Math 865, Topics in Riemannian Geometry

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Contents

1 Introduction

The first part of this course will be a review of some basic concepts in Riemannian geometry. We will then give a fairly basic introduction to the Ricci Flow. We will also study some conformal geometry, and look at Riemannian 4-manifolds in greater depth. If time permits, we will present some basics of hermitian geometry. Some basic references are [Bes87], [CLN06], [Lee97], [Pet06], [Poo81].

2 Lecture 1: September 4, 2007

2.1 Metrics, vectors, and one-forms

Let (M, g) be a Riemannian manifold. The metric $g \in \Gamma(S^2(T^*M))$. In coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i \otimes dx^j, \ \ g_{ij} = g_{ij}, \tag{2.1}$$

and $g_{ij} >> 0$ is a positive definite matrix. The symmetry condition is of course invariantly

$$g(X, Y) = g(Y, X). \tag{2.2}$$

A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M), \tag{2.3}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \tag{2.4}$$

is the coordinate partial. We will use the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

A 1-form is a section of the cotangent bundle, $X \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i \quad \omega_i \in C^\infty(M). \tag{2.5}$$

Remark. Note that components of vector fields have upper indices, while components of 1-forms have lower indices. However, a collection of vector fields will be indexed by lower indices, $\{Y_1, \ldots, Y_p\}$, and a collection of 1-forms will be indexed by upper indices $\{dx^1, \ldots, dx^n\}$. This is one reason why we write the coordinates with upper indices.

2.2 The musical isomorphisms

The metric gives an isomorphism between TM and T^*M,

$$\flat : TM \to T^*M \tag{2.6}$$

defined by

$$\flat(X)(Y) = g(X, Y). \tag{2.7}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp\omega_1, \sharp\omega_2). \tag{2.8}$$

Note that if g has component g_{ij}, then $\langle\cdot,\cdot\rangle$ has components g^{ij}, the inverse matrix of g_{ij}.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{2.9}$$

where

$$X_i = g_{ij} X^j, \tag{2.10}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{2.11}$$

where

$$\omega^i = g^{ij}\omega_j, \tag{2.12}$$

thus the sharp operator "raises" an index.

2.3 Inner product on tensor bundles

The metric induces a metric on $\Lambda^k(T^*M)$. We give 3 definitions, which are all equivalent.

Definition 1: If

$$\begin{aligned} \omega_1 &= \alpha_1 \wedge \cdots \wedge \alpha_k \\ \omega_2 &= \beta_1 \wedge \cdots \wedge \beta_k, \end{aligned} \tag{2.13}$$

then

$$\langle \omega_1, \omega_2 \rangle = \det(\langle \alpha_i, \beta_j \rangle), \tag{2.14}$$

and extend linearly. This is well-defined.

Definition 2: If $\{e_i\}$ is an ONB of T_pM, let $\{e^i\}$ denote the dual basis, defined by $e^i(e_j) = \delta^i_j$. Then declare that

$$e^{i_1} \wedge \cdots \wedge e^{i_k}, \quad 1 \le i_1 < i_2 < \cdots < i_k \le n, \tag{2.15}$$

is an ONB of $\Lambda^k(T^*_pM)$.

Definition 3: If $\omega \in \Lambda^k(T^*M)$, then in coordinates

$$\omega = \sum_{i_1,\dots,i_k=1}^{n} \omega_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{2.16}$$

Then

$$\langle \omega, \omega \rangle = \sum_{i_1<\cdots<i_k} \omega^{i_1\dots i_k}\omega_{i_1\dots i_k} = \frac{1}{k!}\sum_{i_1\dots i_k} \omega^{i_1\dots i_k}\omega_{i_1\dots i_k}, \tag{2.17}$$

where

$$\omega^{i_1 \dots i_k} = g^{i_1 l_i} g^{i_2 l_2} \dots g^{i_k l_k} \omega_{l_1 \dots l_k}. \tag{2.18}$$

To get an inner product on the full tensor bundle, we let

$$\Omega \in \Gamma\Big((TM)^{\otimes_p} \otimes (T^*M)^{\otimes_q}\Big). \tag{2.19}$$

We call such Ω a (p,q)-*tensor field*. Declare

$$e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e^{j_1} \otimes \dots \otimes e^{j_q} \tag{2.20}$$

to be an ONB. If in coodinates,

$$\Omega = \Omega^{i_1 \dots i_p}_{j_1 \dots j_q} \partial_{i_1} \otimes \dots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}, \tag{2.21}$$

then

$$\|\Omega\|^2 = \langle \omega, \omega \rangle = \Omega^{j_1 \dots j_q}_{i_1 \dots i_p} \Omega^{i_1 \dots i_p}_{j_1 \dots j_q}. \tag{2.22}$$

By polarization,

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2}\Big(\|\Omega_1 + \Omega_2\|^2 - \|\Omega_1\|^2 - \|\Omega_2\|^2\Big). \tag{2.23}$$

We remark that one may reduce a (p,q)-tensor field into a $(p-1, q-1)$-tensor field for $p \geq 1$ and $q \geq 1$. This is called a *contraction*, but one must specify which indices are contracted. For example, the contraction of Ω in the first contrvariant index and first covariant index is written invariantly as

$$Tr_{(1,1)}\Omega, \tag{2.24}$$

and in coordinates is given by

$$\delta^{j_1}_{i_1} \Omega^{i_1 \dots i_p}_{j_1 \dots j_q} = \Omega^{l i_2 \dots i_p}_{l j_2 \dots j_q}. \tag{2.25}$$

2.4 Connections on vector bundles

A connection is a mapping $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, with the properties

- $\nabla_X s \in \Gamma(E)$,
- $\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$,
- $\nabla_X (fs) = (Xf)s + f \nabla_X s$.

In coordinates, letting $s_i, i = 1 \dots p$, be a local basis of sections of E,

$$\nabla_{\partial_i} s_j = \Gamma^k_{ij} s_k. \tag{2.26}$$

If E carries an inner product, then ∇ is *compatible* if

$$X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle. \tag{2.27}$$

For a connection in TM, ∇ is called *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM). \tag{2.28}$$

Theorem 2.1. *(Fundamental Theorem of Riemannian Geometry) There exists a unique symmetric, compatible connection in TM.*

Invariantly, the connection is defined by

$$\begin{aligned}\langle \nabla_X Y, Z \rangle = \frac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & -\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\Big).\end{aligned} \tag{2.29}$$

Letting $X = \partial_i, Y = \partial_j, Z = \partial_k$, we obtain

$$\begin{aligned}\Gamma^l_{ij} g_{lk} &= \langle \Gamma^l_{ij} \partial_l, \partial_k \rangle = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle \\ &= \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\Big),\end{aligned} \tag{2.30}$$

which yields the formula

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big)} \tag{2.31}$$

for the Riemannian Christoffel symbols.

2.5 Covariant derivatives of tensor fields

First consider $X \in \Gamma(TM)$, which is a $(1, 0)$ tensor field. We define a $(1, 1)$ tensor field ∇X as follows

$$\nabla X(Y) = \nabla_Y X. \tag{2.32}$$

If S is a $(1, 1)$ tensor field, then we define a $(1, 2)$ tensor field ∇S by

$$\nabla S(X, Y) \equiv (\nabla_X S)Y \equiv \nabla_X(S(Y)) - S(\nabla_X Y). \tag{2.33}$$

With this definition, we have the Leibniz rule

$$\nabla_X(S(Y)) = (\nabla_X S)(Y) + S(\nabla_X Y). \tag{2.34}$$

This serves to define a connection on $\Gamma(TM \otimes T^*M)$.

In general, if S is a $(1, r)$-tensor field then define a $(1, r + 1)$-tensor field ∇S by

$$\nabla S(X, Y_1, \dots, Y_r) = \nabla_X(S(Y_1, \dots, Y_r)) - \sum_{i=1}^{r} S(Y_1, \dots, \nabla_X Y_i, \dots, Y_r), \tag{2.35}$$

and this defines a connection on $\Gamma(TM \otimes (T^*M)^{\otimes r})$.

Next, let S be a $(0, r)$-tensor field, then define a $(0, r + 1)$-tensor field ∇S by

$$\nabla S(X, Y_1, \dots, Y_r) = X(S(Y_1, \dots, Y_r)) - \sum_{i=1}^{r} S(Y_1, \dots, \nabla_X Y_i, \dots, Y_r). \tag{2.36}$$

This defines a connection on $\Gamma((T^*M)^{\otimes r})$.

We next consider the above definitions in components. For the case of a vector field $X \in \Gamma(TM)$, ∇X is a $(1,1)$ tensor field. In coordinates, this is written as

$$\nabla X = \nabla_m X^i dx^m \otimes \partial_i, \tag{2.37}$$

where

$$\nabla_m X^i = \partial_m X^i + X^l \Gamma^i_{ml}. \tag{2.38}$$

For a $(0,r)$-tensor field S, we have

$$\nabla S = \nabla_m S_{i_1,\dots i_r} dx^m \otimes dx^{i_1} \otimes \dots \otimes dx^{i_r}, \tag{2.39}$$

where

$$\nabla_m S_{i_1,\dots i_r} = \partial_m S_{i_1,\dots i_r} - S_{l i_2 \dots i_r} \Gamma^l_{i_1 m} - \dots - S_{i_1 \dots i_{r-1} l} \Gamma^l_{i_r m}. \tag{2.40}$$

Note for a 1-form ω, we have

$$\nabla_m \omega_i = \partial_m \omega_i - \omega_l \Gamma^l_{im}. \tag{2.41}$$

Compare the signs with the covariant derivative of a vector field.

For a general (p,q)-tensor field, in coordinates,

$$\begin{aligned} \nabla_m S^{i_1 \dots i_p}_{j_1 \dots j_q} \equiv \partial_m S^{i_1 \dots i_p}_{j_1 \dots j_q} &+ S^{l i_2 \dots i_p}_{j_1 \dots j_q} \Gamma^{i_1}_{ml} + \dots + S^{i_1 \dots i_{p-1} l}_{j_1 \dots j_q} \Gamma^{i_p}_{ml} \\ &- S^{i_1 \dots i_p}_{l j_2 \dots j_q} \Gamma^l_{m j_1} - \dots - S^{i_1 \dots i_p}_{j_1 \dots j_{q-1} l} \Gamma^l_{m j_q}. \end{aligned} \tag{2.42}$$

We leave it to the reader to write down an invariant definition.

Remark. Some authors instead write covariant derivatives with a semi-colon

$$\nabla_m S^{i_1 \dots i_p}_{j_1 \dots j_q} = S^{i_1 \dots i_p}_{j_1 \dots j_q ; m}. \tag{2.43}$$

However, the ∇ notation fits nicer with our conventions, since the *first* index is the direction of covariant differentiation.

Notice the following calculation,

$$(\nabla g)(X,Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \tag{2.44}$$

so the metric is parallel. Also note that that covariant differentiation commutes with contraction,

$$\nabla_m \left(\delta^{j_1}_{i_1} X^{i_1 i_2 \dots}_{j_1 j_2 \dots} \right) = \delta^{j_1}_{i_1} \nabla_m X^{i_1 i_2 \dots}_{j_1 j_2 \dots} \tag{2.45}$$

Let $I : TM \to TM$ denote the identity map, which is naturally a $(1,1)$ tensor. We have

$$(\nabla I)(X,Y) = \nabla_X (I(Y)) - I(\nabla_X Y) = \nabla_X Y - \nabla_X Y = 0, \tag{2.46}$$

so the identity map is also parallel.

2.6 Gradient and Hessian

For $f \in C^1(M, \mathbb{R})$, the *gradient* is defined as

$$\nabla f = \sharp(df) \tag{2.47}$$

The *Hessian operator* is the endomorphism defined by

$$Hess(f)(X) = \nabla(\nabla f)(X) = \nabla_X(\nabla f) = \nabla_X(\sharp(df)). \tag{2.48}$$

Since the metric is parallel,

$$Hess(f)(X) = \sharp \nabla_X(df). \tag{2.49}$$

The *Hessian* is the $(0, 2)$-tensor defined as

$$\nabla^2 f(X, Y) = \nabla df(X, Y) = X(df(Y)) - df(\nabla_X Y) = X(Yf) - (\nabla_X Y)f. \tag{2.50}$$

Note the Hessian operator is obtained from the Hessian simply by using the sharp operator to convert the Hessian into an endomorphism.

In components, we have

$$\nabla^2 f(\partial_i, \partial_j) = \nabla_i \nabla_j f = \nabla_i (df)_j = \partial_i \partial_j f - \Gamma^k_{ij} f_k. \tag{2.51}$$

The symmetry of the Hessian

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X), \tag{2.52}$$

then follows easily from the symmetry of the Riemannian connection.

3 Lecture 2: September 6, 2007

3.1 Curvature in vector bundles

Let $X, Y \in \Gamma(TM)$, $s \in \Gamma(E)$, where $\pi : E \to M$ is a vector bundle with a connection ∇, and define

$$\mathcal{R}_\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s. \tag{3.1}$$

This is linear over C^∞ functions, so defines a tensor,

$$\mathcal{R}_\nabla \in \Gamma(T^*M \otimes T^*M \otimes End(E)), \tag{3.2}$$

called the *curvature tensor* of the bundle E. Clearly $\mathcal{R}_\nabla$ is skew-symmetric in the first 2 indices, so in fact

$$\mathcal{R}_\nabla \in \Gamma(\Lambda^2(T^*M) \otimes End(E)). \tag{3.3}$$

Let E has an inner product $\langle\cdot,\cdot\rangle$, and assume that ∇ is compatible with this inner product,

$$X\langle s_1, s_2\rangle = \langle\nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle, \tag{3.4}$$

for $X \in \Gamma(TM)$, and $s_1, s_2 \in \Gamma(E)$ (note that, by a partition of unity argument, any vector bundle with inner product admits a compatible connection). In this case,

$$\mathcal{R}_\nabla \in \Gamma(\Lambda^2(T^*M) \otimes \mathfrak{so}(E)), \tag{3.5}$$

where $\mathfrak{so}(E)$ is the bundle of skew-symmetric endomorphisms of E. Equivalently,

$$\langle\mathcal{R}_\nabla(X,Y)s_1, s_2\rangle + \langle\mathcal{R}_\nabla(X,Y)s_2, s_1\rangle = 0, \tag{3.6}$$

for $X, Y \in \Gamma(TM)$, and $s_1, s_2 \in \Gamma(E)$.

Let $f : N \to M$, and consider the pullback bundle

$$f^*E = \{(p,v), p \in N, v \in E : f(p) = \pi(v)\}. \tag{3.7}$$

There are natural projections $\pi_1 : f^*E \to N$, and $\pi_2 : f^*E \to TM$ defined by $\pi_1((p,v)) = p$, and $\pi_2((p,v)) = v$, respectively. A connection in E induces a connection in f^*E. Take $V \in \Gamma(f^*E)$, and vectors $X_p, Y_p \in T_pN$. Then

$$\mathcal{R}_{f^*\nabla}(X_p, Y_p)V_p = \mathcal{R}_\nabla\Big((f_*X)_{f(p)}, (f_*Y)_{f(p)}\Big)\pi_2(V_p)_{f(p)}. \tag{3.8}$$

Written out, this is

$$(f^*\nabla)_{X_p}(f^*\nabla)_{Y_p}V_p - (f^*\nabla)_Y(f^*\nabla)_X V_p = \mathcal{R}_\nabla\Big((f_*X)_{f(p)}, (f_*Y)_{f(p)}\Big)\pi_2(V_p)_{f(p)}. \tag{3.9}$$

This is easily verified in coordinates. This basically says that the curvature of the pull-back connection is the pull-back of the curvature of the connection in TM, and is called the *structure equation* of the connection.

A fantastic reference for the *strict calculus* of connections in a vector bundle is [Poo81, Chapters 2,3].

3.2 Curvature in the tangent bundle

We now restrict the discussion to the tangent bundle TM, and let ∇ be the Riemannian connection. In this case, the curvature tensor will be denoted simply by R. The algebraic symmetries are:

$$\mathcal{R}(X,Y)Z = -\mathcal{R}(Y,X)Z \tag{3.10}$$
$$0 = \mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y \tag{3.11}$$
$$Rm(X,Y,Z,W) = -Rm(X,Y,W,Z) \tag{3.12}$$
$$Rm(X,Y,W,Z) = Rm(W,Z,X,Y), \tag{3.13}$$

where $Rm(X,Y,Z,W) = -\langle \mathcal{R}(X,Y)Z, W\rangle$. In terms of the musical isomorphisms,

$$Rm(X,Y,Z,W) = -\flat\Big(\mathcal{R}(X,Y)Z\Big)W. \tag{3.14}$$

In a coordinate system we write

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l. \tag{3.15}$$

Then

$$\begin{aligned} Rm(\partial_i, \partial_j, \partial_k, \partial_l) &= \langle \mathcal{R}(\partial_i, \partial_j)\partial_k, \partial_l \rangle \\ &= \langle R_{ijk}{}^m \partial_m, \partial_l \rangle \\ &= R_{ijk}{}^m g_{ml}. \end{aligned} \tag{3.16}$$

We now *define*

$$R_{ijlk} = -R_{ijkl} = R_{ijk}{}^m g_{ml}, \tag{3.17}$$

that is, we lower the upper index to the *third* position.

Remark. Some authors choose to lower this index to a different position. One has to be very careful with this, or you might end up proving that S^n has negative curvature!

Therefore as a $(1,3)$ tensor, the curvature tensor is

$$\mathcal{R} = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \tag{3.18}$$

and as a $(0,4)$ tensor,

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{3.19}$$

In coordinates, the algebraic symmetries of the curvature tensor are

$$R_{ijk}{}^l = -R_{jik}{}^l \tag{3.20}$$

$$0 = R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l \tag{3.21}$$

$$R_{ijkl} = -R_{ijlk} \tag{3.22}$$

$$R_{ijkl} = R_{klij}. \tag{3.23}$$

Of course, we can write the first 2 symmetries as a $(0,4)$ tensor,

$$R_{ijkl} = -R_{jikl} \tag{3.24}$$

$$0 = R_{ijkl} + R_{jkil} + R_{kijl}. \tag{3.25}$$

Note that using (3.23), the algebraic Bianchi identity (3.25) may be written as

$$0 = R_{ijkl} + R_{iklj} + R_{iljk}. \tag{3.26}$$

We next compute the curvature tensor in coordinates.

$$
\begin{aligned}
\mathcal{R}(\partial_i, \partial_j)\partial_k &= R_{ijk}{}^{l}\partial_l \\
&= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \partial_i(\Gamma^l_{jk})\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m - \partial_j(\Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m \\
&= \Big(\partial_i(\Gamma^l_{jk}) + \Gamma^m_{jk}\Gamma^l_{im} - \partial_j(\Gamma^l_{ik}) - \Gamma^m_{ik}\Gamma^l_{jm}\Big)\partial_l,
\end{aligned}
\tag{3.27}
$$

which is the formula

$$
\boxed{R_{ijk}{}^{l} = \partial_i(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{ik}) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}}
\tag{3.28}
$$

Fix a point p. Exponential coordinates around p form a normal coordinate system at p. That is $g_{ij}(p) = \delta_{ij}$, and $\partial_k g_{ij}(p) = 0$, which is equivalent to $\Gamma^k_{ij}(p) = 0$. The Christoffel symbol is

$$
\Gamma^l_{jk} = \frac{1}{2} g^{lm}\Big(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\Big).
\tag{3.29}
$$

In normal coordinates at the point p,

$$
\partial_i \Gamma^l_{jk} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} + \partial_i\partial_j g_{km} - \partial_i\partial_m g_{jk}\Big).
\tag{3.30}
$$

We then have at p

$$
R_{ijk}{}^{l} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} - \partial_i\partial_m g_{jk} - \partial_j\partial_k g_{im} + \partial_j\partial_m g_{ik}\Big).
\tag{3.31}
$$

Lowering an index, we have at p

$$
\begin{aligned}
R_{ijkl} &= -\frac{1}{2}\Big(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}\Big) \\
&= -\frac{1}{2}\Big(\partial^2 \owedge g\Big).
\end{aligned}
\tag{3.32}
$$

The $\owedge$ symbol is the Kulkarni-Nomizu product, which takes 2 symmetric $(0,2)$ tensors and gives a $(0,4)$ tensor with the same algebraic symmetries of the curvature tensor, and is defined by

$$
\begin{aligned}
A \owedge B(X,Y,Z,W) =& A(X,Z)B(Y,W) - A(Y,Z)B(X,W) \\
& - A(X,W)B(Y,Z) + A(Y,W)B(X,Z).
\end{aligned}
$$

To remember, first term is $A(X,Z)B(Y,W)$, skew symmetrize in X and Y. Then skew-symmetrize both of these in Z and W.

