow E$. Now let us form the following big diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\delta^{p-1}} & H^p_{c,dR}(E) & \xrightarrow{e} & H^p_{dR}(E) & \xrightarrow{i^*} & H^p_{dR}(S_\epsilon(E)) & \xrightarrow{\delta^p} & H^{p+1}_{c,dR}(E) & \longrightarrow \cdots \\
& \downarrow \pi_* & & \downarrow \iota^* & & \downarrow id & & \downarrow \pi_* & \\
\cdots \xrightarrow{\gamma^{p-1}} & H^{p-k}_{dR}(M) & \xrightarrow{e_E\wedge\cdot} & H^p_{dR}(M) & \xrightarrow{\pi_S^*} & H^p_{dR}(S_\epsilon(E)) & \xrightarrow{\gamma^p} & H^{p+1-k}_{dR}(M) & \longrightarrow \cdots ,
\end{array} \tag{29.5}$$

where $\pi_* : \Omega^p_{cv}(E) \to \Omega^{p-k}(M)$ is integration on the fiber, $\iota : M \to E$ is the zero section. Furthemore, since the vertical maps are isomorphisms, the mapping γ^p is defined using this diagram.

We do not yet know that the lower row is exact. This will follow if we show that all of the squares commute, since all of the vertical mappings are isomorphisms.

For the first square, let us replace the vertical mappings by their inverses:

$$\begin{array}{ccc}
H^p_{c,dR}(E) & \xrightarrow{e} & H^p_{dR}(E) \\
\Phi_E\wedge\pi^*(\cdot) \uparrow & & \uparrow \pi_E^* \\
H^{p-k}_{dR}(M) & \xrightarrow{e_E\wedge\cdot} & H^p_{dR}(M).
\end{array} \tag{29.6}$$

Then if $\omega \in H^{p-k}(M)$, we have

$$e(\Phi_E \wedge \pi_E^*\omega) = e(\Phi_E) \wedge \pi_E^*\omega. \tag{29.7}$$

For the other mapping, we have

$$\pi_E^*(e_E \wedge \omega) = \pi_E^*(\iota^*\Phi_E \wedge \omega) = (\iota \circ \pi_E)^*\Phi_E \wedge \pi_E^*\omega = e(\Phi_E) \wedge \pi_E^*\omega, \tag{29.8}$$

because $\iota \circ \pi_E$ is homotopic to Id_E, so $(\iota \circ \pi_E)^*$ is the identity mapping on de Rham cohomology.

For the second square let us replace the first vertical mapping by its inverse:

$$\begin{array}{ccc}
H^p_{dR}(E) & \xrightarrow{i^*} & H^p_{dR}(S_\epsilon(E)) \\
\pi_E^* \uparrow & & \downarrow id \\
H^p_{dR}(M) & \xrightarrow{\pi_S^*} & H^p_{dR}(S_\epsilon(E)).
\end{array} \tag{29.9}$$

Then $\pi_E \circ i = \pi_S$, so $i^* \circ \pi_E^* = \pi_S^*$, and this is obviously commutative. Therefore, we have proved the following.

Theorem 29.1 (Gysin)**.** *If $\pi : E \to M$ is a rank k orientable vector bundle over a compact manifold M, then there is a long exact sequence*

$$\cdots \xrightarrow{\gamma^{p-1}} H^{p-k}_{dR}(M) \xrightarrow{e_E \wedge \cdot} H^p_{dR}(M) \xrightarrow{\pi^*_S} H^p_{dR}(S_\epsilon(E)) \xrightarrow{\gamma^p} H^{p+1-k}_{dR}(M) \longrightarrow \cdots, \quad (29.10)$$

where γ^p is "integration over the fiber".

This theorem can be used to compute the de Rham cohomology of the total spaces of sphere bundles. However, we will not do any examples right now since we need to move on to the next topic: singular homology.

29.2 Free abelian groups and tensor products

Definition 29.2. Given any set S, the free abelian group $\mathbb{Z}S$ is the abelian group consisting of all formal linear combinations

$$\sum_{s \in S} a_s \cdot s \quad (29.11)$$

where $a_s \in \mathbb{Z}$ and only finitely many coefficients are nonzero in the sum. The group operation is

$$\sum_{s \in S} a_s \cdot s + \sum_{s \in S} b_s \cdot s \equiv \sum_{s \in S} (a_s + b_s) \cdot s. \quad (29.12)$$

The free abelian group is characterized by the following universal property. If $f : S \to G$ is any mapping of S into an abelian group G, then there exists a unique group homomorphism $\tilde{f} : \mathbb{Z}S \to G$ such that $\tilde{f}(1 \cdot s) = f(s)$ for all $s \in S$.

Definition 29.3. If G and H are abelian group, then the tensor product $G \otimes_{\mathbb{Z}} H$ is the quotient of the free abelian group $\mathbb{Z}(G \times H)$ by the subgroup generated by the following elements:

$$(g_1 + g_2, h) - (g_1, h) - (g_2, h), \quad (29.13)$$

$$(g, h_1 + h_2) - (g, h_1) - (g, h_2). \quad (29.14)$$

Denote the natural projection by $\pi : G \times H \to G \otimes H$. We will write $\pi(g, h) = g \otimes h$. The tensor product is characterized by the following universal property. If C is any abelian group, and $f : G \times H \to C$ is a bilinear mapping, that is,

$$f(g_1 + g_2, h) = f(g_1, h) + f(g_2, h) \quad (29.15)$$

$$f(g, h_1 + h_2) = f(g, h_1) + f(g, h_2), \quad (29.16)$$

then there exists a unique group homomorphism $\tilde{f} : G \otimes_{\mathbb{Z}} H \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times H & \xrightarrow{f} & C \\ & \searrow^{\pi} & \uparrow \tilde{f} \\ & & G \otimes_{\mathbb{Z}} H. \end{array}$$

Exercise 29.4. For any group G, show that there is a natural isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$ (natural with respect to group homomorphisms).

29.3 Singular chains

Define the standard p-simplex to be

$$\Delta^p = \Big\{(t_0, \ldots, t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\Big\}. \tag{29.17}$$

The ith face of Δ^p is the $(p-1)$-simplex

$$\Delta_i^p : \Delta^{p-1} \to \Delta^p \tag{29.18}$$

defined by

$$(t_0, \ldots, t_{p-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots t_{p-1}). \tag{29.19}$$

For a topological space X, a continuous mapping

$$c : \Delta^p \to X. \tag{29.20}$$

is called a singular p-simplex.

Definition 29.5. The pth singular chain group $C_p(X; G)$ is the free abelian group generated by a singular p-simplices. Given any abelian group G, the pth singular chain group $C_p(X; G)$ with coefficients in G is $C_p(X; \mathbb{Z}) \otimes_{\mathbb{Z}} G$.

We can think of a singular p-chain $c \in C_p(X; \mathbb{Z})$ as a finite linear combination

$$c = \sum_{i=1}^{N} a_i c_i, \tag{29.21}$$

where $a_i \in \mathbb{Z}$ and c_i are singular p-simplices. Similarly, we can think of $c \in C_p(X; G)$ as a finite sum

$$c = \sum_{i=1}^{N} g_i c_i, \tag{29.22}$$

where $g_i \in G$, since $(a \cdot c) \otimes g = c \otimes (a \cdot g)$.

30 Lecture 30

30.1 The boundary operator and singular homology

Definition 30.1. Define the boundary operator

$$\partial : C_p(X; \mathbb{Z}) \to C_{p-1}(X; \mathbb{Z}) \tag{30.1}$$

by the following given a singular p-simplex $c : \Delta^p \to X$, let

$$\partial c = \sum_{i=0}^{p} (-1)^i c \circ \Delta_i^p, \tag{30.2}$$

and extend to all chains by linearity. For an abelian group G, define

$$\partial : C_p(X;G) \to C_{p-1}(X;G) \tag{30.3}$$

by $\partial(c_p \otimes g) = (\partial c_p) \otimes g$, and extend by linearity.

Proposition 30.2. *We have $\partial^2 = 0$.*

Proof. First, we claim that for all $0 \leq j < i \leq n+1$,

$$\Delta_i^n \circ \Delta_j^{n-1} = \Delta_j^n \circ \Delta_{i-1}^{n-1} \tag{30.4}$$

To see this, the left hand side of (30.4) is

$$\begin{aligned} \Delta_i^n \circ \Delta_j^{n-1}(t_0, \ldots, t_{n-2}) &= \Delta_i^n(t_0, \ldots, t_{j-1}, 0_j, t_j, \ldots, t_{n-2}) \\ &= (t_0, \ldots, t_{j-1}, 0_j, t_j, \ldots, t_{i-2}, 0_i, t_{i-1}, \ldots, t_{n-2}). \end{aligned} \tag{30.5}$$

The right hand side of (30.4) is

$$\begin{aligned} \Delta_j^n \circ \Delta_{i-1}^{n-1}(t_0, \ldots, t_{n-2}) &= \Delta_j^n(t_0, \ldots, t_{i-2}, 0_{i-1}, t_{i-1}, \ldots, t_{n-2}) \\ &= (t_0, \ldots, t_{j-1}, 0_j, t_j, \ldots, t_{i-2}, 0_i, t_{i-1}, \ldots, t_{n-2}). \end{aligned} \tag{30.6}$$

To prove the proposition, clearly we only need to consider the standard $(n+1)$-simplex. We then compute

$$\begin{aligned} \partial_n \circ \partial_{n+1}(\Delta^{n+1}) &= \partial_n\Big(\sum_{i=0}^{n+1}(-1)^i \Delta_i^{n+1}\Big) = \sum_{i=0}^{n+1}(-1)^i \partial_n\Big(\Delta_i^{n+1}\Big) = \\ &= \sum_{i=0}^{n+1}(-1)^i \sum_{j=0}^{n}(-1)^j \Delta_i^{n+1} \circ \Delta_j^n \\ &= \sum_{0 \leq j < i \leq n+1}^{n+1} (-1)^{i+j} \Delta_i^{n+1} \circ \Delta_j^n + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} \Delta_i^{n+1} \circ \Delta_j^n \equiv I + II. \end{aligned} \tag{30.7}$$

By (30.4), we have

$$I = \sum_{0 \leq j < i \leq n+1}^{n+1} (-1)^{i+j} \Delta_j^{n+1} \circ \Delta_{i-1}^n \tag{30.8}$$

Reindex the sum in II by letting $j' = j+1$, and we get

$$II = \sum_{0 \leq i < j' \leq n+1} (-1)^{i+j'-1} \Delta_i^{n+1} \circ \Delta_{j'-1}^n = - \sum_{0 \leq i < j' \leq n+1} (-1)^{i+j'} \Delta_i^{n+1} \circ \Delta_{j'-1}^n. \tag{30.9}$$

Then just by relabeling the indices as $i \to j, j' \to i$, we have

$$II = -\sum_{0 \le j < i \le n+1} (-1)^{i+j} \Delta_j^{n+1} \circ \Delta_{i-1}^n. \tag{30.10}$$

Consequently, by (30.8), we have $I + II = 0$. □

Since $\partial^2 = 0$, we have a *chain complex*

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1}(X;G) \xrightarrow{\partial_{p+1}} C_p(X;G) \xrightarrow{\partial_p} C_{p-1}(X;G) \xrightarrow{\partial_{p-1}} \cdots . \tag{30.11}$$

Define the pth singular homology group by

$$H_p(X;G) = \frac{Ker\{\partial_p : C_p(X;G) \to C_{p-1}(X;G)\}}{Im\{\partial_{p+1} : C_{p+1}(X;G) \to C_p(X;G)\}} \tag{30.12}$$

Remark 30.3. It turns out that the homology groups with $\mathbb{Z}$-coefficients determine the homology groups with any other coefficients G, which is a result known as the universal coefficient theorem for homology. Also, in the following we will denote $C_k(X) \equiv C_k(X;\mathbb{Z})$ and the homology groups $H_k(X) \equiv H_k(X;\mathbb{Z})$.

