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Introduction

In 250A, we will give an introduction to algebraic topology via de Rham cohomology of differentiable manifolds. Topics include the Poincaré Lemma, exact sequences, Mayer-Vietoris sequence, cohomology with compact supports, Poincaré duality, etc.

Then in 250B we will do singular homology and cohomology with integral coefficients, and prove more general things than can be done with just de Rham cohomology.

A guiding reference will be ??, but we will also use [Spi79, War83] for background material.

1 Lecture 1

1.1 Differentiable manifolds

Definition 1.1. A smooth manifold $M^n$ is a second countable Hausdorff space which is locally Euclidean.

"Locally Euclidean" means that for each $p \in M$, there exist an open neighborhood $U$ containing $p$ and a smooth mapping $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ which is a homeomorphism onto its image. Furthermore, if two coordinate charts overlap, then the "overlap mapping"

$$\phi_\alpha \circ \phi^{-1}_\beta : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \quad (1.1)$$

is required to be a diffeomorphism. A collection of coordinate charts $(U_\alpha, \phi_\alpha)$ covering $M$ is called an atlas. The collection of all possible coordinate charts on $M$ compatible with some atlas is called a differentiable structure.

Definition 1.2. A mapping $\Psi : M \rightarrow N$ between smooth manifolds is a continuous mapping such that for each $p \in M$, there exists a coordinate system $\phi$ around $p$, and a coordinate system $\tilde{\phi}$ around $\Psi(p)$ such that

$$\tilde{\phi} \circ \Psi \circ \phi^{-1} \quad (1.2)$$

is smooth in some small neighborhood of $\phi(p)$.

It is easy to see that if $\Psi : M \rightarrow N$, and $\Phi : N \rightarrow M_1$, are smooth then $\Psi \circ \Phi : M \rightarrow M_1$ is smooth.

Definition 1.3. The category of smooth manifolds $\text{Man}^\infty$ has objects as smooth manifolds and morphisms as smooth mappings, where composition of morphisms is just composition of mappings.

Composition of morphisms is obviously associative, i.e.,

$$(\Psi_1 \circ \Psi_2) \circ \Psi_3 = \Psi_1 \circ (\Psi_2 \circ \Psi_3) \quad (1.3)$$
and every manifold has an identity morphism $id_X: X \to X$, which is obviously
smooth, so this is indeed a category.

We could have just consider continuous mappings (instead of smooth mappings)
in all the above definitions, to define the category of topological manifolds $\text{Man}$ and
continuous mappings. There is then a covariant functor between categories

$$F: \text{Man}^\infty \to \text{Man} \quad (1.4)$$
called a forgetful functor which simply maps $F(M) = M$ and $F(\Psi) = \Psi$.

**Example 1.4.** $\mathbb{R}, S^1, S^2, \mathbb{RP}^2, T^2, T^2#T^2, \ldots$ (see lecture note on Canvas page).

## 2 Lecture 2

### 2.1 Tangent vectors

**Definition 2.1.** A germ of a smooth function at $p$ is an equivalence class $[f]_p$ where
$f: U \to \mathbb{R}$ is a smooth function defined on a neighborhood of $p$ and $f_1 \equiv f_2$ if there
exists a neighborhood $U_3 \subset U_1 \cap U_2$ such $f_1 = f_2$ on $U_3$. The set of equivalence classes
is denoted by $C^\infty(p)$.

**Definition 2.2.** A tangent vector at $p$, denoted by $X_p$ is a linear derivation on germs
of smooth functions around a point. That is,

$$X_p: C^\infty(p) \to \mathbb{R} \quad (2.1)$$
is linear over $\mathbb{R},$

$$X_p(c_1[f_1]_p + c_2[f_2]_p) = c_1X_p[f_1]_p + c_2X_p[f_2]_p, \quad (2.2)$$

and

$$X_p([f]_p[g]_p) = X_p([f]_p)g(p) + f(p)X_p([g]_p). \quad (2.3)$$

The collection of all tangent vectors at $p$ is denoted by $T_pM$.

**Exercise 2.3.** Show that $T_pM$ is a vector space over $\mathbb{R}$ and

$$\dim(T_pM) = \dim(M) = n. \quad (2.4)$$

**Definition 2.4.** If $\Psi: M \to N$ is a smooth mapping between manifolds, then
$\Psi_*: T_pM \to T_{\Psi(p)}N$ is the linear map defined by

$$(\Psi_*X_p)[f]_{\Psi(p)} = X_p([f \circ \Psi]_p) \quad (2.5)$$

We next give an alternate definition of a tangent vector. Let $p \in M$ and $c: (-\epsilon, \epsilon) \to M$ be a smooth curve with $c(0) = p$. Let $\phi: U \to \mathbb{R}^n$ be a coordinate system around $p$. Then

$$\phi \circ c: (-\epsilon, \epsilon) \to \mathbb{R}^n, \quad (2.6)$$
and we can consider
\[(\phi \circ c)'(0) = \frac{d}{dt}(\phi \circ c)\big|_{t=0} \in \mathbb{R}^n.\] (2.7)

Given another smooth curve \(\tilde{c} : (-\epsilon, \epsilon) \to M\) with \(c(0) = p\), we say that \(c \equiv c'\) if
\[(\phi \circ c)'(0) = (\phi \circ \tilde{c})'(0).\] (2.8)

We will denote the equivalence class of a smooth curve at \(p\) by \([c]_p\).

Note that if \(\phi_1\) is another coordinate system around \(p\), then
\[(\phi_1 \circ c)'(0) = (\phi_1 \circ \phi^{-1} \circ \phi \circ c)'(0) = (\phi_1 \circ \phi^{-1})_* (\phi \circ c)'(0)\]
\[= (\phi_1 \circ \phi^{-1})_* (\phi \circ \tilde{c})'(0) = (\phi_1 \circ \tilde{c})'(0),\] (2.9)

where \((\phi_1 \circ \phi^{-1})_*\) is the Jacobian matrix of partial derivatives (by the chain rule), so this notion is well-defined.

**Definition 2.5.** The tangent space \(\tilde{T}_p M\) of \(M\) at \(p\) is the collection of all smooth curves \(c : (-\epsilon, \epsilon) \to M\) with \(c(0) = p\), modulo this equivalence relation.

**Definition 2.6.** If \(\Psi : M \to N\) is a smooth mapping between manifolds, then \(\Psi_* : \tilde{T}_p M \to \tilde{T}_{\Psi(p)} N\) is the linear map defined by
\[(\Psi_* [c]_p) = [\Psi \circ c]_{\Psi(p)}.\] (2.10)

**Exercise 2.7.** Show that \(\tilde{T}_p M\) is a vector space over \(\mathbb{R}\) and
\[
\text{dim}(\tilde{T}_p M) = \text{dim}(M) = n, \quad (2.11)
\]
and that there is a natural isomorphism \(\iota_p : \tilde{T}_p M \to T_p M\) given by
\[
\iota_p([c]_p)[f]_p = \frac{d}{dt}(f \circ c)\big|_{t=0}.\] (2.12)

Here “natural” means that if \(\Psi : M \to N\) is a smooth mapping with \(\Psi(p) = q\), then the diagram
\[
\begin{array}{ccc}
\tilde{T}_p M & \xrightarrow{\Psi_*} & \tilde{T}_{\Psi(p)} N \\
\downarrow \iota_p & & \downarrow \iota_{\Psi(p)} \\
T_p M & \xrightarrow{\Psi_*} & T_{\Psi(p)} N
\end{array}
\] (2.13)

commutes.

**Remark 2.8.** Note that the first definition involving derivations on germs did not use any coordinate system, but the second definition did.
2.2 The tangent bundle

**Definition 2.9.** For a smooth manifold \( M \), the tangent bundle \( TM = \bigcup_{p \in M} T_p M \), and \( \pi : TM \to M \) is defined by \( \pi(X_p) = p \).

We next endow \( TM \) with a natural smooth manifold structure so that \( \pi \) is a smooth mapping. Given a coordinate system \( \phi : U \to \mathbb{R}^n \), we write the coordinate functions as \( x^i : U \to \mathbb{R} \) for \( i = 1 \ldots n \). Then \( i \)-th coordinate vector field denoted by

\[
\frac{\partial}{\partial x^i} \equiv \partial_i \tag{2.14}
\]

is defined by

\[
\partial_i[x^j]_p = \delta^j_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{2.15}
\]

the Kronecker delta symbol. Equivalently, using the definition of the tangent space as equivalence classes of curves, we can define

\[
(\partial_i)_p = [\phi^{-1}(x^1, \ldots, t, \ldots x^n)]_p, \tag{2.16}
\]

where \( \phi(p) = (x^1, \ldots, x^n) \). Letting \( \frac{\partial}{\partial t^i} \) denote the usual partial derivative operator in Euclidean space, clearly we have the relation

\[
\phi_*(\frac{\partial}{\partial x^i})_p = \frac{\partial}{\partial t^i}_{\phi(p)} \tag{2.17}
\]

Also note that

\[
\left( \frac{\partial}{\partial x^i} f \right)(p) = \frac{\partial (f \circ \phi)}{\partial t^i}(\phi(p)) \tag{2.18}
\]

Given a coordinate system \( \phi : U \to \mathbb{R}^n \), we define “inverse” local coordinates on \( TM \)

\[
\Phi : \phi(U) \times \mathbb{R}^n \to TM \tag{2.19}
\]

by

\[
\Phi(x, v) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}_{\phi^{-1}(x)}, \tag{2.20}
\]

where \( v = (v^1, \ldots, v^n) \).

Let us consider coordinate systems \( (U_\alpha, \phi_\alpha) \) and \( (U_\beta, \phi_\beta) \) on \( M \) such that \( U_\alpha \cap U_\beta \neq \emptyset \). Then we have

\[
\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}_{\phi_\alpha^{-1}(x)}, \tag{2.21}
\]
and

\[ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \sum_{i=1}^{n} \tilde{v}^i \left. \frac{\partial}{\partial x^i_{\beta}} \phi_{\beta}^{-1}(x) \right| \cdot \quad (2.22) \]

We claim that

\[ \frac{\partial}{\partial x^i_{\beta}} = \sum_{j=1}^{n} \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}} \frac{\partial}{\partial x^j_{\alpha}}. \quad (2.23) \]

The above formula is easily proved by plugging in the function \( x^j_{\alpha} \) into each side, and using the defining property (2.15).

So then

\[ \Phi_{\alpha}^{-1} \circ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \Phi_{\alpha}^{-1} \left( \sum_{i=1}^{n} \tilde{v}^i \left. \frac{\partial}{\partial x^i_{\beta}} \phi_{\beta}^{-1}(x) \right| \right) \]

\[ = \Phi_{\alpha}^{-1} \left( \sum_{i=1}^{n} \tilde{v}^i \sum_{j=1}^{n} \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}} \frac{\partial}{\partial x^j_{\alpha}} \phi_{\beta}^{-1}(x) \right) \]

\[ = \Phi_{\alpha}^{-1} \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \tilde{v}^i \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}} \right) \frac{\partial}{\partial x^j_{\alpha}} \phi_{\beta}^{-1}(x) \right) \]

\[ = \Phi_{\alpha}^{-1} \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \tilde{v}^i \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}} \right) \frac{\partial}{\partial x^j_{\alpha}} \phi_{\alpha}^{-1} \circ \phi_{\beta}^{-1}(x) \right). \quad (2.24) \]

Consequently,

\[ \Phi_{\alpha}^{-1} \circ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \left( \phi_{\alpha} \circ \phi_{\beta}^{-1}(x), \sum_{i=1}^{n} \tilde{v}^i \frac{\partial x^i_{\alpha}}{\partial x^i_{\beta}} \right) \quad (2.25) \]

The functions \( \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}} \) are functions on \( M \), and from (2.18) we have

\[ \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}}(p) = \left( \frac{\partial (x_{\alpha} \circ x_{\beta}^{-1})^j}{\partial t^i_{\beta}} \right)(\phi_{\beta}(p)). \quad (2.26) \]

These functions are smooth since the overlap mappings are smooth. Furthermore, are linear isomorphisms in the second variable (invertibility follows from the chain rule, since \( \phi_{\alpha} \circ \phi_{\beta}^{-1} \) is assumed to be a diffeomorphism). Therefore, the overlap mappings are smooth diffeomorphisms. So we have the following properties of \( TM \):

- If \( M \) is a smooth manifold of dimension \( n \), then \( TM \) is a smooth manifold of dimension \( 2n \).
- For any \( p \in M \), \( T_pM = \pi^{-1}(p) \) is an \( n \)-dimensional vector space.
- For any \( M \), \( TM \) is noncompact.

(It is a vector bundle, which we will discuss in detail later, but it has special properties that any old vector bundle does not possess.)
3 Lecture 3

By the above, a smooth mapping \( f : M \to N \) induces a mapping
\[
f_* : TM \to TN,
\]
(3.1)
called a “push-forward”, which is linear on fibers and which makes the following diagram commutes
\[
\begin{array}{ccc}
TM & \xrightarrow{f_*} & TN \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M & \xrightarrow{f} & N
\end{array}
\]
(3.2)

It is easy to check that \( f_* : TM \to TN \) is a smooth mapping where \( TM \) and \( TN \) are given the smooth manifold structures from the previous lecture.

Note the following important proposition.

**Proposition 3.1 (The chain rule).** If \( f : M \to N \), and \( h : N \to M' \) are smooth maps, then
\[
(h \circ f)_* = h_* \circ f_* : TM \to TM'
\]
(3.3)

**Proof.** For \( f : M \to N \), choose a local coordinate \( \phi \) on \( M \) and \( \psi \) on \( N \) such that the mapping
\[
f_c = \psi \circ f \circ \phi^{-1}
\]
(3.4)

is defined. Then
\[
(f_c)_* \left( \frac{\partial}{\partial t^i} \right) = \sum_{k=1}^{m} \frac{\partial f^k}{\partial t^i} \frac{\partial}{\partial s^k},
\]
(3.5)

for \( i = 1 \ldots n \), where \( n = \dim(M) \) and \( m = \dim(N) \). Using this, the result is then reduced to the ordinary chain rule (details left to the reader).

3.1 Review of theory of vector bundles

**Definition 3.2.** A smooth real vector bundle of rank \( k \) over a smooth manifold \( M^n \) is a topological space \( E \) together with a smooth projection
\[
\pi : E \to M
\]
(3.6)
such that

- For \( p \in M \), \( \pi^{-1}(p) \) is a vector space of dimension \( k \) over \( \mathbb{R} \).
- There exists local trivializations, that is, there are smooth mappings
\[
\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to E
\]
(3.7)

which maps \( p \times \mathbb{R}^k \) linearly onto the fiber \( \pi^{-1}(p) \) for every \( p \in U_\alpha \).
The transition functions of a bundle are defined as follows.

\[ \varphi_{\alpha \beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \quad (3.8) \]

defined by

\[ \varphi_{\alpha \beta}(x)(v) = \pi_2(\Phi^{-1}_\alpha \circ \Phi_\beta(x, v)), \quad (3.9) \]

for \( v \in \mathbb{R}^k \).

On a triple intersection \( U_\alpha \cap U_\beta \cap U_\gamma \), we have the identity

\[ \varphi_{\alpha \gamma} = \varphi_{\alpha \beta} \circ \varphi_{\beta \gamma}. \quad (3.10) \]

Conversely, given a covering \( U_\alpha \) of \( M \) and transition functions \( \varphi_{\alpha \beta} \) satisfying (3.10), there is a vector bundle \( \pi : E \to M \) with transition functions given by \( \varphi_{\alpha \beta} \). (It turns out this bundle is uniquely defined up to bundle equivalence, which we will define below.) If the transitions function \( \varphi_{\alpha \beta} \) are \( C^\infty \), then we say that \( E \) is a smooth vector bundle.

**Example 3.3.** (The tangent bundle redux.) Given a coordinate system \((U_\alpha, x_\alpha)\) on a smooth manifold \( M \), let

\[ \Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^n v_i \frac{\partial}{\partial x^i_\alpha} \bigg|_x. \quad (3.11) \]

On \( U_\beta \), we have

\[ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \sum_{i=1}^n \tilde{v}_i \frac{\partial}{\partial x^i_\beta} \bigg|_x. \quad (3.12) \]

Recall that

\[ \frac{\partial}{\partial x^i_\beta} = \sum_{j=1}^n \frac{\partial x^i_\alpha}{\partial x^j_\beta} \frac{\partial}{\partial x^j_\alpha}, \quad (3.13) \]

so then

\[ \Phi^{-1}_\alpha \circ \Phi_\beta(x, (\tilde{v}^1, \ldots, \tilde{v}^n)) = \Phi^{-1}_\alpha \left( \sum_{i=1}^n \tilde{v}^i \frac{\partial}{\partial x^i_\beta} \right) \]

\[ = \Phi^{-1}_\alpha \left( \sum_{i=1}^n \tilde{v}^i \sum_{j=1}^n \frac{\partial x^i_\alpha}{\partial x^j_\beta} \frac{\partial}{\partial x^j_\alpha} \right) \]

\[ = \Phi^{-1}_\alpha \left( \sum_{j=1}^n \left( \sum_{i=1}^n \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^j_\beta} \right) \frac{\partial}{\partial x^j_\alpha} \right). \quad (3.14) \]

Consequently,

\[ \left( \varphi_{\alpha \beta}(x)(v^1, \ldots, v^n) \right)^j = \sum_{i=1}^n \tilde{v}^i \frac{\partial x^j_\alpha}{\partial x^i_\beta}, \quad (3.15) \]
3.2 Categories

A bundle mapping between vector bundles $E_1$ over $M$ and $E_2$ over $N$ is a mapping $F : E_1 \to E_2$ which maps fibers linearly to fibers and covers a smooth mapping between the base spaces. That is, the diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_2 \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M & \xrightarrow{f} & N
\end{array}
$$

(3.16)

commutes.

**Definition 3.4.** The category $\textbf{Vect}$ of smooth vector bundles over smooth manifolds is the collection of all vector bundle (of any rank) over smooth manifolds. The morphisms are the bundle mappings.

We therefore have a functor $F : \text{Man}^\infty \to \textbf{Vect}$ where $\textbf{Vect}$ is the category of smooth vector bundles over smooth manifolds given by $M \to TM$ and $f : M \to N$ maps to $f_* : TM \to TN$. The mapping $F$ satisfies $F(id_X) = Id_{TM}$ and by by Proposition 3.1 $F(f_1 \circ f_2) = F(f_1) \circ F(f_2)$, so this is a covariant functor.

Next, we define another category.