3.3 Sectional curvature, Ricci tensor, and scalar curvature

Let $\Pi \subset T_pM$ be a 2-plane, and let $X_p, Y_p \in T_pM$ span Π. Then

$$K(\Pi) = \frac{Rm(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{3.33}$$

is independent of the particular chosen basis for Π, and is called the *sectional curvature* of the 2-plane Π. The sectional curvatures in fact determine the full curvature tensor:

Proposition 3.1. *Let Rm and Rm' be two $(0,4)$-curvature tensors which satisfy $K(\Pi) = K'(\Pi)$ for all 2-planes Π, then $Rm = Rm'$.*

From this proposition, if $K(\Pi) = k_0$ is constant for all 2-planes Π, then we must have

$$Rm(X,Y,Z,W) = k_0\Big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\Big), \tag{3.34}$$

That is

$$Rm = \frac{k_0}{2} g \owedge g. \tag{3.35}$$

In coordinates, this is

$$R_{ijkl} = k_0(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{3.36}$$

We define the *Ricci tensor* as the $(0,2)$-tensor

$$Ric(X,Y) = tr(U \to \mathcal{R}(U,X)Y). \tag{3.37}$$

We clearly have

$$Ric(X,Y) = R(Y,X), \tag{3.38}$$

so $Ric \in \Gamma(S^2(T^*M))$. We let R_{ij} denote the components of the Ricci tensor,

$$Ric = R_{ij} dx^i \otimes dx^i, \tag{3.39}$$

where $R_{ij} = R_{ji}$. From the definition,

$$R_{ij} = R_{lij}{}^{l} = g^{lm} R_{limj}. \tag{3.40}$$

Notice for a space of constant curvature, we have

$$\begin{aligned} R_{jl} = g^{ik} R_{ijkl} &= k_0 g^{ik}(g_{ik}g_{jl} - g_{jk}g_{il}) \\ &= (n-1)k_0 g_{jl}, \end{aligned} \tag{3.41}$$

or invariantly

$$Ric = (n-1)k_0 g. \tag{3.42}$$

The *Ricci endomorphism* is defined by

$$Rc(X) \equiv \sharp\Big(Ric(X, \cdot)\Big). \tag{3.43}$$

The *scalar curvature* is defined as the trace of the Ricci endomorphism

$$R \equiv tr(X \to Rc(X)). \tag{3.44}$$

In coordinates,

$$R = g^{pq} R_{pq} = g^{pq} g^{lm} R_{lpmq}. \tag{3.45}$$

Note for a space of constant curvature k_0,

$$R = n(n-1)k_0. \tag{3.46}$$

4 Lecture 3: September 11, 2007

4.1 Differential Bianchi Identity

The differential Bianchi identity is

$$\nabla Rm(X, Y, Z, V, W) + \nabla Rm(Y, Z, X, V, W) + \nabla Rm(Z, X, Y, V, W) = 0. \tag{4.1}$$

This can be easily verified using the definition of the covariant derivative of a $(0, 4)$ tensor field which was given in the last lecture, and using normal coordinates to simplify the computation. In coordinates, this is equivalent to

$$\boxed{\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.} \tag{4.2}$$

Let us raise an index,

$$\nabla_i R_{jkm}{}^{l} + \nabla_j R_{kim}{}^{l} + \nabla_k R_{ijm}{}^{l} = 0. \tag{4.3}$$

Contract on the indices i and l,

$$0 = \nabla_l R_{jkm}{}^{l} + \nabla_j R_{klm}{}^{l} + \nabla_k R_{ljm}{}^{l} = \nabla_l R_{jkm}{}^{l} - \nabla_j R_{km} + \nabla_k R_{jm}. \tag{4.4}$$

This yields the Bianchi identity

$$\boxed{\nabla_l R_{jkm}{}^{l} = \nabla_j R_{km} - \nabla_k R_{jm}.} \tag{4.5}$$

In invariant notation, this is sometimes written as

$$\delta \mathcal{R} = d^{\nabla} Ric, \tag{4.6}$$

where $d^{\nabla} : S^2(T^*M) \to \Lambda^2(T^*M) \otimes T^*M$, is defined by

$$d^{\nabla} h(X, Y, Z) = \nabla h(X, Y, Z) - \nabla h(Y, Z, X), \tag{4.7}$$

and δ is the *divergence operator.*

Next, trace (4.5) on the indices k and m,

$$g^{km}\nabla_l R_{jkm}{}^l = g^{km}\nabla_j R_{km} - g^{km}\nabla_k R_{jm}. \tag{4.8}$$

Since the metric is parallel, we can move the g^{km} terms inside,

$$\nabla_l g^{km} R_{jkm}{}^l = \nabla_j g^{km} R_{km} - \nabla_k g^{km} R_{jm}. \tag{4.9}$$

The left hand side is

$$\begin{aligned}\nabla_l g^{km} R_{jkm}{}^l &= \nabla_l g^{km} g^{lp} R_{jkpm}\\ &= \nabla_l g^{lp} g^{km} R_{jkpm}\\ &= \nabla_l g^{lp} R_{jp} = \nabla_l R_j^l.\end{aligned} \tag{4.10}$$

So we have the Bianchi identity

$$\boxed{2\nabla_l R_j^l = \nabla_j R.} \tag{4.11}$$

Invariantly, this can be written

$$\delta Rc = -\frac{1}{2} dR. \tag{4.12}$$

The minus appears due to the definition of the divergence operator.

Corollary 4.1. *Let (M, g) be a connected Riemannian manifold. If $n > 2$, and there exists a function $f \in C^\infty(M)$ satisfying $Ric = fg$, then $Ric = (n-1)k_0 g$, where k_0 is a constant.*

Proof. Taking a trace, we find that $R = nf$. Using (4.11), we have

$$2\nabla_l R_j^l = 2\nabla_l\Big(\frac{R}{n}\delta_j^l\Big) = \frac{2}{n}\nabla_l R = \nabla_l R. \tag{4.13}$$

Since $n > 2$, we must have $dR = 0$, which implies that R, and therefore f, is constant. $\square$

4.2 Algebraic study of the curvature tensor

Recall that the curvature tensor Rm as a $(0, 4)$-tensor satisfies

$$Rm \in S^2(\Lambda^2 T^*M) \subset \otimes^4 T^*M. \tag{4.14}$$

Define a map $b : S^2\Lambda^2 \to S^2\Lambda^2$ by

$$bRm(x, y, z, t) = \frac{1}{3}\Big(Rm(x, y, z, t) + Rm(y, z, x, t) + Rm(z, x, y, t)\Big), \tag{4.15}$$

this is called the *Bianchi symmetrization map*. Then $S^2(\Lambda^2)$ decomposes as

$$S^2(\Lambda^2) = Ker(b) \oplus Im(b). \tag{4.16}$$

Note that

$$b(\alpha \odot \beta) = \frac{1}{6}\alpha \wedge \beta, \tag{4.17}$$

where $\alpha, \beta \in \Lambda^2(T^*M)$, and $\odot$ denotes the symmetric product, therefore

$$Im(b) = \Lambda^4 T^*M. \tag{4.18}$$

Note that this implies $b \equiv 0$ if $n = 2, 3$, and $dim(Im(b)) = 1$ if $n = 4$.

Next, define

$$\mathcal{C} = Ker(b) \subset S^2(\Lambda^2) \tag{4.19}$$

to be the *space of curvature-like tensors*. Consider the decomposition

$$S^2(\Lambda^2) = \mathcal{C} \oplus \Lambda^4. \tag{4.20}$$

If V is a vector space of dimension p, then

$$dim(S^2(V)) = \frac{p(p+1)}{2}. \tag{4.21}$$

Since

$$dim(\Lambda^2) = \frac{n(n-1)}{2}, \tag{4.22}$$

we find that

$$dimS^2(\Lambda^2) = \frac{1}{8}n(n-1)(n^2-n+2). \tag{4.23}$$

Also,

$$dim(\Lambda^4) = \binom{n}{4}, \tag{4.24}$$

which yields

$$\begin{aligned} dim(\mathcal{C}) &= \frac{1}{8}n(n-1)(n^2-n+2) - \frac{1}{24}n(n-1)(n-2)(n-3) \\ &= \frac{1}{12}n^2(n^2-1). \end{aligned} \tag{4.25}$$

Recall the *Ricci contraction*, $c : \mathcal{C} \to S^2(T^*M)$, defined by

$$(c(Rm))(X,Y) = tr(U \to \sharp Rm(U,X,\cdot,Y)). \tag{4.26}$$

In components, we have

$$c(Rm) = R_{lij}{}^{l} dx^i \otimes dx^j = g^{pq} R_{ipjq} dx^i \otimes dx^j. \tag{4.27}$$

Recall the Kulkarni-Nomizu product $\owedge : S^2(T^*M) \times S^2(T^*M) \to \mathcal{C}$ defined by

$$\begin{aligned} h \owedge k(X,Y,Z,W) = h(X,Z)k(Y,W) - h(Y,Z)k(X,W) \\ - h(X,W)k(Y,Z) + h(Y,W)k(X,Z). \end{aligned} \tag{4.28}$$

Note that $h \owedge k = k \owedge h$.

Proposition 4.1. *The map $\psi : S^2(T^*M) \to \mathcal{C}$ defined by*

$$\psi(h) = h \owedge g, \tag{4.29}$$

is injective for $n > 2$.

Proof. First note that

$$\langle f, h \owedge g \rangle = 4 \langle cf, h \rangle. \tag{4.30}$$

To see this, we compute (in an orthonormal basis)

$$\begin{aligned} & f_{ijkl}(h_{ik}g_{jl} - h_{jk}g_{il} - h_{il}g_{jk} + h_{jl}g_{ik}) \\ &= f_{ijkj}h_{ik} - f_{ijki}h_{jk} - f_{ijjl}h_{il} + f_{ijil}h_{jl} \\ &= 4 f_{ijkj} h_{ik}. \end{aligned} \tag{4.31}$$

Also note that

$$c(h \owedge g) = (n-2)h + (tr(h))g. \tag{4.32}$$

To see this

$$\begin{aligned} c(h \owedge g) &= \sum_{j,l} (h \owedge g)_{ijkl} \\ &= \sum_{j,l} (h_{ik}g_{jl} - h_{jk}g_{il} - h_{il}g_{jk} + h_{jl}g_{ik} \\ &= n h_{ik} - h_{jk}g_{ij} - h_{ij}g_{jk} + (tr(h))g_{ik} \\ &= (n-2)h + (tr(h))g. \end{aligned} \tag{4.33}$$

To prove the proposition, assume that $h \owedge g = 0$. Then

$$\begin{aligned} 0 &= \langle h \owedge g, h \owedge g \rangle \\ &= 4 \langle h, c(h \owedge g) \rangle \\ &= 4 \langle h, (n-2)h + (tr(h))g \rangle \\ &= 4\Big((tr(h))^2 + (n-2)|h|^2 \Big), \end{aligned} \tag{4.34}$$

which clearly implies that $h = 0$ if $n > 2$. □

Corollary 4.2. *For $n = 2$, the scalar curvature determines the full curvature tensor. For $n = 3$, the Ricci curvature determines the full curvature tensor.*

Proof. The $n = 2$ case is trivial, since the only non-zero component of R can be R_{1212}. For any n, define the *Schouten tensor*

$$A = \frac{1}{n-2}\left(Ric - \frac{R}{2(n-1)}g\right). \tag{4.35}$$

We claim that

$$c(Rm - A \owedge g) = 0. \tag{4.36}$$

To see this, we first compute

$$tr(A) = \frac{1}{n-2}\left(R - \frac{nR}{2(n-1)}\right) = \frac{R}{2(n-1)}. \tag{4.37}$$

Then

$$\begin{aligned} c(Rm - A \owedge g) = c(Rm) - c(A \owedge g) &= Ric - \Big((n-2)A + (tr(A))g\Big) \\ &= Ric - \left(Ric - \frac{R}{2(n-1)}g + \frac{R}{2(n-1)}g\right) \\ &= 0. \end{aligned} \tag{4.38}$$

For $n = 3$, we have $dim(\mathcal{C}) = 6$. From the proposition, we also have

$$\psi : S^2(T^*M) \hookrightarrow \mathcal{C}. \tag{4.39}$$

But $dim(S^2(T^*)) = 6$, so ψ is an isomorphism. This implies that

$$Rm = A \owedge g. \tag{4.40}$$

□

Remark. The above argument of course implies that, in any dimension, the curvature tensor can always be written as

$$Rm = W + A \owedge g, \tag{4.41}$$

where $W \in Ker(c)$. The tensor W is called the *Weyl tensor*, which we will study in depth a bit later.

5 Lecture 4: September 13, 2007

5.1 Orthogonal decomposition of the curvature tensor

Last time we showed that the curvature tensor may be decomposed as

$$Rm = W + A \owedge g, \tag{5.1}$$

where $W \in Ker(c)$ is the *Weyl tensor*, and A is the *Schouten tensor*. We can rewrite this as

$$Rm = W + \frac{1}{n-2} E \owedge g + \frac{R}{2n(n-1)} g \owedge g, \tag{5.2}$$

where

$$E = Ric - \frac{R}{n} g \tag{5.3}$$

is the *traceless Ricci tensor*. In general, we will have

$$\begin{aligned} S^2(\Lambda^2(T^*M)) &= \Lambda^4(T^*M) \oplus \mathcal{C} \\ &= \Lambda^4 \oplus \mathcal{W} \oplus \psi(S^2_0(T^*M)) \oplus \psi(\mathbb{R}g), \end{aligned} \tag{5.4}$$

where $\mathcal{W} = Ker(c) \cap Ker(b)$. This turns out to be an irreducible decomposition as an $SO(n)$-module, except in dimension 4. In this case, the $\mathcal{W}$ splits into two irreducible pieces $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-$. We will discuss this in detail later.

Proposition 5.1. *The decomposition (5.2) is orthogonal.*

Proof. From above,

$$\langle W, h \owedge g \rangle = 4 \langle cW, h \rangle = 0, \tag{5.5}$$

so the Weyl tensor is clearly orthogonal to the other 2 terms. Next,

$$\langle E \owedge g, g \owedge g \rangle = \langle E, c(g \owedge g) \rangle = \langle E, 2(n-1)g \rangle = 0. \tag{5.6}$$

□

To compute these norms, note that for any tensor B,

$$\begin{aligned} |B \owedge g|^2 &= \langle B \owedge g, B \owedge g \rangle \\ &= 4 \langle B, c(B \owedge g) \rangle \\ &= 4 \langle B, (n-2)B + tr(B)g \rangle \\ &= 4(n-2)|B|^2 + 4(tr(B))^2. \end{aligned} \tag{5.7}$$

The decomposition (4.41) yields

$$|Rm|^2 = |W|^2 + 4(n-2)|A|^2 + 4(tr(A))^2, \tag{5.8}$$

while the decomposition (5.2) yields

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|E|^2 + \frac{2}{n(n-1)}R^2. \tag{5.9}$$

Note that

$$\begin{aligned} |E|^2 = E_{ij}E_{ij} &= (R_{ij} - \frac{R}{n}g_{ij})(R_{ij} - \frac{R}{n}g_{ij}) \\ &= |Ric|^2 - \frac{2}{n}R^2 + \frac{1}{n}R^2 \\ &= |Ric|^2 - \frac{1}{n}R^2, \end{aligned} \tag{5.10}$$

so we obtain

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|Ric|^2 - \frac{2}{(n-1)(n-2)}R^2. \tag{5.11}$$

5.2 The curvature operator

Consider the curvature

$$\mathcal{R} \in \Gamma\big(\Lambda^2(T^*M) \otimes \mathfrak{so}(TM)\big). \tag{5.12}$$

We know that

$$\mathfrak{so}(TM) = \Lambda^2(T^*M). \tag{5.13}$$

An explicit isomorphism is given as follows. Take $\omega \in \Lambda^2(T^*M)$, and $X \in TM$. Then $\omega(X, \cdot)$ is a 1-form, so ω maps to the endomorphisms $O : TM \to TM$ defined by $X \mapsto \sharp(\omega(X, \cdot))$. This is skew-symmetric:

$$\begin{aligned} \langle O(X), Y\rangle &= \langle \sharp(\omega(X, \cdot)), Y\rangle \\ &= \omega(X, Y) = -\omega(Y, X) = -\langle O(Y), X\rangle. \end{aligned} \tag{5.14}$$

So for the Riemannian connection, we can view the curvature as

$$\mathcal{R} \in \Gamma\big(\Lambda^2(T^*M) \otimes \Lambda^2(T^*M)\big). \tag{5.15}$$

Using the metric, we can identify $\Lambda^2(T^*M) = (\Lambda^2(T^*M))^*$, so we have

$$\mathcal{R} \in \Gamma\Big(End\big(\Lambda^2(T^*M)\big)\Big). \tag{5.16}$$

This is called the *curvature operator*. The identity

$$Rm(X, Y, Z, W) = Rm(Z, W, X, Y) \tag{5.17}$$

implies furthermore that $\mathcal{R}$ is symmetric,

$$\langle \mathcal{R}\omega_1, \omega_2\rangle = \langle \omega_1, \mathcal{R}\omega_2\rangle. \tag{5.18}$$

To see this, compute in an ONB

$$\begin{aligned} \langle \mathcal{R}\alpha, \beta \rangle &= \langle R_{ijkl}\alpha_{ij}, \beta_{kl} \rangle \\ &= \langle \alpha_{ij}, R_{ijkl}\beta_{kl} \rangle \\ &= \langle \alpha_{ij}, R_{klij}\beta_{kl} \rangle \\ &= \langle \alpha, \mathcal{R}\beta \rangle. \end{aligned} \tag{5.19}$$

Since any symmetric matrix can be diagonalized, $\mathcal{R}$ has $n(n-1)/2$ eigenvalues.

5.3 Curvature in dimension three

For $n = 3$, the Weyl tensor vanishes, so the curvature decomposes as

$$Rm = A \owedge g = (Ric - \frac{R}{4}g) \owedge g = Ric \owedge g - \frac{R}{4} g \owedge g, \tag{5.20}$$

in coordinates,

$$R_{ijkl} = R_{ik}g_{jl} - R_{jk}g_{il} - R_{il}g_{jk} + R_{jl}g_{ik} - \frac{R}{2}(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{5.21}$$

The sectional curvature in the plane spanned by $\{e_i, e_j\}$ is

$$\begin{aligned} R_{ijij} &= R_{ii}g_{jj} - R_{ji}g_{ij} - R_{ij}g_{ji} + R_{jj}g_{ii} - \frac{R}{2}(g_{ii}g_{jj} - g_{ji}g_{ij}) \\ &= R_{ii}g_{jj} - 2R_{ij}g_{ij} + R_{jj}g_{ii} - \frac{R}{2}(g_{ii}g_{jj} - g_{ij}g_{ij}). \end{aligned} \tag{5.22}$$

Note we do not sum repeated indices in the above equation! Choose an ONB so that the Rc is diagonalized at a point p,

$$Rc = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{5.23}$$

In this ONB, $R_{ij} = \lambda_i \delta_{ij}$ (again we do not sum!). Then the sectional curvature is

$$\begin{aligned} R_{ijij} &= \lambda_i g_{jj} - 2\lambda_i g_{ij} g_{ij} + \lambda_j g_{ii} - \frac{\lambda_1 + \lambda_2 + \lambda_3}{2}(g_{ii}g_{jj} - g_{ij}g_{ij}) \\ &= \lambda_i - 2\lambda_i \delta_{ij} + \lambda_j - \frac{\lambda_1 + \lambda_2 + \lambda_3}{2}(1 - \delta_{ij}). \end{aligned} \tag{5.24}$$

We obtain

$$\begin{aligned} R_{1212} &= \frac{1}{2}\left(\lambda_1 + \lambda_2 - \lambda_3\right) \\ R_{1313} &= \frac{1}{2}\left(\lambda_1 - \lambda_2 + \lambda_3\right) \\ R_{2323} &= \frac{1}{2}\left(-\lambda_1 + \lambda_2 + \lambda_3\right). \end{aligned} \tag{5.25}$$

We can also express the Ricci eigenvalues in terms of the sectional curvatures

$$Rc = \begin{pmatrix} R_{1212} + R_{1313} & 0 & 0 \\ 0 & R_{1212} + R_{2323} & 0 \\ 0 & 0 & R_{1313} + R_{2323} \end{pmatrix}. \tag{5.26}$$

We note the following, define

$$T_1(A) = -A + tr(A)g = -Ric + \frac{R}{2}g. \tag{5.27}$$

Since Rc is diagonal, $T_1(A)$ takes the form

$$T_1(A) = \begin{pmatrix} R_{2323} & 0 & 0 \\ 0 & R_{1313} & 0 \\ 0 & 0 & R_{1212} \end{pmatrix}. \tag{5.28}$$

That is, the eigenvalue of $T_1(A)$ with eigenvector e_i is equal to the sectional curvature of the 2-plane orthogonal to e_i.

Next, we consider the curvature operator $\mathcal{R} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$. We evaluate in the basis $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$. An easy computation shows that for $i < j$,

$$\mathcal{R}(e_i \wedge e_j) = R_{ijkl} e_k \wedge e_l = 2R_{ijij} e_i \wedge e_j, \tag{5.29}$$

so the curvature operator is also diagonal, and its eigenvalues are just twice the corresponding sectional curvatures.