Proposition 30.4. *If X has path components X_i for $i \in \mathcal{I}$ then*

$$H_k(X) = \bigoplus_{i \in \mathcal{I}} H_k(X_i). \tag{30.13}$$

Proof. First, at the level of chains we have

$$C_k(X) = \bigoplus_{i \in \mathcal{I}} C_k(X_i) \tag{30.14}$$

since the Δ^k is path-connected and the image of Δ^k under a continuous mapping must lie in a path component of X. The boundary operator preserves this decomposition, so the kernels and images of the boundary operator are also direct sums. Therefore, the homology groups also decompose as a direct sum. □

Proposition 30.5. *The lowest homology group $H_0(X)$ is isomorphic to a free abelian group on the set of path components of X.*

Proof. From the previous proposition, we just need to show that a path-connected X satisfies $H_0(X) \cong \mathbb{Z}$. We have that $H_0(X) = C_0(X)/Im(\partial_1)$. A 0-chain is a finite sum of points $c_0 = \sum a_i p_i$. Fix any point $p \in X$, and let $\gamma_i : \Delta^1 \to X$ be paths from p to p_i, which we view as 1-simplices satisfying $\gamma_i(0,1) = p$ and $\gamma(1,0) = p_i$. Then $\partial \gamma_i = p_i - p$. Then

$$c_0 - \partial_1\Big(\sum_i a_i \gamma_i\Big) = \Big(\sum_i a_i\Big) p. \tag{30.15}$$

Consequently, then mapping $\epsilon : c_0 \mapsto \sum_i a_i$ yields an isomorphism $H_0(X) \cong \mathbb{Z}$. □

Example 30.6. The topologist's sine curve X is connected, but $H^0(X) \cong \mathbb{Z}^2$.

We can use the mapping in the above proof to define an "augmented" chain complex $\tilde{C}$

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0. \tag{30.16}$$

and we define the reduced homology $\tilde{H}_p(X)$ to be the homology of this complex.

Exercise 30.7. Show that $\tilde{H}_p(X) = H_p(X)$ for $p \geq 1$ and $H_p(X) \cong \tilde{H}_p(X) \oplus \mathbb{Z}$.

Lemma 30.8. *If $X = \{p\}$ is a one-point space, then*

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases} \tag{30.17}$$

Equivalently, $\tilde{H}_k(X) = 0$ for all $k \geq 0$.

Proof. Any continuous k-simplex is $c_k : \Delta^k \to X$ is a constant mapping, so $C_k(X) = \mathbb{Z}$ for all $k \geq 0$. Fix a generator, and call it c_k. Also, we have

$$\partial_1 c_1 = 0, \quad \partial_2 c_2 = \pm c_1, \quad \partial_3 = 0, \quad \partial_4 c_4 = \pm c_3, \ldots. \tag{30.18}$$

Therefore, the chain complex looks like

$$\cdots \xrightarrow{\partial_4 = \pm 1} \mathbb{Z} \xrightarrow{\partial_3 = 0} \mathbb{Z} \xrightarrow{\partial_2 = \pm 1} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z} \xrightarrow{\partial_0 = 0} 0, \tag{30.19}$$

and the claim follows. □

30.2 Chain complexes

A collection A_p of abelian groups for $p \in \mathbb{Z}$ and operators $\partial_p^A : A_p \to A_{p-1}$ for $p \geq 1$ satisfying $\partial_p^A \partial_{p+1}^A = 0$ is called a *chain complex*.

$$\cdots \xrightarrow{\partial_{p+2}^A} A_{p+1} \xrightarrow{\partial_{p+1}^A} A_p \xrightarrow{\partial_p^A} A_{p-1} \xrightarrow{\partial_{p-1}^A} \cdots. \tag{30.20}$$

Definition 30.9. The pth homology of a chain complex is the abelian group

$$H_p(A) = \frac{Ker\{\partial_p^A : A_p \to A_{p-1}\}}{Im\{\partial_{p+1}^A : A_{p+1} \to A_p\}} \tag{30.21}$$

Definition 30.10. A morphism $\alpha : A \to B$ of chain complexes is a collection of mappings $\alpha_p : A_p \to B_p$ such that $\partial_{p+1}^B \alpha_{p+1} = \alpha_p \partial_{p+1}^A$. In other words, $\alpha : A \to B$ is a morphism if the following diagram commutes

$$\begin{array}{ccc} A_{p+1} & \xrightarrow{\partial_{p+1}^A} & A_p \\ \downarrow{\scriptstyle \alpha_{p+1}} & & \downarrow{\scriptstyle \alpha_p} \\ B_{p+1} & \xrightarrow{\partial_{p+1}^B} & B_p. \end{array} \tag{30.22}$$

Proposition 30.11. *Morphisms satisfy the following properties:*

- *Composition of morphisms: If* $\alpha : A \to B$ *and* $\beta : B \to C$ *are morphisms of chain complexes, then* $\beta \circ \alpha : A \to C$ *is a morphism.*
- *Associativity: If* $\gamma : C \to D$ *is another morphism, then* $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$*.*

Proof. The diagram looks like

$$\begin{array}{ccc} A_{p+1} & \xrightarrow{\partial^A_{p+1}} & A_p \\ \downarrow{\scriptstyle \alpha_{p+1}} & & \downarrow{\scriptstyle \alpha_p} \\ B_{p+1} & \xrightarrow{\partial^B_{p+1}} & B_p. \\ \downarrow{\scriptstyle \beta_{p+1}} & & \downarrow{\scriptstyle \beta_p} \\ C_{p+1} & \xrightarrow{\partial^C_{p+1}} & C_p. \end{array} \tag{30.23}$$

We want to show that

$$\beta_p \circ \alpha_p \circ \partial^A_{p+1} = \partial^C_{p+1} \circ \beta_{p+1} \circ \alpha_{p+1} \tag{30.24}$$

Using commutativity of the top square, the left hand side of (30.24) is

$$\beta_p \circ \alpha_p \circ \partial^A_{p+1} = \beta_p \circ \partial^B_{p+1} \circ \alpha_{p+1}. \tag{30.25}$$

Using commutativity of the bottom square, the right hand side of (30.24) is

$$\beta_p \partial^B_{p+1} \circ \alpha_{p+1}, \tag{30.26}$$

which proves (30.24).

Associtivity is clear: $\gamma_p \circ (\beta_p \circ \alpha_p) = (\gamma_p \circ \beta_p) \circ \alpha_p$ holds for every $p \geq 0$ since composition of mappings is associative. □

Proposition 30.12. *A morphism of chain complexes* $\alpha : A \to B$ *induces mappings* $H_p\alpha : H_p(A) \to H_p(B)$*. Furthermore, if* $\beta : B \to C$ *is another morphism of chain complexes, then*

$$H_p(\beta \circ \alpha) = H_p\beta \circ H_p\alpha. \tag{30.27}$$

Proof. Given $[a_p] \in H_p(A)$ represented by $a_p \in A_p$ satisfying $\partial^A_p a_p = 0$, we have

$$\partial^B_p \alpha_p a_p = \alpha_{p-1} \partial^A_p a_p = 0, \tag{30.28}$$

therefore we can define $(H_p\alpha_p)[a_p] = [\alpha_p a_p]$. To check that this is well-defined,

$$[\alpha_p(a_p + \partial^A_{p+1} a_{p+1})] = [\alpha_p a_p + \alpha_p \partial^A_{p+1} a_{p+1}] = [\alpha_p a_p + \partial^B_{p+1} \alpha_{p+1} a_{p+1}] = [\alpha_p a_p]. \tag{30.29}$$

Next,for $[a_p] \in H_p(A)$ represented by $a_p \in A_p$, we have

$$H_p(\beta \circ \alpha)[a_p] = [(\beta_p \circ \alpha_p) a_p] = [\beta_p(\alpha_p(a_p))] = H_p\beta_p[\alpha_p(a_p)] = H_p\beta(H_p\alpha[a_p]). \tag{30.30}$$

□

30.3 Functorality of homology

We next consider the functorality of homology. If $f : X \to Y$ is a continuous mapping between topological spaces, then we can push forward chains by the following. For a simplex in X, $c : \Delta^p \to X$, we define $(f_*)_p c = f \circ c$, and extend to chains by linearity. This yields mappings

$$(f_*)_p : C_p(X;G) \to C_p(Y;G), \tag{30.31}$$

for $p = 0, 1, 2, \dots$. The following says that the collections of mappings $(f_*)_p$ are a *morphism* of chain complexes.

Proposition 30.13. *The following diagram*

$$\begin{array}{ccc} C_{p+1}(X;G) & \xrightarrow{\partial^X_{p+1}} & C_p(X;G) \\ \downarrow{\scriptstyle (f_*)_{p+1}} & & \downarrow{\scriptstyle (f_*)_p} \\ C_{p+1}(Y;G) & \xrightarrow{\partial^Y_{p+1}} & C_p(Y;G) \end{array} \tag{30.32}$$

commutes.

Proof. Consider a simplex $c : \Delta^{p+1} \to X$. By definition,

$$\partial^X_{p+1} c = \sum_{i=0}^{p+1} (-1)^i c \circ \Delta^{p+1}_i, \tag{30.33}$$

so

$$(f_*)_p \circ \partial^X_{p+1} c = \sum_{i=0}^{p+1} (-1)^i f \circ (c \circ \Delta^{p+1}_i). \tag{30.34}$$

On the other hand, for a simplex $c' : \Delta^{p+1} \to Y$, we have

$$\partial^Y_{p+1} c' = \sum_{i=0}^{p+1} (-1)^i c' \circ \Delta^{p+1}_i. \tag{30.35}$$

Letting $c' = (f_*)_{p+1} c = f \circ c$, we have

$$\partial^Y_{p+1} (f_*)_{p+1} c = \sum_{i=0}^{p+1} (-1)^i (f \circ c) \circ \Delta^{p+1}_i. \tag{30.36}$$

Since composition of mappings is associative, we are done. $\square$

Corollary 30.14. *If $f : X \to Y$ then there are induced mappings*

$$(f_*)_p : H_p(X;G) \to H_p(Y;G). \tag{30.37}$$

If $g : Y \to Z$, then

$$((g \circ f)_*)_p = (g_*)_p \circ (f_*)_p. \tag{30.38}$$

Consequently, if X and Y are homeomorphic, then $H_p(X;G) \cong H_p(Y;G)$ for every $p \geq 0$.

Proof. The first part follows from Proposition 30.13 and Proposition 30.12. If $f : X \to Y$ is a homeomorphism, there exists a continuous inverse $g : Y \to X$ such that

$$g \circ f = id_X, \quad f \circ g = id_Y. \tag{30.39}$$

Since the identity map obviously induces the identity map on homology, we have

$$(g_*)_p \circ (f_*)_p = id_{H_p(X;G)}, \quad (f_*)_p \circ (g_*)_p = id_{H_p(Y;G)}. \tag{30.40}$$

□

Definition 30.15. The topological chain functor is the functor from the category of topological spaces and continuous mappings to the category of chain complexes and morphisms of chain complexes mapping X to $\{C_p(X;G), \partial_p : C_p(X;G) \to C_{p-1}(X;G)\}$ and $f : X \to Y$ maps to $f_* : C_p(X;G) \to C_p(Y;G)$.

The topological pth homology functor is the functor from the category of topological spaces and continuous mapping to the category of abelian groups and homomorphisms given by $X \mapsto H_p(X;G)$ and $f : X \to Y$ maps to $H_p f = (f_*)_p : H_p(X;G) \to H_p(Y;G)$.

Proposition 30.16. *The above functors are covariant functors.*

Proof. This follows since $(g \circ f)_* = g_* \circ f_*$ on the level of chains, and the fact that the composition of covariant functors is a covariant functor. □

31 Lecture 31

31.1 Relative homology

If $A \subset X$ is any subset of the topological space X then $C_p(A) \subset C_p(X)$, and we define $C_p(X, A) = C_p(X)/C_p(A)$. The inclusion mapping $i : A \hookrightarrow X$ induces an injective mapping $i_* : C_p(A) \to C_p(X)$, so for each $p \geq 0$, we have a short exact sequence of abelian groups

$$0 \longrightarrow C_p(A) \xrightarrow{i_*} C_p(X) \longrightarrow C_p(X, A) \longrightarrow 0. \tag{31.1}$$

We note that the boundary operator ∂_X induces an operator

$$\partial_{X,A} : C_p(X, A) \to C_{p-1}(X, A) \tag{31.2}$$

since the boundary of a chain $c \in C_p(A)$ lies in $C_{p-1}(A)$. The above exact sequence is then a short exact sequence of chain complexes. It follows from the zig-zag lemma that there is a long exact sequence

$$\cdots \xrightarrow{\delta_{p+1}} H_p(A) \xrightarrow{H_p i} H_p(X) \longrightarrow H_p(X, A) \xrightarrow{\delta_p} H_{p-1}(A) \longrightarrow \cdots, \tag{31.3}$$

The connecting homomorphism is just $c \mapsto \partial c \in C_{p-1}(A)$.