**Definition 3.5.** For a fixed smooth manifold $M$, the category $\textbf{Vect}(M)$ is the collection of smooth vector bundles over $M$ (of any rank). A morphism in this category is a mapping $F : E_1 \to E_2$ covering the identity mapping, that is, the diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_2 \\
\downarrow{\pi_M} & & \downarrow{\pi_M} \\
M & \xrightarrow{id_M} & M
\end{array}
$$

(3.17)

We say that bundles $E_1$ and $E_2$ over $M$ are isomorphic if there exists an invertible bundle mapping between $E_1$ and $E_2$. If $E$ is isomorphic to the trivial bundle over $M$, $\pi_M : M \times \mathbb{R}^k \to M$ defined by $\pi_M(p, v) = p$, the we say that $E$ is trivial.

We next express the above in coordinates. Assume we have a covering $U_\alpha$ of $M$ such that $E_1$ has trivializations $\Phi_\alpha$ and $E_2$ has trivializations $\Psi_\alpha$. Then any vector bundle mapping gives locally defined functions

$$
f_\alpha : U_\alpha \to Hom(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})
$$

(3.18)

defined by

$$
f_\alpha(x)(v) = \pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)).
$$

(3.19)

It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$
f_\alpha = \varphi_\alpha E_2 f_\beta \varphi_\beta E_1.
$$

(3.20)
equivalently,

\[ \varphi_{\beta\alpha}^{E_2} f_{\alpha} = f_{\beta} \varphi_{\beta\alpha}^{E_1}. \]

(3.21)

Bundles are \( E_1 \) and \( E_2 \) are equivalent if there exists an invertible bundle mapping \( f : E_1 \to E_2 \). Obviously, this means \( \text{rank}(E_1) = \text{rank}(E_2) \) and non-singularity of the local representatives, that is, \( \det(f_{\alpha}) \neq 0 \). A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, \( E \) is trivial if there exist functions

\[ f_\alpha : U_\alpha \to GL(k, \mathbb{R}) \]

(3.22)

such that

\[ \varphi_{\beta\alpha} = f_{\beta}f_{\alpha}^{-1}. \]

(3.23)

**Remark 3.6.** In the above, we only defined morphisms in the category of vector bundle to be mappings covering the identity map. We could have instead morphisms to cover arbitrary diffeomorphisms. This would lead to a coarser notion of equivalence. More on this later.

### 4 Lecture 4: Operations on bundles

#### 4.1 Direct sums

If \( V_1, \ldots, V_k \) are vector spaces over \( \mathbb{R} \), then the direct sum \( V_1 \oplus \cdots \oplus V_k \) is the Cartesian product \( V_1 \times \cdots \times V_k \) with the following vector space structure:

\[ c(v_1, \ldots, v_k) = (cv_1, \ldots, cv_k) \]

(4.1)

\[ (v_1, \ldots, v_k) + (v'_1, \ldots, v'_k) = (v_1 + v'_1, \ldots, v_k + v'_k), \]

(4.2)

for \( c \in \mathbb{R} \). The space \( V_1 \oplus \cdots \oplus V_k \) satisfies the following “universal” mapping property. For \( 1 \leq i \leq k \), let \( \iota_i : V_i \to V_1 \oplus \cdots \oplus V_k \) be the inclusion mapping

\[ \iota_i : v \mapsto (0, \ldots, \underbrace{v}, \ldots, 0). \]

(4.3)

Let \( W \) be any vector space, and \( f_i : V_i \to W \) be linear mappings for \( 1 \leq i \leq k \). Then there is a *unique* linear map \( f : V_1 \oplus \cdots \oplus V_k \to W \) which makes the following diagram

\[
\begin{array}{ccc}
V_i & \xrightarrow{\iota_i} & V_1 \oplus \cdots \oplus V_k \\
\downarrow{f_i} & & \downarrow{f} \\
W & & W
\end{array}
\]

commute for \( 1 \leq i \leq k \).
Exercise 4.1. (i) Show that any vector space $V$ with the above universal mapping property is isomorphic to the direct sum. (ii) Prove that

$$\dim_{\mathbb{R}}(V_1 \oplus \cdots \oplus V_k) = \sum_{i=1}^{k} \dim_{\mathbb{R}}(V_i). \quad (4.4)$$

(iii) Prove that for 3 vector spaces $V_1, V_2, V_3$ we have

$$(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3). \quad (4.5)$$

Definition 4.2. Let $V_i, i \in I$ be any collection of vector spaces. The Cartesian product $\Pi_{i \in I} V_i$ is the collection of all functions

$$f : I \to \bigcup_{i \in I} V_i, \quad (4.6)$$

such that $f(i) \in V_i$ for all $i \in I$. The direct product $\Pi_{i \in I} V_i$ is the Cartesian product with the vector space structure

$$cf(i) = cf(i) \quad (4.7)$$

$$(f + g)(i) = f(i) + g(i). \quad (4.8)$$

The projection $\pi_i : \Pi_{i \in I} V_i \to V_i$ is the mapping $\pi_i(f) = f(i)$. The above definition satisfies the following universal property. If $V$ is any vector space and $\phi_i : V \to V_i$ are linear mappings for $i \in I$, then there is a unique linear mapping $\phi : V \to \Pi_{i \in I} V_i$ such that the diagram

$$\begin{array}{ccc}
V & \xrightarrow{\phi_i} & V_i \\
\downarrow{\phi} & & \downarrow{\pi_i} \\
\Pi_{i \in I} V_i & & \\
\end{array}$$

commutes for each $i \in I$. This property uniquely characterizes the direct product.

Definition 4.3. Let $V_i, i \in I$ be any collection of vector spaces. The direct sum $\bigoplus_{i \in I} V_i$ is the subspace of the direct product consisting of the functions $f$ such that $f(i) \neq 0$ for only finitely many $i \in I$.

Exercise 4.4. (i) Show that the direct sum satisfies the first universal property. (ii) If the index set is finite, then the direct product is isomorphic to the direct sum.

Definition 4.5. The direct sum of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is the vector bundle $\pi : E_1 \oplus E_2 \to M$ defined by $\pi^{-1}(p) = \pi_1^{-1}(p) \oplus \pi_2^{-1}(p)$. If $\Phi_1 : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ and $\Phi_2 : U \times \mathbb{R}^l \to \pi_1^{-1}(U)$ are local trivializations then

$$\Phi : U \times (\mathbb{R}^k \oplus \mathbb{R}^l) \to \pi^{-1}(U) \quad (4.9)$$

defined by

$$\Phi(x, (v_1, v_2)) = (\Phi_1(x, v_1), \Phi_2(x, v_2)) \quad (4.10)$$

is a local trivialization for $E_1 \oplus E_2$. 

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Note, the transition functions satisfy
\[ \varphi_{E_1}^{E_1 \oplus E_2} = \varphi_{E_1}^{E_1} \oplus \varphi_{E_2}^{E_2} \in GL(k + l, \mathbb{R}), \] (4.11)
where this is the “block” matrix
\[ \varphi_{E_1}^{E_1 \oplus E_2}(x)(v, w) = \begin{pmatrix} \varphi_{E_1}^{E_1}(x)v & 0 \\ 0 & \varphi_{E_2}^{E_2}(x)w \end{pmatrix}. \] (4.12)

4.2 Tensor products

**Definition 4.6.** If \( A \) is any set, then the free vector space over \( A \) is
\[ \mathcal{F}(A) = \bigoplus_{a \in A} \mathbb{R}. \] (4.13)
This can be thought of as the vector space with basis elements \( a \in A \). That is, \( \mathcal{F}(A) \) is the set of formal sums
\[ \mathcal{F}(A) = \left\{ \sum_{a \in A} c_a a \mid c_a \neq 0 \text{ for only finitely many } a \in A \right\} \] (4.14)
with vector space structure
\[ c \sum_{a \in A} c_a a = \sum_{a \in A} (cc_a)a \] (4.15)
\[ \sum_{a \in A} c_a a + \sum_{a \in A} c'_a a = \sum_{a \in A} (c_a + c'_a)a. \] (4.16)

**Definition 4.7.** If \( V_1, \ldots, V_k \) are vector spaces over \( \mathbb{R} \), then the tensor product \( V_1 \otimes \cdots \otimes V_k \) is the free real vector space \( \mathcal{F}(V_1 \times \cdots \times V_k) \) modulo the subspace spanned by all elements of the form
\[ (v_1, \ldots, cv_i, \ldots, v_k) - c(v_1, \ldots, v_i, \ldots, v_k) \] (4.17)
\[ (v_i, \ldots, v_i + v'_i, \ldots, v_k) - (v_i, \ldots, v_i, \ldots, v_k) - (v_i, v'_i, \ldots, v_k), \] (4.18)
for \( c \in \mathbb{R} \).

The space \( V_1 \otimes \cdots \otimes V_k \) satisfies the universal mapping property as follows. Let \( W \) be any vector space, and \( F : V_1 \times \cdots V_k \to W \) be a multilinear mapping, i.e., \( F \) is linear when restricted to each factor, with the other variables held fixed. Then there is a unique linear map \( \tilde{F} : V_1 \otimes \cdots \otimes V_k \) which makes the following diagram
\[ V_1 \times \cdots \times V_k \xrightarrow{\pi} V_1 \otimes \cdots \otimes V_k \]
\[ \downarrow F \]
\[ \downarrow \tilde{F} \]
\[ W \]
commutative, where \( \pi \) is the projection to the quotient space, which we write as
\[ \pi(v_1, \ldots, v_k) = v_1 \otimes \cdots \otimes v_k. \] (4.19)
We say that an element in \( V_1 \otimes \cdots \otimes V_k \) of the form \( v_1 \otimes \cdots \otimes v_k \) is decomposable. A general element of \( V_1 \times \cdots \times V_k \) is not decomposable, but can always be written as a sum of decomposable elements.
Exercise 4.8. Prove that
\[
\dim_{\mathbb{R}}(V_1 \otimes \cdots \otimes V_k) = \dim_{\mathbb{R}}(V_1) \cdots \dim_{\mathbb{R}}(V_k).
\] (4.20)
Also, prove that for 3 vector spaces \( V_1, V_2, V_3 \) we have
\[
(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3).
\] (4.21)

Definition 4.9. The tensor product of vector bundles \( \pi_1 : E_1 \to M \) and \( \pi_2 : E_2 \to M \) is the vector bundle \( \pi : E_1 \otimes E_2 \to M \) defined by \( \pi^{-1}(p) = \pi_1^{-1}(p) \otimes \pi_2^{-1}(p) \). If \( \Phi_1 : U \times \mathbb{R}^k \to \pi_1^{-1}(U) \) and \( \Phi_2 : U \times \mathbb{R}^l \to \pi_1^{-1}(U) \) are local trivializations then consider
\[
F : U \times (\mathbb{R}^k \times \mathbb{R}^l) \to \pi^{-1}(U)
\] (4.22)
defined by
\[
F(x, (v_1, v_2)) = \Phi_1(x, v_1) \otimes \Phi_2(x, v_2).
\] (4.23)
This is clearly a multilinear mapping on each fiber, so by the universal property of tensor products, there is a unique induced mapping
\[
\tilde{F} : U \times (\mathbb{R}^k \otimes \mathbb{R}^l) \to \pi^{-1}(U)
\] (4.24)
which, using an isomorphism \( \mathbb{R}^k \otimes \mathbb{R}^l \cong \mathbb{R}^{kl} \), defines a local trivialization for \( E_1 \otimes E_2 \).

We could have equivalently defined the tensor product in terms of transition functions. To do this, note the following. If \( \phi_1 \in GL(k, \mathbb{R}) \) and \( \phi_2 \in GL(l, \mathbb{R}) \) then define
\[
\phi_1 \times \phi_2 : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k \otimes \mathbb{R}^l
\] (4.25)
by
\[
(\phi_1 \times \phi_2)(v_1, v_2) = \phi_1(v_1) \otimes \phi_2(v_2).
\] (4.26)
This is clearly a multilinear mapping, so by the universal property for tensor products, there is a unique induced mapping
\[
\phi_1 \otimes \phi_2 : \mathbb{R}^k \otimes \mathbb{R}^l \to \mathbb{R}^k \otimes \mathbb{R}^l
\] (4.27)
Given transition functions for \( E_1 \)
\[
\phi_{E_1}^{\alpha \beta} : U_{\alpha} \cap U_{\beta} \to GL(k, \mathbb{R}),
\] (4.28)
and transition functions for \( E_2 \)
\[
\phi_{E_2}^{\alpha \beta} : U_{\alpha} \cap U_{\beta} \to GL(l, \mathbb{R}),
\] (4.29)
we define
\[
\phi_{E_1 \otimes E_2}^{\alpha \beta} = \phi_{E_1}^{\alpha \beta} \otimes \phi_{E_2}^{\alpha \beta} \in GL(kl, \mathbb{R}),
\] (4.30)
where we choose some isomorphism \( \mathbb{R}^k \otimes \mathbb{R}^l \cong \mathbb{R}^{kl} \).
5 Lecture 5

5.1 Dual bundles

Definition 5.1. The dual of a vector space $V$ is $V^* = \text{Hom}(V, \mathbb{R})$, which is the space of all linear mappings from $V$ to $\mathbb{R}$.

Exercise 5.2. If $V$ is finite-dimensional, show that $V^* \cong V$ and thus $\dim(V^*) = \dim(V)$.

Definition 5.3. The dual of a vector bundle $\pi : E \to M$ is the vector bundle $\Pi : E^* \to M$ defined by $\Pi^{-1}(p) = (\pi^{-1}(p))^*$. If $\Phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is a local trivialization then

$$\Phi^* : U \times (\mathbb{R}^k)^* \to \pi^{-1}(U)$$

defined by

$$\Phi^*(x, f)(v_p) = f(\pi_2 \circ \Phi^{-1}(v_p))$$

is a local trivialization for $E^*$.

Exercise 5.4. Show that the transition functions of $E^*$ are

$$\varphi_{a\beta}^{E^*} = ((\varphi_{a\beta}^E)^{-1})^T = (\varphi_{\beta a}^E)^T.$$  (5.3)

5.2 Sections of bundles

Definition 5.5. Let $\pi : E \to M$ be a vector bundle. A section of a bundle is a smooth mapping $s : M \to E$ such that $\pi \circ s = \text{id}_M$. The space of sections is denoted by $\Gamma(E)$.

In other words, $s(x) \in E_x$, $s$ maps $x$ to a vector in the fiber over $x$. In terms of local trivializations we have the following. Let

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha)$$

be a local trivialization. Then

$$s_\alpha = \pi_2 \circ \Phi_\alpha^{-1} \circ s : U_\alpha \to \mathbb{R}^k$$

is called a local representative of $s$ with respect to $\Phi_\alpha$. On $U_\beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\Phi_\beta : U_\beta \times \mathbb{R}^k \to \pi^{-1}(U_\beta).$$

Recall that the transition functions of a bundle are

$$\varphi_{a\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$$

(5.7)
defined by
\[ \varphi_{\alpha\beta}(x)(v) = \pi_2(\Phi_{\alpha}^{-1} \circ \Phi_{\beta}(x,v)), \] (5.8)
for \( v \in \mathbb{R}^k \). Then for any \( e_x \in \pi^{-1}(x) \), we have
\[ \phi_{\alpha\beta}(x)(\pi_2 \circ \Phi_{\beta}^{-1}(e_x)) = \pi_2 \circ \Phi_{\alpha}^{-1}(e_x). \] (5.9)
Choosing \( e_x = s(x) \) we have
\[ \phi_{\alpha\beta}(s)(\pi_2 \circ \Phi_{\beta}^{-1} \circ s(x)) = \pi_2 \circ \Phi_{\alpha}^{-1} \circ s(x), \] (5.10)
or simply
\[ \phi_{\alpha\beta}s = s\alpha, \text{ on } U_\alpha \cap U_\beta, \] (5.11)
which is the local transformation law for a section.

Conversely, if a bundle \( \pi : E \to M \) is given to us in terms of transition functions, then any collection of functions
\[ s_\alpha : U_\alpha \to \mathbb{R}^k \] (5.12)
satisfying (5.11) gives a well-defined smooth section \( s : M \to E \).

### 5.3 Riemannian metrics on real vector bundles

If \( \pi : E \to M \) is a real vector bundle, a Riemannian metric on \( E \) is a choice of smoothly varying positive definite symmetric inner product on each fiber. That is \( g \in \Gamma(E^* \otimes E^*) \) satisfying
\[ g(e_1, e_2) = g(e_2, e_1), \] (5.13)
and
\[ g(e, e) > 0 \text{ for } e \neq 0. \] (5.14)

**Proposition 5.6.** If \( E \) is any real vector bundle, then \( E \) admits a Riemannian metric.

**Proof.** Let
\[ \Phi_{\alpha} : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha) \] (5.15)
be a local trivialization for \( U_\alpha \) an open covering of \( M \) which is locally finite. For \( x \in U_\alpha \) and \( e_1, e_2 \in E_x \), define
\[ g_\alpha(e_1, e_2) = \langle \pi_2 \circ \Phi_{\alpha}^{-1}(e_1), \pi_2 \circ \Phi_{\alpha}^{-1}(e_2) \rangle, \] (5.16)
where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product on \( \mathbb{R}^k \). Next, let \( \chi_\alpha \) be a partition of unity subordinate to the cover \( U_\alpha \), that is
\[ \text{supp}(\chi_\alpha) \subset U_\alpha, \quad 0 \leq \chi_\alpha \leq 1, \text{ and } \sum_\alpha \chi_\alpha = 1. \] (5.17)
Define
\[ g(e_1, e_2) = \sum_{\alpha} \phi_\alpha g_\alpha(e_1, e_2). \] (5.18)

This is clearly symmetric since each \( g_\alpha \) is symmetric. It is positive definite since it is a finite sum of positive terms at each point for any non-zero vector.

\[ \square \]

**Corollary 5.7.** For any real vector bundle \( E, E^* \cong E \).

\[ \text{Proof.} \] Choose a Riemannian metric \( g \) on \( E \). Then the mapping \( \flat : E \to E^* \) defined by
\[ \flat(e_1)(e_2) = g(e_1, e_2) \] (5.19)
is an isomorphism on fibers, and covers the identity map. \[ \square \]

**Definition 5.8.** Given vector bundles \( \pi_1 : E_1 \to M \) and \( \pi_2 : E_2 \to M \) over the same base space \( M \), we say that \( E_1 \) is a subbundle of \( E_2 \), written \( E_1 \subset E_2 \) if each fiber \( \pi_1^{-1}(x) \subset \pi_2^{-1}(x) \) is a vector subspace.

In bundle terms, existence of a Riemannian metric implies that there is always a non-zero section of \( E^* \otimes E^* \), which says that
\[ E^* \otimes E^* = A \oplus B \] (5.20)
always admits a trivial 1-dimensional subbundle \( A \). (This is because \( \text{span}(g(x)) \) defines a 1-dimensional subspace of every fiber, and the fact that any 1-dimensional bundle with a non-vanishing section must be a trivial bundle).