6 Lecture 5: September 18, 2007

6.1 Covariant derivatives redux

Let E and E' be vector bundles over M, with covariant derivative operators ∇, and ∇', respectively. The covariant derivative operators in $E \otimes E'$ and $Hom(E, E')$ are

$$\nabla_X(s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s') \tag{6.1}$$
$$(\nabla_X L)(s) = \nabla'_X(L(s)) - L(\nabla_X s), \tag{6.2}$$

for $s \in \Gamma(E), s' \in \Gamma(E')$, and $L \in \Gamma(Hom(E, E'))$. Note also that the covariant derivative operator in $\Lambda(E)$ is given by

$$\nabla_X(s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^{r} s_1 \wedge \cdots \wedge (\nabla_X s_i) \wedge \cdots \wedge s_r, \tag{6.3}$$

for $s_i \in \Gamma(E)$.

These rules imply that if T is an (r, s) tensor, then the covariant derivative ∇T is an $(r, s+1)$ tensor given by

$$\nabla T(X, Y_1, \ldots, Y_s) = \nabla_X(T(Y_1, \ldots Y_s)) - \sum_{i=1}^{s} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s). \tag{6.4}$$

For notation, we will write a double covariant derivative as

$$\nabla^2 T = \nabla\nabla T, \tag{6.5}$$

which is an $(r, s+2)$ tensor.

Proposition 6.1. *For T an (r,s)-tensor field, the double covariant derivative satisfies*

$$\nabla^2 T(X, Y, Z_1, \ldots, Z_s) = \nabla_X(\nabla_Y T)(Z_1, \ldots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \ldots Z_s). \tag{6.6}$$

Proof. We compute

$$\begin{aligned}
\nabla^2 T(X, Y, Z_1, \ldots, Z_s) &= \nabla(\nabla T)(X, Y, Z_1, \ldots, Z_s)\\
&= \nabla_X(\nabla T(Y, Z_1, \ldots, Z_s)) - \nabla T(\nabla_X Y, Z_1, \ldots, Z_s)\\
&\quad - \sum_{i=i}^{s} \nabla T(Y, \ldots, \nabla_X Z_i, \ldots Z_s)\\
&= \nabla_X(\nabla_Y T(Z_1, \ldots, Z_s))\\
&\quad - \nabla_{\nabla_X Y}(T(Z_1, \ldots, Z_s))\\
&\quad + \sum_{i=1}^{s} T(Z_1, \ldots, \nabla_{\nabla_X Y} Z_i, \ldots, Z_s)\\
&\quad - \sum_{i=i}^{s} (\nabla_Y T)(Z_1, \ldots, \nabla_X Z_i, \ldots Z_s).
\end{aligned} \tag{6.7}$$

The right hand side of (6.6) is

$$\begin{aligned}
&\nabla_X(\nabla_Y T)(Z_1, \ldots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \ldots Z_s)\\
&= \nabla_X(\nabla_Y T(Z_1, \ldots, Z_s)) - \sum_{i=1}^{s} (\nabla_Y T)(Z_1, \ldots, \nabla_X Z_i, \ldots, Z_s)\\
&\quad - \nabla_{\nabla_X Y}(T(Z_1, \ldots, Z_s)) + \sum_{i=1}^{s} T(Z_1, \ldots, \nabla_{\nabla_X Y} Z_i, \ldots, Z_s).
\end{aligned} \tag{6.8}$$

Comparing terms, we see that both sides are equal. □

Remark. If we take a normal coordinate system, and $X = \partial_i, Y = \partial_j$, the above proposition says the seemingly obvious fact that, at p,

$$\nabla_i \nabla_j T^{j_1 \ldots j_r}_{i_i \ldots i_s} = \nabla_i(\nabla_j T^{j_1 \ldots j_r}_{i_i \ldots i_s}), \tag{6.9}$$

since the Christoffel symbols vanish at p in normal coordinates.

Equivalently, we could take an ONB at a point p, and parallel translate this frame to a neighborhood of p, to obtain an parallel orthonormal frame field in a neighborhood of p. The above equation would hold for the components of T with respect to this frame.

6.2 Commuting covariant derivatives

Let $X, Y, Z \in \Gamma(TM)$, and compute using the Proposition 6.1

$$\begin{aligned} \nabla^2 Z(X,Y) - \nabla^2 Z(Y,X) &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_X Y - \nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \\ &= \mathcal{R}(X,Y)Z, \end{aligned} \tag{6.10}$$

which is just the definition of the curvature tensor. In coordinates,

$$\boxed{\nabla_i \nabla_j Z^k = \nabla_j \nabla_i Z^k + R_{ijm}{}^k Z^m.} \tag{6.11}$$

We extend this to $(p,0)$-tensor fields:

$$\begin{aligned} &\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X) \\ &= \nabla_X(\nabla_Y(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_X Y}(Z_1 \otimes \cdots \otimes Z_p) \\ &\quad - \nabla_Y(\nabla_X(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_Y X}(Z_1 \otimes \ldots \otimes Z_p \\ &= \nabla_X\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p\Big) - \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_X Y} Z_i \otimes \cdots \otimes Z_p \\ &- \nabla_Y\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p\Big) + \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_Y X} Z_i \otimes \cdots \otimes Z_p \\ &= \sum_{j=1}^p \sum_{i=1, i\neq j}^p Z_1 \otimes \nabla_X Z_j \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p \\ &- \sum_{j=1}^p \sum_{i=1, i\neq j}^p Z_1 \otimes \nabla_Y Z_j \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p \\ &+ \sum_{i=1}^p Z_1 \otimes \cdots \otimes (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z_i \otimes \cdots \otimes Z_p \\ &= \sum_{i=1}^p Z_1 \otimes \cdots \otimes \mathcal{R}(X,Y) Z_i \otimes \cdots \otimes Z_p. \end{aligned} \tag{6.12}$$

In coordinates, this is

$$\boxed{\nabla_i \nabla_j Z^{i_1 \ldots i_p} = \nabla_j \nabla_i Z^{i_i \ldots i_p} + \sum_{k=1}^p R_{ijm}{}^{i_k} Z^{i_1 \ldots i_{k-1} m i_{k+1} \ldots i_p}.} \tag{6.13}$$

Proposition 6.2. *For a 1-form ω, we have*

$$\nabla^2 \omega(X,Y,Z) - \nabla^2 \omega(Y,X,Z) = \omega(\mathcal{R}(Y,X)Z). \tag{6.14}$$

Proof. Using Proposition 6.1, we compute

$$
\begin{aligned}
&\nabla^2\omega(X,Y,Z) - \nabla^2\omega(Y,X,Z) \\
&= \nabla_X(\nabla_Y\omega)(Z) - (\nabla_{\nabla_X Y}\omega)(Z) - \nabla_Y(\nabla_X\omega)(Z) - (\nabla_{\nabla_Y X}\omega)(Z) \\
&= X(\nabla_Y\omega(Z)) - \nabla_Y\omega(\nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y}Z) \\
&\quad - Y(\nabla_X\omega(Z)) + \nabla_X\omega(\nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X}Z) \\
&= X(\nabla_Y\omega(Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y\nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y}Z) \\
&\quad - Y(\nabla_X\omega(Z)) + X(\omega(\nabla_Y Z)) - \omega(\nabla_X\nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X}Z) \\
&= \omega\Big(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z\Big) + X(\nabla_Y\omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z)) \\
&\quad - Y(\nabla_X\omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)).
\end{aligned} \tag{6.15}
$$

The last six terms are

$$
\begin{aligned}
&X(\nabla_Y\omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z)) \\
&- Y(\nabla_X\omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)) \\
&= X\Big(Y(\omega(Z)) - \omega(\nabla_Y Z)\Big) - Y(\omega(\nabla_X Z)) - [X,Y](\omega(Z)) \\
&- Y\Big(X(\omega(Z)) - \omega(\nabla_X Z)\Big) + X(\omega(\nabla_Y Z)) \\
&= 0.
\end{aligned} \tag{6.16}
$$

□

Remark. It would have been a bit easier to assume we were in normal coordinates, and assume terms with $\nabla_X Y$ vanished, but we did the above for illustration.

In coordinates, this formula becomes

$$
\boxed{\nabla_i\nabla_j\omega_k = \nabla_j\nabla_i\omega_k - R_{ijk}{}^p\omega_p.} \tag{6.17}
$$

As above, we can extend this to $(0,s)$ tensors using the tensor product, in an almost identical calculation to the $(r,0)$ tensor case. Finally, putting everything together, the formula in coordinates for a general (r,s)-tensor T is

$$
\begin{aligned}
\nabla_i\nabla_j T^{i_1\dots i_r}_{j_1\dots j_s} = \nabla_j\nabla_i T^{i_1\dots i_r}_{j_1\dots j_s} &+ \sum_{k=1}^{r} R_{ijm}{}^{i_k} T^{i_1\dots i_{k-1} m i_{k+1}\dots i_r}_{j_1\dots j_s} \\
&- \sum_{k=1}^{s} R_{ijj_k}{}^m T^{i_1\dots i_r}_{j_1\dots j_{k-1} m j_{k+1}\dots j_s}.
\end{aligned} \tag{6.18}
$$

6.3 Rough Laplacian and gradient

For (p,q) tensor T, we let

$$
\Delta T = tr(X \to \sharp(\nabla^2 T)(X,\cdot)). \tag{6.19}
$$

This is called the *rough Laplacian*. In coordinates this is

$$\Delta T^{i_1\ldots i_r}_{j_1\ldots j_s} = g^{ij}\nabla_i\nabla_j T^{i_1\ldots i_r}_{j_1\ldots j_s}. \tag{6.20}$$

Equivalently, in an ONB,

$$\Delta T^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{i,j}\delta_{ij}\nabla_i\nabla_j T^{i_1\ldots i_r}_{j_1\ldots j_s}. \tag{6.21}$$

Proposition 6.3. *For a function $f \in C^3(M)$,*

$$\Delta df = d\Delta f + Rc^T(df). \tag{6.22}$$

Proof. We compute in coordinates

$$\begin{aligned}
\Delta\nabla_i f &= g^{lp}\nabla_p\nabla_l\nabla_i f\\
&= g^{lp}\nabla_p\nabla_i\nabla_l f\\
&= g^{lp}(\nabla_i\nabla_p\nabla_l f - R_{pil}{}^q\nabla_q f\\
&= \nabla_i g^{lp}\nabla_p\nabla_l f + R_i^q\nabla_q f\\
&= \nabla_i\Delta f + R_i^q\nabla_q f.
\end{aligned} \tag{6.23}$$

□

Remark. In (6.22), the Laplacian on the left hand side is *not* the Hodge Laplacian on 1-forms. More on this next time.

7 Lecture 6: September 20, 2007

7.1 Commuting Laplacian and Hessian

We compute the commutator of the Laplacian and Hessian, acting on functions.

Proposition 7.1. *Let $f \in C^4(M)$. Then*

$$\Delta\nabla^2 f = \nabla^2\Delta f + Rm * \nabla^2 f + \nabla Rm * \nabla f. \tag{7.1}$$

Proof. We compute

$$\begin{aligned}
(\Delta\nabla^2 f)_{ij} &= g^{kl}\nabla_k\nabla_l\nabla_i\nabla_j f\\
&= g^{kl}\nabla_k(\nabla_l\nabla_i\nabla_j f)\\
&= g^{kl}\nabla_k(\nabla_i\nabla_l\nabla_j f - R_{lij}{}^p\nabla_p f)\\
&= g^{kl}\Big(\nabla_k\nabla_i\nabla_l\nabla_j f\Big) - g^{kl}\nabla_k\Big(R_{lij}{}^p\nabla_p f\Big)\\
&= g^{kl}\Big(\nabla_k\nabla_i\nabla_j\nabla_l f\Big) - g^{kl}\nabla_k\Big(R_{lij}{}^p\nabla_p f\Big)\\
&= I + II.
\end{aligned} \tag{7.2}$$

We have

$$
\begin{aligned}
I &= g^{kl}\Big(\nabla_i\nabla_k\nabla_j\nabla_l f - R_{kil}{}^p\nabla_p\nabla_j f - R_{kij}{}^p\nabla_l\nabla_p f\Big) \\
&= g^{kl}\Big(\nabla_i(\nabla_k\nabla_j\nabla_l f) - R_{kil}{}^p\nabla_p\nabla_j f - R_{kij}{}^p\nabla_l\nabla_p f\Big) \\
&= g^{kl}\Big(\nabla_i\big(\nabla_j\nabla_k\nabla_l f - R_{kjl}{}^p\nabla_p f\big) - R_{kil}{}^p\nabla_p\nabla_j f - R_{kij}{}^p\nabla_l\nabla_p f\Big) \\
&= g^{kl}\Big(\nabla_i\nabla_j\nabla_k\nabla_l f - \nabla_i(R_{kjl}{}^p\nabla_p f) - R_{kil}{}^p\nabla_p\nabla_j f - R_{kij}{}^p\nabla_l\nabla_p f\Big).
\end{aligned}
\tag{7.3}
$$

Lowering some indices, we obtain

$$
I = g^{kl}\Big(\nabla_i\nabla_j\nabla_k\nabla_l f - \nabla_i(R_{kjpl}\nabla^p f) - R_{kipl}\nabla^p\nabla_j f - R_{kipj}\nabla_l\nabla^p f\Big). \tag{7.4}
$$

Since g is parallel,

$$
\begin{aligned}
I &= \nabla_i\nabla_j g^{kl}\nabla_k\nabla_l f - \nabla_i(g^{kl}R_{kjpl}\nabla^p f) - g^{kl}R_{kipl}\nabla^p\nabla_j f - R_{kipj}\nabla^k\nabla^p f \\
&= \nabla_i\nabla_j\Delta f - \nabla_i(-R_{jp}\nabla^p f) + R_{ip}\nabla^p\nabla_j f - R_{kipj}\nabla^k\nabla^p f
\end{aligned}
\tag{7.5}
$$

The second term is

$$
\begin{aligned}
II &= -g^{kl}\nabla_k\Big(R_{lij}{}^p\nabla_p f\Big) \\
&= -g^{kl}\nabla_k\Big(R_{lipj}\nabla^p f\Big) \\
&= -g^{kl}\nabla_k\Big(R_{pjli}\nabla^p f\Big) \\
&= -\nabla_k\Big(R_{pji}{}^k\nabla^p f\Big) \\
&= -\nabla_k R_{pji}{}^k\nabla^p f - R_{pji}{}^k\nabla_k\nabla^p f.
\end{aligned}
\tag{7.6}
$$

Using the contracted differential Bianchi identity (4.5), we write

$$
\begin{aligned}
II &= -(\nabla_p R_{ji} - \nabla_j R_{pi})\nabla^p f - R_{pji}{}^k\nabla_k\nabla^p f \\
&= -(\nabla_p R_{ji} - \nabla_j R_{pi})\nabla^p f - R_{pjki}\nabla^k\nabla^p f
\end{aligned}
\tag{7.7}
$$

Combining everything, we have

$$
\begin{aligned}
\Delta\nabla_i\nabla_j f &= I + II \\
&= \nabla_i\nabla_j\Delta f + (\nabla_i R_{jp})\nabla^p f + R_{jp}\nabla_i\nabla^p f + R_{ip}\nabla^p\nabla_j f - R_{kipj}\nabla^k\nabla^p f \\
&\quad - (\nabla_p R_{ji})\nabla^p f + (\nabla_j R_{pi})\nabla^p f - R_{pjki}\nabla^k\nabla^p f \\
&= \nabla_i\nabla_j\Delta f + R_{jp}\nabla_i\nabla^p f + R_{ip}\nabla^p\nabla_j f - 2R_{kipj}\nabla^k\nabla^p f \\
&\quad + (\nabla_i R_{jp} + \nabla_j R_{pi} - \nabla_p R_{ij})\nabla^p f.
\end{aligned}
\tag{7.8}
$$

□

We can rewrite the formula as

$$
\begin{aligned}
\Delta\nabla_i\nabla_j f &= \nabla_i\nabla_j\Delta f + (R_{jp}g_{ik} + R_{ip}g_{jk} - 2R_{kipj})\nabla^k\nabla^p f \\
&\quad + (\nabla_i R_{jp} + \nabla_j R_{pi} - \nabla_p R_{ij})\nabla^p f.
\end{aligned}
\tag{7.9}
$$

Proposition 7.2. *If g has constant sectional curvature k_0, then*

$$\Delta \nabla_i \nabla_j f = \nabla_i \nabla_j \Delta f + 2nk_0 \nabla_i \nabla_j f - 2k_0 \Delta f g_{ij}. \tag{7.10}$$

Proof. Since g has constant sectional curvature, g is in particular Einstein, so all covariant derivatives of Ricci vanish. The formula becomes

$$\begin{aligned}\Delta \nabla_i \nabla_j f &= \nabla_i \nabla_j \Delta f \\ &\quad + \Big((n-1)k_0 g_{jp} g_{ik} + (n-1)k_0 g_{ip} g_{jk} - 2k_0 (g_{kp} g_{ij} - g_{ip} g_{kj})\Big) \nabla^k \nabla^p f \\ &= \nabla_i \nabla_j \Delta f + 2nk_0 \nabla_i \nabla_j f - 2k_0 \Delta f g_{ij}.\end{aligned} \tag{7.11}$$

□

7.2 An application to PDE

We next give a PDE application of this formula.

Proposition 7.3. *Assume that (M, g) has constant sectional curvature $k_0 \geq 0$, and let $\Omega \subset M$ be a domain with smooth boundary. Let $f \in C^4(\overline{\Omega})$ be a convex function in Ω satisfying*

$$\Delta f = h, \tag{7.12}$$

where $0 < h \in C^2(\overline{\Omega})$ is a positive concave function. Then either (i) f is strictly convex in Ω, or (ii) f satisfies the Monge-Ampére equation

$$\det(\nabla^2 f) = 0, \tag{7.13}$$

everywhere in Ω.

Proof. Consider the function $H = \det^{1/n}(\nabla^2 f)$. Since f is convex, $H \geq 0$. We compute in normal coordinates

$$\begin{aligned}\Delta H &= \sum_l \nabla_l \nabla_l H \\ &= \sum_l \nabla_l (F^{ij} \nabla_l \nabla_i \nabla_j f) \\ &\leq F^{ij} \Delta \nabla_i \nabla_j f \\ &= F^{ij} (\nabla_i \nabla_j \Delta f + 2nk_0 \nabla_i \nabla_j f - 2k_0 \Delta f g_{ij}),\end{aligned} \tag{7.14}$$

where F^{ij} is the linearized operator of $\det^{1/n}$, and we have used the fact that $\det^{1/n}$ is a *concave* function of the eigenvalues, in the positive cone.

Using the equation (7.12), this is

$$\Delta H \leq F^{ij} (\nabla_i \nabla_j h + 2nk_0 \nabla_i \nabla_j f - 2k_0 \Delta f g_{ij}). \tag{7.15}$$

Since f is convex, F^{ij} is positive semi-definite, and since H is concave, $\nabla^2 h$ is negative semi-definite, so

$$\begin{aligned} \Delta H &\leq 2k_0 F^{ij}(n\nabla_i\nabla_j f - \Delta f g_{ij}) \\ &\leq 2k_0 n F^{ij}\nabla_i\nabla_j f \\ &= 2k_0 n H \end{aligned} \tag{7.16}$$

Rewriting, we have shown that

$$\Delta H - 2k_0 n H \leq 0. \tag{7.17}$$

In other words, H is a non-negative super-solution of the operator $\Delta - 2k_0 nI$ in Ω. If H is not strictly positive in Ω, then it must be zero at an interior point. In this case, the strong maximum principle says that H vanishes identically in Ω [GT01, Section 3.2]. This completes the proof. □

Remark. The above result is called a Caffarelli-Friedman type estimate. We also cheated a bit – H is not differentiable at 0, we leave it to the reader to fix this.

8 Lecture 7: Tuesday, September 25.

8.1 Integration and adjoints

If T is an (r, s)-tensor, we define the *divergence* of T, div T to be the $(r, s-1)$ tensor

$$(\text{div } T)(Y_1, \ldots, Y_{s-1}) = tr\Big(X \to \sharp(\nabla T)(X, \cdot, Y_1, \ldots, Y_{s-1})\Big), \tag{8.1}$$

that is, we trace the covariant derivative on the *first* two covariant indices. In coordinates, this is

$$(\text{div } T)^{i_1\ldots i_r}_{j_1\ldots j_{s-1}} = g^{ij}\nabla_i T^{i_1\ldots i_r}_{jj_1\ldots j_{s-1}}. \tag{8.2}$$

If X is a vector field, define

$$(\text{div } X) = tr(\nabla X), \tag{8.3}$$

which is in coordinates

$$\text{div } X = \delta^i_j \nabla_i X^j = \nabla_j X^j. \tag{8.4}$$

For vector fields and 1-forms, these two are of course closely related:

Proposition 8.1. *For a vector field X,*

$$\text{div } X = \text{div } (\flat X). \tag{8.5}$$

Proof. We compute

$$\begin{aligned}\text{div } X &= \delta^i_j \nabla_i X^j \\ &= \delta^i_j \nabla_i g^{jl} X_l \\ &= \delta^i_j g^{jl} \nabla_i X_l \\ &= g^{il} \nabla_i X_l = \text{div } (\flat X).\end{aligned} \tag{8.6}$$

□

If M is oriented, we define the Riemannian volume element dV to be the oriented unit norm element of $\Lambda^n(T^*M_x)$. Equivalently, if $\omega_1, \ldots \omega_n$ is a positively oriented ONB of T^*M_x, then

$$dV = \omega^1 \wedge \cdots \wedge \omega^n. \tag{8.7}$$

In coordinates,

$$dV = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n. \tag{8.8}$$

Recall the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$ defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV_x, \tag{8.9}$$

where $\alpha, \beta \in \Lambda^p$.