Assume that $A \neq \emptyset$. Note we also have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(A) & \xrightarrow{i_*} & C_0(X) & \longrightarrow & C_0(X,A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & , \end{array} \tag{31.4}$$

which has exact rows and columns. Therefore, we also have a long exact sequence in relative homology for the reduced homology terminating in:

$$\cdots \longrightarrow H_1(X,A) \xrightarrow{\delta_1} \tilde{H}_0(A) \xrightarrow{H_p i} \tilde{H}_0(X) \longrightarrow H_0(X,A) \longrightarrow 0. \tag{31.5}$$

By taking $A = \{x_0\}$ where $x_0 \in X$ is any point, we see that

$$\tilde{H}_p(X) \cong H_p(X, x_0). \tag{31.6}$$

The relative homology also enjoys functorial properties.

Proposition 31.1. *If $f : X \to Y$ is continuous and $f(A) \subset B$. Then*

$$\begin{array}{ccc} C_{p+1}(X,A) & \xrightarrow{\partial_{p+1}} & C_p(X,A) \\ \downarrow{\scriptstyle (f_*)_{p+1}} & & \downarrow{\scriptstyle (f_*)_p} \\ C_{p+1}(Y,B) & \xrightarrow{\partial_{p+1}} & C_p(Y,B) \end{array} \tag{31.7}$$

commutes. Conseqently, there is an induced mapping

$$H_p f : H_p(X,A) \to H_p(Y,B). \tag{31.8}$$

Proof. We leave this as an easy exercise. □

Remark 31.2. If A is a "good" subspace, then $H_p(X,A) \cong \tilde{H}_p(X/A)$, for $p \geq 0$, where X/A is the quotient space obtained by identifying all point of A. For example, this holds if there is an open neighborhood of A which deformation retracts onto A. We will discuss this later. This makes the long exact sequence in homology very useful for computing homology groups.

Example 31.3. For example, if we take $X = D^n$, the closed unit disc, and $A = S^{n-1} = \partial D^n$, then we have

$$\cdots \xrightarrow{\delta_{p+1}} \tilde{H}_p(S^{n-1}) \xrightarrow{H_p i} \tilde{H}_p(D^n) \longrightarrow H_p(D^n, S^{n-1}) \xrightarrow{\delta_p} \tilde{H}_{p-1}(S^{n-1}) \longrightarrow \cdots, \tag{31.9}$$

Our next task is to show homotopy invariance of homology. Then since D^n is contractible, $\tilde{H}_p(D^n) = 0$ for all $p \geq 0$. Since D^n/S^{n-1} is homeomorphic to S^n, and S^{n-1} is a deformation retraction of an annulus, by Remark 31.2 and (31.9), we conclude that

$$\tilde{H}_{p-1}(S^{n-1}) \cong H_p(D^n, S^{n-1}) \cong \tilde{H}_p(S^n) \tag{31.10}$$

for all $p \geq 1$.

Since $S^0 = \{-1, 1\}$, by Proposition 30.4 and Lemma 30.8, we know that

$$H_k(S^0) \cong \begin{cases} \mathbb{Z}^2 & k = 0 \\ 0 & k > 0 \end{cases} \tag{31.11}$$

Also by Proposition 30.5 already know that $H_0(S^n) \cong \mathbb{Z}$ for $n > 0$.

An easy induction argument implies that

$$H_p(S^n) = \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & \text{otherwise} \end{cases} \tag{31.12}$$

for $n \geq 1$.

31.2 Exact sequences of coefficients

Consider a short exact sequence of abelian groups

$$0 \longrightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \longrightarrow 0. \tag{31.13}$$

This induces a short exact sequence of chains

$$0 \longrightarrow C_p(X) \otimes_{\mathbb{Z}} G_1 \xrightarrow{id \otimes f} C_p(X) \otimes_{\mathbb{Z}} G_2 \xrightarrow{id \otimes g} C_p(X) \otimes_{\mathbb{Z}} G_3 \longrightarrow 0. \tag{31.14}$$

By the zig-zag lemma, we have a long exact sequence

$$\cdots \longrightarrow H_p(X; G_1) \longrightarrow H_p(X; G_2) \longrightarrow H_p(X; G_3) \longrightarrow H_{p-1}(X; G_1) \longrightarrow \cdots, \tag{31.15}$$

Let $q > 1$ be an integer, and consider the short exact sequence,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/q\mathbb{Z} \longrightarrow 0. \tag{31.16}$$

The corresponding long exact sequence (31.17) shows that

$$\cdots \longrightarrow H_p(X) \xrightarrow{q} H_p(X) \longrightarrow H_p(X; \mathbb{Z}/q\mathbb{Z}) \longrightarrow H_{p-1}(X) \longrightarrow \cdots. \tag{31.17}$$

It follows from this that there is a short exact sequence

$$0 \longrightarrow \frac{H_p(X)}{q_*(H_p(X))} \longrightarrow H_p(X, \mathbb{Z}/q\mathbb{Z}) \longrightarrow Ker\{q_* : H_{p-1}(X) \to H_{p-1}(X)\} \longrightarrow 0. \tag{31.18}$$

For example, we have

$$H_p(S^n; \mathbb{Z}/q\mathbb{Z}) = \begin{cases} \mathbb{Z}/q\mathbb{Z} & p = 0, n \\ 0 & \text{otherwise} \end{cases} \tag{31.19}$$

As another example of this, consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{e^{2\pi i x}} S^1 \longrightarrow 0. \tag{31.20}$$

We then have a long exact sequence

$$\cdots \longrightarrow H_p(X;\mathbb{Z}) \xrightarrow{i_*} H_p(X;\mathbb{R}) \longrightarrow H_p(X;S^1) \longrightarrow H_{p-1}(X;\mathbb{Z}) \longrightarrow \cdots, \tag{31.21}$$

It follows that there is a short exact sequence

$$0 \longrightarrow \frac{H_p(X;\mathbb{R})}{i_*(H_p(X;\mathbb{Z}))} \longrightarrow H_p(X;S^1) \longrightarrow Ker\{i_* : H_{p-1}(X;\mathbb{Z}) \to H_{p-1}(X;\mathbb{R})\} \longrightarrow 0. \tag{31.22}$$

It turns out that $i_*(H_p(X,\mathbb{Z}))$ is always a lattice of full rank inside the vector space $H_p(X,\mathbb{R})$. Therefore the first group is a torus of dimension $b_p(X) = \dim(H_p(X,\mathbb{R}))$. However, the last group could have torsion, so the middle group is not necessarily a torus.

The above are just some special cases of the universal coefficient theorem for homology.

31.3 Chain homotopy between morphisms of chain complexes

Definition 31.4. Let $f : A \to B$, and $g : A \to B$ be two morphisms of chain complexes. We say that f is chain homotopic to g if there exists mappings $S_p : A_p \to B_{p+1}$ such that

$$f_p - g_p = \partial^B_{p+1} S_p + S_{p-1}\partial^A_p. \tag{31.23}$$

Proposition 31.5. *If f is chain homotopic to g then $H_p f = H_p g : H_p(A) \to H_p(B)$.*

Proof. Consider the mapping $H_p f - H_p g$, and take $[a_p] \in H_p(A)$ represented by $a_p \in A_p$ satisfying $\partial^A_p a_p = 0$. Then

$$\begin{aligned}(H_p f - H_p g)[a_p] &= (H_p(f-g)[a_p] = [(f_p - g_p))a_p] \\ &= [\partial^B_{p+1} S_P a_p + S_{p-1}\partial^A_p a_p] = [\partial^B_{p+1} S_P a_p] = 0.\end{aligned} \tag{31.24}$$

□

31.4 The prism operator

Given a topological space X, our goal is to define an operator

$$S_p : C_p(X) \to C_{p+1}(X \times [0,1]) \tag{31.25}$$

such that

$$(\iota_1)_* - (\iota_0)_* = \partial_{p+1} S_p + S_{p-1}\partial_p, \tag{31.26}$$

where $\iota_t : X \to X \times [0,1]$ is the inclusion $\iota_t(x) = (x,t)$.

In other words, S is a chain homotopy between the morphisms $(\iota_0)_*$ and $(\iota_1)_*$ from the singular chain complex on X and the singular chain complex on $X \times [0,1]$.

We only need to define S_p for for singular p-simplices, and extend to all chain by linearity.

31.4.1 $p = 0$

First, we discuss $p = 0$. Recall that

$$\Delta^0 = \{1\} \subset \mathbb{R} \tag{31.27}$$
$$\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1,\ t_i \geq 0, i = 0, 1\}. \tag{31.28}$$

Define

$$p_0^0 : \Delta^1 \to \Delta^0 \times [0, 1] \tag{31.29}$$

by

$$p_0^0((t_0, t_1)) = (t_0 + t_1, t_1) = (1, t_1). \tag{31.30}$$

A 0-simplex is a mapping $c_0 : \Delta^0 \to X$. Consider the mapping

$$c_0 \times id : \Delta^0 \times [0, 1] \to X \times [0, 1] \tag{31.31}$$

given by $(c_0 \times id)(x, s) = (c(x), s)$. Define $S_0 c_0 = c_1$, where $c_1 : \Delta^1 \to X$ is $c_1 = (c_0 \times id) \circ p_0^0$. That is,

$$(S_0 c_0)(t_0, t_1) = (c_0 \times id)(1, t_1) = (c_0(e_0), t_1). \tag{31.32}$$

We need to verify that

$$(\iota_1)_* c_0 - (\iota_0)_* c_0 = \partial_1 S_0 c_0. \tag{31.33}$$

For $e_0 \in \Delta^0$, the left hand side evaluated on e_0 is

$$(\iota_1)_* c_0(e_0) - (\iota_0)_* c_0(e_0) = \iota_1(c_0(e_0)) - \iota_0(c_0(e_0)) = (c_0(e_0), 1) - (c_0(e_0), 0). \tag{31.34}$$

The right hand side is

$$\begin{aligned} \partial_1 S_0 c_0 = \partial_1 c_1 &= \partial_1((c_0 \times id)_* \circ p_0^0) \\ &= (c_0 \times id)_* \partial_1 p_0^0 = (c_0 \times id)_*(p_0^0 \circ \Delta_0^1 - p_0^0 \circ \Delta_1^1). \end{aligned} \tag{31.35}$$

We have

$$p_0^0 \circ \Delta_0^1(1) = p_0^0(0, 1) = (1, 1) \tag{31.36}$$

and

$$p_0^0 \circ \Delta_1^1(1) = p_0^0(1, 0) = (1, 0). \tag{31.37}$$

So we have

$$\partial_1 S_0 c_0(e_0) = (c_0 \times id)_*(1, 1) - (c_0 \times id)_*(1, 0) = (c_0(e_0), 1) - (c_0(e_0), 0), \tag{31.38}$$

and we are done with $p = 0$.