Of course, the metric gives a isomorphism
\[ E^* \otimes E^* \cong E^* \otimes E \cong \text{Hom}(E, E), \] (5.21)
and the latter bundle always admits the identity section. The latter choice is canonical, but the sub-bundle \( A \) is not.

**Remark 5.9.** We will soon see that there is an isomorphism of bundles
\[ E \otimes E \cong \mathbb{R} \oplus S_0^2(E) \oplus \Lambda^2(E), \] (5.22)
which you can think of as decomposing a matrix into a pure trace part, a symmetric traceless part, and a skew-symmetric part.

**Definition 5.10.** If \( E_1 \subset E_2 \) is a subbundle, then the quotient bundle \( E_2/E_1 \) is the vector bundle with fiber \( \pi_2^{-1}(x)/\pi_1^{-1}(x) \) over \( x \).

**Exercise 5.11.** Prove that the quotient bundle is a vector bundle. That is, find local trivializations for \( E_2/E_1 \).

Note the following corollary.
Corollary 5.12. If $E_1 \subset E$ is a sub-bundle, then there exists a subbundle $E_2 \subset E$ such that

$$E \cong E_1 \oplus E_2.$$  \hspace{1cm} (5.23)

Furthermore, the quotient bundle $(E/E_1) \cong E_2$.

Proof. Choose a Riemannian metric $g$ on $E$, and let $E_2 = (E_1)^\perp$. Use Gram-Schmidt to construct local trivializations for $(E_1)^\perp$ to show this is indeed a subbundle. The rest is just linear algebra. \hfill \Box

6 Lecture 6

6.1 Reduction of Structure group

Definition 6.1. If a bundle $\pi : E \to M$ is equivalent to a bundle which has transition functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to K$, where $K$ is a subgroup of $GL(k, \mathbb{R})$, then we say that the structure group of $E$ can be reduced to $K$.

Another way to state the results from the previous section is as follows.

Proposition 6.2. We have the following.

- A bundle is trivial if and only if its structure group can be reduced to $\{Id\}$.
- The structure group of any real vector bundle $\pi : E \to M$ of rank $k$ can be reduced to $O(k)$.

Proof. The first case is obvious. For the second case, from above $E$ admits a Riemannian metric. By Gram-Schmidt, for any point $x \in M$, there exists a neighborhood $U_x$ and a local basis of sections $\{e_1, \ldots, e_k\}$ which are orthonormal at every point in $U_x$. Define local trivializations by

$$\Phi_\alpha(x, (v^1, \ldots, v^n)) = \sum_{i=1}^k v^i e_i.$$  \hspace{1cm} (6.1)

Then overlaps maps then necessarily satisfy

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k),$$  \hspace{1cm} (6.2)

where $O(k)$ is the orthogonal group of $k \times k$ real matrices satisfying $AA^T = I_k$. \hfill \Box
6.2 Real line bundles

Note for a real 1-dimensional line bundle \( \pi : L \to M \), we have that the structure group can be reduced to \( O(1) = \{ \pm 1 \} \). Consider the set
\[
\tilde{M} = \{ v \in L \mid g(v, v) = 1 \}.
\] (6.3)

Since there are exactly two unit norm vectors in any fiber, we have that \( \pi : \tilde{M} \to M \) is a 2-fold covering space. So any real line bundle give an associated 2-fold covering space. Conversely, any 2-fold covering space gives a real line bundle, which is uniquely determined up to equivalence. To see this, note that a 2-fold covering space can be viewed as a fiber bundle with group \( \mathbb{Z}_2 \), and viewing \( \mathbb{Z}_2 = \{ \pm 1 \} \subset GL(1, \mathbb{R}) \), we naturally obtain an associated real line bundle.

**Remark 6.3.** Therefore real line bundles over \( M \) are in one-to-one correspondence with 2-fold covering spaces of \( M \), up to equivalence. Using some covering space theory, the 2-fold coverings correspond to index 2 subgroups of \( \pi_1(M) \), which is
\[
\text{Hom}(\pi_1(M), \mathbb{Z}_2).
\] (6.4)

Later we will see that
\[
\text{Hom}(\pi_1(M), \mathbb{Z}_2) \cong \text{Hom}(H_1(M), \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2),
\] (6.5)
the first cohomology group with \( \mathbb{Z}_2 \) coefficients.

**Remark 6.4.** Another way to understand this is through the following. After reduction the structure group to \( \mathbb{Z}_2 \), the transition functions of the bundle are given by
\[
\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2.
\] (6.6)

The condition on transition functions
\[
\phi_{\alpha\gamma} = \phi_{\alpha\beta} \phi_{\beta\gamma}
\] (6.7)
says that \( \phi_{\alpha\beta} \) form a Čech 1-cocycle, so
\[
\phi_{\alpha\beta} \in \check{H}^1(\mathfrak{U}, \mathbb{Z}_2),
\] (6.8)
the first Čech cohomology group with coefficient in the constant sheaf \( \mathbb{Z}_2 \) with respect to the open covering \( \mathfrak{U} = \{ U_\alpha \}_{\alpha \in \mathcal{I}} \).

For \( \phi_{\alpha\beta} \) to be a co-boundary, note that a 0-cocycle is a collection
\[
f_\alpha : U_\alpha \to \mathbb{Z}_2
\] (6.9)
and
\[
(\delta f)_{\alpha\beta} = f_\beta \cdot f_\alpha^{-1} : U_\alpha \cap U_\beta \to \mathbb{Z}_2.
\] (6.10)
So for $\phi_{\alpha\beta}$ to be a co-boundary, we have $f_\alpha$ so that
\[
\phi_{\alpha\beta} = f_\beta f_\alpha^{-1}
\tag{6.11}
\]
on $U_\alpha \cap U_\beta$ which is exactly the condition for the bundle to the equivalent to a trivial bundle.

For a sufficiently “good” open cover, it turns out that
\[
\tilde{H}^1(M, \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2),
\tag{6.12}
\]
the ordinary first singular cohomology group with $\mathbb{Z}_2$ coefficients.

**Example 6.5.** (Tautological bundle on $\mathbb{R}P^n$) Recall that $\mathbb{R}P^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Equivalently, $\mathbb{R}P^n$ is the space of vectors in $\mathbb{R}^{n+1}$ modulo the equivalence relation
\[
(v_1, \ldots, v_{n+1}) \sim (cv_1, \ldots, cv_{n+1}), \ c \neq 0.
\tag{6.13}
\]
Define
\[
\gamma^1_n = \{(\lfloor x \rfloor, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in \lfloor x \rfloor\}
\tag{6.14}
\]
We claim that $\gamma^1_n$ is a nontrivial 1-dimensional bundle over $\mathbb{R}P^n$. Assume by contradiction that it were the trivial bundle. Then there would exists a nowhere vanishing section $\sigma : \mathbb{R}P^n \to \gamma^1_n$. This is a mapping
\[
\sigma : \mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}^{n+1}
\tag{6.15}
\]
of the form for $x \in S^n$,
\[
\sigma(\lfloor x \rfloor) = (\lfloor x \rfloor, c(x) \cdot x)
\tag{6.16}
\]
For this to be well-defined, we require that $c(x) : S^n \to \mathbb{R}$ is a function satisfying $c(-x) = -c(x)$. Since $c$ must take negative and positive values, by the intermediate value theorem, $c(x_0) = 0$ for some $x_0$, which is a contradiction.

For $n = 1$, we have that $\mathbb{R}P^1 \cong S^1$. There is the trivial bundle $S^1 \times \mathbb{R}$. We also know that there is the Mobius strip $S^1 \times \mathbb{R}$, which we can view as a line bundle over $S^1$.

**Exercise 6.6.** Show that $\gamma^1_1$ is isomorphic to the Mobius bundle.

### 7 Lecture 7

#### 7.1 Exterior powers

Let $V$ be a real vector space. The exterior algebra $\Lambda(V)$ is defined as
\[
\Lambda(V) = \left\{ \bigoplus_{k \geq 0} V^\otimes k \right\} / \mathcal{I} = \bigoplus_{k \geq 0} \left\{ V^\otimes k / \mathcal{I}_k \right\} = \bigoplus_{k \geq 0} \Lambda^k V,
\tag{7.1}
\]
where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $v \otimes v \in V \otimes V$, and $\mathcal{I}_k = V^\otimes k \cap \mathcal{I}$. The wedge product of $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$ is just the multiplication induced by the tensor product in this algebra, that is, lift $v$ and $w$ to $\tilde{v} \in V^\otimes p$, and $\tilde{w} \in V^\otimes q$, and define $v \wedge w = \pi(\tilde{v} \otimes \tilde{w})$, where $\pi : V^\otimes p+q \to \Lambda^{p+q}V$ is the projection. This is easily seen to be well-defined. We say that an element in $\Lambda^k(V)$ of the form $v_1 \wedge \cdots \wedge v_k$ is decomposable. A general element of $\Lambda^k(V)$ is not decomposable, but can always be written as a sum of decomposable elements.

The space $\Lambda^k(V)$ satisfies the universal mapping property as follows. Let $W$ be any vector space, and let

\begin{equation}
F : V \times \cdots \times V \to W
\end{equation}

be an alternating multilinear mapping. That is, $F$ is multilinear and $F(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

\begin{equation}
\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{\pi} & \Lambda^k(V) \\
\downarrow F \quad & & \downarrow \tilde{F} \\
W & &
\end{array}
\end{equation}

commutative, where $\pi$ is the projection, which we denote as

\begin{equation}
\pi(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k.
\end{equation}

**Exercise 7.1.** Prove the following properties of the wedge product.

- **Bilinearity**: $(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w$, and $(cv) \wedge w = c(v \wedge w)$ for $c \in \mathbb{R}$.
- If $v \in \Lambda^p(V)$ and $w \in \Lambda^q(V)$, then $v \wedge w = (-1)^{pq}w \wedge v$.
- **Associativity** $(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3)$.

**Exercise 7.2.** If $\dim_{\mathbb{R}}(V) = n$, prove that $\Lambda^k(V) = \{0\}$ if $k > n$,

\begin{equation}
\dim(\Lambda^k(V)) = \binom{n}{k} \text{ if } 0 \leq k \leq n,
\end{equation}

and

\begin{equation}
\dim(\Lambda(V)) = 2^n,
\end{equation}

**Definition 7.3.** For a real vector bundle $\pi : E \to M$, we define $\Pi : \Lambda^p(E) \to M$ by $\Pi^{-1}(x) = \Lambda^p(\pi^{-1}(x))$. If $\Phi : U \times \mathbb{R}^k \to \pi_1^{-1}(U)$ is a local trivialization for $E$, then consider the mapping

\begin{equation}
F : U \times \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \Pi^{-1}(U)
\end{equation}
defined by

\[ F(x, (v_1, \ldots, v_p)) = \Phi(x, v_1) \wedge \cdots \wedge \Phi(x, v_k) \quad (7.7) \]

This is clearly an alternating multilinear mapping on fibers, so by the universal property, there is an unique induced mapping

\[ \tilde{F} : U \times \Lambda^p(\mathbb{R}^k) \to \Pi^{-1}(U) \quad (7.8) \]

which is a local trivialization for \( \Lambda^p(E) \).

We can equivalently define the \( p \)th exterior power in terms of transition functions. To do this, note that for any linear map \( f : \mathbb{R}^k \to \mathbb{R}^k \), there is a naturally induced mapping

\[ \Lambda^p f : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \quad (7.9) \]

define as follows. Define

\[ F : \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \Lambda^p(\mathbb{R}^k) \quad (7.10) \]

by

\[ F(v_1, \ldots, v_p) = f(v_1) \wedge \cdots \wedge f(v_p) \quad (7.11) \]

This is clearly an alternating multilinear mapping, so by the universal property, there exists a unique mapping

\[ \Lambda^p f = \tilde{F} : \Lambda^p(\mathbb{R}^k) \to \Lambda^p(\mathbb{R}^k) \quad (7.12) \]

therefore for any vector bundle \( E \), the \( p \)th exterior power \( \Lambda^p(E) \) is defined to be the bundle with transition functions

\[ \varphi^{\Lambda^p(E)}_{\alpha\beta} = \Lambda^p(\varphi^E_{\alpha\beta}) \quad (7.13) \]

Putting all of these together, we can define the following.

**Definition 7.4.** For a real vector bundle \( \pi : E \to M \), define the exterior algebra bundle \( \Lambda(E) = \bigoplus_{p=0}^{k} \Lambda^p(E) \).

Note in the above discussion, if we sum together all of the \( \Lambda^p f \) mappings, we get an induced mapping between the exterior algebras

\[ \Lambda(f) : \Lambda(\mathbb{R}^k) \to \Lambda(\mathbb{R}^k) \quad (7.14) \]

which satisfies

\[ \Lambda(f)(\alpha \wedge \beta) = \Lambda(f)(\alpha) \wedge \Lambda(f)(\beta) \quad (7.15) \]

Therefore, the wedge product gives an algebra structure on each fiber of \( \Lambda(E) \).
7.2 Orientability of real bundles

Note that if $V$ is an $n$-dimensional vector space, then $\Lambda^n V$ is 1-dimensional. So if $L : V \to V$ is a linear transformation then $\Lambda^n L : \Lambda^n V \to \Lambda^n V$ is an endomorphism of a 1-dimensional vector space. Therefore $\Lambda^n(\omega) = c \cdot \omega$ for some scalar $c$. So we can make the following definition:

**Definition 7.5.** For a linear transformation $L : V \to V$, define $\det(L)$ to be the real number so that

$$\Lambda^n L(\omega) = \det(L) \cdot \omega. \quad (7.16)$$

**Exercise 7.6.** Show that this definition of determinant agrees with the usual linear algebra definition of determinant.

**Proposition 7.7.** Let $\pi : E \to M$ be a real vector bundle of rank $k$. The following are equivalent.

- The line bundle $\Lambda^k(E)$ is trivial.
- $\Lambda^k(E)$ admits a nowhere zero section.
- The double cover $\tilde{M}$ corresponding to $\Lambda^k(E)$ is a trivial 2-fold covering space.
- The structure group of $E$ can be reduced to

$$GL_+(k, \mathbb{R}) \equiv \{ A \in GL(k, \mathbb{R}) \mid \det(A) > 0 \} \quad (7.17)$$

- The structure group of $E$ can be reduced to $SO(k)$

**Proof.** The proof follows from the above discussion, with the following remarks. If $e_1, \ldots, e_k$ is a local basis of sections, we say that $\{ e_1, \ldots, e_k \}$ is oriented if

$$e_1 \wedge \cdots \wedge e_k = f \omega, \quad (7.18)$$

with $f > 0$ and $\omega \in \Lambda^k(E)$ is the nowhere zero section. Restricting to local trivializations arising from oriented local bases of sections will give a reduction of structure group to $GL_+(k, \mathbb{R})$. \qed

**Definition 7.8.** We say that a real vector bundle $\pi : E \to M$ is **orientable** if any of the equivalent conditions in Proposition 7.7 are satisfied.

**Remark 7.9.** If we use the 2-fold covering notion, then we see that if $\pi_1(M) = \{ e \}$ then every vector bundle over $M$ is orientable. This is because any covering of a simply connected space is trivial. (Actually, we just need to assume that $H^1(M, \mathbb{Z}_2) = 0$.) Thus, every vector bundle over $S^n$ is orientable for $n \geq 2$.

**Example 7.10.** Returning to $\mathbb{R}P^n$, since $\mathbb{R}P^n$ is double covered by $S^n$, we have $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$. Therefore there are exactly 2 real line bundles over $\mathbb{R}P^n$, the trivial bundle and the tautological line bundle. Note that if we put a Riemannian metric on the tautological bundle $\pi : \gamma_1^n \to \mathbb{R}P^n$, then the total space of the unit sphere bundle is just $S^n$. But for the trivial bundle over $\mathbb{R}P^n$, the unit sphere bundle is just 2 copies of $\mathbb{R}P^n$. 23
8 Lecture 8

8.1 Induced mappings

Recall that if \( L : V \rightarrow W \) is a linear mapping between vector spaces, then there is a mapping, \( L^* : W^* \rightarrow V^* \) called the transpose, defined by the following. If \( \omega \in W^* \), and \( v \in V \), then

\[
(L^*\omega)(v) = \omega(Lv). \tag{8.1}
\]

This is called the transpose for the following reason. Let \( \dim(V) = n \), and \( \dim(W) = m \). Let \( e_1, \ldots, e_n \) be a basis of \( V \) and \( f_1, \ldots, f_m \) be a basis of \( W \). Let \( e^1, \ldots, e^n \), and \( f^1, \ldots, f^m \) denote the dual bases, that is

\[
e^i(e_j) = \delta^i_j, \quad 1 \leq i, j \leq n \tag{8.2}
\]

\[
f^i(f_j) = \delta^i_j, \quad 1 \leq i, j \leq m. \tag{8.3}
\]

We define the quantities \( L^i_j \), \( 1 \leq i \leq n \), \( 1 \leq j \leq m \), by

\[
Le_i = L^i_j f_j. \tag{8.4}
\]

Note that if we write \( v \in V \) as \( v = v^i e_i \), and \( w \in W \) as \( w = w^i f_i \), then

\[
Lv = L(v^i e_i) = v^i L(e_i) = (v^i L^i_j) f_j. \tag{8.5}
\]

So the components of a vector transform like

\[
\{v^i\} \mapsto \{L^i_j v^i\}, \tag{8.6}
\]

which is the matrix corresponding to the transformation \( L \).