Proposition 8.2. *(i) The Hodge star is an isometry from Λ^p to Λ^{n-p}.*
(ii) $*(\omega^1 \wedge \cdots \wedge \omega^p) = \omega^{p+1} \wedge \cdots \wedge \omega^n$ if $\omega_1, \ldots \omega_n$ is a positively oriented ONB of T^*M_x. In particular, $*1 = dV$, and $*dV = 1$.
(iii) On Λ^p, $*^2 = (-1)^{p(n-p)}$.
(iv) For $\alpha, \beta \in \Lambda^p$,

$$\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta) = *(\beta \wedge *\alpha). \tag{8.10}$$

(v) If $\{e_i\}$ and $\{\omega^i\}$ are dual ONB of T_xM, and T^*_xM, respectively, and $\alpha \in \Lambda^p$, then

$$*(\omega^j \wedge \alpha) = (-1)^p i_{e_j}(*\alpha), \tag{8.11}$$

where $i_X : \Lambda^p \to \Lambda^{p-1}$ is interior multiplication *defined by*

$$i_X \alpha(X_1, \ldots, X_p) = \alpha(X, X_1, \ldots, X_p). \tag{8.12}$$

Proof. The proof is left to the reader. □

Remark. In general, locally there will be two different Hodge star operators, depending upon the two different choices of local orientation. Each will extend to a *global* Hodge star operator if and only if M is orientable. However, one can still construct *global* operators using the Hodge star, even if M is non-orientable, an example of which will be the Laplacian.

We next give a formula relating the exterior derivative and covariant differentiation.

Proposition 8.3. *The exterior derivative $d : \Omega^p \to \Omega^{p+1}$ is given by*

$$d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{8.13}$$

(the notation means that the $\hat{X}_j$ term is omitted). That is, the exterior derivative $d\omega$ is the skew-symmetrization of $\nabla\omega$, we write $d\omega = Sk(\nabla\omega)$. If $\{e_i\}$ and $\{\omega^i\}$ are dual ONB of T_xM, and T_x^*M, then this may be written

$$d\omega = \sum_i \omega^i \wedge \nabla_{e_i}\omega. \tag{8.14}$$

Proof. Recall the formula for the exterior derivative [War83, Theorem ?],

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{8.15}$$

Since both sides of the equation (8.13) are tensors, we may assume that $[X_i, X_j]_x = 0$, at a fixed point x. Since the connection is Riemannian, we also have $\nabla_{X_i}X_j(x) = 0$. We then compute at the point x.

$$\begin{aligned} d\omega(X_0, \ldots, X_p)(x) &= \sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \Big)(x) \\ &= \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p)(x), \end{aligned} \tag{8.16}$$

using the definition of the covariant derivative. This proves the first formula. For the second, note that

$$\nabla_{X_j}\omega = \nabla_{(X_j)^i e_i}\omega = \sum_{i=1}^{n} \omega^i(X_j) \cdot (\nabla_{e_i}\omega), \tag{8.17}$$

so we have

$$\begin{aligned} d\omega(X_0, \ldots, X_p)(x) &= \sum_{j=0}^{p} (-1)^j \sum_{i=1}^{n} \omega^i(X_j) \cdot (\nabla_{e_i}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p)(x) \\ &= \sum_i (\omega^i \wedge \nabla_{e_i}\omega)(X_0, \ldots, X_p)(x). \end{aligned} \tag{8.18}$$

□

Remark. One has to choose an identification of $\Lambda(T^*M)$ with $\Lambda(TM)^*$, in order to view forms as multilinear alternating maps on the tangent space. We choose the identification as in [War83, page 59]: if $\omega = e^1 \wedge \cdots \wedge e^p \in \Lambda^p(T^*M)$, and $e = e_1 \wedge \cdots \wedge e_p \in \Lambda^p(TM)$, then

$$\omega(e) = \det[e^i(e_j)]. \tag{8.19}$$

This makes the wedge product defined as follows. If $\alpha \in \Omega^p$, and $\beta \in \Omega^q$, then

$$\alpha \wedge \beta(X_1, \ldots, X_{p+q}) = \frac{1}{p!\, q!} \sum_{\sigma} \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \cdot \beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}), \tag{8.20}$$

and the sum is over all permutations of length $p + q$.

Proposition 8.4. *For a vector field X,*

$$*(\text{div } X) = (\text{div } X)dV = d(i_X dV) = \mathcal{L}_X(dV). \tag{8.21}$$

Proof. Fix a point $x \in M$, and let $\{e_i\}$ be an orthonormal basis of T_xM. In a small neighborhood of x, parallel translate this frame along radial geodesics. For such a frame, we have $\nabla_{e_i} e_j(x) = 0$. Such a frame is called an *adapted* moving frame field at x. Let $\{\omega^i\}$ denote the dual frame field. We have

$$\begin{aligned}
\mathcal{L}_X(dV) &= (di_X + i_X d)dV = d(i_X dV) \\
&= \sum_i \omega^i \wedge \nabla_{e_i}\big(i_X(\omega^1 \wedge \cdots \wedge \omega^n)\big) \\
&= \sum_i \omega^i \wedge \nabla_{e_i}\Big((-1)^{j-1} \sum_{j=1}^{n} \omega^j(X)\omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^n\Big) \\
&= \sum_{i,j} (-1)^{j-1} e_i\big(\omega^j(X)\big)\omega^i \wedge \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^n \\
&= \sum_i \omega^i(\nabla_{e_i} X)dV \\
&= (\text{div } X)dV = *(\text{div } X).
\end{aligned} \tag{8.22}$$

□

Corollary 8.1. *Let (M, g) be compact, orientable and without boundary. If X is a C^1 vector field, then*

$$\int_M (\text{div } X)dV = 0. \tag{8.23}$$

Proof. Using Stokes' Theorem and Proposition 8.4,

$$\int_M (\text{div } X)dV = \int d(i_X dV) = \int_{\partial M} i_x dV = 0. \tag{8.24}$$

□

Using this, we have an integration formula for (r,s)-tensor fields.

Corollary 8.2. *Let (M,g) be as above, T be an (r,s)-tensor field, and S be a $(r,s+1)$ tensor field. Then*

$$\int_M \langle \nabla T, S\rangle dV = -\int_M \langle T, \text{div } S\rangle dV. \tag{8.25}$$

Proof. Let us view the inner product $\langle T, S\rangle$ as a 1-form ω. In coordinates

$$\omega = \langle T, S\rangle = T^{j_1\dots j_s}_{i_1\dots i_r} S^{i_1\dots i_r}_{jj_1\dots j_s} dx^j. \tag{8.26}$$

Note the indices on T are reversed, since we are taking an inner product. Taking the divergence, since g is parallel we compute

$$\begin{aligned}\text{div } (\langle T, S\rangle) &= \nabla^j (T^{j_1\dots j_s}_{i_1\dots i_r} S^{i_1\dots i_r}_{jj_1\dots j_s})\\ &= \nabla^j (T^{j_1\dots j_s}_{i_1\dots i_r}) S^{i_1\dots i_r}_{jj_1\dots j_s} + T^{j_1\dots j_s}_{i_1\dots i_r} \nabla^j S^{i_1\dots i_r}_{jj_1\dots j_s}\\ &= \langle \nabla T, S\rangle + \langle T, \text{div } S\rangle.\end{aligned} \tag{8.27}$$

The result then follows from Proposition 8.1 and Corollary 8.1. □

Remark. Some authors define $\nabla^* = -\text{div}$, for example [Pet06].

Recall the adjoint of d, $\delta : \Omega^p \to \Omega^{p-1}$ defined by

$$\delta\omega = (-1)^{n(p+1)+1} * d * \omega. \tag{8.28}$$

Proposition 8.5. *The operator δ is the L^2 adjoint of d,*

$$\int_M \langle \delta\alpha, \beta\rangle dV = \int_M \langle \alpha, d\beta\rangle dV, \tag{8.29}$$

where $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$.

Proof. We compute

$$\begin{aligned}\int_M \langle \alpha, d\beta\rangle dV &= \int_M d\beta \wedge *\alpha\\ &= \int_M \Big(d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big)\\ &= \int_M (-1)^{p+(n-p+1)(p-1)} \beta \wedge * * d * \alpha\\ &= \int_M \langle \beta, (-1)^{n(p+1)+1} * d * \alpha\rangle dV\\ &= \int_M \langle \beta, \delta\alpha\rangle dV.\end{aligned} \tag{8.30}$$

□

Proposition 8.6. *On Ω^p, $\delta = -\text{div}$.*

Proof. Let $\omega \in \Omega^p$. Fix $x \in M$, and dual ONB $\{e_i\}$ and $\{\omega^i\}$. We compute at x,

$$\begin{aligned}(\text{div } \omega)(x) &= \sum_j i_{e_j} \nabla_{e_j} \omega \\ &= \sum_j (-1)^{p(n-p)} \Big(i_{e_j} \big(* * (\nabla_{e_j} \omega) \big) \Big) \\ &= (-1)^{p(n-p)} \sum_j (-1)^{n-p} * (\omega^j \wedge * \nabla_{e_j} \omega) \\ &= (-1)^{(p+1)(n-p)} \sum_j * \big(\omega^j \wedge \nabla_{e_j} (*\omega) \big) \\ &= (-1)^{n(p+1)} (* d * \omega)(x).\end{aligned} \tag{8.31}$$

□

Remark. Formula (8.6) requires a bit of explanation. The divergence is defined on tensors, while δ is defined on differential forms. What we mean is defined on the first line of (8.31), where the covariant derivative is the induced covariant derivative on forms.

An alternative proof of the proposition could go as follows.

$$\begin{aligned}\int_M \langle \alpha, \delta\beta \rangle dV &= \int_M \langle d\alpha, \beta \rangle dV \\ &= \int_M \langle Sk(\nabla\alpha), \beta \rangle dV \\ &= \int_M \langle \nabla\alpha, \beta \rangle dV \\ &= \int_M \langle \alpha, -\text{div } \beta \rangle dV.\end{aligned} \tag{8.32}$$

Thus both δ and $-\text{div}$ are L^2 adjoints of d. The result then follows from uniqueness of the L^2 adjoint.

9 Lecture 8: September 23, 2007

9.1 Bochner and Weitzenböck formulas

For T an (r, s)-tensor, the rough Laplacian is given by

$$\Delta T = \text{div } \nabla T. \tag{9.1}$$

For $\omega \in \Omega^p$ we define the *Hodge laplacian* $\Delta_H : \Omega^p \to \Omega^p$ by

$$\Delta_H \omega = (d\delta + \delta d)\omega. \tag{9.2}$$

We say a p-form is *harmonic* if it is in the kernel of Δ_H.

Proposition 9.1. *For T and S both (r,s)-tensors,*

$$\int_M \langle \Delta T, S \rangle dV = -\int_M \langle \nabla T, \nabla S \rangle dV = \int_M \langle T, \Delta S \rangle dV. \tag{9.3}$$

For $\alpha, \beta \in \Omega^p$,

$$\int_M \langle \Delta_H \alpha, \beta \rangle dV = \int_M \langle \alpha, \Delta_H \beta \rangle dV. \tag{9.4}$$

Proof. Formula (9.3) is an application of (9.1) and Corollary (8.2). For the second, from Proposition 8.5,

$$\begin{aligned}
\int_M \langle \Delta_H \alpha, \beta \rangle dV &= \int_M \langle (d\delta + \delta d)\alpha, \beta \rangle dV \\
&= \int_M \langle d\delta\alpha, \beta \rangle dV + \int_M \langle \delta d\alpha, \beta \rangle dV \\
&= \int_M \langle \delta\alpha, \delta\beta \rangle dV + \int_M \langle d\alpha, d\beta \rangle dV \\
&= \int_M \langle \alpha, d\delta\beta \rangle dV + \int_M \langle \alpha, \delta d\beta \rangle dV \\
&= \int_M \langle \alpha, \Delta_H \beta \rangle dV.
\end{aligned} \tag{9.5}$$

□

Note that Δ maps alternating $(0,p)$ tensors to alternating $(0,p)$ tensors, therefore it induces a map $\Delta : \Omega^p \to \Omega^p$ (note that on [Poo81, page 159] it is stated that the rough Laplacian of an r-form is in general not an r-form, but this seems to be incorrect). On p-forms, Δ and Δ_H are two self-adjoint linear second order differential operators. How are they related? Consider the case of 1-forms.

Proposition 9.2. *Let $\omega \in \Omega^1(M)$. If $d\omega = 0$, then*

$$\Delta \omega = -\Delta_H(\omega) + Rc^T(\omega). \tag{9.6}$$

Proof. In Proposition 6.3 above, we showed that on functions,

$$\Delta df = d\Delta f + Rc^T(df). \tag{9.7}$$

But on functions, $\Delta f = -\Delta_H f$. Clearly Δ_H commutes with d, so we have

$$\Delta(df) = -\Delta_H(df) + Rc^T(df). \tag{9.8}$$

Given any closed 1-form ω, by the Poincaré Lemma, we can locally write $\omega = df$ for some function f. This proves the formula. □

Corollary 9.1. *If (M,g) has non-negative Ricci curvature, then any harmonic 1-form is parallel. In this case $b_1(M) \leq n$. If, in addition, Rc is positive definite at some point, then any harmonic 1-form is trivial. In this case $b_1(M) = 0$.*

Proof. Formula (9.6) is

$$\Delta\omega = Rc^T(\omega). \tag{9.9}$$

Take inner product with ω, and integrate

$$\int_M \langle \Delta\omega, \omega \rangle = -\int_M |\nabla\omega|^2 dV = \int_M Ric(\sharp\omega, \sharp\omega) dV \tag{9.10}$$

This clearly implies that $\nabla\omega \equiv 0$, thus ω is parallel. If in addition Rc is strictly positive somewhere, ω must vanish identically. The conclusion on the first Betti number follows from the Hodge Theorem. □

We next generalize this to p-forms.

Definition 1. *For $\omega \in \Omega^p$, we define a $(0,p)$-tensor field ρ_ω by*

$$\rho_\omega(X_1, \dots, X_p) = \sum_{i=1}^{n} \sum_{j=1}^{p} \Big(\mathcal{R}_{\Lambda^p}(e_i, X_j)\omega \Big)(X_1, \dots, X_{j-1}, e_i, X_{j+1}, \dots, X_p), \tag{9.11}$$

where $\{e_i\}$ is an ONB at $x \in M$.

Remark. Recall what this means. The Riemannian connection induces a metric connection in the bundle $\Lambda^p(T^*M)$. The curvature of this connection therefore satisfies

$$\mathcal{R}_{\Lambda^p} \in \Gamma\Big(\Lambda^2(T^*M) \otimes \mathfrak{so}(\Lambda^p(T^*M))\Big). \tag{9.12}$$

We leave it to the reader to show that (9.11) is well-defined.

The relation between the Laplacians is given by

Theorem 9.1. *Let $\omega \in \Omega^p$. Then*

$$\Delta_H \omega = -\Delta\omega + \rho_\omega. \tag{9.13}$$

We also have the formula

$$\langle \Delta_H \omega, \omega \rangle = \frac{1}{2}\Delta_H |\omega|^2 + |\nabla\omega|^2 + \langle \rho_\omega, \omega \rangle. \tag{9.14}$$

Proof. Take $\omega \in \Omega^p$, and vector fields $X, Y_1, \dots, Y_p$. We compute

$$(\nabla\omega - d\omega)(X, Y_1, \dots, Y_p) = (\nabla_X \omega)(Y_1, \dots, Y_p) - d\omega(X, Y_1, \dots, Y_p) \tag{9.15}$$

$$= \sum_{j=1}^{p} (\nabla_{Y_j}\omega)(Y_1, \dots, Y_{j-1}, X, Y_{j+1}, \dots, Y_p), \tag{9.16}$$

using Proposition 8.3. Fix a point $x \in M$. Assume that $(\nabla Y_j)_x = 0$, by parallel translating the values of Y_j at x. Also take e_i to be an adapted moving frame at p. Using Proposition 8.6, we compute at x

$$
\begin{aligned}
(\text{div } \nabla\omega + \delta d\mu)(Y_1, \ldots, Y_p) &= \text{div } (\nabla\omega - d\omega)(Y_1, \ldots, Y_r) \\
&= \sum_{i=1}^{n} \Big(\nabla_{e_i}(\nabla\omega - d\omega)\Big)(e_i, Y_1, \ldots, Y_p) \\
&= \sum_{i=1}^{n} \Big(e_i(\nabla\omega - d\omega)\Big)(e_i, Y_1, \ldots, Y_p) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{p} e_i\Big((\nabla_{Y_j}\omega)(Y_1, \ldots, Y_{j-1}, e_i, Y_{j+1}, \ldots, Y_p) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{p} (\nabla_{e_i}\nabla_{Y_j}\omega)(Y_1, \ldots, Y_{j-1}, e_i, Y_{j+1}, \ldots, Y_p)
\end{aligned}
\tag{9.17}
$$

We also have

$$
\begin{aligned}
d\delta\omega(Y_1, \ldots, Y_p) &= \sum_{j=1}^{p}(-1)^{j+1}(\nabla_{Y_j}\delta\omega)(Y_1, \ldots, \hat{Y}_j, \ldots, Y_p) \\
&= \sum_{j=1}^{p}(-1)^{j} Y_j \left(\sum_{i=1}^{n}(\nabla_{e_i}\omega)(e_i, Y_1, \ldots, \hat{Y}_j, \ldots, Y_p)\right) \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{p}(\nabla_{Y_j}\nabla_{e_i}\omega)(Y_1, \ldots, Y_{j-1}, e_i, Y_{j+1}, \ldots, Y_p).
\end{aligned}
\tag{9.18}
$$

The commutator $[e_i, Y_j](x) = 0$, since $\nabla_{e_i} Y_j(x) = 0$, and $\nabla_{Y_j} e_i(x) = 0$, by our choice. Consequently,

$$
(\Delta_H\omega + \Delta\omega)(Y_1, \ldots, Y_p) = (\Delta_H\omega + \text{div } \nabla\omega)(Y_1, \ldots, Y_p) = \rho_\omega(Y_1, \ldots, Y_p). \tag{9.19}
$$

This proves (9.13). For (9.14), we compute at x

$$
\begin{aligned}
\text{div } \nabla\omega(Y_1, \ldots, Y_p) &= \sum_i \nabla_{e_i}(\nabla\omega)(e_1, Y_1, \ldots, Y_p) \\
&= \sum_i e_i(\nabla_{e_i}\omega)(Y_1, \ldots, Y_p) \\
&= \sum_i (\nabla_{e_i}\nabla_{e_i}\omega)(Y_1, \ldots, Y_p).
\end{aligned}
\tag{9.20}
$$

Next, again at x,

$$\begin{aligned}
\langle -\mathrm{div}\nabla\omega, \omega\rangle &= -\sum_i \langle \nabla_{e_i}\nabla_{e_i}\omega, \omega\rangle \\
&= -\sum_i e_i\left(\langle \nabla_{e_i}\omega, \omega\rangle - \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega\rangle\right) \\
&= -\frac{1}{2}\sum_i (e_i e_i|\omega|^2) + |\nabla\omega|^2 \\
&= \frac{1}{2}\Delta_H|\omega|^2 + |\nabla\omega|^2.
\end{aligned} \tag{9.21}$$

□

Remark. The rough Laplacian is therefore "roughly" the Hodge Laplacian, up to curvature terms. Note also in (9.14), the norms are tensor norms, since the right hand side has the term $|\nabla\omega|^2$ and $\nabla\omega$ is not a differential form. We are using (8.19) to identify forms and alternating tensors. This is an important point. For example, as an element of $\Lambda^2(T^*M)$, $e^1 \wedge e^2$ has norm 1 if e^1, e^2 are orthonormal in T^*M. But under our identification with tensors, $e^1 \wedge e^2$ is identified with $e^1 \otimes e^2 - e^2 \otimes e^1$, which has norm $\sqrt{2}$ with respect to the tensor inner product. Thus our identification in (8.19) is *not* an isometry, but is a constant multiple of an isometry.

10 Lecture 9: October 2, 2007

10.1 Manifolds with positive curvature operator

We begin with a general property of curvature in exterior bundles.