31.4.2 $p = 1$

We next consider $p = 1$. Let c_1 be a 1-simplex $c_1 : \Delta^1 \to X$. We let $e_0 = (1, 0)$, and $e_1 = (0, 1)$ be the unit basis vectors in $\mathbb{R}^2$. Recall that

$$\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 \mid t_0 + t_1 + t_2 = 1,\ t_i \geq 0, i = 0, 1, 2\}. \tag{31.39}$$

We will divide $\Delta^1 \times [0, 1]$ into 2 2-simplices, which is geometrically cutting a square into two triangles. For $i = 0, 1$, define

$$p_i^1 : \Delta^2 \to \Delta^1 \times [0, 1] \tag{31.40}$$

by

$$p_0^1 : (t_0, t_1, t_2) \mapsto (t_0 + t_1, t_2, t_1 + t_2) \tag{31.41}$$
$$p_1^1 : (t_0, t_1, t_2) \mapsto (t_0, t_1 + t_2, t_2). \tag{31.42}$$

Note that p_0^1 maps the following

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_0, 1), \quad e_2 \mapsto (e_1, 1), \tag{31.43}$$

so the image of p_0^1 is the "upper" triangle of the square. The mapping p_1^1 maps

$$e_0 \mapsto (e_0, 0), \quad e_1 \mapsto (e_1, 0), \quad e_2 \mapsto (e_1, 1) \tag{31.44}$$

so the image of p_1^1 is the "lower" triangle of the square.
Define

$$S_1(c_1) = (c_1 \times id) \circ p_0^1 - (c_1 \times id) \circ p_1^1. \tag{31.45}$$

We want to verify the formula

$$(\iota_1)_* c_1 - (\iota_0)_* c_1 = \partial_2 S_1 c_1 + S_0 \partial_1 c_1 \tag{31.46}$$

The left hand side is

$$\iota_1 \circ c_1 - \iota_0 \circ c_1 = (c_1, 1) - (c_1, 0). \tag{31.47}$$

Next, we have

$$\begin{aligned}
\partial_2 S_1 c_1 &= \partial_2((c_1 \times id) \circ p_0^1) - \partial_2((c_1 \times id) \circ p_1^1) \\
&= \partial_2((c_1 \times id)_* p_0^1) - \partial_2((c_1 \times id)_* p_1^1) \\
&= (c_1 \times id)_*(\partial_2 p_0^1 - \partial_2 p_1^1) \\
&= (c_1 \times id)_* \Big(\sum_{j=0}^{2} (-1)^j p_0^1 \circ \Delta_j^2 - \sum_{j=0}^{2} (-1)^j p_1^1 \circ \Delta_j^2 \Big) \\
&= (c_1 \times id)_*(p_0^1 \circ \Delta_0^2 - p_0^1 \circ \Delta_1^2 + p_0^1 \circ \Delta_2^2 - p_1^1 \circ \Delta_0^2 + p_1^1 \circ \Delta_1^2 - p_1^1 \circ \Delta_2^2).
\end{aligned} \tag{31.48}$$

We next compute terms inside the parenthesis. First,

$$p_0^1 \circ \Delta_0^2(t_0, t_1) = p_0^1(0, t_0, t_1) = (t_0, t_1, t_0 + t_1) \tag{31.49}$$
$$p_0^1 \circ \Delta_1^2(t_0, t_1) = p_0^1(t_0, 0, t_1) = (t_0, t_1, t_1) \tag{31.50}$$
$$p_0^1 \circ \Delta_2^2(t_0, t_1) = p_0^1(t_0, t_1, 0) = (t_0 + t_1, 0, t_1). \tag{31.51}$$

Next,

$$p_1^1 \circ \Delta_0^2(t_0, t_1) = p_0^1(0, t_0, t_1) = (0, t_0 + t_1, t_1) \tag{31.52}$$
$$p_1^1 \circ \Delta_1^2(t_0, t_1) = p_0^1(t_0, 0, t_1) = (t_0, t_1, t_1) \tag{31.53}$$
$$p_1^1 \circ \Delta_2^2(t_0, t_1) = p_0^1(t_0, t_1, 0) = (t_0, t_1, 0). \tag{31.54}$$

So we have (with a slight abuse of notation)

$$\begin{aligned}\partial_2 S_1 c_1 = (c_1 \times id)_* \Big(&(t_0, t_1, t_0 + t_1) - (t_0, t_1, t_1) + (t_0 + t_1, 0, t_1) \\ &- (0, t_0 + t_1, t_1) + (t_0, t_1, t_1) - (t_0, t_1, 0) \Big) \\ = &(c_1, 1) + (c_1(e_0), t_1) - (c_1(e_1), t_1) - (c_1, 0).\end{aligned} \tag{31.55}$$

The term $S_0 \partial_1 c_1$ is

$$\begin{aligned}S_0 \partial_1 c_1 &= S_0(c_1 \circ \Delta_0^1 - c_1 \circ \Delta_1^1) \\ &= S_0(c_1(e_1) - c_1(e_0)) = (c_1(e_1), t_1) - (c_1(e_0), t_1).\end{aligned} \tag{31.56}$$

So summing everything up gives

$$\begin{aligned}\partial_2 S_1 c_1 + S_0 \partial_1 c_1 &= (c_1, 1) + (c(e_0), t_1) - (c(e_1), t_1) - (c_1, 0) + (c_1(e_1), t_1) - (c_1(e_0), t_1) \\ &= (c_1, 1) - (c_1, 0) = \iota_1 \circ c_1 - \iota_0 \circ c_1.\end{aligned} \tag{31.57}$$

Next time we will do the general case.

32 Lecture 32

32.1 Homotopy invariance of homology

Definition 32.1. Let X and Y be topological spaces. Continuous mappings $f, g : X \to Y$ are said to be homotopic if there exists a continuous mapping $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Proposition 32.2. *If $f, g : X \to Y$ are homotopic then*

$$H_k f = H_k g : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}) \tag{32.1}$$

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy between f and g. Let $\iota_t : X \to X \times [0,1]$ be the mapping $\iota_t(x) = (x,t)$. In the previous lecture, we constructed mappings

$$S_k : C_k(X,\mathbb{Z}) \to C_{k+1}(X \times [0,1],\mathbb{Z}) \tag{32.2}$$

such that

$$(\iota_1)_* - (\iota_0)_* = \partial_{k+1} S_k + S_{k-1}\partial_k, \tag{32.3}$$

for $k = 0, 1$. Below, we will construct this operator in the general case, but for now, we assume such an operator exists. Equation (32.3) implies that

$$H_k\iota_0 = H_k\iota_1 : H_k(X,\mathbb{Z}) \to H_k(X \times [0,1],\mathbb{Z}). \tag{32.4}$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, and H_k is a covariant functor, we have

$$H_k f = H_k F \circ H_k\iota_0, \quad H_k g = H_k F \circ H_k\iota_1, \tag{32.5}$$

therefore $H_k f = H_k g : H_k(X,\mathbb{Z}) \to H_k(Y,\mathbb{Z})$. □

Corollary 32.3. *The homology groups of* $\mathbb{R}^n$ *are given by*

$$H_k(\mathbb{R}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases}. \tag{32.6}$$

Proof. Define $f : \mathbb{R}^n \to \mathbb{R}^0$ by $f(x) = \{0\}$ and $g : \mathbb{R}^0 \to \mathbb{R}^n$ by $f(\{0\}) = \{0^n\}$. Then $f \circ g = Id_{\mathbb{R}^0}$, so we have

$$H_k f \circ H_k g = Id_{H^k(\mathbb{R}^0)}. \tag{32.7}$$

Also, $g \circ f(x) = 0^n$. The mapping $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ defined by

$$F(x,t) = tx. \tag{32.8}$$

is a homotopy from $g \circ f$ to the identity map $Id : \mathbb{R}^n \to \mathbb{R}^n$. Proposition 32.2 says that

$$H_k g \circ H_k f = Id_{H^k(\mathbb{R}^n)} \tag{32.9}$$

Consequently, $H_k(\mathbb{R}^n;\mathbb{Z}) \cong H_k(\mathbb{R}^0;\mathbb{Z})$. The latter space was determined above in Lemma 30.8, so we are done. □

32.2 Homotopy type

Definition 32.4. Topological spaces X and Y have the same homotopy type if there exist continuous mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.

Corollary 32.5. *If* X *and* Y *have the same homotopy type, then* $H_*(X,\mathbb{Z}) \cong H_*(Y,\mathbb{Z})$.

Proof. From Proposition 32.2, we have

$$H_*g \circ H_*f = Id_{H_*(X,\mathbb{R})} \tag{32.10}$$
$$H_*f \circ H_*g = Id_{H_*(Y,\mathbb{R})}, \tag{32.11}$$

so H_*f and H_*g are isomorphisms. □

Some special cases of this are the following.

Definition 32.6. A space X is contractible if X has the same homotopy type as a point.

Corollary 32.7. *If X is contractible, then*

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases}. \tag{32.12}$$

Definition 32.8. A subset $i : A \hookrightarrow X$ is a deformation retraction of X if there exists a mapping $r : X \to X$ such that

$$r \circ i = id_A, \tag{32.13}$$

and $i \circ r$ is homotopic to Id_X.

Corollary 32.9. *If A is a deformation retraction of X then $H_k(A;\mathbb{Z}) \cong H_k(X;\mathbb{Z})$ for all $k \geq 0$.*

Example 32.10. Consider $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \subset \mathbb{R}^n$ given by $r(x) = x/|x|$. The mapping $F(x,t) = (1-t)x + t(x/|x|)$ is a smooth homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, so S^{n-1} is a smooth deformation retraction of $\mathbb{R}^n \setminus \{0\}$ and we therefore have

$$H_k(S^{n-1};\mathbb{Z}) \cong H_k(\mathbb{R}^n \setminus \{0\};\mathbb{Z}). \tag{32.14}$$

Corollary 32.11 (Invariance of dimension)**.** *If $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.*

Proof. Assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism, then $f : \mathbb{R}^m \setminus \{0^m\} \to \mathbb{R}^n \setminus \{f(0^m)\}$ is also a homeomorphism. From Example 32.10, we obtain that

$$H^k(S^{m-1};\mathbb{Z}) \cong H^k(\mathbb{R}^m \setminus \{0^m\};\mathbb{Z}) \cong H^k(\mathbb{R}^n \setminus \{f(0^m)\};\mathbb{Z}) \cong H^k(S^{n-1};\mathbb{Z}). \tag{32.15}$$

Example 31.3 then implies that $m = n$. (But recall we still need to prove the claim in Remark 31.2.) □

32.3 Prism operator: general case

We will divide $\Delta^n \times [0,1]$ into $(n+1)$ $(n+1)$-simplices. For $i = 0, \ldots, n$, define

$$p_i^n : \Delta^{n+1} \to \Delta^n \times [0,1] \tag{32.16}$$

by

$$(t_0, \dots, t_{n+1}) \mapsto ((t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}), t_{i+1} + \cdots + t_{n+1}) \tag{32.17}$$

We will view $\Delta^n \times [0,1] \subset \mathbb{R}^{n+2}$, so will henceforth omit the inner parenthesis.

Define $S_n : C_n(X;\mathbb{Z}) \to C_{n+1}(X \times [0,1];\mathbb{Z})$ by

$$S_n(c_n) = \sum_{i=0}^{n} (-1)^i (c_n \times id) \circ p_i^n, \tag{32.18}$$

and extend to all chains by linearity.

We want to verify the formula

$$(\iota_1)_* c_n - (\iota_0)_* c_n = \partial_{n+1} S_n c_n + S_{n-1} \partial_n c_n \tag{32.19}$$

The left hand side of (32.19) is

$$\iota_1 \circ c_n - \iota_0 \circ c_n = (c_n, 1) - (c_n, 0). \tag{32.20}$$

The first term on the right hand side of (32.19) is

$$\begin{aligned}
\partial_{n+1} S_n c_n &= \partial_{n+1} \Big(\sum_{i=0}^{n} (-1)^i (c_n \times id) \circ p_i^n \Big) \\
&= \sum_{i=0}^{n} (-1)^i \partial_{n+1} \Big((c_n \times id)_* p_i^n \Big) \\
&= (c_n \times id)_* \sum_{i=0}^{n} (-1)^i \partial_{n+1} (p_i^n) \\
&= (c_n \times id)_* \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n+1} (-1)^j p_i^n \circ \Delta_j^{n+1} \\
&= (c_n \times id)_* \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} p_i^n \circ \Delta_j^{n+1}.
\end{aligned} \tag{32.21}$$

The second term on the right hand side of (32.19) is

$$\begin{aligned}
S_{n-1} \partial_n c_n &= S_{n-1} \Big(\sum_{i=0}^{n} (-1)^i c_n \circ \Delta_i^n \Big) \\
&= \sum_{i=0}^{n} (-1)^i S_{n-1} (c_n \circ \Delta_i^n) \\
&= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \Big((c_n \circ \Delta_i^n) \times id \Big) \circ p_j^{n-1} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (c_n \times id) \circ (\Delta_i^n \times id) \circ p_j^{n-1} \\
&= (c_n \times id)_* \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (\Delta_i^n \times id) \circ p_j^{n-1}.
\end{aligned} \tag{32.22}$$

Combining these yields

$$\begin{aligned}&\partial_{n+1} S_n c_n + S_{n-1}\partial_n c_n \\ &= (c_n \times id)_* \Big(\sum_{i=0}^{n}\sum_{j=0}^{n+1} (-1)^{i+j} p_i^n \circ \Delta_j^{n+1} + \sum_{i=0}^{n}\sum_{j=0}^{n-1} (-1)^{i+j} (\Delta_i^n \times id) \circ p_j^{n-1} \Big).\end{aligned} \tag{32.23}$$

To analyze these sums, we need a few lemmas. Let $\iota_s : \Delta^n \to \Delta^n \times [0,1]$ be the mapping $\iota_s(x) = (x, s)$.