We define the quantities \( (L^*)^i_j \), \( 1 \leq i \leq m \), \( 1 \leq j \leq n \), by

\[
L^* f^i = (L^*)^i_j e^j. \tag{8.7}
\]

Plugging in the dual bases, we compute

\[
(L^* f^i)(e_k) = (L^*)^i_j e^j(e_k) = (L^*)^i_j \delta^j_k = (L^*)^i_k. \tag{8.8}
\]

However, by the definition of the transpose mapping, we have

\[
(L^* f^i)(e_k) = f^i(L e_k) = f^i L^i_k f_j = L^i_k f^i(f_j) = L^i_k \delta^j_j = L^i_k. \tag{8.9}
\]

So if we write \( \omega \in V^* \) as \( \omega^i e^i \) and \( \eta \in W^* \) as \( \eta^j f^j \), the components of a dual vector transform like

\[
\{\eta^j\} \mapsto \{L^i_j \eta^i\}. \tag{8.10}
\]

So the matrix corresponding to \( L^* \) in the dual basis is indeed the transpose matrix.
The mapping $L^* : W^* \to V^*$ induces a mapping

$$(L^*)^p : W^* \times \cdots \times W^* \to (V^*)^{\otimes p} \quad (8.11)$$

by

$$(L^*)^p (\alpha^1, \ldots, \alpha^p) \equiv (L^* \alpha^1) \otimes \cdots \otimes (L^* \alpha^p). \quad (8.12)$$

This mapping is a multilinear mapping, so by the universal property of tensor products, this induces a unique mapping

$$(L^*)^{\otimes p} : (W^*)^{\otimes p} \to (V^*)^{\otimes p}. \quad (8.13)$$

By composing with the projection $\pi : (V^*)^{\otimes p} \to \Lambda^p(V^*)$, we obtain an alternating multilinear mapping

$$(L^*)^p : (W^*)^{\otimes p} \to \Lambda^p(V^*). \quad (8.14)$$

Now by the universal property of exterior products, this induces a mapping

$$\Lambda^p(L^*) : \Lambda^p(W^*) \to \Lambda^p(V^*). \quad (8.15)$$

Note that by taking the direct sum on all $p$-s, we obtain a mapping between the full exterior algebras

$$\Lambda(L^*) : \Lambda(W^*) \to \Lambda(V^*) \quad (8.16)$$

which is an algebra homomorphism, that is

$$\Lambda(L^*)(\alpha \wedge \beta) = (\Lambda(L^*)\alpha) \wedge (\Lambda(L^*)\beta). \quad (8.17)$$

### 8.2 Pull-back bundles

If $M$ and $N$ are smooth manifolds, and $\pi_N : E \to N$ is a vector bundle over $N$, then given a smooth mapping $f : M \to N$, define

$$f^*E = \{(p, v) \in M \times E \mid f(p) = \pi_N(v)\}. \quad (8.18)$$

**Proposition 8.1.** The pullback $f^*E$ is a vector bundle over $M$, with projection given by $\pi_1(p, v) = p$, and the fiber $f^*E$ over $p \in M$ is identified with the fiber $E_{f(p)}$, i.e., the following diagram commutes

$$
\begin{array}{ccc}
  f^*(E) & \xrightarrow{\pi_1} & E \\
  \downarrow{\pi_2} & & \downarrow{\pi_N} \\
  M & \xrightarrow{f} & N.
\end{array}
$$

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Proof. Let \( \Phi : U \times \mathbb{R}^k \to \pi_N^{-1}(U) \) be a local trivialization for \( E \). The set \( f^{-1}(U) \) is open since \( f \) is continuous, and define
\[
f^*\Phi : f^{-1}(U) \times \mathbb{R}^k \to \pi_1^{-1}(f^{-1}(U))
\] (8.20)
by
\[
f^*\Phi(x, v) = (x, \Phi(f(x), v)).
\] (8.21)
The reader can verify that this is a local trivialization for \( f^*E \).

Next we note that sections can be pulled back to sections of the pullback bundle.

**Definition 8.2.** Let \( f : M \to N \) be a smooth mapping between smooth manifolds, and \( \pi : E \to N \) be a vector bundle over \( N \). If \( \sigma : N \to E \) is a section of \( E \), then \((\sigma \circ f)(x) = (x, \sigma(f(x)))\) is a section of \( \pi_1 : f^*E \to M \) and is called the pullback of \( \sigma \) under \( f \).

The fact that this is a section of the pullback bundle is almost obvious, we just need to check that
\[
\pi_1(\sigma \circ f)(x) = \pi_1(x, \sigma(f(x))) = x.
\] (8.22)

### 8.3 Push-forward of vector fields

Next, we restrict to tangent bundles. Let \( f : M \to N \) be a smooth mapping between smooth manifolds. Then \( f^*TN \) is a vector bundle over \( M \). Define
\[
(f_*)_B : TM \to f^*TN
\] (8.23)
by
\[
(f_*)_B(v_p) = (p, f_*v).
\] (8.24)
(the subscript \( B \) is short for “bundle mapping”). We have the commutative diagram
\[
\begin{array}{ccc}
TM & \xrightarrow{(f_*)_B} & f^*TN \\
\downarrow\pi_M & & \downarrow\pi_1 \\
M & \xrightarrow{id} & M.
\end{array}
\] (8.25)

**Definition 8.3.** If \( X \in \Gamma(TM) \), then we can define \( f_*X \in \Gamma(f^*TN) \), by
\[
f_*X \equiv (f_*)_B \circ X.
\] (8.26)

In words: under smooth mappings, vector fields push-forward to sections of the pull-back bundle.

**Remark 8.4.** Note that for \( f : M \to N \), although we can push-forward individual tangent vectors, in general there is not a mapping
\[
f_* : \Gamma(TM) \to \Gamma(TN).
\] (8.27)
For example, \( f \) might not even be surjective. This is one reason we had to consider pull-back bundles in the above discussion.
8.4 Pull-back of differential forms

Noting that \((f^*(TN))^*\) is isomorphic to \(f^*(T^*N)\), let us dualize the diagram \((8.9)\) to obtain

\[
\begin{array}{ccc}
  f^*(T^*N) & \xrightarrow{f_B^*} & T^*M \\
  \downarrow \pi_1 & & \downarrow \pi_M \\
  M & \xrightarrow{id} & M.
\end{array}
\] (8.28)

Note that if \(\omega \in \Gamma(T^*N)\), then \(\omega \circ f \in \Gamma(f^*(T^*N))\). Then, if \(\omega \in \Gamma(T^*N)\), we can compose with the bundle mapping in \((8.28)\) to define the following.

**Definition 8.5.** If \(f : M \to N\) is a smooth mapping between smooth manifolds, then

\[
f^*\omega \equiv f_B^* \circ (\omega \circ f) \in \Gamma(T^*M)
\] (8.29)

is called the pullback of \(\omega\) under \(f\).

More generally, we have the following.

**Definition 8.6.** A differential form is a section of \(\Lambda^p(T^*M)\). That is, a differential form is a smooth mapping \(\omega : M \to \Lambda^p(T^*M)\) such that \(\pi \circ \omega = \text{Id}_M\), where \(\pi : \Lambda^p(T^*M) \to M\) is the bundle projection map. We will write \(\omega \in \Gamma(\Lambda^p(T^*M))\), or \(\omega \in \Omega^p(M)\).

If \(f : M \to N\) is a smooth mapping, then by the diagram \((8.28)\) and the above discussion, we obtain induced mappings

\[
\begin{array}{ccc}
  f^*(\Lambda^p(T^*N)) & \xrightarrow{N(f_B^*)} & \Lambda^p(T^*M) \\
  \downarrow \pi_1 & & \downarrow \pi_M \\
  M & \xrightarrow{id} & M,
\end{array}
\] (8.30)

which is linear on fibers, and therefore \(f_B^*\) is a smooth mapping.

**Definition 8.7** (Pull-back of a differential form). If \(f : M \to N\) is a smooth mapping, and \(\omega \in \Lambda^p(T^*N)\), then define \(\omega \circ f \in \Gamma(f^*(\Lambda^p(T^*N)))\) by \(\omega \circ f(p) = (p, \omega_{f(p)})\). Then define

\[
f^*\omega \equiv f_B^* (\omega \circ f) \in \Gamma(\Lambda^p(T^*M)).
\] (8.31)

For any manifold \(M\), define

\[
\Omega(M) = \Gamma(\Lambda(T^*M)) = \bigoplus_{p \geq 0} \Gamma(\Lambda^p(T^*M)) = \bigoplus_{p \geq 0} \Omega^p(M).
\] (8.32)

By taking the direct sum of the exterior powers, we obtain a mapping

\[
f^* : \Omega(N) \to \Omega(M),
\] (8.33)

which by \((8.17)\) satisfies

\[
f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta).
\] (8.34)

The proof of the following proposition is left as an exercise.
Proposition 8.8 (The chain rule). If \( f : M \to N \) and \( h : N \to M' \) are smooth maps, then
\[
(h \circ f)^* = f^* \circ h^* : \Omega(M') \to \Omega(M).
\] (8.35)

9 Lecture 9

9.1 The exterior derivative

Given a function \( f \in C^\infty(M, \mathbb{R}) \) we define \( df \in \Omega^1(M) \) in two ways. First, viewing vector fields as derivations on smooth functions, we can define
\[
\text{df}(X) \equiv X(f).
\] (9.1)

Alternatively, since \( f : M \to \mathbb{R} \), we have \( f_* : TM \to T\mathbb{R} \). But there is a natural identification \( T_p\mathbb{R} \cong \mathbb{R} \) for any \( p \in \mathbb{R} \), so we can view
\[
f_* : TU \to \mathbb{R},
\] (9.2)
which is naturally an element in \( df \in \Omega^1(U) \).

Exercise 9.1. Verify that these two definitions agree.

For a coordinate system \((U, x)\), and let \( \frac{\partial}{\partial x^i} \) denote the coordinate vector field. Recall that viewing vector fields as derivations on germs of functions, this is characterized by
\[
\frac{\partial}{\partial x^i}(x^j) = \delta^j_i.
\] (9.3)

We then define a local basis of 1-forms \( dx^i \) by
\[
dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.
\] (9.4)

Note this is just the dual basis, but these are also \( d(x^i) \) as defined above in (9.1).

An element \( \alpha \in \Omega^p(U) \) can be written as
\[
\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
\] (9.5)

where the coefficients \( \alpha_{i_1 \cdots i_p} : U \to \mathbb{R} \) are well-defined functions. Note these coefficients are only defined for strictly increasing sequences \( i_1 < \cdots < i_p \).

We next define the exterior derivative operator \([War83, \text{Theorem 2.20}]\).

Proposition 9.2. There exists an exterior derivative operator
\[
d : \Omega^p(M) \to \Omega^{p+1}(M)
\] (9.6)
which is the unique linear mapping satisfying
• For $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.

• $d^2 = 0$.

• If $f \in C^\infty(M, \mathbb{R})$ then $df$ is the differential of $f$ defined above.

Proof. Note that the differential of a function is given locally by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$  \hfill (9.7)

This is obviously well-defined and independent of the coordinate system. Given a $p$-form $\alpha$, write $\alpha$ locally as in (9.5), and then define

$$d\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^{n} d\alpha_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \sum_{i=1}^{n} \frac{\partial \alpha_{i_1 \cdots i_p}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$ \hfill (9.8)

The first “anti-derivation” property is easily verified by computation. The second property holds on functions, because

$$d(df) = d\left( \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^i} dx^i \right) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0,$$ \hfill (9.9)

since the Hessian of a smooth function is symmetric.

For existence, we need to check that this definition is independent of the coordinate system. Let $d'$ be the operator defined with respect to another coordinate system $x' : U \to \mathbb{R}^n$. Then

$$d'(\alpha) = d' \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \right)$$

$$= \sum_{|I|=p} (d'\alpha_{i_1 \cdots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$+ \sum_{|I|=p} \alpha_{i_1 \cdots i_p} \sum_{k} (-1)^{k-1} dx^{i_1} \wedge \cdots \wedge d'(dx^{i_k}) \wedge \cdots \wedge dx^{i_p}$$

$$= \sum_{|I|=p} (d\alpha_{i_1 \cdots i_p}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = d(\alpha),$$ \hfill (9.10)

since $d$ and $d'$ agree on functions, and since $d'dx^i = d'd'x^i = 0$.

Then for any $p$-form $\alpha$,

$$d(d\alpha) = d \left( \sum_{|I|=p} (d\alpha_I) \wedge dx^I \right) = \sum_{|I|=p} (d^2 \alpha_I) \wedge dx^I - d\alpha_I \wedge d(dx^I) = 0.$$ \hfill (9.11)

Uniqueness is left as an exercise. \hfill \square
An important fact is that $d$ commutes with pull-back.

**Proposition 9.3.** If $f : M \to N$ is a smooth mapping, and $\omega \in \Omega^p(N)$, then

$$f^*(d_N\omega) = d_M(f^*\omega). \quad (9.12)$$

**Proof.** If $\omega$ is a 0-form, which is a function, then $f^*\omega = \omega \circ f$. So by above, we have

$$d(f^*\omega) = d(\omega \circ f) = (\omega \circ f)_*. \quad (9.13)$$

By the chain rule, we then have

$$d(f^*\omega) = \omega_* \circ f_* \quad (9.14)$$

On the other hand, we have that

$$f^*(d\omega)(X) = d\omega(f_*^*(X)) = \omega_* \circ f_*(X). \quad (9.15)$$

So the claim is true on functions. Then if $\omega$ is a $p$-form, write

$$\omega = \sum_{|I| = p} \omega_I dx^I. \quad (9.16)$$

Since the pull-back operation is an algebra homomorphism, we have

$$f^*\omega = \sum_{|I| = p} (f^*\omega_I)f^*dx^I = \sum_{|I| = p} (\omega_I \circ f_*)d(x^I \circ f). \quad (9.17)$$

Then

$$d(f^*\omega) = \sum_{|I| = p} d(\omega_I \circ f) \wedge d(x^I \circ f). \quad (9.18)$$

On the other hand, we have

$$d\omega = \sum_{|I| = p} (d\omega_I) \wedge dx^I, \quad (9.19)$$

so

$$f^*(d\omega) = \sum_{|I| = p} f^*(d\omega_I) \wedge f^*dx^I = \sum_{|I| = p} d(f^*\omega_I) \wedge d(f^*x^I)$$

$$= \sum_{|I| = p} d(\omega_I \circ f) \wedge d(x^I \circ f) = d(f^*\omega). \quad (9.20)$$

$$\square$$
9.2 de Rham cohomology

Let $M$ be a smooth manifold. Since $d^2 = 0$, we have a “cochain” complex

$$
\cdots \xrightarrow{d^{p-2}} \Omega^{p-1}(M) \xrightarrow{d^{p-1}} \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \xrightarrow{d^{p+1}} \cdots.
$$

(9.21)

which terminates at $\Omega^n(M)$, where $n = \dim(M)$. Clearly we have that $\text{Im}(d^{p-1}) \subset \text{Ker}(d^p)$, so we can define the following vector spaces.

**Definition 9.4.** For $0 \leq p \leq n$, the $p$th de Rham cohomology group is

$$
H^p_{dR}(M) = \frac{\text{Ker}\{d^p : \Omega^p(M) \rightarrow \Omega^{p+1}(M)\}}{\text{Im}\{d^{p-1} : \Omega^{p-1}(M) \rightarrow \Omega^p(M)\}}.
$$

(9.22)

Note that

$$
H^*_{dR}(M) \equiv \bigoplus_{p=0}^n H^p_{dR}(M)
$$

(9.23)

has an algebra structure induced by the wedge product. To see this, for $[\alpha] \in H^p_{dR}(M)$ and $[\beta] \in H^q_{dR}(M)$, represented by $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, we have that

$$
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0,
$$

(9.24)

so we define

$$
[\alpha] \wedge [\beta] = [\alpha \wedge \beta].
$$

(9.25)

To see that this is well-defined, we have

$$
(\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d\gamma \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta),
$$

(9.26)

since $\beta$ is closed, so

$$
[(\alpha + d\gamma) \wedge \beta] = [\alpha \wedge \beta].
$$

(9.27)

Well-definedness in the other factor is similar, or just use the skew-symmetry property of the wedge product. Therefore we have

$$
\wedge : H^p_{dR}(M) \otimes H^q_{dR}(M) \rightarrow H^{p+q}_{dR}(M).
$$

(9.28)

Note that from Proposition 9.2, we have

$$
[\alpha] \wedge [\beta] = (-1)^{pq}[\beta] \wedge [\alpha].
$$

(9.29)

Next, let $f : X \rightarrow Y$ be a smooth mapping between smooth manifolds. As discussed before, we have a pullback operation on differential forms, $f^* : \Omega^*(Y) \rightarrow \Omega^*(X)$, which makes the following diagram commute

$$
\begin{array}{ccc}
\Omega^p(Y) & \xrightarrow{d^p} & \Omega^{p+1}(Y) \\
(f^*)^p \downarrow & & \downarrow (f^*)^{p+1} \\
\Omega^p(X) & \xrightarrow{d^p} & \Omega^{p+1}(X)
\end{array}
$$

(9.30)

That is the collection of mappings $(f^*)^p$ is a morphism of cochain complexes.

The de Rham cohomology algebra is a diffeomorphism invariant.
Corollary 9.5. If \( f : X \to Y \) then there are induced mappings
\[
(f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X).
\] (9.31)

If \( g : Y \to Z \), then
\[
((g \circ f)^*)^p = (f^*)^p \circ (g^*)^p.
\] (9.32)

Consequently, if \( X \) and \( Y \) are diffeomorphic, then \( H^p_{dR}(X) \cong H^p_{dR}(Y) \) for every \( p \geq 0 \), and moreover, the cohomology algebras are isomorphic \( H^*_d(X) \cong H^*_d(Y) \).

**Proof.** We first note that any smooth mapping \( f : X \to Y \) induces a well-defined mapping on cohomology \( (f^*)^p : H^p_{dR}(Y) \to H^p_{dR}(X) \) by the following. If \([\alpha] \in H^p_{dR}(Y)\) is represented by a form \( \alpha \), such that \( d^p Y \alpha = 0 \), then we have
\[
d^p X (f^*)^p \alpha = (f^*)^{p+1} d^p Y \alpha = (f^*)^{p+1} 0 = 0,
\] (9.33)
so we can define \( f^*[\alpha] = [f^*\alpha] \), that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,
\[
(f^*)^p (\alpha + d^{p-1} Y \beta) = (f^*)^p \alpha + (f^*)^p d^{p-1} Y \beta = (f^*)^p \alpha + d^{p-1} X (f^*)^{p-1} \beta,
\] (9.34)
so \([ (f^*)^p (\alpha + d^{p-1} Y \beta) ] = [(f^*)^p \alpha + d^{p-1} X (f^*)^{p-1} \beta ] = [(f^*)^p \alpha ] \).

If \( f \) is a diffeomorphism, then \( f^{-1} \) exists and is smooth, so we have
\[
f \circ f^{-1} = id_Y, \quad f^{-1} \circ f = id_X,
\] (9.35)
and from Proposition 8.8, the induced mappings on cohomology satisfy
\[
f^* \circ (f^{-1})^* = id_{H^*_d(X)}, \quad (f^{-1})^* \circ f^* = id_{H^*_d(Y)},
\] (9.36)
Finally, since \( f^* (\alpha \wedge \beta) = f^* \alpha \wedge f^* \beta \), together these mappings form an algebra homomorphism on cohomology algebras, which will be an algebra isomorphism if \( X \) and \( Y \) are diffeomorphic. \( \square \)

### 10 Lecture 10

Let us formalize some of the above notions.