Proposition 10.1. *Let ∇ be a connection in a vector bundle $\pi : E \to M$. As before, extend ∇ to a connection in $\Lambda^p(E)$ by defining it on decomposable elements*

$$\nabla_X(s_1 \wedge \cdots \wedge s_p) = \sum_{i=1}^{p} s_1 \wedge \cdots \wedge \nabla_X s_i \wedge \cdots \wedge s_p. \tag{10.1}$$

For vector fields X, Y, $\mathcal{R}_{\Lambda^p(E)}(X,Y) \in End(\Lambda^p(E))$ acts as a derivation

$$\mathcal{R}_{\Lambda^p(E)}(X,Y)(s_1 \wedge \cdots \wedge s_p) = \sum_{i=1}^{p} s_1 \wedge \cdots \wedge \mathcal{R}_\nabla(X,Y)(s_i) \wedge \cdots \wedge s_p. \tag{10.2}$$

Proof. We prove for $p = 2$, the case of general p is left to the reader. Since this is a tensor equation, we may assume that $[X,Y] = 0$. We compute

$$\begin{aligned}
&\mathcal{R}_{\Lambda^2(E)}(X,Y)(s_1 \wedge s_2) = \nabla_X\nabla_Y(s_1 \wedge s_2) - \nabla_Y\nabla_X(s_1 \wedge s_2) \\
&= \nabla_X\Big((\nabla_Y s_1) \wedge s_2 + s_1 \wedge (\nabla_Y s_2)\Big) - \nabla_Y\Big((\nabla_X s_1) \wedge s_2 + s_1 \wedge (\nabla_X s_2)\Big) \\
&= (\nabla_X\nabla_Y)s_1 \wedge s_2 + \nabla_Y s_1 \wedge \nabla_X s_2 + \nabla_X s_1 \wedge \nabla_Y s_2 + s_1 \wedge (\nabla_X\nabla_Y)s_2 \\
&- (\nabla_Y\nabla_X)s_1 \wedge s_2 - \nabla_X s_1 \wedge \nabla_Y s_2 - \nabla_Y s_1 \wedge \nabla_X s_2 - s_1 \wedge (\nabla_Y\nabla_X)s_2 \\
&= \Big(\mathcal{R}_\nabla(X,Y)(s_1)\Big) \wedge s_2 + s_1 \wedge \Big(\mathcal{R}_\nabla(X,Y)(s_2)\Big).
\end{aligned} \tag{10.3}$$

□

We apply this to the bundle $E = \Lambda^p(T^*M)$. Recall for a 1-form ω,

$$\nabla_i \nabla_j \omega_l = \nabla_j \nabla_i \omega_l - R_{ijl}{}^k \omega_k. \tag{10.4}$$

In other words,

$$(\mathcal{R}(\partial_i, \partial_j)\omega)_l = -R_{ijl}{}^k \omega_k, \tag{10.5}$$

where the left hand side means the curvature of the connection in T^*M, but the right hand side is the Riemannian curvature tensor. For a p-form $\omega \in \Omega^p$, with components $\omega_{i_1 \dots i_p}$, Proposition 10.1 says that

$$\Big(\mathcal{R}_{\Lambda^p}(e_\alpha, e_\beta)\omega\Big)_{i_1 \dots i_p} = -\sum_{k=1}^{p} R_{\alpha\beta i_k}{}^l \omega_{i_1 \dots i_{k-1} l i_{k+1} \dots i_p}, \tag{10.6}$$

where the left hand side now means the curvature of the connection in $\Lambda^p(T^*M)$.

Next, we look at ρ_ω in coordinates. It is written

$$(\rho_\omega)_{i_i \dots i_p} = g^{\alpha l} \sum_{j=1}^{p} \big(\mathcal{R}_{\Lambda^p}(\partial_\alpha, \partial_{i_j})\omega\big)_{i_1 \dots i_{j-1} l i_{j+1} \dots i_p}. \tag{10.7}$$

Using (10.6), we may write ρ_ω as

$$\begin{aligned} (\rho_\omega)_{i_i \dots i_p} = &-g^{\alpha l} \sum_{j=1}^{p} \sum_{k=1, k\neq j}^{p} R_{\alpha i_j i_k}{}^m \omega_{i_1 \dots i_{j-1} l i_{j+1} \dots i_{k-1} m i_{k+1} \dots i_p} \\ &- g^{\alpha l} \sum_{j=1}^{p} R_{\alpha i_j l}{}^m \omega_{i_1 \dots i_{j-1} m i_{j+1} \dots i_p} \end{aligned} \tag{10.8}$$

Let us rewrite the above formula in an orthonormal basis,

$$\begin{aligned} (\rho_\omega)_{i_i \dots i_p} = &-\sum_{l,m=1}^{n} \sum_{j=1}^{p} \sum_{k=1, k\neq j}^{p} R_{l i_j m i_k} \omega_{i_1 \dots i_{j-1} l i_{j+1} \dots i_{k-1} m i_{k+1} \dots i_p} \\ &+ \sum_{m=1}^{n} \sum_{j=1}^{p} R_{i_j m} \omega_{i_1 \dots i_{j-1} m i_{j+1} \dots i_p}. \end{aligned} \tag{10.9}$$

Using the algebraic Bianchi identity (3.25), this is

$$R_{l i_j m i_k} + R_{l m i_k i_j} + R_{l i_k i_j m} = 0, \tag{10.10}$$

which yields

$$R_{l i_j m i_k} - R_{m i_j l i_k} = R_{l m i_j i_k}. \tag{10.11}$$

Substituting into (10.9) and using skew-symmetry,

$$\begin{aligned}(\rho_\omega)_{i_i\ldots i_p} &= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p}(R_{li_jmi_k} - R_{mi_jli_k})\omega_{i_1\ldots i_{j-1}li_{j+1}\ldots i_{k-1}mi_{k+1}\ldots i_p}\\ &\quad+\sum_{m=1}^{m}\sum_{j=1}^{p}R_{i_jm}\omega_{i_1\ldots i_{j-1}mi_{j+1}\ldots i_p}\\ &= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p}R_{lmi_ji_k}\omega_{i_1\ldots i_{j-1}li_{j+1}\ldots i_{k-1}mi_{k+1}\ldots i_p}\\ &\quad+\sum_{m=1}^{m}\sum_{j=1}^{p}R_{i_jm}\omega_{i_1\ldots i_{j-1}mi_{j+1}\ldots i_p}.\end{aligned} \tag{10.12}$$

Theorem 10.1. *If (M^n, g) is closed and has non-negative curvature operator, then any harmonic form is parallel. In this case, $b_1(M) \leq \binom{n}{k}$. If in addition, the curvature operator is positive definite at some point, then any harmonic p-form is trivial for $p = 1 \ldots n-1$. In this case, $b_p(M) = 0$ for $p = 1 \ldots n-1$.*

Proof. Let ω be a harmonic p-form. Integrating the Weitzenböck formula (9.14), we obtain

$$0 = \int_M |\nabla\omega|^2 dV + \int_M \langle\rho_\omega, \omega\rangle dV. \tag{10.13}$$

It turns out the the second term is positive if the manifold has positive curvature operator [Poo81, Chapter 4], [Pet06, Chapter 7]. Thus $|\nabla\omega| = 0$ everywhere, so ω is parallel. A parallel form is determined by its value at a single point, so using the Hodge Theorem, we obtain the first Betti number estimate. If the curvature operator is positive definite at some point, then we see that ω must vanish at that point, which implies the second Betti number estimate. Note this only works for $p = 1 \ldots n-1$, since ρ_ω is zero in these cases. □

This says that all of the real cohomology of a manifold with positive curvature operator vanishes except for H^n and H^0. We say that M is a rational homology sphere (which necessarily has $\chi(M) = 2$). If M is simply-connected and has positive curvature operator, then is M diffeomorphic to a sphere? In dimension 3 this was answered affirmatively by Hamilton in [Ham82]. Hamilton also proved the answer is yes in dimension 4 [Ham86]. Very recently, Böhm and Wilking have shown that the answer is yes in all dimensions [BW06]. The technique is using the Ricci flow, which we will discuss shortly.

We also mention that recently, Brendle and Schoen have shown that manifolds with 1/4-pinched curvature are diffeomorphic to space forms, again using the Ricci flow. If time permits, we will also discuss this later [BS07].

Remark. On 2-forms, the Weitzenböck formula is

$$(\Delta_H\omega)_{ij} = -(\Delta w)_{ij} - \sum_{l,m} R_{lmij}\omega_{lm} + \sum_m R_{im}\omega_{mj} + \sum_m R_{jm}\omega_{im}. \tag{10.14}$$

Through a careful analysis of the curvature terms, M. Berger was able to prove a vanishing theorem for $H^2(M,\mathbb{R})$ provided that the sectional curvature is pinched between 1 and $2(n-1)/(8n-5)$ [Ber60].

11 Lecture 10: October 4, 2007

11.1 Killing vector fields

For a vector field X, the covariant derivative ∇X is a $(1,1)$ tensor. Equivlently, $\nabla X \in \Gamma(End(TM))$. Any endomorphism T of an inner product space can be decomposed into its symmetric and skew-symmetric parts via

$$\begin{aligned} \langle Tu, v\rangle &= \frac{1}{2}\left(\langle Tu, v\rangle + \langle u, Tv\rangle\right) + \left(\langle Tu, v\rangle - \langle u, Tv\rangle\right) \\ &= \langle T_{sym}u, v\rangle + \langle T_{sk}u, v\rangle. \end{aligned} \tag{11.1}$$

Furthermore, the symmetric part may be decomposed into its pure trace and traceless components

$$T_{sym} = \frac{trT}{n}I + \overset{\circ}{T} = \frac{trT}{n}I + \left(T - \frac{trT}{n}I\right). \tag{11.2}$$

The decomposition

$$T = \frac{trT}{n}I + \overset{\circ}{T} + T_{sk}, \tag{11.3}$$

is irreducible under the action of the orthogonal group $O(n)$.

Proposition 11.1. *For a vector field X,*

$$g((\nabla X)_{sym}Y, Z) = \frac{1}{2}\mathcal{L}_X g(Y, Z) \tag{11.4}$$

$$g((\nabla X)_{sk}Y, Z) = \frac{1}{2}d(\flat X)(Y, Z), \tag{11.5}$$

and the diagonal part is

$$\frac{\text{div } X}{n}I. \tag{11.6}$$

Proof. Recalling the formula for the Lie derivative of a $(0,2)$ tensor,

$$\begin{aligned} \mathcal{L}_X g(Y, Z) &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= X(g(Y, Z)) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) + X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= 2g((\nabla X)_{sym}Y, Z), \end{aligned} \tag{11.7}$$

which proves (11.4). Using the formula for d,

$$\begin{aligned}
d(\flat X)(Y,Z) &= Y((\flat X)(Z)) - Z((\flat X)Y) - (\flat X)([Y,Z]) \\
&= Y(g(X,Z)) - Z(g(X,Y)) - g(X,[Y,Z]) \\
&= g(\nabla_Y X, Z) + g(X, \nabla_Y Z) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) - g(X,[Y,Z]) \\
&= g(\nabla_Y X, Z) - g(\nabla_Z X, Y) + g(X, \nabla_Y Z - \nabla_Z Y - [Y,Z]) \\
&= 2g((\nabla X)_{sk} Y, Z),
\end{aligned} \tag{11.8}$$

which proves (11.5). Formula (11.6) is just the definition of the divergence as the trace of the covariant derivative. □

A vector field X is a *Killing field* if the 1-parameter group of local diffeomorphisms generated by X consists of local isometries of g.

Proposition 11.2. *A vector field is a Killing field if and only if $\mathcal{L}_X g = 0$, which is equivalent to the skew-symmetry of ∇X.*

Proof. Let ϕ_t denote the 1-parameter group of X,

$$\begin{aligned}
\frac{d}{ds}(\phi_s^* g)\Big|_t &= \frac{d}{ds}(\phi_{s+t}^* g)\Big|_0 \\
&= \phi_t^* \frac{d}{ds}(\phi_s^* g)\Big|_0 \\
&= \phi_t^* \mathcal{L}_X g.
\end{aligned} \tag{11.9}$$

It follows that $\phi_t^* g = g$ for every t if and only if $\mathcal{L}_X g = 0$. The skew-symmetry of ∇X follows from Proposition 11.1 □

Note that, in particular, a Killing field is divergence free. We next have a formula due to Bochner

Proposition 11.3. *Let X be a vector field. Then*

$$2g(\Delta X, X) + 2|\nabla X|^2 + \Delta |X|^2 = 0. \tag{11.10}$$

Proof. Let e_i be an adapted moving frame at $x \in M$. We compute at x

$$\begin{aligned}
&2g(\Delta X, X) + 2|\nabla X|^2 + \Delta |X|^2 \\
&= 2\sum_{i=1}^n \Big(g(\nabla_{e_i}\nabla_{e_i} X, X) + |\nabla_{e_i} X|^2 \Big) - \text{div } (\nabla |X|^2)(x) \\
&= 2\sum_{i=1}^n \Big(e_i(g(\nabla_{e_i} X, X)) - g(\nabla_{e_i} X, \nabla_{e_i} X) + |\nabla_{e_i} X|^2 \Big) - \sum_{i=1}^n \nabla_{e_i}\nabla_{e_i}(g(X,X)) \\
&= 2\sum_{i=1}^n e_i(g(\nabla_{e_i} X, X)) - 2\sum_{i=1}^n \nabla_{e_i}(g(\nabla_{e_i} X, X)) = 0.
\end{aligned} \tag{11.11}$$

□

We next have

Proposition 11.4. *If X is a Killing field, then*

$$\Delta X + Rc(X) = 0. \tag{11.12}$$

If in addition M is compact and Rc is negative semidefinite, then X is parallel and $Ric(X, X) = 0$.

Proof. Let e_i be an adapted frame at $x \in M$, and let Y be a vector field with $(\nabla Y)_x = 0$. At the point x,

$$\begin{aligned}
g(&\Delta X, Y) + Ric(X, Y) \\
&= \sum_{i=1}^{n} g(\nabla_{e_i}\nabla_{e_i}X, Y) + \sum_{i=1}^{n} g(\nabla_{e_i}\nabla_Y X - \nabla_Y \nabla_{e_i} X, e_i) \\
&= \sum_{i=1}^{n} e_i(g(\nabla_{e_i}X, Y)) + \sum_{i=1}^{n} e_i(g(\nabla_Y X, e_i)) - \sum_{i=1}^{n} Y(g(\nabla_{e_i}X, e_i)) \quad (11.13)\\
&= -\sum_{i=1}^{n} e_i(g(e_i, \nabla_Y X)) + \sum_{i=1}^{n} e_i(g(\nabla_Y X, e_i)) - Y(\text{div } X), \\
&= 0.
\end{aligned}$$

This proves 11.12. Using (11.10), we have

$$-Ric(X, X) + 2|\nabla X|^2 + \Delta |X|^2 = 0. \tag{11.14}$$

If M is compact, using Corollary 8.1, we obtain

$$-\int_M Ric(X, X) dV + 2\int_M |\nabla X|^2 dV = 0. \tag{11.15}$$

If Ric is negative semidefinite, then clearly $\nabla X = 0$, so X is parallel. □

Corollary 11.1. *Let (M, g) be compact, and let $Iso(M, g)$ denote the isometry group of (M, g). If (M, g) has negative semi-definite Ricci tensor, then $dim(Iso(M, g)) \leq n$. If, in addition, the Ricci tensor the Ricci tensor is negative definite at some point, then $Iso(M, g)$ is finite.*

Proof. If the isometry group is not finite, then there exists a non-trivial 1-parameter group $\{\phi_t\}$ of isometries. By Proposition 11.2, this generates a non-trivial Killing vector field. From Proposition 11.4, X is parallel and $Ric(X, X) = 0$. Since X is parallel, it is determined by its value at a single point, so the dimension of the space of Killing vector fields is less than n, which implies that $dim(\text{Iso}(M, g)) \leq n$. If Ric is negative definite at some point x, then $Ric(X_x, X_x) = 0$, which implies that $X_x = 0$, and thus $X \equiv 0$ since it is parallel. Consequently, there are no nontrival 1-parameter groups of isometries, so $\text{Iso}(M, g)$ must be finite. □

Note that an n-dimensional flat torus $S^1 \times \cdots \times S^1$ attains equality in the above inequality. Note also that by Gauss-Bonnet, any metric on a surface of genus $g \geq 2$ must have a point of negative curvature, so any non-positively curved metric on a surface of genus $g \geq 2$ must have finite isometry group.

11.2 Isometries

Since there were a few non-trivial points about isometries used above, we present here some standard facts about isometries. There are 2 notions of isometry. The first definition is a map which is surjective and distance preserving, viewing a Riemannian manifold as a length space. The other is a diffeomorphism $\phi : M \to M$ which satisfies $\phi^* g = g$. These two notions coincide [Hel78, Theorem I.11.1].

Theorem 11.1. *The isometry group Iso(M, g) of a connected Riemannian manifold is a Lie group with respect to the compact-open topology. Furthermore, If M is compact, then Iso(M, g) is also compact.*

Proof. Consider the bundle of orthonormal frames

$$O(n) \to F(M) \to M, \tag{11.16}$$

which is a principal $O(n)$ bundle over M. Fix a point $x \in M$, and a frame $V_x = \{e_1, \ldots, e_n\}$ based at x. Any isometry ϕ of (M, g) lifts to a bundle automorphism $\tilde{\phi} : F(M) \to F(M)$, which preserves the canonical $\mathbb{R}^n$-valued 1-form ω, and the $\mathfrak{so}(n)$-valued connection form α. The mapping $\phi \to \tilde{\phi}(V_x)$ defines an embedding. This is injective since any isometry which preserve a point, and induces the identity map in the tangent space at that point, is globally the identity map [Hel78, Lemma I.11.2]. Furthermore, the image of Iso(M, g) is a closed submanifold, this is proved in [Kob95, Theorem 3.2]. If M is compact, then so is $F(M)$. A closed submanifold of a compact space is itself compact. □

Note that as a corollary of the above proof, we obtain for any (M^n, g),

$$\dim(\text{Iso}(M, g)) \leq \frac{n(n+1)}{2}. \tag{11.17}$$

If equality is attained in the above inequality, then (M, g) must have constant sectional curvature. To see this, by the above imbedding, we would have $Iso(M, g) = F(M)$, or one of the 2 connected components of $F(M)$. The isometry group must act transitively on 2-planes in any tangent space (since $SO(n)$ does), therefore (M, g) must have constant sectional curvature. With a little more work, one can show that (M, g) is isometric to one of the following spaces of constant curvature (a) $\mathbb{R}^n$, (b) S^n, (c) $\mathbb{RP}^n$, (d) hyperbolic space $\mathbf{H}^n$, see [Kob95, Theorem II.3.1].

Remark. We remark that if (M, g) is complete, then every Killing vector field is complete (that is, the local 1-parameter group is *global*, and the 1-parameter subgroups are defined for all time) [Kob95, Theorem II.2.5]. Thus the Lie algebra of Iso(M, g) can be identified with the space of Killing vector fields, even in the complete non-compact case.