Lemma 32.12. *We have*

$$p_0^n \circ \Delta_0^{n+1} = \iota_1 \tag{32.24}$$

$$p_n^n \circ \Delta_{n+1}^{n+1} = \iota_0 \tag{32.25}$$

$$p_i^n \circ \Delta_i^{n+1} = p_{i-1}^n \circ \Delta_i^{n+1} \quad 1 \le i \le n. \tag{32.26}$$

Proof. The mapping p_0^n is given by

$$p_0^n(t_0, \ldots, t_{n+1}) = (t_0 + t_1, t_2, \ldots, t_{n+1}, t_1 + \cdots + t_{n+1}), \tag{32.27}$$

so we have

$$\begin{aligned} p_0^n \circ \Delta_0^{n+1}(t_0, \ldots, t_n) &= p_0^n(0, t_0, \ldots, t_n) = (t_0, t_1, \ldots, t_n, t_0 + \cdots + t_n) \\ &= (t_0, \ldots, t_n, 1), \end{aligned} \tag{32.28}$$

which proves (32.24). Next, the mapping p_n^n is given by

$$p_n^n(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_n + t_{n+1}, t_{n+1}), \tag{32.29}$$

so we have

$$p_n^n \circ \Delta_{n+1}^{n+1}(t_0, \ldots, t_n) = p_n^n(t_0, \ldots, t_n, 0) = (t_0, \ldots, t_n, 0), \tag{32.30}$$

which proves (32.25).

Next, the left hand side of (32.26) is the mapping

$$\begin{aligned} p_i^n \circ \Delta_i^{n+1}(t_0, \ldots, t_n) &= p_i^n(t_0, \ldots, t_{i-1}, 0_i, t_i, \ldots, t_n) \\ &= (t_0, \ldots, t_{i-1}, 0_i + t_i, t_{i+1}, \ldots, t_n, t_i + \cdots + t_n) \\ &= \Big(t_0, \ldots, t_n, \sum_{j=i}^{n} t_j \Big). \end{aligned} \tag{32.31}$$

Next, the right hand side of (32.26) is the mapping

$$\begin{aligned} p_{i-1}^n \circ \Delta_i^{n+1}(t_0, \ldots, t_n) &= p_{i-1}^n(t_0, \ldots, t_{i-1}, 0_i, t_i, \ldots, t_n) \\ &= (t_0, \ldots, t_{i-2}, t_{i-1} + 0_i, t_i, \ldots, t_n, t_i + \cdots + t_n) \\ &= \Big(t_0, \ldots, t_n, \sum_{j=i}^{n} t_j \Big). \end{aligned} \tag{32.32}$$

□

Next, we have the following.

Lemma 32.13. *For $n \geq 0$, and $0 \leq i \leq n$ the mapping*

$$p_i^n : \Delta^{n+1} \to \Delta^n \times [0,1] \tag{32.33}$$

is the unique affine mapping satisfying

$$p_i^n(e_k) = \begin{cases} (e_k, 0) & 0 \leq k \leq i \\ (e_{k-1}, 1) & i < k \leq n+1. \end{cases} \tag{32.34}$$

Also, for $0 \leq i \leq n+1$, the i-th face of the standard $(n+1)$-simplex is the unique affine mapping $\Delta_i^{n+1} : \Delta^n \to \Delta^{n+1}$ satisfying

$$\Delta_i^{n+1}(e_k) = \begin{cases} e_k & 0 \leq k < i \\ e_{k+1} & i \leq k \leq n. \end{cases} \tag{32.35}$$

Proof. The mappings p_i^n is uniquely determined by its action on the vertices for the following reason. The mapping $p_i^n : \Delta^{n+1} \to \Delta^n \times [0,1] \subset \mathbb{R}^{n+2}$ is the restriction of an affine mapping $p_i^n : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$, which is of the form

$$p_i^n = L_i^n + c_i^n, \tag{32.36}$$

where L_i^n is a linear mapping, and c_i^n is a constant vector. Any $t \in \Delta^{n+1}$ can be written as a linear combination

$$t = \sum_{j=0}^{n+1} t_j e_j, \tag{32.37}$$

where $t_j \geq 0$ and $\sum_{j=0}^{n} t_j = 1$, so we have

$$p_i^n(t) = L_i^n\Big(\sum_{j=0}^{n+1} t_j e_j\Big) + c_i^n = \sum_{j=0}^{n+1} t_i L_i^n(e_j) + c_i^n, \tag{32.38}$$

so p_i^n is determined by its action on the vertices, as claimed.

Since

$$p_i^n : (t_0, \ldots, t_{n+1}) \mapsto ((t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), t_{i+1} + \cdots + t_{n+1}), \tag{32.39}$$

formula (32.34) follows immediately.

Similar to above, the mapping Δ_i^{n+1} is uniquely determined by its action on the vertices. The face mapping is $\Delta_i^{n+1} : \Delta^n \to \Delta^{n+1}$ defined by

$$\Delta_i^{n+1}(t_0, \ldots, t_{n1}) = (t_0, \ldots, t_{i-1}, 0_i, t_i, \ldots, t_n), \tag{32.40}$$

and formula (32.35) follows from this. Finally, similar to above, the mapping Δ_i^{n+1} is uniquely determined by its action on the vertices.

∎

The next lemma is crucial.

Lemma 32.14. *We have*

$$p^n_{j+1} \circ \Delta^{n+1}_i = (\Delta^n_i \times id) \circ p^{n-1}_j \quad j \geq i \tag{32.41}$$

$$p^n_j \circ \Delta^{n+1}_{i+1} = (\Delta^n_i \times id) \circ p^{n-1}_j \quad j < i. \tag{32.42}$$

Proof. Since both sides of each equation are affine maps with domain the standard n-simplex, we just need to check the equation on the vertices e_k, for $0 \leq k \leq n$. For this, we use (32.34) and (32.35).

For (32.41), we assume that $j \geq i$. If $k < i$, then

$$p^n_{j+1} \circ \Delta^{n+1}_i(e_k) = p^n_{j+1}(e_k) = (e_k, 0), \tag{32.43}$$

and

$$(\Delta^n_i \times id) \circ p^{n-1}_j(e_k) = (\Delta^n_i \times id)(e_k, 0) = (e_k, 0). \tag{32.44}$$

Next, if $i \leq k \leq j$, then

$$p^n_{j+1} \circ \Delta^{n+1}_i(e_k) = p^n_{j+1}(e_{k+1}) = (e_{k+1}, 0), \tag{32.45}$$

and

$$(\Delta^n_i \times id) \circ p^{n-1}_j(e_k) = (\Delta^n_i \times id)(e_k, 0) = (e_{k+1}, 0). \tag{32.46}$$

Next, if $k \geq j$, then

$$p^n_{j+1} \circ \Delta^{n+1}_i(e_k) = p^n_{j+1}(e_{k+1}) = (e_k, 1), \tag{32.47}$$

and

$$(\Delta^n_i \times id) \circ p^{n-1}_j(e_k) = (\Delta^n_i \times id)(e_{k-1}, 1) = (e_k, 1). \tag{32.48}$$

The formula (32.42) is proved similarly, the details are left as an exercise. □

Exercise 32.15. Prove formula (32.42).

Now we return to analyzing the formula

$$\begin{aligned}
&\partial_{n+1} S_n c_n + S_{n-1} \partial_n c_n \\
&= (c_n \times id)_* \Big(\sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} p^n_i \circ \Delta^{n+1}_j + \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big).
\end{aligned} \tag{32.49}$$

We split up the second sum on the right hand side as follows

$$\begin{aligned}
\sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big) &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big) \\
&+ \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} (\Delta^n_i \times id) \circ p^{n-1}_j \Big) \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} p^n_j \circ \Delta^{n+1}_{i+1} \\
&+ \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} p^n_{j+1} \circ \Delta^{n+1}_i,
\end{aligned} \tag{32.50}$$

where we used (32.41) and (32.42) on the last line.

Flipping i and j, we write the other sum as

$$\begin{aligned}\sum_{j=0}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}p_j^n\circ\Delta_i^{n+1} &= \Big(\sum_{0\le i<j\le n}+\sum_{i=j=0}^{n}+\sum_{i=j+1=1}^{n+1}+\sum_{1\le j+1<i\le n+1}\Big)(-1)^{i+j}p_j^n\circ\Delta_i^{n+1}\\ &= \sum_{0\le i<j\le n}(-1)^{i+j}p_j^n\circ\Delta_i^{n+1}+\sum_{i=0}^{n}p_i^n\circ\Delta_i^{n+1}-\sum_{i=1}^{n+1}p_{i-1}^n\circ\Delta_i^{n+1}\\ &+\sum_{1\le j+1\le i\le n+1}(-1)^{i+j}p_j^n\circ\Delta_i^{n+1}.\end{aligned}\tag{32.51}$$

Using (32.24), (32.25), and (32.26), the middle two sums are

$$\sum_{i=0}^{n}p_i^n\circ\Delta_i^{n+1}-\sum_{i=1}^{n+1}p_{i-1}^n\circ\Delta_i^{n+1}=p_0^n\circ\Delta_0^{n+1}-p_n^n\circ\Delta_i^{n+1}=\iota_1-\iota_0.\tag{32.52}$$

(This is the cancellation of interior overlapping faces, leaving only to top face minus the bottom face).

The other two sums are

$$\begin{aligned}&\sum_{0\le i<j\le n}(-1)^{i+j}p_j^n\circ\Delta_i^{n+1}+\sum_{1\le j+1<i\le n+1}(-1)^{i+j}p_j^n\circ\Delta_i^{n+1}\\ &\quad=-\sum_{0\le i\le j\le n}(-1)^{i+j}p_{j+1}^n\circ\Delta_i^{n+1}-\sum_{0\le j<i\le n}(-1)^{i+j}p_j^n\circ\Delta_{i+1}^{n+1}\end{aligned}\tag{32.53}$$

which follows from making the reindexing $j'=j-1$ in the first sum, and $i'=i-1$ in the second sum. These terms cancel out those in (32.50).

33 Lecture 33

This section closely follows [**?**, Chapter 17].

33.1 Barycentric subdivision

Consider the standard n-simplex $\Delta^n\subset\mathbb{R}^{n+1}$. We are going to subdivide Δ^n into $(n+1)!$ n-simplices through an inductive definition.

Definition 33.1. The barycenter of the standard p-simplex is the point $b=(t_0,\dots,t_{p+1})$ with $t_i=\frac{1}{n+1}$.

Let $c:\Delta^p\to\Delta^n$ be an *affine* p-simplex. Since c is affine, it is determined by its values on the basis vectors $e_i, 0\le i\le p$, and write $c(e_i)=v_i$. For shorthand notation, we can write $c=[v_0v_1\cdots v_p]$. Choose any $v\in\Delta^n$. Define the cone on c with basepoint v to be the affine $(p+1)$-simplex $\mathcal{C}_v(c)$ which maps $\mathcal{C}_v(c)(e_0)=v$ and $\mathcal{C}_v(c)(e_i)=v_{i-1}$. In shorthand, $\mathcal{C}_v(c)=[vv_0\cdots v_p]$. The free abelian group on affine p-simplices in Δ^n is denoted by $A^p(\Delta^n)$.