#### 10.1 Cochain complexes

A collection \( A^p \) of vector spaces for \( p \geq 0 \) and operators \( \delta^p_A : A^p \to A^{p+1} \) for \( p \geq 0 \) satisfying \( \delta^{p+1}_A \delta^p_A = 0 \) is called a **cochain complex**.

\[
\cdots \xrightarrow{\delta^{p-2}_A} A^{p-1} \xrightarrow{\delta^{p-1}_A} A^p \xrightarrow{\delta^p_A} A^{p+1} \xrightarrow{\delta^{p+1}_A} \cdots.
\] (10.1)
**Definition 10.1.** The $p$th cohomology of a chain complex is the vector space

$$H^p(A) = \frac{\ker\{\delta^p_A : A^p \to A^{p+1}\}}{\text{im}\{\delta^{p-1}_A : A^{p-1} \to A^p\}}$$  \hspace{1cm} (10.2)

**Definition 10.2.** A morphism $\alpha : A \to B$ of cochain complexes is a collection of mappings $\alpha^p : A^p \to B^p$ such that $\delta^p_B\alpha^p = \alpha^{p+1}\delta^p_A$ for $p \geq 0$. In other words, $\alpha : A \to B$ is a morphism if the following diagram commutes

$$
\begin{array}{ccc}
A^p & \xrightarrow{\delta^p_A} & A^{p+1} \\
\downarrow{\alpha^p} & & \downarrow{\alpha^{p+1}} \\
B^p & \xrightarrow{\delta^p_B} & B^{p+1} \\
\end{array}
$$  \hspace{1cm} (10.3)

**Proposition 10.3.** Morphisms satisfy the following properties:

- **Composition of morphisms:** If $\alpha : A \to B$ and $\beta : B \to C$ are morphisms of chain complexes, then $\beta \circ \alpha : A \to C$ is a morphism.

- **Associativity:** If $\gamma : C \to D$ is another morphism, then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.

Consequently, the collection of cochain complexes and morphisms of cochain complexes forms a category.

**Proof.** The diagram looks like

$$
\begin{array}{ccc}
A^p & \xrightarrow{\delta^p_A} & A^{p+1} \\
\downarrow{\alpha^p} & & \downarrow{\alpha^{p+1}} \\
B^p & \xrightarrow{\delta^p_B} & B^{p+1} \\
\downarrow{\beta^p} & & \downarrow{\beta^{p+1}} \\
C^p & \xrightarrow{\delta^p_C} & C^{p+1} \\
\end{array}
$$  \hspace{1cm} (10.4)

We want to show that

$$\beta^{p+1} \circ \alpha^{p+1} \circ d^p_A = d^p_C \circ \beta^p \circ \alpha^p.$$  \hspace{1cm} (10.5)

Using commutativity of the top square, the left hand side of (10.5) is

$$\beta^{p+1} \circ \alpha^{p+1} \circ \delta^p_A = \beta^{p+1} \circ \delta^p_B \circ \alpha^p.$$  \hspace{1cm} (10.6)

Using commutativity of the bottom square, the right hand side of (10.5) is

$$d^p_C \circ \beta^p \circ \alpha^p = \beta^{p+1} \circ d^p_B \circ \alpha^p,$$  \hspace{1cm} (10.7)

which proves (10.5).

Associativity is clear: $\gamma^p \circ (\beta^p \circ \alpha^p) = (\gamma^p \circ \beta^p) \circ \alpha^p$ holds for every $p \geq 0$ since composition of mappings is associative.
Proposition 10.4. A morphism of chain complexes \( \alpha : A \to B \) induces mappings \( H^p\alpha : H^p(A) \to H^p(B) \). Furthermore, if \( \beta : B \to C \) is another morphism of chain complexes, then
\[
H^p(\beta \circ \alpha) = H^p\beta \circ H^p\alpha.
\] (10.8)

Proof. Given \([a^p] \in H^p(A)\) represented by \(a^p \in A^p\) satisfying \(\delta_A^p a^p = 0\), we have
\[
\delta_B^p \alpha^p a^p = \alpha^{p+1} \delta_A^p a^p = 0,
\] (10.9)
therefore we can define \((H^p\alpha)^p[a^p] = [\alpha^p a^p]\). To check that this is well-defined,
\[
[\alpha^p(a^p + \delta_A^{p-1} a^{p-1})] = [\alpha^p a^p + \alpha^p \delta_A^{p-1} a^{p-1}] = [\alpha^p a^p + \delta_B^{p-1} \alpha^{p-1} a^{p-1}] = [\alpha^p a^p].
\] (10.10)

Next, for \([a^p] \in H^p(A)\) represented by \(a^p \in A^p\), we have
\[
H^p(\beta \circ \alpha)[a^p] = [(\beta \circ \alpha)^p a^p] = [\beta^p(\alpha^p(a^p))] = H^p\beta^p[\alpha^p(a^p)] = H^p\beta(H^p\alpha[a^p]).
\] (10.11)

Definition 10.5. For \(p \geq 0\), the \(p\)th cohomology functor \(H^p\) is the mapping between the category of chain complexes to the category of vector spaces (with morphisms being linear mappings) given by \(A \mapsto H^p(A)\).

Proposition 10.6. The functor \(H^p\) is a covariant functor.

Proof. The functor \(H^p\) maps objects to objects, just by mapping the chain complex \(C\) to the vector space \(H^p(C)\). Also for each morphism \(\alpha : A \to B\) between chain complexes, we associate the morphism \(H^p\alpha : H^p(A) \to H^p(B)\). The covariant property is \([\alpha^p(a^p + \delta_A^{p-1} a^{p-1})] = [\alpha^p a^p + \alpha^p \delta_A^{p-1} a^{p-1}] = [\alpha^p a^p + \delta_B^{p-1} \alpha^{p-1} a^{p-1}] = [\alpha^p a^p].\) (10.10)

Definition 10.7. The \(p\)th de Rham cohomology functor is the functor from the category of smooth manifolds and smooth mappings to vector spaces the category of vector spaces and linear mappings given by \(M \mapsto H^p_{dR}(M, \mathbb{R})\) and \(f : X \to Y\) maps to \(H^p f = (f^*)^p : H^p(Y) \to H^p(X)\).

Proposition 10.8. The \(p\)th de Rham cohomology functor is a contravariant functor.

Proof. This follows from \((g \circ f)^* = f^* \circ g^*\), and the fact that the composition of a covariant functor and a contravariant functor is a contravariant functor.

10.2 Cochain homotopy between morphisms of cochain complexes

Definition 10.9. Let \(f : A \to B\), and \(g : A \to B\) be two morphisms of cochain complexes. We say that \(f\) is cochain homotopic to \(g\) if there exists mappings \(S^p : A^p \to B^{p-1}\) such that
\[
f^p - g^p = \delta_B^{p-1} S^p + S^{p+1} \delta_A^p.
\] (10.12)
Proposition 10.10. If \( f \) is cochain homotopic to \( g \) then \( H^p f = H^p g : H^p(A) \to H^p(B) \).

Proof. Consider the mapping \( H^p f - H^p g \), and take \([a^p] \in H^p(A)\) represented by \( a^p \in A^p \) satisfying \( \delta_A a^p = 0 \). Then

\[
(H^p f - H^p g)[a^p] = (H^p(f - g))[a^p] = [(H^p(f - g))a^p]
= [\delta_B^{-1} S^p a^p + S^{p+1} \delta_A a^p = [\delta_B^{-1} S^p a^p] = 0. \tag{10.13}
\]

\[\square\]

10.3 Homotopy invariance of de Rham cohomology

Let \( M \) be a smooth manifold, possibly noncompact. Let \( \Omega^p(M) \) denote the smooth \( p \)-forms on \( M \). Recall that we have a cochain complex

\[\cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots, \tag{10.14}\]

and \( H^p_{dR}(M) \) is defined to be the cohomology of this complex.

Let \( M \) be a differentiable \( n \)-manifold, and consider \( N = M \times [0,1] \). Let \( \pi : N \to M \) be the projection \( \pi(x,t) = x \). Also, let \( \iota_t : M \to M \times [0,1] \) be the inclusion \( \iota_t(x) = (x,t) \).

Remark 10.11. The object \( N = M \times [0,1] \) is a manifold with boundary. The reader can check that all the properties of differential forms proved above hold in the category of manifolds with boundary.

We next define a mapping

\[ I^k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M) \tag{10.15} \]

by the following. Any \( k \)-form on \( N \) can be written as

\[ \omega = h(x,t)\pi^* \phi_k + f(x,t)dt \wedge (\pi^* \phi_{k-1}), \tag{10.16} \]

where \( \phi_k \in \Omega^k(M) \) and \( \phi_{k-1} \in \Omega^{k-1}(M) \), but \( h, f \in \Omega^0(M \times [0,1]) \). Define

\[ I^k(\omega) = \left( \int_0^1 f(p,t)dt \right) \phi_{k-1}. \tag{10.17} \]

Proposition 10.12. For \( \omega \in \Omega^k(N) \), we have

\[ (\iota_1)^* \omega - (\iota_0)^* \omega = d_M I^k \omega + I^{k+1} d_N \omega. \tag{10.18} \]

In other words, \( I^k \) is a cochain homotopy between \((\iota_0)^*\) and \((\iota_1)^*\).
Proof. Writing $\omega$ in the form (10.16), since $\iota^*_u dt = 0$, and $\pi \circ \iota_t = \text{id}_M$, the left hand side of (10.18) is

\[(\iota_1)^* \omega - (\iota_0)^* \omega = (\iota_1)^* h(x, t) \pi^* \phi_k - (\iota_0)^* h(x, t) \pi^* \phi_k = (h(x, 1) - h(x, 0)) \phi_k \tag{10.19}\]

Next, assume that $\omega$ is just of the form

$\omega = h(x, t) \pi^* \phi_k$. \tag{10.20}$

Then

\[d_N \omega = \left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial t} dt\right) \wedge \pi^* \phi_k + h(x, t) \pi^* d_M \phi_k. \tag{10.21}\]

By definition of $I^*$,

\[d_M I^k \omega = 0, \tag{10.22}\]

and

\[I^{k+1} d_N \omega = I^{k+1} \left\{ \left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial t} dt\right) \wedge \pi^* \phi_k + h(x, t) \pi^* d_M \phi_k \right\} \]

\[= I^{k+1} \left\{ \frac{\partial h}{\partial t} dt \wedge \pi^* \phi_k \right\} = \left(\int_0^1 \frac{\partial h}{\partial t} dt\right) \phi_k = (h(x, 1) - h(x, 0)) \phi_k. \tag{10.23}\]

So the proposition holds for forms of this type.

Next, assume that $\omega$ is just of the form

$\omega = f(x, t) dt \wedge (\pi^* \phi_{k-1})$. \tag{10.24}$

From (10.19) above, we have

\[(\iota_1)^* \omega - (\iota_0)^* \omega = 0. \tag{10.25}\]

Note that

\[d_N \omega = \frac{\partial f}{\partial x} dx \wedge dt \wedge (\pi^* \phi_{k-1}) - f(x, t) dt \wedge \pi^* (d_M \phi_{k-1}) \]

\[= - \frac{\partial f}{\partial x} dt \wedge \pi^* (dx \wedge \phi_{k-1}) - f dt \wedge \pi^* (d_M \phi_{k-1}). \tag{10.26}\]

By definition of $I^k$,

\[d_M I^k \omega = d_M \left\{ \left(\int_0^1 f(x, t) dt\right) \phi_{k-1} \right\} \]

\[= \left(\int_0^1 \frac{\partial f}{\partial x} dt\right) dx \wedge \phi_{k-1} + \left(\int_0^1 f dt\right) d_M \phi_{k-1}. \tag{10.27}\]
Next, by definition of $I^{k+1}$ and (10.26), we have

$$I^{k+1}d_N\omega = -\left(\int_0^1 \frac{\partial f}{\partial x} dt\right) dx \wedge \phi_{k-1} - \left(\int_0^1 f dt\right) d_M\phi_{k-1}. \quad (10.28)$$

So on forms of this type, we have

$$d_M I^k \omega + I^{k+1} d_N \omega = 0. \quad (10.29)$$

So the proposition is true for forms of the second type. By linearity, the proposition holds for all forms, and we are done.

**Definition 10.13.** Let $X$ and $Y$ be smooth manifolds. Smooth mappings $f, g : X \to Y$ are said to be smoothly homotopic if there exists a smooth mapping $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

**Proposition 10.14.** Let $X$ and $Y$ be smooth manifolds. If $f, g : X \to Y$ are smoothly homotopic then

$$f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X) \quad (10.30)$$

**Proof.** Let $F : X \times [0, 1] \to Y$ be a homotopy between $f$ and $g$. Let $\iota_t : X \to X \times [0, 1]$ be the mapping $\iota_t(x) = (x, t)$, and note that

$$(\iota_t)^* : \Omega^*(X \times [0, 1]) \to \Omega^*(X). \quad (10.31)$$

In Proposition 10.12, we constructed a cochain homotopy between $\iota_1^*$ and $\iota_0^*$,

$$I^k : \Omega^k(X \times [0, 1]) \to \Omega^{k-1}(X) \quad (10.32)$$

satisfying

$$(\iota_1)^* - (\iota_0)^* = I^{k+1} d_{X \times [0, 1]} + d_X I^k. \quad (10.33)$$

By Proposition 10.10, we have that

$$(\iota_0)^* = (\iota_1)^* : H^k_{dR}(X \times [0, 1]) \to H^k_{dR}(X). \quad (10.34)$$

Since $f = F \circ \iota_0$ and $g = F \circ \iota_1$, and $H^k_{dR}$ is a contravariant functor, we have

$$f^* = (\iota_0)^* \circ F^*, \quad g^* = (\iota_1)^* \circ F^*, \quad (10.35)$$

therefore $f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X)$.
11 Lecture 11

11.1 Homotopy type

Definition 11.1. Smooth manifolds $X$ and $Y$ have the same smooth homotopy type if there exist smooth mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is smoothly homotopic to $Id_X$ and $f \circ g$ is smoothly homotopic to $id_Y$.

Corollary 11.2. If $X$ and $Y$ have the same smooth homotopy type, then $H^*_dR(X) \cong H^*_dR(Y)$.

Proof. From Proposition 10.14 we have
\[ f^* \circ g^* = Id_{H^*_dR(X)} \quad (11.1) \]
\[ g^* \circ f^* = Id_{H^*_dR(Y)} \],
so $f^*$ and $g^*$ are isomorphisms. □

Some special cases of this are the following.

Definition 11.3. A smooth manifold $X$ is smoothly contractible if $X$ has the same smooth homotopy type as a point.

Corollary 11.4. If $X$ is smoothly contractible, then
\[ H^k_dR(X) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & 0 < k \end{cases} \] (11.3)

Proof. If $X = \{p\}$ is a single point. Then $\Omega^0(X)$ are functions $f : p \to \mathbb{R}$, so $\Omega^0(X) = \mathbb{R}$, and since $df = 0$, we have $H^0_{dR}(X) = \mathbb{R}$. There are no $k$-forms on $X$ for $k > 0$, so the corollary follows. □

Example 11.5. A domain $A \subset \mathbb{R}^n$ is star-shaped if there exists a $p \in A$ such that for any $x \in A$, the line segment between $p$ and $x$ is contained in $A$. In this case, let $F : A \times [0,1] \to \mathbb{R}^n$ be the mapping $F(x,t) = (1-t)x + tp$. This shows that $A$ is (smoothly) contractible, so $A$ has the same de Rham cohomology groups as a point.

Definition 11.6. A subset (submanifold) $i : A \hookrightarrow X$ is a (smooth) deformation retraction of $X$ if there exists a (smooth) mapping $r : X \to X$ such that
\[ r \circ i = Id_A \],
and $i \circ r$ is (smoothly) homotopic to $Id_X$.

Corollary 11.7. If $A$ is a smooth deformation retraction of $X$ then
\[ H^k_dR(A) \cong H^k_dR(X), \] (11.5)
for all $k \geq 0$.

Example 11.8. Consider $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \subset \mathbb{R}^n$ given by $r(x) = x/|x|$. The mapping $F(x,t) = (1-t)x + t(x/|x|)$ is a smooth homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, so $S^{n-1}$ is a smooth deformation retraction of $\mathbb{R}^n \setminus \{0\}$ and we therefore have
\[ H^k_dR(S^{n-1}) = H^k_dR(\mathbb{R}^n \setminus \{0\}), \] (11.6)
11.2 Exact sequences of cochain complexes

Definition 11.9. A sequence of vector spaces $A, B, C$, with linear mappings $\alpha : A \to B, \beta : B \to C$

$$
0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0
$$

(11.7)
is called exact if the kernel of each mapping is equal to the image of the previous mapping. That is $Ker(\alpha) = \{0\}$ if and only if $\alpha$ is injective. Next, $Ker(\beta) = Im(\alpha)$. Finally, $Im(\beta) = C$, if and only if $\beta$ is surjective.

Let $C_i$ be a co-complex of vector spaces for $i = 1, 2, 3.$

$$
\cdots \xrightarrow{d_i^{p-2}} C_i^{p-1} \xrightarrow{d_i^{p-1}} C_i^p \xrightarrow{d_i^p} C_i^{p+1} \xrightarrow{d_i^{p+1}} \cdots
$$

(11.8)

with $d^2 = 0$. A morphism from $C_i$ to $C_j$ are mappings $\alpha^k : C_i^k \to C_j^k$ such that the following diagram commutes for every $p$

$$
\begin{array}{ccc}
C_i^p & \xrightarrow{d_i^p} & C_i^{p+1} \\
\downarrow{\alpha^p} & & \downarrow{\alpha^{p+1}} \\
C_j^p & \xrightarrow{d_j^p} & C_j^{p+1}
\end{array}
$$

(11.9)

For co-complexes $C_1, C_2, C_3$, and morphisms $\alpha : C_1 \to C_2$ and $\beta : C_2 \to C_3$. We say that a sequence of co-complexes is exact if

$$
0 \to C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \to 0
$$

(11.10)

if the sequence

$$
0 \to C_1^p \xrightarrow{\alpha^p} C_2^p \xrightarrow{\beta^p} C_3^p \to 0
$$

(11.11)
is exact for every $p$.

Lemma 11.10 (The zig-zag lemma for cochain complexes). If

$$
0 \to C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \to 0
$$

(11.12)
is a short exact sequence of co-complexes, then there exist connecting homomorphisms

$$
\delta^p : H^p(C_3) \to H^{p+1}(C_1)
$$

(11.13)

for every $p$ such that the sequence

$$
\cdots \xrightarrow{\delta^{p-1}} H^p(C_1) \xrightarrow{\alpha^p} H^p(C_2) \xrightarrow{\beta^p} H^p(C_3) \xrightarrow{\delta^p} H^{p+1}(C_1) \xrightarrow{} \cdots
$$

(11.14)
is exact.
Proof. We look at the huge commutative diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & C^p_1 & \overset{\alpha_{p-1}}{\longrightarrow} & C^p_2 & \overset{\beta_{p-1}}{\longrightarrow} & C^p_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^p_1 & \longrightarrow & C^p_2 & \longrightarrow & C^p_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{p+1}_1 & \overset{\alpha_{p+1}}{\longrightarrow} & C^{p+1}_2 & \overset{\beta_{p+1}}{\longrightarrow} & C^{p+1}_3 & \longrightarrow & 0 \\
\end{array}
\]

(11.15)

which has all horizontal rows exact.