12 Lecture 11: October 9, 2007

12.1 Linearization of Ricci tensor

We recall the formula for the Christoffel symbols in coordinates

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \Big), \tag{12.1}$$

and the formula for the curvature tensor in terms of the Christoffel symbols,

$$R_{ijk}{}^l = \partial_i (\Gamma_{jk}^l) - \partial_j (\Gamma_{ik}^l) + \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m. \tag{12.2}$$

We let $h \in \Gamma(S^2(T^*M))$ be a symmetric $(0,2)$ tensor and linearize the Christoffel symbols in the direction of h. We will let primes denote derivatives, for example

$$(\Gamma_{ij}^k)' \equiv \frac{\partial}{\partial t} \Gamma_{ij}^k (g + th). \tag{12.3}$$

Recall the formula for the derivative of an inverse matrix

$$(g^{-1})' = -g^{-1} g' g^{-1}. \tag{12.4}$$

Proposition 12.1. *The linearization of the Christoffel symbols is given by*

$$(\Gamma_{ij}^k)' = \frac{1}{2} g^{kl} \Big(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \Big). \tag{12.5}$$

Proof. It is easy to see that the difference of any two Riemannian connections $\nabla - \tilde{\nabla}$ is a tensor, satisfying

$$\nabla - \tilde{\nabla} = \Gamma(TM \otimes S^2(T^*M)). \tag{12.6}$$

This clearly implies that $\Gamma' \in \Gamma(TM \otimes S^2(T^*M))$ is also a tensor. Since (12.5) is a tensor equation, we prove in a normal coordinate system centered at $x \in M$,

$$(\Gamma_{ij}^k)' = \frac{1}{2} (g^{kl})' \Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \Big) + \frac{1}{2} g^{kl} \Big(\partial_i g'_{jl} + \partial_j g'_{il} - \partial_l g'_{ij} \Big). \tag{12.7}$$

At the point x, we have

$$\begin{aligned} (\Gamma_{ij}^k)'(x) &= \frac{1}{2} g^{kl} \Big(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij} \Big)(x) \\ &= \frac{1}{2} g^{kl} \Big(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \Big)(x), \end{aligned} \tag{12.8}$$

and we are done. □

Proposition 12.2. *The linearization of the Ricci tensor is given by*

$$\begin{aligned} (Ric')_{ij} = \frac{1}{2} \Big(& - \Delta h_{ij} + \nabla_i (\text{div } h)_j + \nabla_j (\text{div h})_i - \nabla_i \nabla_j (trh) \\ & - 2 R_{iljp} h^{lp} + R_i^p h_{jp} + R_j^p h_{ip} \Big). \end{aligned} \tag{12.9}$$

Proof. Tracing (12.2) on i and l, we find the formula

$$(Ric)_{jk} = \partial_l(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{lk}) + \Gamma^m_{jk}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{jm}. \tag{12.10}$$

In normal coordinates, at x we have

$$\begin{aligned}
(Ric')_{ij}(x) &= \nabla_l(\Gamma^l_{ij})' - \nabla_i(\Gamma^l_{lj})' = \frac{1}{2}\nabla_l\Big(g^{lm}\big(\nabla_i h_{jm} + \nabla_j h_{im} - \nabla_m h_{ij}\big)\Big)\\
&\quad - \frac{1}{2}\nabla_i\Big(g^{lm}\big(\nabla_l h_{jm} + \nabla_j h_{lm} - \nabla_m h_{lj}\big)\Big)\\
&= \frac{1}{2}g^{lm}\big(\nabla_l\nabla_i h_{jm} - \nabla_i\nabla_l h_{jm} + \nabla_l\nabla_j h_{im} - \nabla_i\nabla_j h_{lm} - \nabla_l\nabla_m h_{ij} + \nabla_i\nabla_m h_{lj}\big)\\
&= \frac{1}{2}g^{lm}\big(-R_{lij}{}^p h_{pm} - R_{lim}{}^p h_{jp}\big) - \frac{1}{2}\nabla_i\nabla_j(trh) - \frac{1}{2}\Delta h_{ij}\\
&\quad + \frac{1}{2}g^{lm}\big(\nabla_l\nabla_j h_{im} + \nabla_i\nabla_m h_{lj}\big)\\
&= \frac{1}{2}\Big(-R_{lij}{}^p h^l_p + R^p_i h_{jp} - \nabla_i\nabla_j(trh) - \Delta h_{ij}\Big)\\
&\quad + \frac{1}{2}g^{lm}\big(\nabla_j\nabla_l h_{im} - R_{lji}{}^p h_{pm} - R_{ljm}{}^p h_{ip} + \nabla_i\nabla_m h_{lj}\big)\\
&= \frac{1}{2}\Big(-R_{lij}{}^p h^l_p + R^p_i h_{jp} - \nabla_i\nabla_j(trh) - \Delta h_{ij}\Big)\\
&\quad + \frac{1}{2}\Big(\nabla_j(\text{div h})_i - R_{lji}{}^p h^l_p + R^p_j h_{ip} + \nabla_i(\text{div } h)_j\Big).
\end{aligned} \tag{12.11}$$

Using the symmetry of h,

$$\begin{aligned}
-R_{lij}{}^p h^l_p - R_{lji}{}^p h^l_p &= -R_{lipj}h^{pl} - R_{ljpi}h^{pl}\\
&= -R_{lipj}h^{pl} - R_{pilj}h^{lp} = -2R_{iljp}h^{lp}.
\end{aligned} \tag{12.12}$$

Collecting all the terms, we have proved (12.9). □

Remark. Equation (12.9) is often written as

$$(Ric')_{ij} = \frac{1}{2}\Big(\Delta_L h_{ij} + \nabla_i(\text{div } h)_j + \nabla_j(\text{div h})_i - \nabla_i\nabla_j(trh)\Big), \tag{12.13}$$

where Δ_L is the *Lichneriowicz Laplacian* defined by

$$\Delta_L = -\Delta h_{ij} - 2R_{iljp}h^{lp} + R^p_i h_{jp} + R^p_j h_{ip}. \tag{12.14}$$

Proposition 12.3. *The linearization of the scalar curvature is given by*

$$R' = -\Delta(trh) + \text{div}^2 h - R_{lp}h^{lp}. \tag{12.15}$$

Proof. Using (12.4) and Proposition 12.2, we compute

$$\begin{aligned}
R' &= (g^{ij}(Ric)_{ij})' = (g^{ij})'R_{ij} + g^{ij}(Ric')_{ij}\\
&= -(g^{-1}hg^{-1})^{ij}R_{ij} - \Delta(trh) + \text{div}^2 h\\
&= -\Delta(trh) + \text{div}^2 h - R_{lp}h^{lp}.
\end{aligned} \tag{12.16}$$

□

12.2 The total scalar curvature functional

Let $\mathcal{M}$ denote the space of Riemannian metrics on M. We define the Einstein-Hilbert functional $E : \mathcal{M} \to \mathbb{R}$

$$E(g) = \int_M R_g dV_g. \tag{12.17}$$

This is a *Riemannian functional* in the sense that it is invariant under diffeomorphisms.

Proposition 12.4. *If M is closed and $n \geq 3$, then a metric $g \in \mathcal{M}$ is critical for E if and only if g is Ricci-flat.*

Proof. We compute the first variation of E. For the volume element,

$$\begin{aligned}(dV_g)' &= (\sqrt{\det(g)}dx^1 \wedge \cdots \wedge dx^n)' \\ &= \frac{1}{2}(\det(g))^{-1/2}T^{ij}_{n-1}h_{ij}dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2}\Big(\frac{1}{\det(g)}T^{ij}_{n-1}\Big)h_{ij}dV_g \\ &= \frac{1}{2}tr_g(h)dV_g,\end{aligned} \tag{12.18}$$

where T^{ij}_{n-1} is the cofactor matrix of g_{ij}. We then have

$$\begin{aligned}E'(g) &= \int_M \Big(R' + \frac{R}{2}tr_g(h)\Big)dV_g \\ &= \int_M \Big(-\Delta(trh) + \mathrm{div}^2 h - R^{lp}h_{lp} + \frac{R}{2}tr_g(h)\Big)dV_g \\ &= \int_M \Big((-R^{lp} + \frac{R}{2}g^{lp})h_{lp}\Big)dV_g.\end{aligned} \tag{12.19}$$

If this vanishes for all variations h, then

$$Ric = \frac{R}{2}g. \tag{12.20}$$

Taking a trace, we find that $R = 0$, so (M, g) is Ricci-flat. □

Remark. If $n = 2$ the above proof shows that E has zero variation, thus is constant. This is not surprising in view of the Gauss-Bonnet Theorem.

Remark. Notice that the Euler-Lagrange equations are the vanishing of the Einstein tensor, which is divergence free. This is actually a consequence of the invariance of E under the diffeomorphism group, so this property will hold for *any* Riemannian functional.

Lemma 12.1. *Let $\lambda > 0$, and let $g(\lambda) = \lambda g$. Then*

$$\begin{aligned} \mathcal{R}_{g(\lambda)} &= \mathcal{R}_g \\ Rm(g(\lambda)) &= \lambda Rm(g) \\ Ric(g(\lambda)) &= Ric(g) \\ R(g(\lambda)) &= \lambda^{-1} R(g) \\ dV_{g(\lambda)} &= \lambda^{n/2} dV_g. \end{aligned} \tag{12.21}$$

Proof. This is clear directly from the definitions of the various curvatures and volume element given above. ∎

The functional E is not scale-invariant for $n \geq 3$. To fix this we define

$$\overline{E}(g) = Vol(g)^{\frac{2-n}{n}} \int_M R_g dV_g. \tag{12.22}$$

To see that this is scale-invariant, replace g with $g(\lambda)$

$$\begin{aligned} \overline{E}(g(\lambda)) &= Vol(g(\lambda))^{\frac{2-n}{n}} \int_M R_{g(\lambda)} dV_{g(\lambda)} \\ &= (\lambda^{n/2} Vol(g))^{\frac{2-n}{n}} \int_M \lambda^{-1} R_g \lambda^{n/2} dV_g \\ &= E(g). \end{aligned} \tag{12.23}$$

Proposition 12.5. *If M is closed and $n \geq 3$, then a metric $g \in \mathcal{M}$ is critical for $\overline{E}$ if and only if g is Einstein. A metric g is critical for $\overline{E}$ under all conformal variations if and only if g has constant scalar curvature.*

Proof. We compute

$$\begin{aligned} \overline{E}'(g) &= \frac{2-n}{n} Vol(g)^{\frac{2-n}{n}-1} (Vol(g))' \int_M R_g dV_g \\ &\quad + Vol(g)^{\frac{2-n}{n}} \int_M \Big(- R^{lp} + \frac{R}{2} g^{lp}) h_{lp} \Big) dV_g \\ &= Vol(g)^{\frac{2-n}{n}} \Big(\frac{2-n}{n} Vol(g)^{-1} \int_M \frac{1}{2} (tr_g h) dV_g \cdot \int_M R_g dV_g \Big) \\ &\quad + Vol(g)^{\frac{2-n}{n}} \int_M \Big(- R^{lp} + \frac{R}{2} g^{lp} \Big) h_{lp} dV_g. \end{aligned} \tag{12.24}$$

Consider only conformal variations, that is $g(t) = f(t)g$, then $h = g'(t) = f'(t)g =$

$\frac{tr_g h}{n} g$ is diagonal. For these variations, we have

$$\begin{aligned}
\overline{E}'(g) &= Vol(g)^{\frac{2-n}{n}} \Big(\frac{2-n}{n} Vol(g)^{-1} \int_M \frac{1}{2}(tr_g h) dV_g \cdot \int_M R_g dV_g \Big) \\
&\quad + Vol(g)^{\frac{2-n}{n}} \int_M \Big(-R^{lp} + \frac{R}{2} g^{lp} \Big) \frac{tr_g h}{n} g_{lp} \Big) dV_g \\
&= Vol(g)^{\frac{2-n}{n}} \Big(\frac{2-n}{n} Vol(g)^{-1} \int_M \frac{1}{2}(tr_g h) dV_g \cdot \int_M R_g dV_g \Big) \\
&\quad - \frac{2-n}{2n} Vol(g)^{\frac{2-n}{n}} \int_M R_g (tr_g h) dV_g \\
&= \frac{n-2}{2n} Vol(g)^{\frac{2-n}{n}} \Big(\int_M (tr_g h)(R_g - \overline{R}) dV_g \Big),
\end{aligned} \tag{12.25}$$

where $\overline{R}$ denotes the average scalar curvature

$$\overline{R} = Vol(g)^{-1} \int_M R_g dV_g. \tag{12.26}$$

If this is zero for an arbitrary function $tr_g h$, then R_g must be constant. The full variation then simplifies to

$$\overline{E}'(g) = Vol(g)^{\frac{2-n}{n}} \int_M \Big(-R^{lp} + \frac{R}{n} g^{lp} \Big) h_{lp} dV_g. \tag{12.27}$$

If this vanishes for all variations, then the traceless Ricci tensor must vanish, so (M, g) is Einstein. □

Remark. Instead of looking at the scale invariant functional $\overline{E}$, one could instead restrict E to the space of unit volume metrics $\mathcal{M}_1$. This introduces a Lagrange multiplier term, and the resulting Euler-Lagrange equations are equivalent to those of the scale invariant functional.

13 Lecture 12: October 11, 2007.

13.1 Ricci flow: short-time existence

In the previous section, we saw that critical points of the Einstein-Hilbert functional are Einstein. In order to find Einstein metrics, one would first think of looking at the gradient flow on the space of Riemannian metric. This is

$$\frac{\partial}{\partial t} g = Ric_g - \frac{R_g}{n} g, \quad g(0) = g_0. \tag{13.1}$$

It turns out that this is not parabolic. Undeterred by this fact, Hamilton considered the modified flow

$$\frac{\partial}{\partial t} g = -2Ric_g, \quad g(0) = g_0. \tag{13.2}$$

This turns out to be *almost* strictly parabolic. The problem is with the action of the diffeomorphism group. For example, consider a Ricci flat metric on a compact manifold. If the Ricci flow were strictly parabolic, then the space of steady state solutions would be finite dimensional. But the space of Ricci flat metrics is invariant under the diffeomorphism group, so is infinite dimensional. Nevertheless, we have

Proposition 13.1. *Let (M, g) be a compact Riemannian manifold. Then there exists an $\epsilon > 0$ such that a solution of the Ricci flow exists on $M \times [0, \epsilon)$. Furthermore, the solution is unique.*

The remainder of this section will be devoted to the proof. First assume we have a solution of the Ricci flow defined on some short time interval. For any nonlinear system of PDEs, we say it is parabolic at a solution u_t provided the linearized operator at u_t is parabolic. As mentioned above, the Ricci flow is degenerate parabolic. To see this, recall the linearization of the Ricci tensor,

$$(Ric')_{ij} = \frac{1}{2}\Big(- \Delta h_{ij} + \nabla_i (div\ h)_j + \nabla_j (div\ h)_i - \nabla_i \nabla_j (trh)\Big) + \text{ lower order terms.} \tag{13.3}$$

Fix a point $x \in M$, and consider normal coordinates at p. We may write the above operator at x,

$$(Ric')_{ij} = \frac{1}{2}\sum_{k,l=1}^{n}\Big(- \frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{lj}}{\partial x^i \partial x^l} + \frac{\partial^2 h_{li}}{\partial x^j \partial x^l} - \frac{\partial^2 h_{kl}}{\partial x^i \partial x^j}\Big) + \text{ lower order terms.} \tag{13.4}$$

We would like the first term to be the Laplacian, so let $E = -2Ric$, and we have

$$(E')_{ij} = \sum_{k,l=1}^{n}\Big(\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 h_{lj}}{\partial x^i \partial x^l} - \frac{\partial^2 h_{li}}{\partial x^j \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^j}\Big) + \text{ lower order terms.} \tag{13.5}$$

The linearization of E at g is a mapping

$$E'(g) : \Gamma(S^2(T^*M)) \to \Gamma(S^2(T^*M)). \tag{13.6}$$

The symbol of E' at x is a mapping

$$\sigma(E')(x) : T_x^*M \times S^2(T_x^*M) \to S^2(T_x^*M), \tag{13.7}$$

and is formed by replacing partial derivatives with the corresponding cotangent directions in only the highest order terms. We obtain

$$\sigma(E')(x)(\xi, h) = \sum_{k,l=1}^{n} \Big(\xi_k \xi_l h_{ij} - \xi_i \xi_l h_{lj} - \xi_j \xi_l h_{li} + \xi_i \xi_j h_{kl}\Big). \tag{13.8}$$

Let us assume that $\xi = (1, 0, \dots, 0)$ satisfies $\xi_1 = 1$, and $\xi_i = 0$ for $i > 1$. A simple computation shows that

$$\begin{aligned} \big(\sigma(E')(x)(\xi, h)\big)_{ij} &= h_{ij} \text{ if } i \neq 1, j \neq 1 \\ \big(\sigma(E')(x)(\xi, h)\big)_{1j} &= 0 \text{ if } j \neq 1, \\ \big(\sigma(E')(x)(\xi, h)\big)_{11} &= \sum_{k=2}^{n} h_{kk}. \end{aligned} \tag{13.9}$$

The symbol in the direction ξ has a zero eigenvalue, so the Ricci flow cannot possibly be strictly parabolic. To remedy this, we will define a modified flow which *is* strictly parabolic. Define the 1 form V by

$$V = \text{div } h - \frac{1}{2}\nabla(trh), \tag{13.10}$$

and rewrite E' as

$$(E')_{ij} = \Delta h_{ij} - \nabla_i V_j - \nabla_j V_i + \text{ lower order terms.} \tag{13.11}$$

We will next find another operator whose linearization is the negative the second two terms on the right hand side, up to lower order terms. To this end, define a vector field

$$W^k = g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}), \tag{13.12}$$

where $\tilde{\Gamma}$ are the Christoffel symbols of a reference connection. Since the difference of two connections is a tensor, this defines a global vector field W. Consider the operator $P : \mathcal{M} \to \Gamma(S^2(T^*M))$, defined by

$$P(g) = \mathcal{L}_W g. \tag{13.13}$$

Recall from Proposition 11.1 that the Lie derivative of the metric with respect to a vector field is the symmetric part of the covariant derivative. In coordinates, this is

$$(\mathcal{L}_W g)_{ij} = \nabla_i W_j + \nabla_j W_i, \tag{13.14}$$

where W_i are the components of the dual 1-form $\flat W$. We linearize in the direction of h, and use normal coordinates at x:

$$\begin{aligned} \big(P'(g)(h)\big)_{ij} &= \partial_i W'_j + \partial_j W'_i \\ &= \frac{1}{2}\sum_{p,q}\Big(\partial_i(\partial_p h_{qj} + \partial_q h_{pj} - \partial_j h_{pq}) + \partial_j(\partial_p h_{qi} + \partial_q h_{pi} - \partial_i h_{pq})\Big) \\ &= \nabla_i(\text{div } h)_j + \nabla_j(\text{div } h)_i - \nabla_i\nabla_j(trh). \end{aligned} \tag{13.15}$$

This shows that

$$\big(P'(g)(h)\big)_{ij} = \nabla_i V_j + \nabla_j V_i + \text{ lower order terms.} \tag{13.16}$$

So we define the *Ricci-DeTurck* flow by

$$\frac{\partial}{\partial t} g = -2Ric_g + \delta^*(\flat W), \quad g(0) = g_0, \tag{13.17}$$

where $\delta^* : \Gamma(T^*M) \to \Gamma(S^2(T^*M))$ is the operator defined by

$$(\delta^*\omega)_{ij} = \nabla_i \omega_j + \nabla_j \omega_i, \tag{13.18}$$

and W is the vector field defined in (13.12) above. The computations above show this is now a *strictly* parabolic system, since the leading term is just the rough Laplacian, which has diagonal symbol

$$\sigma(\Delta)(x)(\xi, h) = |\xi|_x^2 h. \tag{13.19}$$

Short time existence for the modified flow follows from [Èĭd69, Chapter ?] using an iteration procedure, see also [Lie96, Theorem VIII,8.2], by using the Schauder fixed point theorem. We will discuss this next lecture.

We next show how to go from a solution of the Ricci-DeTurck flow back to a solution of the Ricci flow. Define a 1-parameter family of maps $\phi_t : M \to M$ by

$$\frac{\partial}{\partial t} \phi_t(x) = -W(\phi_t(x), t), \;\; \phi_0 = Id_M. \tag{13.20}$$

The maps ϕ_t exists and are diffeomorphisms as long as the solution $g(t)$ exists, this is proved in [CK04, Section 3.3.1].

We claim that $\tilde{g}(t) = \phi_t^* g(t)$ is a solution to the Ricci flow. First, $\tilde{g}(0) = g(0)$ since $\phi_0 = Id_M$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}(\phi_t^* g(t)) &= \frac{\partial}{\partial s}(\phi_{s+t}^* g(s+t))\Big|_{s=0} \\
&= \phi_t^*\Big(\frac{\partial}{\partial t} g(t)\Big) + \frac{\partial}{\partial s}(\phi_{s+t}^* g(t))\Big|_{s=0} \\
&= \phi_t^*\big(-2Ric(g(t)) + \mathcal{L}_{W(t)} g(t)\big) + \frac{\partial}{\partial s}\Big((\phi_t^{-1} \circ \phi_{t+s})^* \phi_t^* g(t)\Big)\Big|_{s=0} \\
&= -2Ric(\phi_t^* g(t)) + \phi_t^*(\mathcal{L}_{W(t)} g(t)) - \mathcal{L}_{(\phi_t^{-1})_* W(t)}(\phi_t^* g(t)) \\
&= -2Ric(\phi_t^* g(t)),
\end{aligned} \tag{13.21}$$

using the fact that

$$\frac{\partial}{\partial s}(\phi_t^{-1} \circ \phi_{t+s})\Big|_{s=0} = (\phi_t^{-1})_*\Big(\frac{\partial}{\partial s}\phi_{t+s}\Big|_{s=0}\Big) = (\phi_t^{-1})_* W(t). \tag{13.22}$$

We will discuss uniqueness next time.

14 Lecture 13: October 16, 2007

14.1 Uniqueness

Last time we showed how go to Ricci-DeTurck flow back to a solution of Ricci flow. The procedure was: given a solution $g(t)$ to the Ricci-DeTurck flow defined on $M \times [0, \epsilon]$, define the vector field W by

$$W^k = g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}), \tag{14.1}$$

and let ϕ_t be the 1-parameter family of diffeomorphisms solving

$$\frac{\partial}{\partial t}\phi_t(x) = -W(\phi_t(x), t), \;\; \phi_0 = Id_M. \tag{14.2}$$

Then $\tilde{g}(t) = \phi_t^* g(t)$ is a solution of the Ricci flow. To go in the other direction, we look at harmonic maps. Let (M, g) and (N, h) be Riemannian manifolds. For a smooth map $f : M \to N$, view the derivative of f as a section

$$f_* \in \Gamma(T^*M \otimes f^*TN). \tag{14.3}$$

Since both TM and TN are equipped with their respective Riemannian connections, the bundle on the right hand side also carries the induced connection. We then write

$$\nabla(f_*) \in \Gamma(T^*M \otimes T^*M \otimes f^*TN). \tag{14.4}$$

We define the *harmonic map Laplacian* as

$$\Delta_{g,h} f = tr_g\big(\nabla(f_*)\big) \in \Gamma(f^*TN). \tag{14.5}$$

In coordinates, this is

$$(\Delta_{g,h} f)^\alpha = g^{ij}\Big\{\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - (\Gamma_g)^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \big((\Gamma_h)^\alpha_{\beta\gamma} \circ f\big)\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}.\Big\} \tag{14.6}$$

We define the *harmonic map flow*

$$\frac{\partial f}{\partial t} = \Delta_{g,h} f, \quad f(0) = f_0. \tag{14.7}$$

This strictly parabolic equation was first studied by Eells and Sampson. In the case the target has non-positive sectional curvature, they proved that the flow exists for all time and converges exponentially fast to a harmonic map [ES64].