Definition 33.2 (Barycentric subdivision operator)**.** For any affine simplex in $c : \Delta^p \to \Delta^n$, we inductively define

$$S_p(c) = \begin{cases} C_b(S_{p-1}\partial_p c) & p > 0 \\ c & p = 0, \end{cases} \tag{33.1}$$

and then extend to any affine p-chain by linearity.

Proposition 33.3. *For $c \in A_p(\Delta^n)$, we have*

$$\partial(\mathcal{C}_v c) = \begin{cases} c - \mathcal{C}_v(\partial c) & p > 0 \\ c - \epsilon(c)[v] & p = 0 \end{cases} \tag{33.2}$$

where ϵ is the augmentation.

Proof. Let $c : \Delta^p \to \Delta^n$ be an affine simplex, and let $v_i = c(e_i)$. Then we denote c by $c = [v_0, \dots, v_p]$. For $p = 0$, we have

$$\partial(\mathcal{C}_v[v_0]) = \partial([v, v_0]) = [v_0] - [v]. \tag{33.3}$$

For $p > 1$,

$$\begin{aligned} \partial[v, v_0, \dots, v_p] &= [v_0, \dots, v_p] - \sum_{i=0}^{p}(-1)^i[v, v_0, \dots, \hat{v}_i, v_{i+1}, \dots, v_p] \\ &= c - \mathcal{C}_v(\partial c). \end{aligned} \tag{33.4}$$

□

Corollary 33.4. *If $c \in A_p(\Delta^n)$ then*

$$\partial_p S_p c = S_{p-1}\partial_p c. \tag{33.5}$$

I.e., $S_p : A_p(\Delta^n) \to A_p(\Delta^n)$ is a morphism of chain complexes.

Proof. We prove this inductively. The $p = 0$ case is an exercise. Assume true up to $p - 1$, then for an affine p-simplex c,

$$\begin{aligned} \partial_p S_p c &= \partial_p(\mathcal{C}_b(S_{p-1}\partial_p c)) \\ &= S_{p-1}\partial_p c - \mathcal{C}_b \partial_{p-1} S_{p-1}\partial_p c \\ &= S_{p-1}\partial_p c - \mathcal{C}_b S_{p-2}\partial_{p-1}\partial_p c = S_{p-1}\partial_p c. \end{aligned} \tag{33.6}$$

□

Proposition 33.5. *There exists homomorphisms $T_p : A_p(\Delta^n) \to A_{p+1}(\Delta^n)$ so that*

$$S_p - Id = \partial_{p+1}T_p + T_{p-1}\partial_p. \tag{33.7}$$

In other words S_p is chain homotopic to the identity mapping.

Proof. Given an affine p-simplex $c : \Delta^p \to \Delta^n$, we define T inductively by

$$Tc = \begin{cases} \mathcal{C}_b(S_p c - c - T_{p-1}\partial_p c) & p > 0 \\ 0 & p = 0 \end{cases}. \tag{33.8}$$

We leave the $p = 0, 1$ cases as an exercise. Assume the result is true up to $p - 1$. Then

$$\begin{aligned} \partial_{p+1} T_p c &= \partial_{p+1}\big(\mathcal{C}_b(S_p c - c - T_{p-1}\partial_p c)\big) \\ &= S_p c - c - T_{p-1}\partial_p c - \mathcal{C}_b\big(\partial_p(S_p c - c - T_{p-1}\partial_p c)\big) \\ &= S_p c - c - T_{p-1}\partial_p c - \mathcal{C}_b(S_{p-1}\partial_p c - \partial_p c - \partial_p T_{p-1}\partial_p c) \\ &= S_p c - c - T_{p-1}\partial_p c - \mathcal{C}_b(T_{p-2}\partial_{p-1}\partial_p c) = S_p c - c - T_{p-1}\partial_p c. \end{aligned} \tag{33.9}$$

□

Given a topological space X, we define

$$S_p : C_p(X) \to C_p(X), \quad T_p : C_p(X) \to C_{p+1}(X), \tag{33.10}$$

by the following. For a simplex $c : \Delta^p \to X$, we define

$$S_p c = c_* S_p \Delta^p, \quad T_p c = c_* T_p \Delta^p, \tag{33.11}$$

and then extend to all chains by linearity. We then have

Proposition 33.6. *The operators S_p and T_p satisfy*

$$\partial_p S_p c = S_{p-1}\partial_p c, \tag{33.12}$$

that is, S_p is a morphism of chain complexes, and

$$S_p - Id = \partial_{p+1} T_p + T_{p-1}\partial_p, \tag{33.13}$$

that is, S_p is chain homotopic to the identity mapping.

Proof. Follows from the above discussion and functorality. □

Corollary 33.7. *For any integer $k \geq 1$, the morphism $S_p^k : C_p(X) \to C_p(X)$ is a morphism of chain complexes which is chain homotopic to the identity by a chain homotopy $T_{p,k}$.*

Proof. For $k = 1$, this is the previous proposition. For $k > 1$, write

$$\begin{aligned} S_p^k - Id &= (S_p^{k-1} + S_p^{k-2} + \cdots + Id)(S_p - Id) \\ &= G_{p,k}(\partial_{p+1} T_p + T_{p-1}\partial_p) \\ &= \partial_{p+1} G_{p+1,k} T_p + G_{p,k} T_{p-1}\partial_p. \end{aligned} \tag{33.14}$$

In other words $T_{p,k} = G_{p+1,k} T_p$ is the desired chain homotopy. □

Next, let $\mathcal{U}$ be a collection of sets whose interiors form an open covering of X. Consider the subcomplex $C_p(\mathcal{U}) \subset C_p(X)$ consisting only of chains whose images are contained in some element of $\mathcal{U}$. The main result is the following

Theorem 33.8. *The inclusion mapping $i : C_p(\mathcal{U}) \subset C_p(X)$ induces an isomorphism*

$$i_* : H_p(C_*(\mathcal{U})) \cong H_p(X). \tag{33.15}$$

Proof. First, we show the mapping is injective. Let $c_p \in C_p(\mathcal{U})$ with $\partial_p c_p = 0$. Assume that $c_p = \partial_{p+1} c_{p+1}$ where $c_{p+1} \in C_p(X)$. Since the diameter of simplices in the barycentric subdivision process limit to zero as $k \to \infty$, it follows from the Lebesgue number lemma that there is a $k \geq 1$ so that $S^k_{p+1} c_{p+1} \in C_{p+1}(\mathcal{U})$. We then have

$$\begin{aligned} S^k_{p+1} c_{p+1} - c_{p+1} &= \partial_{p+2} T_{p+1,k} c_{p+1} + T_{p,k} \partial_{p+1} c_{p+1} \\ &= \partial_{p+2} T_{p+1,k} c_{p+1} + T_{p,k} c_p. \end{aligned} \tag{33.16}$$

Taking the boundary of this gives

$$\partial_{p+1} S^k_{p+1} c_{p+1} - c_p = \partial_{p+1} T_{p,k} c_p, \tag{33.17}$$

or

$$c_p = \partial_{p+1} (S^k_{p+1} c_{p+1} - T_{p,k} c_p) \tag{33.18}$$

Since, $S^k_{p+1} c_{p+1} - T_{p,k} c_p \in C_{p+1}(\mathcal{U})$, we are done with injectivity.

Next, for surjectivity, Let $c_p \in C_p(X)$ with $\partial_p c_p = 0$. Again, by the Lebesgue number lemma, there is a $k \geq 1$ so that $S^k_p c_p \in C_p(\mathcal{U})$. Then we have

$$S^k_p c_p - c_p = \partial_{p+1} T_{p,k} c_p + T_{p-1,k} \partial_p c_p = \partial_{p+1} T_{p,k} c_p. \tag{33.19}$$

This says exactly that c_p is the image of $S^k_p c_p$, and we are done. □

34 Lecture 34

34.1 Excision for singular homology

We next prove another one of the main axioms of singular homology theory.

Theorem 34.1. *Consider $U \subset A \subset X$ such that the closure of U is contained in the interior of A. Then for all $k \geq 0$, the inclusion mapping $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism on singular homology*

$$H_k i : H_k(X \setminus U, A \setminus U) \to H_k(X, A). \tag{34.1}$$

Proof. From the assumption that $\overline{U} \subset A$, we have that $\mathcal{U} = \{A, B = X \setminus U\}$ is a covering of X. Then

$$(X \setminus U, A \setminus U) = (B, A \cap B), \tag{34.2}$$

so we want to show that that the inclusion $(B, A\cap B) \subset (X, A)$ induces isomorphisms in homology. Note we have the short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(\mathcal{U}) & \longrightarrow & C_*(\mathcal{U})/C_*(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X,A) & \longrightarrow & 0.
\end{array} \tag{34.3}$$

We know the middle maps induced isomorphisms in homology, and the first one obviously does, so by the five lemma we conclude that

$$H_p(X, A) = H_p(C_*(\mathcal{U})/C_*(A)) \tag{34.4}$$

We have that

$$C_*(\mathcal{U}) = C_*(A) + C_*(B) \subset C_*(X). \tag{34.5}$$

Lemma 34.2 (Noether's second isomorphism)**.** *Let G_1, G_2 be subgroups of an Abelian group G such that $G_1 + G_2 = G$. Then there is an isomorphism*

$$\frac{G_2}{G_1 \cap G_2} \cong \frac{G}{G_1}. \tag{34.6}$$

Using Lemma 34.2, we therefore have

$$\frac{C_*(B)}{C_*(A\cap B)} = \frac{C_*(B)}{C_*(A)\cap C_*(B)} \cong \frac{C_*(A) + C_*(B)}{C_*(A)} = \frac{C_*(\mathcal{U})}{C_*(A)}. \tag{34.7}$$

This completes the proof. □

34.2 Relative homology

We next give a generalization of the long exact sequence in relative homology. Consider a triple (X, A, B), where $B \subset A \subset X$. For each $p \geq 0$, we have a short exact sequence of abelian groups

$$0 \longrightarrow C_p(A, B) \longrightarrow C_p(X, B) \longrightarrow C_p(X, A) \longrightarrow 0, \tag{34.8}$$

where all the mappings are induced by inclusions. The above exact sequence is then a short exact sequence of chain complexes. It follows from the zig-zag lemma that there is a long exact sequence

$$\cdots \xrightarrow{\delta_{p+1}} H_p(A, B) \xrightarrow{H_p i} H_p(X, B) \longrightarrow H_p(X, A) \xrightarrow{\delta_p} H_{p-1}(A, B) \longrightarrow \cdots, \tag{34.9}$$

Now we can turn Remark 31.2 from above into a theorem.

Theorem 34.3. *If $A \subset X$ is a nontrivial closed subspace which is a deformation retraction of an open set $U \subset X$, then $H_p(X, A) \cong \tilde{H}_p(X/A)$, for $p \geq 0$, where X/A is the quotient space obtained by identifying all points of A.*

Proof. We begin with the following commutative diagram

$$\begin{array}{ccccc} H_p(X,A) & \longrightarrow & H_p(X,U) & \longleftarrow & H_p(X \setminus A, U \setminus A) \\ \downarrow & & \downarrow & & \downarrow \\ H_p(X/A, A/A) & \longrightarrow & H_p(X/A, U/A) & \longleftarrow & H_p(X/A \setminus A/A, U/A \setminus A/A), \end{array} \tag{34.10}$$

where the vertical mappings are induced by the quotient maps, and the horizontal mapping are induced by inclusions. The long exact squence for the triple (X, U, A)

$$\cdots \xrightarrow{\delta_{p+1}} H_p(U,A) \xrightarrow{H_p i} H_p(X,A) \longrightarrow H_p(X,U) \xrightarrow{\delta_p} \tilde{H}_{p-1}(U,A) \longrightarrow \cdots \tag{34.11}$$

yields $H_p(X,A) \cong H_p(X,U)$. So the upper left arrow in (34.10) is an isomorphism. The upper right arrow is an isomorphism by excision. Since U deformation retracts onto A, it follows that U/A deformation retracts onto A/A, and the lower left arrow is an isomorphism. The lower right arrow is an isomorphism by excision.