To define the connecting homomorphism, take \( c^p_3 \in C^p_3 \) with \( d^p_3 c^p_3 = 0 \). By exactness of the middle row, \( \beta^p \) is surjective, so \( c^p_3 = \beta^p(c^p_2) \) for some \( c^p_2 \in C^p_2 \). Then since the diagram commutes, we have

\[
\beta^{p+1} d^{p+1}_2 c^p_2 = d^{p+1}_3 \beta^p c^p_2 = d^{p+1}_3 c^p_3 = 0.
\]

(11.16)

By exactness of the bottom row, we have \( d^{p+1}_2 c^p_2 = \alpha^{p+1} c^{p+1}_1 \) for some \( c^{p+1}_1 \in C^{p+1}_1 \). Since \( C_1 \) is a co-complex, and by commutativity of the diagram, we have

\[
0 = d^{p+1}_2 c^p_2 = d^{p+1}_2 \alpha^{p+1} c^{p+1}_1 = \alpha^{p+2} d^{p+1}_1 c^{p+1}_1
\]

(11.17)

which implies that \( d^{p+1}_1 c^{p+1}_1 = 0 \), since \( \alpha^{p+2} \) is injective. So we define \( \delta^p(c^p_3) = [c^{p+1}_1] \), the homology class of \( c^{p+1}_1 \) in \( H^{p+1}(C_1) \).

To prove this mapping is well-defined, assume that we started with \( c^p_2 \in C^p_2 \) which was of the form \( c^p_2 = d^{p-1}_3 c^{p-1}_3 \). Then we can write \( c^{p-1}_3 = \beta^{p-1} c^{p-1}_2 \), and the element \( c^p_2 = d^{p-1}_2 c^{p-1}_2 \) satisfies \( \beta^p(\tilde{c}^p_2) = c^p_3 \). But this element is exact, so the next step clearly gives zero. Independence of the choice of \( c^p_2 \) is similarly established.

Exactness of the resulting sequence is left as an exercise in diagram chasing.

\[
\square
\]

12 Lecture 12

12.1 Mayer-Vietoris for de Rham cohomology

Write \( M = U \cup V \) as the union of two open sets in \( M \). Then the following sequence is exact:

\[
0 \longrightarrow \Omega^p(U \cup V) \overset{\beta^p}{\longrightarrow} \Omega^p(U) \oplus \Omega^p(V) \overset{\alpha^p}{\longrightarrow} \Omega^p(U \cap V) \longrightarrow 0
\]

(12.1)

where

\[
\beta^p(\omega) = ((i_{U \to M})^* \omega, (i_{V \to M})^* \omega).
\]

(12.2)
and

\[ \alpha^p(\omega_U, \omega_V) = (i_{U \cap V \hookrightarrow U})^* \omega_U - (i_{U \cap V \hookrightarrow V})^* \omega_V \quad (12.3) \]

To see this, \( \beta^p \) is obviously injective. For exactness at the middle step, obviously \( \alpha^p \beta^p \omega = 0 \). If \( \beta^p(\omega_U, \omega_V) = 0 \), then \( \omega_U = \omega_V \) on \( U \cap V \), so then \( (\omega_U, \omega_V) \) is a well-defined global form on \( M \).

To show that \( \alpha \) is onto, let \( \omega \in \Omega^p(U \cap V) \). Let \( \phi_U, \phi_V \) be a partition of unity subordinate to the covering \( \{ U, V \} \). Then \( \omega = \alpha(\phi_V \omega, -\phi_U \omega) \).

By the zig-zag lemma for cohomology, we obtain a long exact sequence

\[ \cdots \rightarrow H^p_{dR}(U \cup V) \rightarrow H^p_{dR}(U) \oplus H^p_{dR}(V) \rightarrow H^p_{dR}(U \cap V) \rightarrow \delta^p \rightarrow \cdots \quad (12.4) \]

Let us review the definition of the mapping \( \delta^p \). Given a cohomology class \([\omega] \in H^p_{dR}(U \cap V)\), represented by \( \omega \in \Omega^p(U \cap V) \) with \( d\omega = 0 \), we first write \( \omega = \alpha^p(\phi_V \omega, -\phi_U \omega) \), then we apply the exterior derivative to get

\[ (d(\phi_V \omega), -d(\phi_U \omega)) = (d\phi_V \wedge \omega, -d\phi_U \wedge \omega) \in \Omega^p(U) \oplus \Omega^p(V). \quad (12.5) \]

Note that on \( U \cap V \), we have \( (\phi_U + \phi_V) \omega = \omega \), so applying \( d \) to this equation, we have that \( d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0 \) on \( U \cap V \), so together these define a global form

\[ \delta^p \omega = \begin{cases} d\phi_V \wedge \omega & \text{in } U \\ -d\phi_U \wedge \omega & \text{in } V \end{cases} \quad (12.6) \]

and we take the cohomology class of this form.

**Remark 12.1.** This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

The following lemma will be useful.

**Lemma 12.2.** If

\[ 0 \rightarrow V_1 \xrightarrow{\alpha} V_2 \rightarrow \cdots \rightarrow V_{k-1} \rightarrow V_k \rightarrow 0. \quad (12.7) \]

is exact, then

\[ 0 = \dim(V_1) - \dim(V_2) + \dim(V_3) + \cdots + (-1)^{k-1} \dim(V_k). \quad (12.8) \]

**Proof.** Induction. \( \square \)

**Example 12.3.** \( S^n \): Cover with 2 open sets \( U, V \), with \( U \cong \mathbb{R}^n \cong V \) and \( U \cap V \cong S^{n-1} \), use the Mayer-Vietoris sequence and induction to get

\[ H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases} \quad (12.9) \]

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First, consider the case of $S^1$.

$$
0 \longrightarrow H^0_{dR}(S^1) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \longrightarrow 0. \tag{12.10}
$$

But $U \cap V$ is contractible to 2 points, so this is

$$
0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta^0} H^1_{dR}(S^1) \xrightarrow{\beta^1} 0. \tag{12.11}
$$

The lemma then says that $H^1_{dR}(S^1) \cong \mathbb{R}$.

Next, for $n > 1$, look at the beginning of the Mayer-Vietoris sequence

$$
0 \longrightarrow H^0_{dR}(S^n) \xrightarrow{\beta^0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\alpha^0} H^0_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots \tag{12.12}
$$

But now $U \cap V$ is connected, so this is

$$
0 \longrightarrow \mathbb{R} \xrightarrow{\beta^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha^0} \mathbb{R} \xrightarrow{\delta^0} \cdots. \tag{12.13}
$$

Since $\beta$ is injective, the kernel of $\alpha^0$ is 1-dimensional. But $\alpha^0$ has a 2-dimensional domain, so the image of $\alpha^0$ is 1-dimensional, that is $\alpha^0$ is surjective. So we can move to the next level and get

$$
0 \longrightarrow H^1_{dR}(S^n) \xrightarrow{\beta^0} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\alpha^0} H^1_{dR}(U \cap V) \xrightarrow{\delta^0} \cdots, \tag{12.14}
$$

Since $U$ and $V$ are contractible, this says that $H^1_{dR}(S^n) = 0$ for $n \geq 2$.

Next, we look at the upper portion of the Mayer-Vietoris sequence

$$
\cdots \longrightarrow H^{n-2}_{dR}(U) \oplus H^{n-2}_{dR}(V) \xrightarrow{\alpha^{n-2}} H^{n-2}_{dR}(S^{n-1}) \xrightarrow{\delta^{n-2}} H^{n-1}_{dR}(S^{n-1}) \xrightarrow{\beta^{n-1}} 0 \longrightarrow 0. \tag{12.15}
$$

This yields

$$H^n_{dR}(S^n) \cong H^{n-1}_{dR}(S^{n-1}) \cong \mathbb{R}, \tag{12.16}
$$

and

$$H^k_{dR}(S^n) \cong H^{k-1}_{dR}(S^{n-1}) = 0, \tag{12.17}
$$

for $2 \leq k \leq n - 1$, so this finishes the proof.

**Example 12.4.** Torus $T^2$. See lecture notes for details. The conclusion was that $1 - \dim(H^1_{dR}(T^2)) + \dim(H^2_{dR}(T^2)) = 0$. 

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13 Lecture 13

Definition 13.1. We say that a manifold $M$ has a good cover $U_i$ each non-trivial finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ has the same de Rham cohomology as $\mathbb{R}^n$.

Corollary 13.2. If $M$ has a finite good cover, then the de Rham cohomology of $M$ is finite-dimensional.

Proof. Note that if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at $B$, then

$$B \cong \ker(g) \oplus \text{Im}(g) \cong \text{Im}(f) \oplus \text{Im}(g).$$

Consequently, if $A$ and $C$ are both finite-dimensional, then $B$ is also finite-dimensional.

We prove the corollary using induction on the number of open sets in a finite good cover. To see this, let $k$ be the number of sets in a good cover. For $k = 1$, we know the corollary is true. Assume the corollary is true up to $k$, and let $\{U_1, \ldots, U_{k+1}\}$ be a good cover of a manifold $M$. Let $U = U_1 \cup \cdots \cup U_k$, and let $V = U_{k+1}$. Then $U$ and $V$ have good covers with fewer than $k + 1$ open sets, so their de Rham cohomology is finite-dimensional. Also, $U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}$ is a good cover of $U \cap V$, so the theorem is true for $U \cap V$ as well.

Now we look at the following portion of the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\alpha^{p-1}} H_{dR}^{p-1}(U \cap V) \xrightarrow{\delta^{p-1}} H_{dR}^p(U \cup V) \xrightarrow{\beta^p} H_{dR}^p(U) \oplus H_{dR}^p(V) \xrightarrow{\alpha^p} \cdots$$

The above observation then implies that $H_{dR}^p(U \cup V)$ is finite-dimensional.

Corollary 13.3. If $M$ is compact, then the de Rham cohomology of $M$ is finite-dimensional.

Proof. Using a Riemannian metric, there exists a covering of $M$ by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map at any point, a geodesically convex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$, which we previously showed has the same de Rham cohomology as $\mathbb{R}^n$. It follows that every compact manifold admits a finite good cover.

Exercise 13.4. (For those who do not like Riemannian geometry.) If $M$ is compact and admits a triangulation, then show that $M$ admits a finite good cover.

13.1 Mayer-Vietoris for cohomology with compact supports

Let $M$ be a manifold, possibly noncompact. Let $\Omega_c^p(M)$ denote the smooth $p$-forms with compact support. We have a complex

$$\cdots \xrightarrow{d} \Omega_c^{p-1}(M) \xrightarrow{d} \Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \cdots$$

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and $H^p_{c,dR}(M)$ is defined to be the cohomology of this complex. Of course, if $M$ is compact then $H^p_{c,dR}(M) = H^p_{dR}(M)$.

Write $M = U \cup V$ as the union of two open sets in $M$. Note that if $U_1 \subset U_2$ and $\omega \in \Omega^k_c(U_1)$ then $\omega$ extends to be a compactly supported form in $U_2$. Letting $\iota : U_1 \hookrightarrow U_2$ denote the inclusion mapping, we denote by $i_*\omega$ this extension map on forms. We claim that the following sequence is exact:

$$0 \longrightarrow \Omega^p_c(U \cap V) \xrightarrow{\delta^p} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{\beta^p} \Omega^p_c(U \cup V) \longrightarrow 0 \quad (13.5)$$

where

$$\tilde{\alpha}^p(\omega_{U \cap V}) = \left((i_{U \cap V} \hookrightarrow U)_*\omega_{U \cap V}, -(i_{U \cap V} \hookrightarrow V)_*\omega_{U \cap V}\right) \quad (13.6)$$

and

$$\tilde{\beta}^p(\omega_U, \omega_V) = (i_{U \hookrightarrow M})_*\omega_U + (i_{V \hookrightarrow M})_*\omega_V. \quad (13.7)$$

To see this, $\tilde{\alpha}^p$ is obviously injective. For exactness at the middle step, obviously $\tilde{\beta}^p \tilde{\alpha}^p = 0$. If $\tilde{\beta}^p(\omega_U, \omega_V) = 0$, then $\omega_U = -\omega_V$. This implies that the support of both forms is contained in $U \cap V$, and since they are equal there, take $\omega_{U \cap V} = \omega_U$, and then $(\omega_U, \omega_V) = \tilde{\alpha}^p(\omega_U)$. To show that $\tilde{\beta}$ is onto, let $\omega \in \Omega^p_c(M)$. Let $\phi_U, \phi_V$ be a partition of unity subordinate to the covering $\{U, V\}$. Then $\omega = \tilde{\beta}^p(\phi_U \omega, \phi_V \omega)$.

Consequently, from the ziz-zag Lemma, we obtain a long exact sequence

$$\cdots \longrightarrow \tilde{\delta}^{p-1} H^p_{c,dR}(U \cap V) \xrightarrow{\tilde{\alpha}^p} H^p_{c,dR}(U) \oplus H^p_{c,dR}(V) \xrightarrow{\tilde{\beta}^p} H^p_{c,dR}(U \cup V) \xrightarrow{\tilde{\delta}^p} \cdots \quad (13.8)$$

Let us review the definition of the mapping $\tilde{\delta}^p$. Given a cohomology class $[\omega] \in H^p_{c,dR}(U \cup V)$, represented by $\omega \in \Omega^p_c(U \cup V)$ with $d\omega = 0$, we first write $\omega = \tilde{\beta}^p(\phi_U \omega, \phi_V \omega)$, then we apply the exterior derivative to get

$$(d(\phi_U \omega), d(\phi_V \omega)) = (d\phi_U \wedge \omega, d\phi_V \wedge \omega) \in \Omega^p_c(U) \oplus \Omega^p_c(V) \quad (13.9)$$

Either of these elements is supported in $U \cap V$ and then since $d\phi_U \wedge \omega + d\phi_V \wedge \omega = 0$,

$$\tilde{\delta}^p \omega = \left[d\phi_U \wedge \omega\right] = \left[-d\phi_V \wedge \omega\right] \in H^{p+1}_{c,dR}(U \cap V). \quad (13.10)$$

**Remark 13.5.** This mapping appears to depend upon the choice of partition of unity, but recall that when viewed as a cohomology class, it is actually independent of such choice.

### 13.2 Integration of differential forms

Another important fact is that we can integrate top-dimensional differential forms on a compact manifold. But we need to recall orientability. First, an orientation on a $n$-dimensional vector space $V$ is a choice of ordered basis $(v_1, \ldots, v_n)$ with equivalence
relation if 2 ordered bases are related by a change of basis matrix with positive determinant. There are exactly 2 such equivalence classes, and if \( M \) is a manifold, the oriented double cover of \( M \) denoted by \( \tilde{M} \) is the double cover obtained by replacing a point \( p \) with the 2 orientations on \( T_p M \).

**Definition 13.6.** A manifold \( M \) is orientable if any of the following equivalent conditions are satisfied.

- \( M \) admits an coordinate atlas \((U_\alpha, \phi_\alpha)\) such that the overlap maps are orientation-preserving \( \phi_\alpha \circ \phi_\beta^{-1} \), that is, the Jacobian \((\phi_\alpha \circ \phi_\beta^{-1})_*\) has positive determinant.
- \( M \) admits a nowhere-zero \( n \)-form.
- The oriented double cover \( \tilde{M} \to M \) is trivial, i.e., it has 2 components.

If \( M \) is orientable, the choice of one of the components of \( \tilde{M} \) is called an **orientation** on \( M \).

On an oriented \( n \)-dimensional manifold, the integral of \( \omega \in \Omega^n(M) \) is defined as follows. Choose an oriented coordinate atlas \((U_\alpha, \phi_\alpha)\). First, assume that \( \omega \in \Omega^n(M) \) has compact support in a single coordinate system \( U_\alpha \). Then

\[
(\phi_\alpha)_*(\omega) = f dx^1 \wedge \cdots \wedge dx^n,
\]

(13.11)

where \( f : \phi_\alpha(U_\alpha) \to \mathbb{R} \) has compact support. Define

\[
\int_M \omega \equiv \int_{\phi_\alpha(U_\alpha)} f dx^1 \ldots dx^n.
\]

(13.12)

By the change-of-variables formula for integrals, this definition is independent of coordinate system containing the support of \( \omega \).

Next, if \( M \) is compact, or if \( \omega \) has compact support, let \( \chi_\alpha \) be a partition of unity subordinate to \( U_\alpha \), and define

\[
\int_M \omega = \sum_\alpha \int_M \chi_\alpha \omega.
\]

(13.13)

Since the sum is finite, this definition is independent of the choice of coordinate atlas and choice of partition of unity. To see this, let \( U_\alpha \) and \( V_\beta \) be open covers with subordinate partitions of unity \( \rho_\alpha, \chi_\beta \), respectively. Then

\[
\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \sum_\beta \chi_\beta \rho_\alpha \omega = \sum_{\alpha, \beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \omega = \sum_{\alpha, \beta} \int_{V_\beta} \rho_\alpha \chi_\beta \omega;
\]

(13.14)

since \( \rho_\alpha \chi_\beta \) is supported in \( V_\beta \). Therefore

\[
\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \sum_\alpha \rho_\alpha \chi_\beta \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega.
\]

(13.15)
14 Lecture 14

14.1 Stokes’ Theorem

Integration by parts on manifolds is the following.

**Theorem 14.1** (Stokes’ Theorem for manifolds with boundary). Let \((M, \partial M)\) be an oriented manifold with boundary of dimension \(n\). If \(\omega \in \Omega^{n-1}(M)\) has compact support, then

\[
\int_{\partial M} \omega = \int_M d\omega,
\]  

(14.1)

where the boundary has the orientation induced from the outer normal, i.e., if \(v_i \in T_p(\partial M)\), then the ordered basis \((v_1, \ldots, v_{n-1})\) is oriented if \((v, v_1, \ldots, v_{n-1})\) is positively oriented, for any outward pointing normal vector \(v\).

**Proof.** A manifold with boundary, by definition, can be covered by usual coordinate charts in the interior together with coordinate charts \((U_i, \phi_i)\), where \(\phi_i : U_i \to H^n\), where

\[
H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\},
\]  

(14.2)

is the upper half space in \(\mathbb{R}^n\), such that

\[
\phi_i : U_i \cap \partial M \to \mathbb{R}^{n-1}
\]  

(14.3)

is a coordinate chart on \(\partial M\) viewed as an \((n-1)\)-dimensional smooth manifold.