Returning to the Ricci flow, assume we have a solution $\overline{g}(t)$ to the Ricci flow, both defined on $M \times [0, \epsilon)$. Let ϕ_t be the solution to the harmonic map heat flow

$$\frac{\partial \phi_t}{\partial t} = \Delta_{\overline{g}(t), \tilde{g}} \phi_t, \;\; \phi(0) = Id_M. \tag{14.8}$$

where $\tilde{g}$ is any reference metric. By direct computation, it can be shown that $g(t) = (\phi_t)_* \overline{g}(t)$ solves Ricci-DeTurck flow. To prove uniqueness, if you have 2 solutions $\overline{g}_1(t)$ and $\overline{g}_2(t)$ of Ricci flow with the same initial data. Using the harmonic map heat flow, we obtain 2 solutions of Ricci-DeTurck flow with the same initial data. By uniqueness of solution to Ricci-Deturck flow, they are the same. But the diffeomorphisms defined in (14.2) must be the same, so $\overline{g}_1(t) = \overline{g}_2(t)$. For more details, see [CK04, Section 3.4.4].

14.2 Linear parabolic equations

We recall the definition of parabolic Hölder norms. Let us endow $M \times \mathbb{R}$ with the distance function

$$d(z_1, z_2) = d(x_1, x_2) + |t_1 - t_2|^{1/2}, \quad z_i = (x_i, t_i). \tag{14.9}$$

Let $0 < \alpha \leq 1$, and let $\Omega \subset M \times \mathbb{R}$ be a domain. For $f : \Omega \to \mathbb{R}$, define

$$[f]_{\alpha;\Omega} = \sup_{z_1 \neq z_2, z_i \in \Omega} \frac{|f(z_1) - f(z_2)|}{d(z_1, z_2)^\alpha} \tag{14.10}$$

$$|f|_{\alpha;\Omega} = |f|_{0,\Omega} + [f]_{\alpha;\Omega}. \tag{14.11}$$

Remark. Roughly, the Hölder exponent in t is half of the spatial Hölder exponent. This is because a heat equation is $u_t = u_{xx}$, so we only require "half" of the regularity in the time direction as we do in the spatial direction.

If this norm is finite, we say f is Hölder continuous with exponent α, and write $f \in C^\alpha(\Omega)$. We next define

$$[f]_{2,\alpha;\Omega} = [f_t]_{\alpha;\Omega} + \sum_{i,j=1}^{n} [(D_x^2)_{i,j} f]_{\alpha;\Omega} \tag{14.12}$$

$$|f|_{2,\alpha;\Omega} = |f|_{0;\Omega} + |Df|_{0;\Omega} + |f_t|_{0;\Omega} + \sum_{i,j=1}^{n} |(D_x^2)_{i,j} f|_{0;\Omega} + [f]_{2,\alpha;\Omega}. \tag{14.13}$$

If this norm is finite, we write $f \in C^{2,\alpha}(\Omega)$. The spaces $C^\alpha(\Omega)$, and $C^{2,\alpha}(\Omega)$ are Banach spaces under their corresponding norms $|\cdot|$. Note that a $C^{k,\alpha}$ norm can be defined analogously for any integer $k \geq 0$.

We consider parabolic linear operators of the form

$$Lu = -u_t + a^{ij}(x,t) D_{ij} u + b^i(x,t) D_i u + cu, \tag{14.14}$$

as expressed in a coordinate system. The following is a fundamental theorem on existence of solutions to *linear* parabolic equations.

Theorem 14.1. *([Kry96]). Let $\Omega = M \times [0, t)$, for some $t > 0$. Assume that for some $0 < \alpha < 1$, there exists a constant Λ such that*

$$|a^{ij}|_{\alpha;\Omega} + |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega} < \Lambda. \tag{14.15}$$

Also, assume that L is strictly parabolic, that is, for some constant $\lambda > 0$,

$$a^{ij}(x,t)\xi_i\xi_j \geq \lambda |\xi|^2. \tag{14.16}$$

Given $f \in C^\alpha$, and $\phi \in C^{2,\alpha}$, there exists a unique solution to

$$Lu = 0, \ u(x, 0) = \phi, \tag{14.17}$$

on $M \times [0, t)$. Furthermore, there exists a constant $C = C(\alpha, \lambda, \Lambda, n)$ such that

$$|u|_{2+\alpha;\Omega} \leq C(|f|_{\alpha;\Omega} + |\phi|_{2+\alpha;\Omega}). \tag{14.18}$$

Proof. The idea of the proof is simple, although we do not have time to write down a complete proof here. First prove (14.18) for an equation with constant coefficients. Using the Hölder condition on the coefficients, the equation is locally close to a constant coefficient heat equation, so the estimate will hold locally. The global estimate is then obtained by a patching argument. The existence follows from the estimate (14.18) and the method of continuity. □

Note that the constant is independent of t. Thus if the coefficients are bounded Hölder for all time, then it is not hard to show the solution exists for all time. Furthermore, the above theorem holds for linear parabolic *systems*, with the ellipticity assumption meaning that the symbol is non-degenerate, see [Èĭd69, page 4]. An important point: this theorem is true for linear systems, but NOT necessarily true for nonlinear systems such as the Ricci flow.

Remark. For elliptic equations, to prove uniqueness it is usually necessary to assume the zeroth order term has a sign $c \leq 0$. For parabolic equations with bounded coefficients, such an assumption is not necessary. To see this, if u solves a parabolic equation $u_t = Lu + f$, with bounded $c \leq \lambda$, then the function $v(x,t) = u(x,t)e^{-\lambda t}$ satisfies the equation $v_t = Lv - \lambda v + fe^{-\lambda t}$. The maximum principle can be applied to this latter equation.

14.3 Quasilinear parabolic systems

We just saw that linear parabolic systems have long-time existence. This is NOT true for nonlinear systems, but the above result for linear equations can actually be applied to prove short-time existence for nonlinear systems. We will consider quasilinear equations of the form

$$u_t = Pu, \quad u(x,0) = \phi, \tag{14.19}$$

where P has the form

$$Pu = a^{ij}(x,t,u,Du)D^2_{ij}u + h(x,t,u,Du). \tag{14.20}$$

where a^{ij} and h are smooth functions, and parabolicity assumption

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad t < \epsilon. \tag{14.21}$$

where $\lambda > 0$ is a constant.

Proposition 14.1. *Let M be compact, and assume (14.21) is satisfied. if $\phi \in C^{2,\alpha}$, then there exists an $\epsilon > 0$ such that the equation (14.19) has a unique solution defined on $M \times [0,\epsilon)$. If ϕ is smooth, then so is this solution.*

Proof. Choose θ so that $0 < \theta < \delta < 1$, and define

$$\mathcal{S} = \{v \in C^{1,\theta}(M \times [0,\epsilon)) : |v|_{1,\theta} \leq M_0\}, \tag{14.22}$$

where $M_0 = 1 + |\phi|_{2,\alpha}$, and ϵ is to be chosen later. Define the map $J : \mathcal{S} \to C^{2,\theta}$ by $u = Jv$ is the unique solution of the *linear* problem

$$\begin{aligned} u_t &= a^{ij}(x,t,v,Dv)D^2_{ij}u + h(x,t,v,Dv) \\ u(x,0) &= \phi. \end{aligned} \tag{14.23}$$

Such a solution exists by Theorem 14.1, and satisfies

$$\begin{aligned} |u|_{2,\theta} &\le C(|h(x,t,v,Dv)|_{\theta;\Omega} + |\phi|_{2,\theta;\Omega}) \\ &\le CM_0, \end{aligned} \tag{14.24}$$

on $M \times [0,\epsilon)$. In particular $|u|_{0,1} < CM_0$, which says that

$$|u(x,t) - \phi(x)| = |u(x,t) - u(x,0)| \le CM_0 t^{1/2} \le CM_0\epsilon^{1/2}. \tag{14.25}$$

Using interpolation, for any $\delta > 0$,

$$|u|_{1,\theta} \le \delta|u|_{2,\theta} + C|u|_0 \le \delta M_0 + CM_0\epsilon^{1/2} = (\delta + C\epsilon^{1/2})M_0, \tag{14.26}$$

so by choosing ϵ sufficiently small, we see that $J : \mathcal{S} \to \mathcal{S}$. Finally, $\mathcal{S}$ is a convex, compact subset of the Banach space C_1, and J is continuous, so by the Schauder fixed point theorem [Lie96, Theorem VIII.8.1], J has a fixed point. Such a fixed point is clearly a solution of the original nonlinear equation. If ϕ is smooth, the the solution will also be smooth by parabolic regularity. □

Remark. For simplicity, we just considered parabolic equations, but the above proof is also valid in the case of parabolic systems [Èĭd69, Section 3.4].

15 Lecture 14

15.1 Maximum principle for scalar parabolic equations

We begin with the most basic parabolic maximum principle. Recall that a C^2 function on a domain in space time is C^2 in space, but C^1 in time. The results in this section can be found in [CK04, Chapter 4].

Proposition 15.1. *Let g_t be a smooth 1-parameter family of metrics on $M \times [0,T)$. If $u(x,t) : M \times [0,T) \to \mathbb{R}$ is a C^2 supersolution of the heat equation*

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u, \tag{15.1}$$

that is,

$$\frac{\partial u}{\partial t} \ge \Delta_{g(t)} u, \tag{15.2}$$

satisfying $C_1 \le u(x,0)$, for some constant C_1, then $C_1 \le u(x,t)$ for all $t \in [0,T)$. If $u(x,t) : M \times [0,T) \to \mathbb{R}$ is a C^2 subsolution of (15.1),

$$\frac{\partial u}{\partial t} \le \Delta_{g(t)} u, \tag{15.3}$$

satisfying $u(x,0) \le C_2$, for some constant C_2, then $u(x,t) \le C_2$ for all $t \in [0,T)$.

Proof. We will prove the supersolution case, the other case is similar. Let $F : M \times [0, T)$ be any C^2 function. Suppose that (x_0, t_0) is a point satisfying

$$F(x_0, t_0) = \min_{M \times [0, t_0]} F. \tag{15.4}$$

That is (x_0, t_0) is a point at which F attains its minimum, taken over all earlier spacetime points. Then clearly

$$\frac{\partial F}{\partial t}(x_0, t_0) \leq 0 \tag{15.5}$$
$$\nabla F(x_0, t_0) = 0 \tag{15.6}$$
$$\Delta F(x_0, t_0) \geq 0. \tag{15.7}$$

Now $F(x, t) = u(x, t) - C_1 + \epsilon t + \epsilon$, for any $\epsilon > 0$. For $t = 0$, we have $F \geq \epsilon > 0$. F satisfies the inequality

$$\frac{\partial F}{\partial t} = \frac{\partial u}{\partial t} + \epsilon \geq \Delta_{g(t)} u + \epsilon = \Delta_{g(t)} F + \epsilon. \tag{15.8}$$

If we prove that $F > 0$ for all $t \in [0, T)$, for any $\epsilon > 0$, then we will clearly be done. To prove this, assume by contradiction that $F \leq 0$ for some $(x_1, t_1) \in M \times [0, T)$. Since M is compact, and $F > 0$ at $t = 0$, then there is a first time $t_0 \in (0, t_1]$ such that there exists a point $x_0 \in M$ with $F(x_0, t_0) = 0$. Note that

$$u(x_0, t_0) = C_1 - \epsilon t_0 - \epsilon < C_1. \tag{15.9}$$

Using the above inequalities, we have

$$0 \geq \frac{\partial F}{\partial t}(x_0, t_0) \geq \Delta_{g(t)} F(x_0, t_0) + \epsilon \geq \epsilon > 0, \tag{15.10}$$

which is a contradiction. □

Remark. Note from (15.9) we only needed to assume u is a subsolution whenever $u < C_1$. Also, the above theorem holds if the right hand side of the equation has a gradient term, clearly this will not affect the argument since the gradient will vanish at a minimum point.

We next consider the case that the equation has a zeroth order term.

Proposition 15.2. *Let g_t be a smooth 1-parameter family of metrics on $M \times [0, T)$, and let $\beta : M \times [0, T)$ be bounded from above. If $u(x, t) : M \times [0, T) \to \mathbb{R}$ is a C^2 supersolution*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \beta(x, t) \cdot u, \tag{15.11}$$

satisfying $0 \leq u(x, 0)$, then $0 \leq u(x, t)$ for all $t \in [0, T)$. If u is a C^2 subsolution

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \beta(x, t) \cdot u, \tag{15.12}$$

satisfying $u(x, 0) \leq 0$, then $u(x, t) \leq 0$ for all $t \in [0, T)$.

Proof. Let $v(x,t) = e^{-Ct}u(x,t)$ where C is a lower bound for $\beta(x,t)$. We compute

$$\begin{aligned}\frac{\partial v}{\partial t} &= -Ce^{-Ct}u(x,t) + e^{-Ct}\frac{\partial u}{\partial t} \\ &\geq -Ce^{-Ct}u(x,t) + e^{-Ct}\left(\Delta_{g(t)}u + \beta(x,t)\cdot u\right) \\ &= \Delta_{g(t)}v + (\beta - C)v.\end{aligned} \tag{15.13}$$

From the remark above, we need only verify that v is a subsolution whenever $u < 0$. If we let C be an upper bound for β, then for $v < 0$,

$$\frac{\partial v}{\partial t} \geq \Delta_{g(t)}v + (\beta - C)v \geq \Delta_{g(t)}v. \tag{15.14}$$

Since $0 \leq v(t,0)$, and v is a supersolution of the heat equation, applying Proposition 15.1 we are done. In the subsolution case, we need to verify that v is a supersolution whenever $v > 0$. If C again denotes an upper bound for β, then for $v > 0$,

$$\frac{\partial v}{\partial t} \leq \Delta_{g(t)}v + (\beta - C)v < 0, \tag{15.15}$$

and the result again follows from Proposition 15.1. □

Remark. Proposition 15.2 implies uniqueness of solution to linear parabolic equations with zeroth order term bounded from above. Furthermore, the method of proof can be used to derive *a priori estimates* for linear equations, see [Lie96, Theorem 2.11].

Next, we consider the case of a *nonlinear* zeroth order term.

Proposition 15.3. *Let g_t be a smooth 1-parameter family of metrics on $M \times [0,T)$, and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Let $u(x,t) : M \times [0,T) \to \mathbb{R}$ be a C^2 supersolution*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)}u + F(u), \tag{15.16}$$

satisfying $C_1 \leq u(x,0)$. Let ϕ_1 be the solution to the ODE

$$\frac{d}{dt}\phi_1 = F(\phi_1), \quad \phi_1(0) = C_1, \tag{15.17}$$

then $\phi_1(t) \leq u(x,t)$ for all $x \in M$ and for all $t \in [0,T)$ such that $\phi_1(t)$ exists.

Let $u(x,t) : M \times [0,T) \to \mathbb{R}$ be a C^2 subsolution

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)}u + F(u), \tag{15.18}$$

satisfying $u(x,0) \leq C_2$. Let ϕ_2 be the solution to the ODE

$$\frac{d}{dt}\phi_2 = F(\phi_2), \quad \phi_2(0) = C_2, \tag{15.19}$$

then $u(x,t) \leq \phi_2(t)$ for all $x \in M$ and for all $t \in [0,T)$ such that $\phi_2(t)$ exists.

Proof. We just consider the supersolution case. We have

$$\frac{\partial}{\partial t}(u - \phi_1) \geq \Delta_{g(t)} u + F(u) - F(\phi_1) = \Delta_{g(t)}(u - \phi_1) + F(u) - F(\phi_1). \quad (15.20)$$

By assumption $u \geq \phi_1$ at $t = 0$. Take $t_0 \in (0, T)$. Since M is compact, there exists a constant C_{t_0} such that $|u(x,t)| \leq C_{t_0}$ and $|\phi_1(t)| \leq C_{t_0}$ for all $(x,t) \in M \times [0, t_0]$. By the Lipschitz assumption on F, there exists a constant C' such that

$$|F(r) - F(s)| \leq C'|s - t| \text{ for all } r, s \in [-C_{t_0}, C_{t_0}]. \quad (15.21)$$

On $M \times [0, t_0], u - \phi_1$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}(u - \phi_1) &\geq \Delta_{g(t)}(u - \phi_1) + F(u) - F(\phi_1) \\ &\geq \Delta_{g(t)}(u - \phi_1) - C'\text{sign}(u - \phi_1) \cdot (u - \phi_1). \end{aligned} \quad (15.22)$$

This says that $u - \phi_1$ is a supersolution of an equation of the form in Proposition 15.2 with $\beta = -C'\text{sign}(u - \phi_1)$, so we conclude that $u - \phi_1 \geq 0$ on $M \times [0, t_0]$. Since $t_0 \in (0, T)$ was arbitrary, we are done. □

16 Lecture 15

16.1 Evolution of scalar curvature under the Ricci flow

Proposition 16.1. *Under the Ricci flow, $g' = -2Ric$, the evolution of the scalar curvature is given by*

$$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2. \quad (16.1)$$

Proof. From Proposition 12.3 the linearization of the scalar curvature is

$$R' = -\Delta(trh) + \text{div}^2 h - R_{lp} h^{lp}. \quad (16.2)$$

We let $h = -2Ric$, and use the twice contracted differential Binachi identity

$$\text{div } Ric = \frac{1}{2} dR, \quad (16.3)$$

to obtain

$$\frac{\partial}{\partial t} R = 2\Delta R + \text{div}^2(-2Ric) + 2R_{lp}R^{lp} = \Delta R + 2|Ric|^2. \quad (16.4)$$

□

Corollary 16.1. *If the solution the Ricci flow exists on a time interval $[0, T)$, and the scalar curvature of the metric $g(0)$ satisfies $R_{g(0)} \geq C_1$ for some constant C_1, then $R_{g(t)} \geq C_1$ for all $t \in [0, T)$. Furthermore, if $R_{g(0)}$ is nonnegative and strictly positive at some point, then $R_{g(t)}$ is strictly positive $t \in (0, T)$.*

Proof. We clearly have the inequality

$$\frac{\partial}{\partial t} R \geq \Delta R, \tag{16.5}$$

which says that R is a supersolution of the scalar heat equation. So the first statement follows from the maximum principle as stated in Proposition 15.1. The second statement follows from the *strong* maximum principle, see [Lie96, Theorem 2.9]. □

We can furthermore obtain a more quantitative estimate on the scalar curvature from below.

Proposition 16.2. *If the solution the Ricci flow exists on a time interval* $[0, T)$*, then*

$$R_{g(t)} \geq \frac{R_{min}(g_0)}{1 - (2/n)t \cdot R_{min}(g_0)}. \tag{16.6}$$

Proof. Let E be the traceless Ricci tensor. From the obvious inequality $|E|^2 \geq 0$, we obtain

$$|Ric|^2 - (2/n)R^2 + (1/n)R^2 \geq 0, \tag{16.7}$$

which is the inequality

$$|Ric|^2 \geq (1/n)R^2. \tag{16.8}$$

From (16.1), we obtain

$$\frac{\partial}{\partial t} R \geq \Delta R + (2/n)R^2. \tag{16.9}$$

Let ϕ_1 be the solution to the ODE

$$\frac{d}{dt}\phi_1 = (2/n)\phi_1^2, \tag{16.10}$$

with initial value $\phi_1(0) = R_{min}(g_0)$. The exact solution is

$$\phi_1 = \frac{1}{R_{min}(g_0)^{-1} - (2/n)t}, \tag{16.11}$$

if $R_{min}(g_0) \neq 0$. From Proposition 15.3, we conclude that

$$R_{g(t)} \geq \frac{1}{R_{min}(g_0)^{-1} - (2/n)t}. \tag{16.12}$$

□

Corollary 16.2. *If If the solution the Ricci flow exists on a time interval* $[0, T)$*, and* $R_{g(0)}$ *is strictly positive, then* $T \leq (n/2) \cdot R_{min}(g_0)^{-1}$.

Proof. Clearly (16.6) says that the scalar curvature would have to blow-up at time $T_0 = (n/2) \cdot R_{min}(g_0)^{-1}$, so existence time of the Ricci flow must be less than T_0. □

16.2 Einstein metrics

Assume we have a solution to the Ricci flow which is of the form

$$g(t) = f(t)g(0), \tag{16.13}$$

where $f(t)$ is a positive function. We compute

$$g'(t) = f'(t)g(0) = \frac{f'(t)}{f(t)} f(t)g(0) = (\log f)'(t)g(t). \tag{16.14}$$

For this to be a solution of Ricci Flow, we require

$$-2Ric(g(t)) = (\log f)'(t)g(t), \tag{16.15}$$

which says that $g(t)$ must be an Einstein metric. Letting $Ric(g(0)) = \lambda g(0)$, since Ricci is scale invariant, we have

$$-2Ric(g(t)) = -2Ric(g(0)) = -2\lambda g(0) = f'(t)g(0), \tag{16.16}$$

which has solution

$$f(t) = -2\lambda t + C. \tag{16.17}$$

If $f(0) = 1$, then

$$g(t) = (1 - 2\lambda t)g(0). \tag{16.18}$$

We have the following trichotomy for the Ricci flow with initial data an Einstein metric:

(i) If $\lambda < 0$ then the solution to Ricci flow exists for all time, the solution eternally expands.