The right vertical arrow is an isomorphism since the quotient mapping

$$q : X \setminus A \to X/A \setminus A/A \tag{34.12}$$

is a homeomorphism which maps $U \setminus A$ onto $U/A \setminus A/A$. Since the diagram commutes, the left vertical arrow must be an isomorphism. To finish, we note that for $p = 0$, from (31.6) above,

$$H_0(X/A, A/A) = H_0(X/A, \{pt\}) \cong \tilde{H}_0(X/A). \tag{34.13}$$

□

34.3 Mayer-Vietoris sequence for singular homology

Write $X = U \cup V$ as the union of two open sets in X. Then the following sequence is exact:

$$0 \longrightarrow C_p(U \cap V) \xrightarrow{\alpha_p} C_p(U) \oplus C_p(V) \xrightarrow{\beta_p} C_p(U) + C_p(V) \longrightarrow 0 \tag{34.14}$$

where

$$\alpha(c_p) = \big((i_{U\cap V \hookrightarrow U})_* c_p, (i_{U \cap V \hookrightarrow V})_* c_p\big) \tag{34.15}$$

and

$$\beta(a_p, b_p) = a_p - b_p. \tag{34.16}$$

From Theorem 33.8, we know the homology of the complex $H_*(C_p(U) + C_p(V)) \cong H_*(X)$. Consequently, by the zig-zag lemma, we obtain a long exact sequence

$$\cdots \xrightarrow{\partial_{p+1}} H_p(U \cap V) \xrightarrow{\alpha_p} H_p(U) \oplus H_p(V) \xrightarrow{\beta_p} H_p(U \cup V) \xrightarrow{\partial_p} \cdots \tag{34.17}$$

34.4 Singular cohomology

We now define singular cohomology. Let G be any abelian group, and define

$$C^p(X;G) = Hom(C_p(X), G), \tag{34.18}$$

to be the group of homomorphisms from $C_p(X)$ to G, and let $\delta^p : C^p(X;G) \to C^{p+1}(X;G)$ denote the dual to the boundary operator $\partial_{p+1} : C_{p+1}(X) \to C_p(X)$, defined as follows. For $c^p \in C^p(X;G)$ and $c_{p+1} \in C_{p+1}(X)$,

$$(\delta^p c^p)(c_{p+1}) = c^p(\partial_{p+1} c_{p+1}). \tag{34.19}$$

Since $\partial_p \circ \partial_{p+1} = 0$, we have $\delta^{p+1} \circ \delta^p = 0$, so we have a *cochain complex*

$$\cdots \xrightarrow{\delta^{p-2}} C^{p-1}(X;G) \xrightarrow{\delta^{p-1}} C^p(X;G) \xrightarrow{\delta^p} C^{p+1}(X;G) \xrightarrow{\delta^{p+1}} \cdots. \tag{34.20}$$

Define the pth singular cohomology group with coefficients in G by

$$H^p(X;G) = \frac{Ker\{\delta^p : C^p(X;G) \to C^{p+1}(X;G)\}}{Im\{\delta^{p-1} : C^{p-1}(X;G) \to C^p(X;G)\}}. \tag{34.21}$$

Next, let $f : X \to Y$ be a continuous mapping between topological spaces. The mapping on chains $(f_*)_p : C_p(X) \to C_p(Y)$ induces the dual mapping on cochains

$$f^* : C^p(Y;G) \to C^p(X;G) \tag{34.22}$$

by the following. For $c^p \in C^p(Y;G)$ and $c_p \in C_p(X)$, define

$$(f^* c^p)(c_p) = c^p(f_* c_p) \tag{34.23}$$

Dualizing the diagram (30.32), we have the following commutative diagram

$$\begin{array}{ccc} C^p(Y;G) & \xrightarrow{\delta^p_Y} & C^{p+1}(Y;G) \\ \Big\downarrow{\scriptstyle (f^*)^p} & & \Big\downarrow{\scriptstyle (f^*)^{p+1}} \\ C^p(X;G) & \xrightarrow{\delta^p_X} & C^{p+1}(X;G). \end{array} \tag{34.24}$$

That is the collection of mappings $(f^*)^p$ is a *morphism* of cochain complexes.

The singular cohomology spaces are a topological invariant.

Corollary 34.4. *If $f : X \to Y$ is continuous, then there are induced mappings*

$$(f^*)^p : H^p(Y;G) \to H^p(X;G). \tag{34.25}$$

If $g : Y \to Z$, then

$$((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p. \tag{34.26}$$

Consequently, if X and Y are homeomorphic, then $H^p(X;G) \cong H^p(Y;G)$ for every $p \geq 0$.

Proof. Exactly the same as the proof of Corollary 30.14, with ∂_X, ∂_Y replaced by δ_X, δ_Y. $\square$

In the category of abelian groups, Lemma 17.2 does not hold. However, the following does hold.

Lemma 34.5. *Let G be an abelian group, and let G_1, G_2, G_3 be free abelian groups. If the sequence*

$$0 \longrightarrow G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \longrightarrow 0. \tag{34.27}$$

then the dual sequence

$$0 \longrightarrow Hom(G_3, G) \xrightarrow{\beta^*} Hom(G_2, G) \xrightarrow{\alpha^*} Hom(G_1, G) \longrightarrow 0. \tag{34.28}$$

is exact.

Proof. The proof is the same as the proof of Lemma 17.2 for vector spaces, since we have assumed the groups in the exact sequence are free abelian groups. $\square$

Remark 34.6. More generally, if there is a short exact sequence of abelian groups (not assumed to be free abelian)

$$0 \longrightarrow G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \longrightarrow 0, \tag{34.29}$$

then there is an exact sequence

$$\begin{aligned} 0 \longrightarrow Hom(G_3, G) \longrightarrow Hom(G_2, G) \longrightarrow Hom(G_1, G) \\ \longrightarrow Ext(G_3, G) \longrightarrow Ext(G_2, G) \longrightarrow Ext(G_1, G) \longrightarrow 0, \end{aligned} \tag{34.30}$$

where $Ext(G', G)$ is a canonically defined functor, defined by the above exact sequence for any 2-step free resolution of G'. That is, given an abelian group G', take any short exact sequence

$$0 \longrightarrow G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G' \longrightarrow 0, \tag{34.31}$$

where G_1 and G_2 are free, and then define $Ext(G', G)$ by

$$0 \longrightarrow Hom(G', G) \longrightarrow Hom(G_2, G) \longrightarrow Hom(G_1, G) \longrightarrow Ext(G', G) \longrightarrow 0. \tag{34.32}$$

(This is well-defined, but needs proof). The term $Ext(G', G) = 0$ provided that G' is a free abelian group.

35 Lecture 35

35.1 The axioms for singular cohomology

Homotopy invariance. If $f : X \to Y$ is homotopic to $g : X \to Y$ then $H^p f = H^p g : H^p(Y;G) \to H^p(X;G)$. Consequently, if X is homotopy equivalent to Y then $H^p(X;G) \cong H^p(Y;G)$ for all $p \geq 0$. The proof follows by dualizing the chain homotopy constructed in the proof of homotopy invariance of homology.

Long exact sequence in relative cohomology. If $A \subset X$, then dualizing the relative exact sequence on chains yields

$$0 \longrightarrow Hom(C_p(X,A),G) \longrightarrow Hom(C_p(X),G) \longrightarrow Hom(C_p(A),G) \longrightarrow 0. \tag{35.1}$$

We define the relative cohomology to be the cohomology of the first co-complex. The zig-zag lemma shows that there is a long exact sequence

$$\cdots \xrightarrow{\delta^{p-1}} H^p(X,A;G) \longrightarrow H^p(X;G) \xrightarrow{i^*} H^p(A;G) \xrightarrow{\delta^p} H^{p+1}(X,A;G) \longrightarrow \cdots, \tag{35.2}$$

Excision. Consider $U \subset A \subset X$ such that the closure of U is contained in the interior of A. Then for all $k \geq 0$, the inclusion mapping $i : (X \setminus U, A \setminus U) \to (X,A)$ induces an isomorphism on singular homology

$$H^k i : H^k(X,A;G) \to H^k(X \setminus U, A \setminus U; G). \tag{35.3}$$

The proof follows from dualizing the maps in the proof of the homology isomorphisms. Alternatively, this follows from the statement of excision on homology, and an algebraic lemma which says that if a mapping induces isomorphism on homology in all degrees, then the dual mapping also induce isomorphisms on cohomology in all degrees. (This is basically the universal coefficient theorem for cohomology, which we do not have time to explain.)

Mayer-Vietoris sequence. If $X = U \cup V$ where U and V are open, then

$$\cdots \xrightarrow{\delta^{p-1}} H^p(U \cup V;G) \xrightarrow{\beta_p^*} H^p(U;G) \oplus H^p(V;G) \xrightarrow{\alpha_p^*} H^p(U \cap V;G) \xrightarrow{\delta^p} \cdots \tag{35.4}$$

The proof follows directly by dualizing the operators in the proof of Mayer-Vietoris for homology.

Remark 35.1. Now we have enough tools to compute the integral cohomology of many examples. Since we are out of time this quarter, we will have to leave it to the student to "upgrade" all of our previous de Rham cohomology computations to integral cohomology. We will next finish up the course by proving de Rham's Theorem.

35.2 Smooth singular cohomology

Next, we restrict to the category of smooth manifolds. For a smooth manifold space X, a smooth mapping

$$c : \Delta^p \to X. \tag{35.5}$$

is called a smooth singular p-simplex.

Definition 35.2. The smooth pth singular chain group $C_p^\infty(X, \mathbb{R})$ is the free vector space over $\mathbb{R}$ generated by smooth singular p-simplices.

We note that the usual boundary operator maps

$$\partial_{p+1} : C_{p+1}^\infty(X; \mathbb{R}) \to C_p^\infty(X; \mathbb{R}) \tag{35.6}$$

To define smooth singular cohomology, let $C_\infty^p(X; \mathbb{R})$ denote the smooth singular cochains, which are dual to smooth singular chains, i.e.,

$$C_\infty^p(X; \mathbb{R}) = Hom(C_p^\infty(X; \mathbb{R}), \mathbb{R}), \tag{35.7}$$

and let $\delta^p : C_\infty^p(X; \mathbb{R}) \to C_\infty^{p+1}(X; \mathbb{R})$ denote the dual to the boundary operator defined as before. For $c^p \in C_\infty^p(X; \mathbb{R})$ and $c_{p+1} \in C_p^\infty(X; \mathbb{R})$,

$$(\delta^p c^p)(c_{p+1}) = c^p(\partial_{p+1} c_{p+1}). \tag{35.8}$$

Since $\partial_p \circ \partial_{p+1} = 0$, we have $\delta^{p+1} \circ \delta^p = 0$, so we have a *cochain complex*

$$\cdots \xrightarrow{\delta^{p-2}} C_\infty^{p-1}(X; \mathbb{R}) \xrightarrow{\delta^{p-1}} C_\infty^p(X; \mathbb{R}) \xrightarrow{\delta^p} C_\infty^{p+1}(X; \mathbb{R}) \xrightarrow{\delta^{p+1}} \cdots . \tag{35.9}$$

Define the pth smooth singular cohomology group by

$$H_\infty^p(X; \mathbb{R}) = \frac{Ker\{\delta^p : C_\infty^p(X; \mathbb{R}) \to C_\infty^{p+1}(X; \mathbb{R})\}}{Im\{\delta^{p-1} : C_\infty^{p-1}(X; \mathbb{R}) \to C_\infty^p(X; \mathbb{R})\}}. \tag{35.10}$$

The smooth singular cohomology satisfies the same axioms as the topological singular cohomology.