We first consider forms compactly supported in such a coordinate chart. Then just consider an \((n-1)\)-form of the form

\[
\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.
\]  

(14.4)

Note that

\[
d\omega = (-1)^{i-1} \partial_i f dx^1 \wedge \cdots \wedge dx^n
\]  

(14.5)

If \(i < n\), then \(\omega\) restricted to the boundary is zero, and

\[
\int_{H^n} d\omega = (-1)^{i-1} \int_{H^n} \partial_i f dx^1 \cdots dx^n = 0,
\]  

(14.6)

by Fubini’s Theorem and the fundamental theorem of calculus, since \(f\) has compact support. If \(i = n\), then

\[
\int_{H^n} d\omega = (-1)^{n-1} \int_{H^n} \partial_n f dx^1 \cdots dx^n
\]

\[
= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_n f dx^1 \cdots dx^n
\]

\[
= (-1)^n \int_{\mathbb{R}^{n-1}} \omega(x^1, \ldots, x^n, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{\partial H^n} \omega,
\]  

(14.7)
since the outward normal is \(-e_n\), so \({-e_n, e_1, \ldots, e_{n-1}}\) is oriented, which is equivalent to \((-1)^n\) times \(\{e_1, \ldots, e_n\}\). In general \(\omega\) is a sum of \(n\)-terms of the above type, so this proves Stokes’ Theorem for \(\omega \in \Omega^{n-1}(H^n)\) with compact support.

Next, we choose a partition of unity \(\chi_i\) subordinate to the cover \((U_i, \phi_i)\), \(\phi_i : U_i \to \mathbb{R}^n\), and write \(\omega = \sum_i \chi_i \omega\). Let \(\omega_i = \chi_i \omega\). Then for each \(i\) in the index set, we have

\[
\int_M d\omega_i = \int_{U_i} d\omega_i = \int_{\phi_i^{-1}(U_i)} (\phi_i^{-1})^*(d\omega_i) = \int_{\phi_i^{-1}(U_i)} d(\phi_i^{-1})^*(\omega_i)
\]

where the last equality holds since \(\phi_i|_{\partial M}\) is a coordinate chart on \(\partial M\) as a \((n-1)\)-dimensional manifold. Finally, we have

\[
\int_M d\omega = \int_M d\left(\sum_i \omega_i\right) = \sum_i \int_M d\omega_i = \sum_i \int_{\partial M} \omega_i = \int_{\partial M} \sum_i \omega_i = \int_{\partial M} \omega. \tag{14.9}
\]

14.2 Poincaré Lemma for cohomology with compact supports

Let \(M\) be a manifold, possibly noncompact. Let \(\Omega^p_c(M)\) denote the smooth \(p\)-forms with compact support. We have a complex

\[
\cdots \xrightarrow{d} \Omega^{p-1}_c(M) \xrightarrow{d} \Omega^p_c(M) \xrightarrow{d} \Omega^{p+1}_c(M) \xrightarrow{d} \cdots, \tag{14.10}
\]

and \(H^p_{c,dR}(M)\) is defined to be the cohomology of this complex. Of course, if \(M\) is compact then \(H^p_{c,dR}(M) = H^p_{dR}(M)\).

**Theorem 14.2** (Poincaré Lemma for compact supported cohomology). Let \(M\) be a differentiable \(n\)-manifold, then

\[
H^k_{c,dR}(M \times \mathbb{R}) \cong H^{k-1}_{c,dR}(M). \tag{14.11}
\]

First, we define a mapping “integration over the fiber” by

\[
\pi_* : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M) \tag{14.12}
\]

by the following. Any \(k\)-form on \(M \times \mathbb{R}\) can be written as

\[
\omega = h(p, t) \pi^* \phi_k + f(p, t)(\pi^* \phi_{k-1}) \wedge dt, \tag{14.13}
\]

where \(\phi_k \in \Omega^k(M)\) and \(\phi_{k-1} \in \Omega^{k-1}(M)\), but \(h, f \in \Omega^0_c(M \times \mathbb{R})\). Define

\[
\pi_*(\omega) = \left( \int_{-\infty}^{\infty} f(p, t) dt \right) \phi_{k-1}, \tag{14.14}
\]
noting that the integral is defined because $\omega$ is assumed to have compact support, and this form has compact support since $f$ has compact support.

We claim that
\[ d_M \circ \pi_* = \pi_* \circ d_{M \times \mathbb{R}}. \]  
To see this, the left hand side of (14.15) is
\[
d_M \circ \pi_* \omega = d_M \left( \int_{-\infty}^{\infty} f(p, t) dt \phi_{k-1} \right)
= \left( \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} dt \right) dx \wedge \phi_{k-1} + \left( \int_{-\infty}^{\infty} f(p, t) dt \right) d_M \phi_{k-1},
\]
(14.16)

The right hand side of (14.15) is
\[
\pi_* \circ d_N \omega = \pi_* \left( \frac{\partial h}{\partial t} dt \wedge \pi^* \phi_k + \frac{\partial f}{\partial x} dx \wedge \pi^* \phi_{k-1} \wedge dt + f(p, t) \pi^*(d_M \phi_{k-1}) \wedge dt \right)
= \pi_* \left( \frac{\partial f}{\partial x} dx \wedge \pi^* \phi_{k-1} \wedge dt + f(p, t) \pi^*(d_M \phi_{k-1}) \wedge dt \right)
= \left( \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} dt \right) dx \wedge \phi_{k-1} + \left( \int_{-\infty}^{\infty} f(p, t) dt \right) d_M \phi_{k-1},
\]
(14.17)

since the term involving $h$ is zero because $h$ has compact support, and using the fundamental theorem of calculus. Therefore $\pi_*$ induces a mapping
\[ \pi_* : H^{k}_{c,dR}(M \times \mathbb{R}) \rightarrow H^{k-1}_{c,dR}(M). \]  
(14.18)

Next, we choose $e \in \Omega^1_c(\mathbb{R})$ with $\int_{\mathbb{R}} e = 1$, and define
\[ e_* : \Omega^k_c(M) \rightarrow \Omega^{k+1}_c(M \times \mathbb{R}) \]  
(14.19)

by
\[ e_*(\omega) = (\pi^* \omega) \wedge e. \]  
(14.20)

It is not hard to see that
\[ d_{M \times \mathbb{R}} \circ e_* = e_* \circ d_M. \]  
(14.21)

To see this,
\[ d_N \circ e_*(\omega) = d_N \pi^* \omega \wedge e = (d_N \pi^* \omega) \wedge e = \pi^*(d_M \omega) \wedge e = e_* \circ d_M(\omega). \]  
(14.22)

Therefore $e_*$ induces a mapping
\[ e_* : H^{k}_{c,dR}(M) \rightarrow H^{k+1}_{c,dR}(M \times \mathbb{R}). \]  
(14.23)

Let us write $e = \chi dt$, then
\[ \pi_* \circ e_*(\omega) = \pi_* \left( \chi(t)(\pi^* \omega) \wedge dt \right) = \left( \int_{-\infty}^{\infty} \chi(t) dt \right) \omega = \omega \]  
(14.24)

Therefore, we have $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$, so $\pi_* \circ e_* = 1$ on $H^{k}_{c,dR}(M)$. 

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Proposition 14.3. We have $e_* \circ \pi_* = 1$ on $H^k_{c,dR}(M \times \mathbb{R})$. Consequently, $\pi_*$ and $e_*$ are isomorphisms on compactly supported cohomology.

Proof. Again writing
\[ \omega = h(p, t)\pi^* \phi_k + f(p, t)(\pi^* \phi_{k-1}) \wedge dt, \] (14.25)
define a mapping
\[ K : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R}) \] (14.26)
by
\[ K(\omega) = \pi^* \phi_{k-1} \left( \int_{-\infty}^t f(x, s)ds - \left( \int_{-\infty}^t e \right) \int_{-\infty}^\infty f(x, s)ds \right). \] (14.27)

Note that the right hand side is indeed a $(k-1)$-form on $M \times \mathbb{R}$ with compact support, the $ds$'s are not 1-forms in this formula. We claim that if $\omega \in \Omega^k_c(M \times \mathbb{R})$ then
\[ (1 - e_* \pi_*) \omega = (-1)^{k-1} (dK - Kd) \omega, \] (14.28)
which can be separately verified for $\omega = h(p, t)\pi^* \phi_k$, and for forms of type $\omega = f(p, t)dt \wedge \pi^* \phi_{k-1}$.

For forms of the first type, we obviously have
\[ (1 - e_* \pi_*) h(p, t)\pi^* \phi_k = h(p, t)\pi^* \phi_k. \] (14.29)

On the other hand, since $K$ is zero on forms of this type,
\[ (dK - Kd)(h(p, t)\pi^* \phi_k) = -K \left( \left( \frac{\partial h}{\partial x} \right) dx \wedge \pi^* \phi_k + \left( \frac{\partial h}{\partial t} \right) dt \wedge \pi^* \phi_k + h(p, t)\pi^* d\phi_k \right) \]
\[ = -K \left( \left( \frac{\partial h}{\partial t} \right) dt \wedge \pi^* \phi_k \right) \]
\[ = (-1)^{k-1} K \left( \left( \frac{\partial h}{\partial t} \right) (\pi^* \phi_k) \wedge dt \right) \]
\[ = (-1)^{k-1} \pi^* \phi_k \left( \int_{-\infty}^t \frac{\partial h}{\partial t} ds - \left( \int_{-\infty}^t e \right) \int_{-\infty}^\infty \frac{\partial h}{\partial t} ds \right) \]
\[ = (-1)^{k-1} (\pi^* \phi_k) h(p, t). \] (14.30)

For forms of the second type, we have
\[ (1 - e_* \pi_*) f(p, t)\pi^* \phi_{k-1} \wedge dt = f(p, t)\pi^* \phi_{k-1} \wedge dt - \left( \int_{-\infty}^\infty f(p, t)dt \right) (\pi^* \phi_{k-1}) \wedge e \]
\[ = \pi^* \phi_{k-1} \wedge \left( f(p, t)dt - \left( \int_{-\infty}^\infty f(p, t)dt \right) e \right) \]
\[ = (-1)^{k-1} \left( f(p, t) - \left( \int_{-\infty}^\infty f(p, t)dt \right) \chi(t) \right) \pi^* \phi_{k-1} \wedge dt \] (14.31)

The verification that this is equal to $(-1)^{k-1}(dK - Kd)$ is left as an exercise.

Using Proposition 10.10 this formula then implies that $e_* \circ \pi_* = 1$ as a mapping on $H^k_{c,dR}(M \times \mathbb{R})$, and the proposition follows. \qed
Corollary 14.4. We have

\[
H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} 
\mathbb{R} & k = n \\
0 & k \neq n 
\end{cases}
\]  

(14.32)

and a generator for \(H^n_{c,dR}(\mathbb{R}^n)\) is given by any compactly supported \(n\)-form \(\mu\) with \(\int_{\mathbb{R}^n} \mu = 1\).

Proof. We start with \(M = \{p\}\) a single point. The above shows that

\[
H^1_{c,dR}(\mathbb{R}) \cong H^0_{c,dR}(\{p\}) \cong \mathbb{R}.
\]  

(14.33)

Furthermore, the proof shows that a generator of the left hand side is \(\chi(x^1)dx^1\). Next, we have

\[
H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R},
\]  

(14.34)

and a generator of the left hand side is \(\chi(x^1)dx^1 \wedge \chi(x^2)dx^2\). In general, a generator is given by will be

\[
\chi(x^1) \cdots \chi(x^n)dx^1 \wedge \cdots \wedge dx^n.
\]  

(14.35)

Next, we use the fact that \(\pi_*\) is an isomorphism. The isomorphism

\[
H^1_{c,dR}(\mathbb{R}) \cong H^0_{c,dR}(\{p\}) \cong \mathbb{R}
\]  

(14.36)

is given by

\[
\phi_1 \mapsto \int_{\mathbb{R}} \phi_1 dx^1.
\]  

(14.37)

Then the isomorphism

\[
H^2_{c,dR}(\mathbb{R}^2) \cong H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R},
\]  

(14.38)

is given by

\[
f(x^1, x^2)dx^1 \wedge dx^2 \mapsto \left( \int_{\mathbb{R}} f(x^1, x^2)dx^2 \right) dx^1
\]  

(14.39)

Composing these isomorphisms and using Fubini’s Theorem, we get

\[
f(x^1, x^2)dx^1 \wedge dx^2 \mapsto \int_{\mathbb{R}^2} f(x^1, x^2)dx^1 \wedge dx^2.
\]  

(14.40)

In general, the isomorphism is given by

\[
f(x^1, \ldots, x^n)dx^1 \wedge \cdots \wedge dx^n \mapsto \int_{\mathbb{R}^n} f(x^1, \ldots, x^n)dx^1 \wedge \cdots \wedge dx^n.
\]  

(14.41)
15 Lecture 15

Recall last time we showed that

\[ H^k_{c,dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases} \]  

(15.1)

**Remark 15.1.** This shows that \( H^*_c,dR(M) \) is not a homotopy invariant, since (14.32) is not the same as the cohomology of a point. But of course, \( H^*_c,dR(M) \) is a diffeomorphism invariant.

If \( M \) is any oriented manifold of dimension \( n \), then we have a pairing

\[ \Omega^k(M) \times \Omega^{n-k}_c(M) \to \mathbb{R}, \]  

(15.2)

given by

\[ (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta. \]  

(15.3)

By Stokes’ Theorem, this mapping descends to cohomology, and since this mapping is bilinear, we obtain a pairing

\[ PD : H^k_{dR}(M) \otimes H^{n-k}_c(M) \to \mathbb{R}. \]  

(15.4)

In the case \( M = \mathbb{R}^n \), note that \( H^k_{c,dR}(\mathbb{R}^n) \cong H^{n-k}_{dR}(\mathbb{R}^n) \). Furthermore, we have an isomorphism

\[ PD : H^k_{dR}(\mathbb{R}^n) \to (H^{n-k}_{c,dR}(\mathbb{R}^n))^* \]  

(15.5)

given by \( PD(\alpha)(\beta) = \int_{\mathbb{R}^n} \alpha \wedge \beta \). This is because we showed that an isomorphism of \( H^k_{c,dR}(\mathbb{R}^n) \) and \( \mathbb{R} \) is obtained from composing the isomorphisms

\[ H^n_{c,dR}(\mathbb{R}^n) \xrightarrow{\pi^n} H^{n-1}_{c,dR}(\mathbb{R}^{n-1}) \xrightarrow{\pi^{n-1}} \cdots \xrightarrow{\pi^1} H^0_{c,dR}(\mathbb{R}^0), \]  

(15.6)

where \( \pi^k : H^k_{c,dR}(\mathbb{R}^k) \to H^{k-1}_{c,dR}(\mathbb{R}^{k-1}) \) is the mapping induced by writing \( \mathbb{R}^k = \mathbb{R}^{k-1} \times \mathbb{R} \) and using coordinates \((x^1, \ldots, x^{k-1}, t)\), writing \( \omega \in \Omega^k_c(\mathbb{R}^k) \) as

\[ \omega = f(x^1, \ldots, x^{k-1}, t) \, dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dt \]  

(15.7)

and then

\[ \pi^k_*(\omega) = \left( \int_{-\infty}^{\infty} f(x^1, \ldots, x^{k-1}, t) \, dt \right) dx^1 \wedge \cdots \wedge dx^{n-1}, \]  

(15.8)
so the iterated map is

$$
\pi^1 \circ \cdots \circ \pi^n(\omega) = \pi^1 \circ \cdots \circ \pi^{n-1}(\int_{-\infty}^{\infty} f(x^1, \ldots, x^n)dx^n)dx^1 \wedge \cdots \wedge dx^{n-1}
$$

$$
= \pi^1 \circ \cdots \circ \pi^{n-2}(\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x^1, \ldots, x^{n-1}, x^n)dx^n \right)dx^{n-1})dx^1 \wedge \cdots \wedge dx^{n-2}
$$

$$
= \int_{-\infty}^{\infty} \left( \cdots \left( \int_{-\infty}^{\infty} f(x^1, \ldots, x^{n-2}, x^n)dx^n \right)dx^{n-2} \right)dx^1
$$

$$
= \int_{\mathbb{R}^n} f(x^1, \ldots, x^n)dx^1 \cdots dx^n = \int_{\mathbb{R}^n} \omega
$$

(15.9)

by Fubini’s Theorem.

15.1 Star-shaped open sets in \( \mathbb{R}^n \)

Let \( U \) be a star-shaped open set in \( \mathbb{R}^n \). Let us assume that the special point is the origin. Then we write

$$
U = \{ tv \mid v \in S^{n-1}, 0 \leq t < \rho(v) \}
$$

(15.10)

where \( \rho : S^{n-1} \rightarrow \mathbb{R}_+ \).

**Lemma 15.2.** If \( \rho \) is smooth, then \( U \) is diffeomorphic to the unit ball in \( \mathbb{R}^n \), and thus

$$
H^{k}_{c,dR}(U) = \begin{cases} 
\mathbb{R} & k = n \\
0 & k \neq n
\end{cases}
$$

(15.11)

**Proof.** Since \( U \) is an open set, clearly \( B(0, \epsilon) \subset U \) for some \( \epsilon > 0 \), so \( \rho(v) \geq \epsilon \). Choose a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) so that \( 0 \leq f \leq 1 \),

$$
f(t) = \begin{cases} 
0 & t \leq \epsilon/2 \\
1 & t \geq \epsilon
\end{cases}
$$

(15.12)

and \( f'(t) > 0 \) for \( 0 < t \leq \epsilon \). Define \( h : B(0, 1) \rightarrow U \) by

$$
h(tv) = \left( t + (\rho(v) - \epsilon)f(t) \right)v
$$

(15.13)

for \( 0 \leq t \leq 1 \).

Then \( h \) is one-one and surjective. In a neighborhood of the origin, it is smooth with \( h_{*} \) invertible, because it is the identity map there. The radial derivative of \( h \) is \( 1 + (\rho(v) - \epsilon)f'(t) > 0 \). Therefore \( h_{*} \) is invertible everywhere, so \( h \) is a diffeomorphism. \( \square \)

**Lemma 15.3.** In general, \( \rho : S^{n-1} \rightarrow \mathbb{R} \) is upper semi-continuous. That is for any \( v \in S^{n-1} \) and \( \epsilon > 0 \), there is a neighborhood \( U_v \subset S^{n-1} \) of \( v \) such that \( \rho(w) > \rho(v) - \epsilon \) for all \( w \in U_v \).
Proof. Choose a $t$ so that $tv \in U$ with $\rho(v) - \epsilon < t$. Since $U$ is an open set, there is a open ball $B(tv, \delta) \subset U$. Then there is a neighborhood $U_v \subset S^{n-1}$ of $v$ such that $ty \in B(tv, \delta) \subset U$ for $y \in U_v$. This implies that $\rho(y) \geq t > \rho(v) - \epsilon$.

The main result of the subsection is the following.

**Corollary 15.4.** If $U$ is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{c,dR}(U) \cong H^k_{c,dR}(\mathbb{R}^n)$ for all $0 \leq k \leq n$. Furthermore, an isomorphism of $H^k_{c,dR}(U)$ and $\mathbb{R}$ is given by integration.

Proof. At the end of the proof we will show that given any compact subset $K \subset U$, there is a star-shaped open set $V$ with $K \subset V \subset U$ and such that $\rho V$ is $C^\infty$, and thus $V$ is diffeomorphic to $\mathbb{R}^n$ by Lemma $[15.2]$. If $0 \leq k < n$, then $\omega = d\eta$, where $\eta \in \Omega_c^{k-1}(U)$, but we just view $\eta \in \Omega_c^{k-1}(V)$, which proves that $H^k_{c,dR}(U) = \{0\}$. For $k = n$, then we know that $\omega = d\eta + c\mu$, where $\mu$ is a compactly supported form in $V$ with $\int_{\mathbb{R}^n} \mu = 1$ and $c \in \mathbb{R}$. Again we can view $\mu$ as a compactly supported form in $U$, which proves that dim($H^n_{c,dR}(U)$) $\leq 1$. If $\mu = d\gamma$ with $\gamma \in \Omega_c^{n-1}(U)$, then we can view $\gamma \in \Omega_c^{n-1}(V)$ for a nice $V$. Then Stokes’ Theorem says that

$$\int_{\mathbb{R}^n} \mu = \int_{\mathbb{R}^n} d\gamma = \int_{\partial V} \gamma = 0,$$

(15.14)

a contradiction. Therefore dim($H^n_{c,dR}(U)$) $= 1$.

Finally, we will find the star-shaped domain $V$ claimed above. For each $v \in S^{n-1}$, there is a number $t_v < \rho(v)$ so that all the points in $K$ of the form $uv$ must have $u < t_v$. That is, the segment $\{rv \mid t_v < r < \rho(v)\}$ does not hit $K$. This is because $K$, being compact, must stay at a positive distance from the boundary in the radial direction. Furthermore since $K$ is closed, there is a neighborhood of $W_v \subset S^{n-1}$ of $v$ such that the set $\{ry \mid t_v < r < \rho(v)\}$ does not hit $K$ either. From Lemma $[15.3]$ $\rho$ is upper semicontinuous, so choosing $\epsilon = \rho(v) - t_v$, there is a neighborhood $U_v \subset S^{n-1}$ of $v$ such that $\rho(w) > \rho(v) - (\rho(v) - t_v) = t_v$ for all $w \in U_v$. Therefore, by taking the intersection of these neighborhoods, there is a neighborhood of $v$, $W_v \subset S^{n-1}$ so that $t_v$ works as $t_y$ for all $y \in W_v$. Since $S^{n-1}$ is compact, we can cover by finitely many $W_{v_1}, \ldots, W_{v_k}$. Let $\chi_i$ be a partition of unity subordinate to this cover. Define

$$\tilde{\rho} = t_{v_1} \chi_1 + \cdots + t_{v_k} \chi_k.$$  

(15.15)

Since $\chi_1 + \cdots + \chi_k = 1$, and each $t_{v_j} < \rho(x)$, we have $\tilde{\rho}(x) < \rho(x)$. Letting

$$V = \{tv \mid v \in S^{n-1}, \ 0 \leq t < \tilde{\rho}(v)\}$$

(15.16)

then $K \subset V \subset U$, and $\rho V = \tilde{\rho}$ is smooth.

Remark 15.5. It is actually true that $U$ is diffeomorphic to $\mathbb{R}^n$, but this is more difficult to show, and we do not need such a strong result.
16 Lecture 16

16.1 Good covers

Next, let us modify our definition of good cover.

**Definition 16.1.** We say that a manifold $M$ has a good cover $U_i$ each non-trivial finite intersection $U_i \cap \cdots \cap U_i$ has the same de Rham cohomology as $\mathbb{R}^n$, the same compactly supported de Rham cohomology as $\mathbb{R}^n$.

Recall we proved earlier that if $M$ has a finite good cover, then the de Rham cohomology is finite-dimensional. We next extend this to compactly supported cohomology.

**Corollary 16.2.** If $M$ has a finite good cover, then the compactly supported de Rham cohomology is finite-dimensional.

**Proof.** Recall that if

$$A \xrightarrow{f} B \xrightarrow{g} C \quad (16.1)$$

is exact at $B$, then

$$B \cong \text{Ker}(g) \oplus \text{Im}(g) \cong \text{Im}(f) \oplus \text{Im}(g). \quad (16.2)$$

Consequently, if $A$ and $C$ are both finite-dimensional, then $B$ is also finite-dimensional.

We prove the corollary using induction on the number of open sets in a finite good cover. To see this, let $k$ be the number of sets in a good cover. For $k = 1$, we know the corollary is true. Assume the corollary is true up to $k$, and let $\{U_1, \ldots, U_{k+1}\}$ be a good cover of a manifold $M$. Let $U = U_1 \cup \cdots \cup U_k$, and let $V = U_{k+1}$. Then $U$ and $V$ have good covers with fewer than $k + 1$ open sets, so their compactly supported de Rham cohomology is finite-dimensional. Also, $U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}$ is a good cover of $U \cap V$, so the theorem is true for $U \cap V$ as well.

Now we look at the following portion of the compactly supported Mayer-Vietoris sequence

$$\cdots \xrightarrow{\tilde{\alpha}_p} H_{c,dR}^p(U) \oplus H_{c,dR}^p(V) \xrightarrow{\tilde{\beta}_p} H_{c,dR}^p(U \cup V) \xrightarrow{\delta^p} H_{c,dR}^{p+1}(U \cap V) \xrightarrow{\tilde{\alpha}_{p+1}} \cdots \quad (16.3)$$

The above observation then implies that $H_{c,dR}^p(U \cup V)$ is finite-dimensional.

**Corollary 16.3.** If $M$ is compact, then $M$ admits a finite good cover.

**Proof.** Using a Riemannian metric, there exists a covering of $M$ by geodesically convex neighborhoods. Any nontrivial intersection of such sets is also geodesically convex. Using the exponential map at any point, a geodesically convex set is diffeomorphic to a star-shaped domain $\mathbb{R}^n$. This is contractible, so from the Poincaré Lemma, it has the same de Rham cohomology as $\mathbb{R}^n$. Corollary 15.4 tells us that it also has the same compactly supported de Rham cohomology as $\mathbb{R}^n$, so we are done.
17 Lecture 17

17.1 Poincaré Duality

Lemma 17.1 (The Five Lemma). Assume the diagram

\[
\begin{array}{cccccc}
V_1 & \to & V_2 & \to & V_3 & \to & V_4 & \to & V_5 \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \downarrow \phi_4 & & \downarrow \phi_5 \\
W_1 & \to & W_2 & \to & W_3 & \to & W_4 & \to & W_5
\end{array}
\]

(17.1)

commutes, and has exact rows. If \(\phi_1, \phi_2, \phi_4, \phi_5\) are isomorphisms, then \(\phi_3\) is also an isomorphism.

Proof. Injectivity of \(\phi_3\): If \(\phi_3(v_3) = 0\), then \(\beta_3(\phi_3(v_3)) = 0 = \phi_4\alpha_3(v_3)\). Since \(\phi_4\) is injective, \(\alpha_3(v_3) = 0\). By exactness, \(v_3 = \alpha_2(v_2)\). Then \(\phi_3\alpha_2(v_2) = 0 = \beta_2\phi_2(v_2)\). By exactness, \(\phi_2(v_2) = \beta_1(\alpha_1(v_1))\). By surjectivity of \(\phi_1\), \(w_1 = \phi_1(v_1)\). Then

\[
\phi_2(v_2) = \beta_1\phi_1(v_1) = \phi_2\alpha_1(v_1),
\]

(17.2)

but since \(\phi_2\) is injective, this implies that \(v_2 = \alpha_1(v_1)\). Finally, \(v_2 = \alpha_2(v_2) = \alpha_2\alpha_1(v_1) = 0\), by exactness.

The proof of surjectivity is similar, and left to the reader. \(\square\)

Lemma 17.2. If the sequence

\[
W_1 \to W_2 \to W_3
\]

(17.3)
is exact at \(W_2\), then the dual sequence

\[
W_3^* \to W_2^* \to W_1^*
\]

(17.4)
is exact at \(W_2^*\).

Proof. First, if \(w_3^* \in W_3^*\), and \(w_1 \in W_1\), then

\[
\alpha^*(\beta^* w_3^*)(w_1) = (\beta^* w_3^*) (\alpha(w_1)) = w_3^* (\beta \alpha(w_1)) = 0,
\]

(17.5)
since \(\beta \circ \alpha = 1\) by assumption. This proves that \(\text{Im}(\beta^*) \subset \text{Ker}(\alpha^*)\). For the other direction, if \(w_2^* \in \text{Ker}(\alpha^*)\), then for all \(w_1 \in W_1\), \(\alpha^*(w_2^*)(w_1) = w_2^* (\alpha(w_1))\). So the element \(0 = w_2^* \circ \alpha \in W_2^*\). We want to find \(w_3^* \in W_3^*\) such that \(w_2^* = \beta^* w_3^*\). For all \(w_2 \in W_2\), this is \(w_2^* (w_2) = w_3^* \beta w_2\), which is just \(w_2^* = w_3^* \circ \beta\). So if \(w_3 \in W_3\) is of the form \(\beta(w_2)\) then define

\[
w_3^*(w_3) \equiv w_2^*(w_2).
\]

(17.6)

If \(w_3 = \beta(w_2)\), then \(\beta(w_2 - w_2') = 0\), so \(w_2 - w_2' = \alpha(w_1)\). Then

\[
w_3^*(w_2 - w_2') = w_3^* (\alpha(w_1)) = (w_3^* \alpha)(w_1) = 0.
\]

(17.7)

So we have defined \(w_3^*\) on the subspace \(\text{Im}(\beta) \subset W_3\). To extend to a linear mapping on all of \(W_3\), just take any subspace so that \(W_3 = \text{Im}(\beta) \oplus W\), and define \(w_3^*\) to vanish on \(W\). Then the condition \(w_2^* = w_3^* \circ \beta\) is obviously satisfied. \(\square\)
**Theorem 17.3.** If \( M^n \) is orientable and has a finite good cover, then

\[
PD : H^k_{dR}(M) \to (H^{n-k}_{c,dR}(M))^* \tag{17.8}
\]

is an isomorphism for all \( 0 \leq k \leq n \).

**Proof.** Let \( m = n - k \), and consider the diagram

\[
\begin{array}{cccccc}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) & \xrightarrow{\alpha^{k-1}} & H^{k}_{dR}(U \cap V) & \xrightarrow{\delta^{k-1}} & H^{k}_{dR}(U) & \xrightarrow{\delta^{k}} & H^{m}_{c,dR}(U) \\
\downarrow{PD \oplus PD} & & \downarrow{PD} & & \downarrow{PD} & & \downarrow{PD} \\
H^{m+1}_{c,dR}(U) \oplus H^{m+1}_{d,dR}(V) & \xrightarrow{(\tilde{\delta})^{m+1}} & H^{m}_{c,dR}(U \cap V) & \xrightarrow{(\tilde{\delta})^{m}} & (H^{m}_{c,dR}(U) \oplus H^{m}_{c,dR}(V))^* & \xrightarrow{(\tilde{\delta})^{m}} & H^{m}_{c,dR}(U \cap V) \\
\end{array}
\]

(17.9)

The top horizontal row is exact since it is the usual Mayer-Vietoris sequence. The bottom horizontal row is exact since is the dual exact sequence of the Mayer-Vietoris sequence with compact support. We next claim that this diagram commutes up to sign, so by changing some of the vertical maps to their negatives if necessary, we obtain a commutative diagram.

Consider the square

\[
\begin{array}{cc}
H^{k-1}_{dR}(U \cap V) & \xrightarrow{\delta^{k-1}} & H^{k}_{dR}(U \cup V) \\
\downarrow{PD} & & \downarrow{PD} \\
H^{m+1}_{c,dR}(U \cap V) & \xrightarrow{(\tilde{\delta})^{m}} & H^{m}_{c,dR}(U \cup V) \\
\end{array}
\]

(17.10)

For the mapping

\[
PD \circ \delta^{k-1} : H^{k-1}_{dR}(U \cap V) \to H^{m}_{c,dR}(U \cup V)^* \tag{17.11}
\]

let’s take an element \([\omega] \in H^{k-1}_{dR}(U \cap V)\), and an element \([\tau] \in H^{m}_{c,dR}(U \cup V)\). Then

\[
(PD \circ \delta^{k-1}[\omega])[\tau] = PD(\delta^{k-1}[\omega])[\tau] = \int_M (\delta^{k-1}\omega) \wedge \tau. \tag{17.12}
\]

Recall from our discussion of the Mayer-Vietoris sequence that

\[
\delta^{k-1}\omega = \begin{cases} 
d\phi_V \wedge \omega & \text{in } U \\
-d\phi_U \wedge \omega & \text{in } V 
\end{cases} \tag{17.13}
\]

This form is supported in \( U \cap V \), so we have

\[
(PD \circ \delta^{k-1}[\omega])[\tau] = \int_{U \cap V} (\delta^{k-1}\omega) \wedge \tau = \int_{U \cap V} (-d\phi_U \wedge \omega) \wedge \tau. \tag{17.14}
\]

Next, we look at the mapping

\[
(\tilde{\delta})^{m} \circ PD : H^{k-1}_{dR}(U \cap V) \to H^{m}_{c,dR}(U \cup V)^*. \tag{17.15}
\]
We then have
\[
((\tilde{\delta}^m)^* \circ PD[\omega])[\tau] = PD[\omega](\tilde{\delta}^m[\tau]) = \int_{U \cap V} \omega \wedge \tilde{\delta}^m \tau. \tag{17.16}
\]

Recall from our discussion of the compactly supported Mayer-Vietoris sequence that
\[
\tilde{\delta}^m \tau = [d\phi_U \wedge \tau] = [-d\phi_V \wedge \tau] \in H^{m+1}_{c,dR}(U \cap V). \tag{17.17}
\]

So we have
\[
((\tilde{\delta}^m)^* \circ PD[\omega])[\tau] = \int_{U \cap V} \omega \wedge d\phi_U \wedge \tau = (-1)^{k-1} \int_{U \cap V} d\phi_U \wedge \omega \wedge \tau. \tag{17.18}
\]

So we see that
\[
(\tilde{\delta}^m)^* \circ PD = (-1)^k PD \circ \delta^{k-1} \tag{17.19}
\]

Next, we look at the square
\[
\begin{array}{c}
H^k_{dR}(U \cup V) \xrightarrow{\beta^k} H^k_{dR}(U) \oplus H^k_{dR}(V) \\
\downarrow PD \quad \quad \quad \downarrow PD \oplus PD
\end{array}
\]
\[
\begin{array}{c}
H^m_{c,dR}(U \cup V)^* \xrightarrow{(\tilde{\delta}^m)^*} (H^m_{c,dR}(U) \oplus H^m_{c,dR}(V))^* \\
\end{array}
\tag{17.20}
\]

Next, a lemma

**Lemma 17.4.** Let \( f : A \to B \oplus C \) be a linear map between finite dimensional vector spaces. Write \( f = (f_B, f_C) \), where \( f : A \to B \) and \( f_C : A \to C \). Then \( f^* : (B \oplus C)^* \to A^* \) is given by
\[
f^*(b^*, c^*)(a) = b^*(f_B(a)) + c^*(f_C(a)), \tag{17.21}
\]

where we used the isomorphism \((B \oplus C)^* \cong B^* \oplus C^*\).

**Proof.** We leave the proof of this as an exercise. \( \square \)

For the mapping
\[
(PD \oplus PD) \circ \beta^k : H^k_{dR}(U \cup V) \to (H^m_{c,dR}(U) \oplus H^m_{c,dR}(V))^*, \tag{17.22}
\]
choose \([\omega] \in H^k_{dR}(U \cup V)\), \([\tau_1] \in H^m_{c,dR}(U)\) and \([\tau_2] \in H^m_{c,dR}(V)\), and we have
\[
((PD \oplus PD) \circ \beta^k[\omega])([\tau_1], [\tau_2]) = (PD_U \circ \beta^k_U[\omega])([\tau_1]) + PD_V \circ \beta^k_V[\omega])([\tau_2]) = \int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2. \tag{17.23}
\]

Next, we look at the mapping
\[
(\tilde{\delta}^m)^* \circ PD : H^k_{dR}(U \cup V) \to (H^m_{c,dR}(U) \oplus H^m_{c,dR}(V))^*, \tag{17.24}
\]

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for which we have
\[
\left((\tilde{\beta}^m)^* \circ PD[\omega]\right)([\tau_1], [\tau_2]) = PD[\omega](\tilde{\beta}^m((\tau_1, \tau_2))) = PD[\omega](\tau_1 + \tau_2)
\]
\[
= \int_M \omega \wedge (\tau_1 + \tau_2) = \int_M \omega \wedge \tau_1 + \int_M \omega \wedge \tau_2 = \int_U \omega \wedge \tau_1 + \int_V \omega \wedge \tau_2,
\]
(17.25)
since \(\tau_1\) has compact support on \(U\) and \(\tau_2\) has compact support on \(V\). So we have that
\[
(PD \oplus PD) \circ \beta^k = (\tilde{\beta}^m)^* \circ PD.
\]
(17.26)

We leave the remaining \(\alpha\) square(s) as an exercise.

By the five lemma, if the outer 4 vertical maps are isomorphisms, then so is the central vertical map. The proof is completed by induction on the number of open sets in the good cover, since we know it is true for \(\mathbb{R}^n\) from the previous lecture. 

**Corollary 17.5.** If \(M^n\) is a connected and orientable \(n\)-manifold with a finite good cover, then \(H^n_{cdR}(M) \cong \mathbb{R}\). If \(M\) is moreover compact, then \(H^n_{dR}(M) \cong \mathbb{R}\).

**Corollary 17.6.** If \(M^n\) is a connected and orientable \(n\)-manifold with a finite good cover then \(H^k_{dR}(M)\) and \(H^{n-k}_{cdR}(M)\) have the same dimension. If \(M\) is moreover compact, then \(H^k_{dR}(M)\) and \(H^{n-k}_{dR}(M)\) have the same dimension.

**Corollary 17.7.** If \(M^n\) is a compact oriented odd-dimensional manifold, then the Euler characteristic \(\chi(M) = 0\).

**Remark 17.8.** Poincaré duality is also true for singular homology with \(\mathbb{Z}\) coefficients on a orientable manifold. If \(M\) is not orientable, then it is still true for \(\mathbb{Z}/2\mathbb{Z}\) coefficients.

Next topics in the Spring: Poincare duality on nonorientable manifolds, Kunneth formula, Thom isomorphism, singular homology and cohomology, de Rham’s Theorem.

**References**