Proposition 35.3 (Smooth homotopy invariance)**.** *If $f : X \to Y$ is smoothly homotopic to $g : X \to Y$ then $H^p f = H^p g : H_\infty^p(Y; \mathbb{R}) \to H_\infty^p(X; \mathbb{R})$. Consequently, if X is homotopy equivalent to Y then $H_\infty^p(X; \mathbb{R}) \cong H_\infty^p(Y; \mathbb{R})$ for all $p \geq 0$.*

Proof. The proof is the same as in the topological case, because the co-chain homotopy maps

$$S^k : C_\infty^k(X \times [0, 1]; \mathbb{R}) \mapsto C_\infty^{k-1}(X; \mathbb{R}), \tag{35.11}$$

that is, the chain homotopy constructed there preserves smoothness. □

Proposition 35.4 (Mayer-Vietoris)**.** *If $X = U \cup V$ where U and V are open, then there is a long exact sequence*

$$\cdots \xrightarrow{\delta^{p-1}} H^p_\infty(U \cup V; \mathbb{R}) \xrightarrow{\beta_p^*} H^p_\infty(U; \mathbb{R}) \oplus H^p_\infty(V; \mathbb{R}) \xrightarrow{\alpha_p^*} H^p_\infty(U \cap V; \mathbb{R}) \xrightarrow{\delta^p} \cdots \tag{35.12}$$

Proof. The proof of the topological Mayer-Vietors relies on the cone construction, which unfortunately does not map smooth simplices to smooth simplices since the cone is not differentiable at the cone point. However, this is easily fixed using a cutoff function to map to a simplex which is constant near the cone point. Details left as an easy exercise. □

Proposition 35.5. *If M is a smooth manifold which has a finite good cover, then*

$$H^p_\infty(M; \mathbb{R}) \cong H^p(M; \mathbb{R}) \tag{35.13}$$

Proof. Both cohomology theories agree on contractible spaces, and both satisfy a Mayer-Vietoris sequence. There is an obvious chain mapping from smooth co-chains to topological co-chains. By the same argument as before using the five lemma and induction on the number of elements in a good cover, the cohomology groups are isomorphic in any degree. □

35.3 de Rham's Theorem

Define the standard n-simplex to be

$$\Delta^p = \Big\{(t_0, \dots, t_p) \in \mathbb{R}^{p+1}, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\Big\}. \tag{35.14}$$

We orient Δ_p with respect to the normal $\hat{n} = (1, \dots, 1)$. I.e., $(v_1, \dots, v_p) \in T_x\Delta^p$ is oriented if $(\hat{n}, v_1, \dots, v_p)$ is oriented equivalent to $(e_0, \dots, e_p)$ in $\mathbb{R}^{p+1}$. The ith face of Δ^p is the $(p-1)$-simplex

$$\Delta^p_i : \Delta^{p-1} \to \Delta^p \tag{35.15}$$

defined by

$$(t_0, \dots, t_{p-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{p-1}). \tag{35.16}$$

For a smooth manifold space X, a smooth mapping

$$c : \Delta^p \to X. \tag{35.17}$$

is called a smooth singular p-simplex. If $\omega \in \Omega^p(X)$, and c is a smooth singular p-simplex, define

$$\int_c \omega = \int_{\Delta^p} c^*\omega. \tag{35.18}$$

A smooth singular p-chain is a finite linear combination

$$c = \sum_{i=1}^{N} a_i c_i, \tag{35.19}$$

where $a_i \in \mathbb{R}$ and c_i are smooth singular p-simplices. For $\omega \in \Omega^p(M)$, define

$$\int_c \omega = \sum_{i=1}^{N} a_i \int_{c_i} \omega. \tag{35.20}$$

Theorem 35.6 (Stokes' Theorem on chains)**.** *Let M be a compact oriented manifold. If $\omega \in \Omega^{p-1}(M)$, then for any chain $c \in C_p^{\infty}(X;\mathbb{R})$,*

$$\int_{\partial c} \omega = \int_c d\omega. \tag{35.21}$$

Proof. The standard n-simplex is a "manifold with corners", on which Stokes' Theorem remains valid. The sign in the definition of the boundary operator gives the correct orientation on each face. □

We are now in a position to state the theorem of de Rham relating de Rham cohomology and singular cohomology with real coefficients of a smooth manifold M. Morever, we can write the explicit mapping. Consider the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{p-2}} & \Omega^{p-1}(M) & \xrightarrow{d^{p-1}} & \Omega^p(M) & \xrightarrow{d^p} & \Omega^{p+1}(M) & \xrightarrow{d^{p+1}} & \cdots \\
 & & \downarrow{\scriptstyle \mathcal{F}^{p-1}} & & \downarrow{\scriptstyle \mathcal{F}^{p}} & & \downarrow{\scriptstyle \mathcal{F}^{p+1}} & & \\
\cdots & \xrightarrow{\delta^{p-2}} & C_\infty^{p-1}(M;\mathbb{R}) & \xrightarrow{\delta^{p-1}} & C_\infty^{p}(M;\mathbb{R}) & \xrightarrow{\delta^{p}} & C_\infty^{p+1}(M;\mathbb{R}) & \xrightarrow{\delta^{p+1}} & \cdots ,
\end{array} \tag{35.22}$$

where the vertical maps are defined as follows. If $\omega \in \Omega^p(M)$, and c_p is a smooth p-chain, then let

$$(\mathcal{F}^p\omega)(c_p) = \int_{c_p} \omega. \tag{35.23}$$

Proposition 35.7. *The diagram* (35.22) *commutes. Consequently, there are induced mappings*

$$H^p\mathcal{F}^p : H^p_{dR}(M) \to H^p_{\infty}(M;\mathbb{R}). \tag{35.24}$$

Proof. Commutativity says that

$$\delta^p\mathcal{F}^p = \mathcal{F}^{p+1}d^p \tag{35.25}$$

Given $\omega \in \Omega^p(M)$, and $(p+1)$-chain c_{p+1}, the left hand side of (35.23) evaluates to

$$\delta^p(\mathcal{F}^p(\omega))(c_{p+1}) = \mathcal{F}^p(\omega)(\partial_{p+1}c_{p+1}) = \int_{\partial_{p+1}c_{p+1}} \omega. \tag{35.26}$$

The right hand side of (35.23) evaluates to

$$\mathcal{F}^{p+1} d^p\omega(c_{p+1}) = \int_{c_{p+1}} d^p\omega, \tag{35.27}$$

These are equal by Theorem 35.6, Stokes' Theorem on chains.

Consequently, $\mathcal{F}^*$ is a morphism of co-chain complexes, so there are well-defined induced maps on the cohomology groups.

□

We can now prove the main result.

Theorem 35.8 (de Rham)**.** *If M has a finite good cover then the mappings*

$$\mathcal{F}^p : H^p_{dR}(M) \to H^p_\infty(M;\mathbb{R}), \tag{35.28}$$

are isomorphisms for all $p \geq 0$.

Proof. Note we have the following morphism of short exact sequences of co-complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^p(U\cup V) & \xrightarrow{\beta^p} & \Omega^p(U)\oplus\Omega^p(V) & \xrightarrow{\alpha^p} & \Omega^p(U\cap V) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathcal{F}^p} & & \big\downarrow{\scriptstyle \mathcal{F}^p\oplus\mathcal{F}^p} & & \big\downarrow{\scriptstyle \mathcal{F}^p} & & \\ 0 & \longrightarrow & (C^\infty_p(U;\mathbb{R})+C^\infty_p(V;\mathbb{R}))^* & \xrightarrow{\beta^*_p} & C^p_\infty(U;\mathbb{R})\oplus C^p(V;\mathbb{R}) & \xrightarrow{\alpha^*_p} & C^p_\infty(U\cap V;\mathbb{R})^* & \longrightarrow & 0. \end{array} \tag{35.29}$$

The above diagram is easily seen to commute at every square. It follows that the following diagram of associated Mayer-Vietoris exact sequences commutes at every square

$$\begin{array}{ccccccccc} H^{k-1}_{dR}(U)\oplus H^{k-1}_{dR}(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_{dR}(U\cap V) & \xrightarrow{\delta^k} & H^k_{dR}(U\cup V) & \xrightarrow{\beta^k} & H^k_{dR}(U)\oplus H^k_{dR}(V) & \xrightarrow{\alpha^k} & H^k_{dR}(U\cap V) \\ \big\downarrow{\scriptstyle \mathcal{F}^{k-1}\oplus\mathcal{F}^{k-1}} & & \big\downarrow{\scriptstyle \mathcal{F}^{k-1}} & & \big\downarrow{\scriptstyle \mathcal{F}^k} & & \big\downarrow{\scriptstyle \mathcal{F}^k\oplus\mathcal{F}^k} & & \big\downarrow{\scriptstyle \mathcal{F}^k} \\ H^{k-1}_s(U)\oplus H^{k-1}_s(V) & \xrightarrow{\alpha^{k-1}} & H^{k-1}_s(U\cap V) & \xrightarrow{\delta^k} & H^k_s(U\cup V) & \xrightarrow{\beta^k} & H^k_s(U)\oplus H^k_s(V) & \xrightarrow{\alpha^k} & H^k_s(U\cap V) \end{array} \tag{35.30}$$

If there is only 1 element in the covering, then we are done by the homotopy invariance of both theories. By the Five Lemma, if the result is true for U, V and $U\cap V$, then it is also true for $U\cup V$. By induction on the number of elements in a finite good cover, the theorem is then true for any manifold which admits a finite good cover. □

From Proposition 35.5 above, we have

Theorem 35.9. *If X is a smooth manifold with a finite good cover, then*

$$H^p_{dR}(X) \cong H^p(X;\mathbb{R}). \tag{35.31}$$

Consequently, the de Rham cohomology groups are a topological invariant. That is if smooth manifolds X and Y are homeomorphic, then $H^p_{dR}(X)\cong H^p_{dR}(Y)$.

Remark 35.10. There do exist examples of homeomorphic but non-diffeomorphic smooth manifolds! But de Rham cohomology will never be able to tell these apart – for this one needs more refined invariants.

35.4 Products in cohomology

Cochains have some extra ring structure: we next define the cup product

$$\cup : C^p(X;\mathbb{R}) \times C^q(X;\mathbb{R}) \to C^{p+q}(X;\mathbb{R}), \tag{35.32}$$

by the following. If c_{p+q} is a singular $(p+q)$-simplex, and c^p and c^q are singular cochains, then define

$$(c^p \cup c^q)(c_{p+q}) = c^p(c_{p+q} \circ (t_0, \cdots, t_p, 0, \ldots, 0)) \cdot c^q(c_{p+q} \circ (0, \ldots, 0, t_p, \ldots, t_{p+q})). \tag{35.33}$$

We will see below that the cup product is bilinear, so by the universal property of the tensor product, the cup product descends to a linear mapping

$$\cup : C^p(X;\mathbb{R}) \otimes C^q(X;\mathbb{R}) \to C^{p+q}(X;\mathbb{R}). \tag{35.34}$$

The cup product enjoys the following properties, of which we omit the proof.

Proposition 35.11. *The cup product is associative*

$$(c^p \cup c^q) \cup c^r = c^p \cup (c^q \cup c^r). \tag{35.35}$$

The cup product is bilinear. That is, for $c \in \mathbb{R}$,

$$a^p \cup (cb^q) = c(a^p \cup b^q), \quad a^p \cup (b^q + c^q) = a^p \cup b^q + a^p \cup c^q \tag{35.36}$$

$$(ca^p) \cup b^q = c(a^p \cup b^q), \quad (a^p + b^p) \cup c^q = a^p \cup c^q + b^p \cup c^q. \tag{35.37}$$

Cup products are functorial, i.e., for $f : X \to Y$ continuous,

$$f^*(c^p \cup c^q) = f^*c^p \cup f^*c^q. \tag{35.38}$$

The coboundary operator is an anti-derivation with respect to the cup product

$$\delta(c^p \cup c^q) = (\delta c^p) \cup c^q + (-1)^p c^p \cup (\delta c^q). \tag{35.39}$$

This implies the following corollary.

Corollary 35.12. *The cup product on cochains induces a cup product on cohomology*

$$\cup : H^p(X;\mathbb{R}) \otimes H^q(X;\mathbb{R}) \to H^{p+q}(X;\mathbb{R}). \tag{35.40}$$

If $f : X \to Y$ is continuous, then

$$f^* : H^*(Y;\mathbb{R}) \to H^*(X;\mathbb{R}) \tag{35.41}$$

is an algebra homeomorphism. Consequently, if X and Y are homeomorphic, then the singular cohomology algebras are isomorphic.

Recall that the de Rham cohomology also has an algebra structure induced by the wedge product. One can show also the following:

Theorem 35.13. *The de Rham isomorphism*

$$\mathcal{F}^* : H^*_{dR}(M) \to H^*_{\infty}(M;\mathbb{R}), \tag{35.42}$$

is moreover an isomorphism of algebras.

This is not at all obvious!

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