(ii) If $\lambda = 0$, then the solution is static.

(iii) If $\lambda > 0$, then the solution to Ricci flow maximally exists on the time interval $[0, (2\lambda)^{-1}]$, and the solution shrinks to a point in finite time.

We compare case (iii) to the conclusion of Corollary 16.2. The scalar curvature of g is equal to $n\lambda$, and we indeed have

$$(n/2) \cdot R_{min}(g_0)^{-1} = (n/2)(n\lambda)^{-1} = (2\lambda)^{-1}. \tag{16.19}$$

Indeed, this had to be the same, since we used the inequality $|E|^2 \geq 0$, which is an equality for Einstein metrics.

16.3 Normalized versus unnormalized Ricci flow

There standard way to modify Ricci flow so that all Einstein metrics are static solutions. Let

$$r = \frac{\int_M R_g dV_g}{\int_M dV_g}, \tag{16.20}$$

denote the average scalar curvature. The flow is

$$\frac{\partial}{\partial t}g = -2Ric + \frac{2}{n}r \cdot g. \tag{16.21}$$

Indeed, an Einstein metric is a static solution since

$$\frac{\partial}{\partial t}g = -2(R/n)g + \frac{2}{n}Rg = 0. \tag{16.22}$$

The main point is that the normalized Ricci flow preserves the volume. To see this, from equation (12.24),

$$\frac{d}{dt}(Vol(g(t)) = \int_M \frac{1}{2}tr_g\Big(- 2Ric + \frac{2}{n}rg\Big)dV_g = \frac{1}{2}\int_M (-2R + 2r)dV_g = 0. \tag{16.23}$$

The normalized Ricci flow and unnormalized Ricci flow are essentially the same flow, they just differ by scaling factor in space, and a re-parametrization of time.

Assume we have a solution of Ricci flow on some time interval $[0, T)$,

$$\frac{\partial}{\partial t}g = -2Ric. \tag{16.24}$$

The corresponding solution of normalized Ricci flow is found as follows. First, choose $\psi(t) > 0$ so that the metrics $\bar{g}(t) = \psi(t)g(t)$ have unit volume, and define

$$\bar{t} = \int_0^{\tau} \psi(\tau)d\tau. \tag{16.25}$$

We compute

$$\begin{aligned} \frac{\partial}{\partial \bar{t}}\bar{g} &= \frac{dt}{d\bar{t}}\frac{\partial}{\partial t}(\psi(t)g(t)) \\ &= \frac{1}{\psi(\bar{t})}\left(\psi(t)\frac{\partial}{\partial t}g + \frac{d\psi}{dt}\cdot g(t)\right) \\ &= -2\overline{Ric} + \left(\frac{1}{\psi^2}\frac{d\psi}{dt}\right)\bar{g}. \end{aligned} \tag{16.26}$$

Since the metrics $\bar{g}$ have unit volume, we have

$$\begin{aligned} 0 &= \frac{1}{2}\int tr_{\bar{g}}\left(-2\overline{Ric} + \left(\frac{1}{\psi^2}\frac{d\psi}{dt}\right)\bar{g}\right)dV_{\bar{g}} \\ &= \int \Big(- \overline{R} + \frac{n}{2\psi^2}\frac{d\psi}{dt}\Big)dV_{\bar{g}} \\ &= -\bar{r} + \frac{n}{2\psi^2}\frac{d\psi}{dt}. \end{aligned} \tag{16.27}$$

Substituting this into the above, we obtain

$$\frac{\partial}{\partial \bar{t}}\bar{g} = -2\overline{Ric} + (2/n)\bar{r}\bar{g}, \tag{16.28}$$

which is the normalized flow. To go from the normalized flow to unnormalized flow, Use the ODE

$$(2/n)\bar{r} = \frac{1}{\psi^2}\frac{d\psi}{dt} = \frac{1}{\psi}\frac{d\psi}{d\bar{t}} \tag{16.29}$$

to define ψ as a function of $\bar{t}$, and reverse the above computations.

16.4 Evolution of scalar under normalized Ricci flow

Proposition 16.3. *Under the normalized Ricci flow, $g' = -2Ric + (2/n)rg$, the evolution of the scalar curvature is given by*

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2 - (1/n)Rr. \tag{16.30}$$

Proof. We just repeat the computation we did in the case of unnormalized flow. Again the general linearization of the scalar curvature is

$$R' = -\Delta(trh) + \mathrm{div}^2 h - R_{lp}h^{lp}. \tag{16.31}$$

We let $h = -2Ric + (2/n)rg$, and again use the twice contracted differential Binachi identity

$$\frac{\partial}{\partial t}R = 2\Delta R + \mathrm{div}^2(-2Ric) + 2R_{lp}(R^{lp} - (1/n)rg^{lp}) = \Delta R + 2|Ric|^2 - (1/n)Rr. \tag{16.32}$$

□

17 Lecture 16

17.1 Parabolic maximum principles for tensors

In dealing with the Ricci flow, one requires the maximum principle for parabolic tensor systems, rather than just on scalar functions. The following is the first version

Proposition 17.1. *Let g_t be a smooth 1-parameter family of metrics on $M \times [0,T)$. Let $\alpha(t)$ be a symmetric $(0,2)$ tensor which is a supersolution*

$$\frac{\partial}{\partial t}\alpha \geq \Delta_{g(t)}\alpha + \beta(\alpha, g, t), \tag{17.1}$$

where β is a symmetric $(0,2)$ tensor which is locally Lipschitz in all of its arguments. Furthermore, assume that β satisfies the null eigenvector assumption

$$\beta(V,V)(x,t) \geq 0, \tag{17.2}$$

whenever $V(x,t)$ satisfies $\alpha(V,\cdot) = 0$, that is V is a null eigenvector for $\alpha(x,t)$. If $\alpha(x,0)$ is positive semidefinite, then $\alpha(x,t)$ is positive semidefinite for all $(x,t) \in M \times [0,T)$.

Proof. Suppose (x_0, t_0) is a first spacetime point at which α acquires a zero eigenvector V. Extend V to a vector field in a spacetime neighborhood so that

$$\begin{aligned} \frac{\partial}{\partial t}V(x_0,t_0) &= 0 \\ \nabla V(x_0,t_0) &= 0. \end{aligned} \tag{17.3}$$

This can be done by parallel translating $V_{(x_0,t_0)}$ along radial geodesics in the $g(t_0)$ metric, and then by extending to be independent of time. Then locally around (x_0, t_0), we compute

$$\begin{aligned}\frac{\partial}{\partial t}\big(\alpha(V,V)\big) &= \frac{\partial}{\partial t}(\alpha)(V,V)\\ &\geq (\Delta\alpha + \beta)(V,V).\end{aligned} \tag{17.4}$$

By choice of V, we have $\alpha(V,V)(x_0,t_0) = 0$, and $\alpha(V,V)(x,t_0) \geq 0$ in a neighborhood of x_0, which implies that

$$\Delta\big(\alpha(V,V)\big)(x_0,t_0) \geq 0. \tag{17.5}$$

We next compute

$$\begin{aligned}\Delta\big(\alpha(V,V)\big) &= \Delta\big(\alpha_{ij}V^iV^j\big)\\ &= g^{pq}\nabla_p\nabla_q\big(\alpha_{ij}V^iV^j\big)\\ &= g^{pq}\nabla_p\Big(\nabla_q(\alpha_{ij})V^iV^j + 2\alpha_{ij}(\nabla_qV^i)V^j\Big)\\ &= g^{pq}\Big(\nabla_p\nabla_q\alpha_{ij}\Big)V^iV^j + 2g^{pq}\big(\nabla_q\alpha_{ij}\big)\big(\nabla_pV^i\big)V^j + 2g^{pq}\big(\alpha_{ij}\nabla_pV^i\nabla_qV^j\big)\\ &\quad + 2g^{pq}\alpha_{ij}(\nabla_p\nabla_qV^i)V^j.\end{aligned} \tag{17.6}$$

Using the equations (17.3), since V is a null eigenvector, we have at (x_0, t_0),

$$\Delta\big(\alpha(V,V)\big) = (\Delta\alpha)(V,V). \tag{17.7}$$

By the null eigenvector assumption on β, we therefore have

$$\frac{\partial}{\partial t}\big(\alpha(V,V)\big) \geq (\Delta\alpha + \beta)(V,V)(x_0,t_0) \geq 0, \tag{17.8}$$

which shows that α on a null eigendirection cannot decrease. If we had assumed that β is strictly positive definite, we would have strict inequality, which would imply that any zero eigendirection immmediately becomes positive. For the full proof, one argues as in the proof of Proposition 15.1, by considering the modified tensor

$$\alpha_\epsilon(x,t) = \alpha(x,t) + (\epsilon t + \epsilon)g(x,t), \tag{17.9}$$

to make things strictly positive definite. This is the main idea, but there are some extra details which we omit, see [CK04, Theorem 4.6]. □

17.2 Evolution of Ricci tensor under Ricci flow

Proposition 17.2. *Under the Ricci flow, $g' = -2Ric$, the evolution of the Ricci tensor is given by*

$$\frac{\partial}{\partial t}R_{ij} = \Delta_L R_{ij} = \Delta R_{ij} + 2R_{iljp}R^{lp} - 2R_i^pR_{jp}. \tag{17.10}$$

Proof. Recall from Proposition 12.2 the linearization of the Ricci tensor:

$$\begin{aligned}(Ric')_{ij} = \frac{1}{2}\Big(& -\Delta h_{ij} + \nabla_i(\text{div } h)_j + \nabla_j(\text{div } h)_i - \nabla_i\nabla_j(trh) \\ & - 2R_{iljp}h^{lp} + R_i^p h_{jp} + R_j^p h_{ip}\Big)\end{aligned} \tag{17.11}$$

For the Ricci flow, we have $g' = -2Ric$, and we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ij} = & \Delta R_{ij} - \nabla_i(\text{div } Ric)_j - \nabla_j(\text{div } Ric)_i + \nabla_i\nabla_j(R) \\ & + 2R_{iljp}R^{lp} - R_i^p R_{jp} - R_j^p R_{ip}.\end{aligned} \tag{17.12}$$

From the Bianchi identity

$$\text{div } Ric = \frac{1}{2}dR, \tag{17.13}$$

the terms containing derivatives cancel out, and we obtain

$$\frac{\partial}{\partial t}Ric_{ij} = \Delta R_{ij} + 2R_{iljp}R^{lp} - R_i^p R_{jp} - R_j^p R_{ip} = \Delta_L R_{ij} \tag{17.14}$$

□

Remark. Examining the above proof, it is easy to see that the exact same evolution formula holds for the normalized Ricci flow.

We see that the evolution of the Ricci tensor contains terms which depend upon the full curvature tensor. Expanding the curvature tensor, we obtain

Proposition 17.3. *Under the Ricci flow, $g' = -2Ric$, the evolution of the Ricci tensor is given by*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ij} = & \Delta R_{ij} + 2W_{iljp}R^{lp} - \frac{2n}{n-2}R_i^p R_{jp} + \frac{2n}{(n-1)(n-2)}RR_{ij} \\ & + \frac{2}{n-2}\Big(|Ric|^2 - \frac{1}{n-1}R^2\Big)g_{ij}\end{aligned} \tag{17.15}$$

Proof. We use the decomposition of the curvature tensor given from Section (4.2),

$$Rm = W + A \owedge g, \tag{17.16}$$

which written out is

$$R_{ijkl} = W_{ijkl} + A_{ik}g_{jl} - A_{jk}g_{il} - A_{il}g_{jk} + A_{jl}g_{ik}. \tag{17.17}$$

We substitue this into (17.11) and simplify:

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ij} &= \Delta R_{ij} + 2R_{iljp}R^{lp} - 2R_i^p R_{jp} \\
&= \Delta R_{ij} + 2(W_{iljp} + A_{ij}g_{lp} - A_{lj}g_{ip} - A_{ip}g_{lj} + A_{lp}g_{ij})R^{lp} - 2R_i^p R_{jp} \\
&= \Delta R_{ij} + 2W_{iljp}R^{lp} + \frac{2}{n-2}\Big(R_{ij}g_{lp} - R_{lj}g_{ip} - R_{ip}g_{lj} + R_{lp}g_{ij}\Big)R^{lp} \\
&\quad - \frac{R}{(n-1)(n-2)}\Big(g_{ij}g_{lp} - g_{lj}g_{ip} - g_{ip}g_{lj} + g_{lp}g_{ij}\Big)R^{lp} - 2R_i^p R_{jp} \\
&= \Delta R_{ij} + 2W_{iljp}R^{lp} + \frac{2}{n-2}\Big(RR_{ij} - 2R_{lj}R_i^l + |Ric|^2 g_{ij}\Big) \\
&\quad - \frac{R}{(n-1)(n-2)}\Big(2Rg_{ij} - 2R_{ij}\Big) - 2R_i^p R_{jp}.
\end{aligned} \tag{17.18}
$$

Collecting terms, we obtain (17.15). □

Specializing to dimension 3, we obtain

Corollary 17.1. *In dimension 3, under the Ricci flow $g' = -2Ric$, the evolution of the Ricci tensor is given by*

$$
\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} - 6R_i^p R_{jp} + 3RR_{ij} + \big(2|Ric|^2 - R^2\big)g_{ij}. \tag{17.19}
$$

Proof. From Corollary 4.2, the Weyl tensor is identically zero in dimension 3. □

Proposition 17.4. *If the solution the Ricci flow exists on a time interval $[0, T)$, and the Ricci tensor of the metric $g(0)$ is positive (semi-)definite, then the Ricci tensor of $g(t)$ remains positive (semi-)definite for for all $t \in [0, T)$. Furthermore, if the Ricci tensor of $g(0)$ is nonnegative and has a strictly positive definite at some point, then the Ricci tensor of $g(t)$ is strictly positive for all $t \in (0, T)$.*

Proof. To apply Proposition 17.1, we need to verify the null-eigenvector assumption (17.2). So let V be a null eigenvector for the Ricci tensor. We look at

$$
\Big(-6R_i^p R_{jp} + 3RR_{ij} + \big(2|Ric|^2 - R^2\big)g_{ij}\Big)V^iV^j = \big(2|Ric|^2 - R^2\big)|V|^2 \geq 0, \tag{17.20}
$$

by the inequality (16.8) for $n = 2$ (since Ric has a zero eigenvalue), so we are done. The last statement follows from the strong maximum principle. □

Proposition 17.5. *If the solution the Ricci flow exists on a time interval $[0, T)$, and the metric $g(0)$ has positive (nonnegative) sectional curvature, then $g(t)$ has positive (nonnegative) sectional curvature for for all $t \in [0, T)$. Furthermore, if the sectional curvature of $g(0)$ is nonnegative and is strictly positive at some point, then the sectional curvature of $g(t)$ is strictly positive for all $t \in (0, T)$.*

Proof. Recall from Section 5.3 that positivity (nonnegativity) of the sectional curvature in dimension 3 is equivalent positivity (nonnegativity) of the tensor $T = T_1(A)$, which is

$$T = -Ric + \frac{1}{2}Rg. \tag{17.21}$$

We have

$$\frac{\partial}{\partial t}T = -\frac{\partial}{\partial t}Ric + \frac{1}{2}\Big(\frac{\partial}{\partial t}R\Big)g - R \cdot Ric. \tag{17.22}$$

Using (17.19), and (16.1) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij} &= -\Delta R_{ij} + 6R_i^pR_{jp} - 3RR_{ij} - \big(2|Ric|^2 - R^2\big)g_{ij} + \frac{1}{2}(\Delta R + 2|Ric|^2)g_{ij} - RR_{ij}\\ &= -\Delta T + 6R_i^pR_{jp} - 4RR_{ij} - \big(|Ric|^2 - R^2\big)g_{ij}.\end{aligned} \tag{17.23}$$

We rewrite the right hand side in terms of T.

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij} &= \Delta T + 6(-T_i^p + (1/2)Rg_i^p)(-T_{jp} + (1/2)Rg_{jp}) - 4R(-T_{ij} + (1/2)Rg_{ij})\\ &\qquad - \big(\big| - T + (1/2)Rg\big|^2 - R^2\big)g_{ij}\\ &= \Delta T + 6T_i^pT_{jp} - 6RT_{ij} + (3/2)R^2g_{ij} + 4RT_{ij} - 2R^2g_{ij}\\ &\qquad - \big(|T|^2 - (1/2)R^2g + (3/4)R^2 - R^2\big)g_{ij}\\ &= \Delta T + 6T_i^pT_{jp} - 2RT_{ij} + (1/4)R^2g_{ij} - |T|^2g_{ij}.\end{aligned} \tag{17.24}$$

Assuming T has a zero eigenvalue, we need to verify

$$(1/4)R^2 - |T|^2 \geq 0. \tag{17.25}$$

Since $tr(T) = R/2$, we can rewrite this as

$$(tr\ T)^2 - |T|^2 \geq 0. \tag{17.26}$$

This is obvious – since T is symmetric, assume it is diagonal. The last statement again follows from the strong maximum principle. □

18 Lecture 17

18.1 Evolution of curvature tensor under Ricci flow

We begin with a general proposition about the linearization of the curvature tensor.

Proposition 18.1. *The linearization of the* $(1,3)$ *curvature tensor in the direction* $g' = h$ *is given by*

$$(R')_{ijk}{}^{l} = \frac{1}{2} g^{lm} \Big(\nabla_i \nabla_k h_{jm} - \nabla_i \nabla_m h_{jk} - \nabla_j \nabla_k h_{im} + \nabla_j \nabla_m h_{ik} \\ - R_{ijk}{}^{p} h_{pm} - R_{ijm}{}^{p} h_{kp} \Big). \tag{18.1}$$

Proof. Recall from Proposition 12.1, the linearization of the Christoffel symbols:

$$(\Gamma^k_{ij})' = \frac{1}{2} g^{kl} \Big(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \Big). \tag{18.2}$$

Also recall the formula (3.28) for the curvature tensor in terms of the Christoffel symbols:

$$R_{ijk}{}^{l} = \partial_i (\Gamma^l_{jk}) - \partial_j (\Gamma^l_{ik}) + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}. \tag{18.3}$$

We compute in normal coordinates at a point x,

$$\begin{aligned} (R')_{ijk}{}^{l} &= \nabla_i (\Gamma^l_{jk})' - \nabla_j (\Gamma^l_{ik})' \\ &= \frac{1}{2} g^{lm} \nabla_i \Big(\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk} \Big) - \frac{1}{2} g^{lm} \nabla_j \Big(\nabla_i h_{km} + \nabla_k h_{im} - \nabla_m h_{ik} \Big) \\ &= \frac{1}{2} g^{lm} \Big(\nabla_i \nabla_k h_{jm} - \nabla_i \nabla_m h_{jk} - \nabla_j \nabla_k h_{im} + \nabla_j \nabla_m h_{ik} \\ &\qquad + \nabla_i \nabla_j h_{km} - \nabla_j \nabla_i h_{km} \Big) \\ &= \frac{1}{2} g^{lm} \Big(\nabla_i \nabla_k h_{jm} - \nabla_i \nabla_m h_{jk} - \nabla_j \nabla_k h_{im} + \nabla_j \nabla_m h_{ik} \\ &\qquad - R_{ijk}{}^{p} h_{pm} - R_{ijm}{}^{p} h_{kp} \Big) \end{aligned} \tag{18.4}$$

□

Proposition 18.2. *For any Riemannian metric, we have*

$$\Delta R_{ijk}{}^{l} = g^{lm} \Big(\nabla_i \nabla_m R_{kj} - \nabla_i \nabla_k R_{mj} - \nabla_j \nabla_m R_{ki} + \nabla_j \nabla_k R_{mi} \Big) - R_i^r R_{jrk}{}^{l} - R_j^r R_{rik}{}^{l} \\ + g^{pq} \Big(R_{ijp}{}^{r} R_{qrk}{}^{l} + R_{pik}{}^{r} R_{jqr}{}^{l} - R_{pir}{}^{l} R_{jqk}{}^{r} + R_{pjk}{}^{r} R_{qir}{}^{l} - R_{pjr}{}^{l} R_{qik}{}^{r} \Big). \tag{18.5}$$

Proof. We compute

$$\begin{aligned} \Delta R_{ijk}{}^{l} &= g^{pq} \nabla_p \nabla_q R_{ijk}{}^{l} \\ &= g^{pq} \nabla_p (- \nabla_i R_{jqk}{}^{l} - \nabla_j R_{qik}{}^{l}) \qquad \text{(Differential Bianchi)} \\ &= g^{pq} \Big(- \nabla_i \nabla_p R_{jqk}{}^{l} + R_{pij}{}^{r} R_{rqk}{}^{l} + R_{piq}{}^{r} R_{jrk}{}^{l} + R_{pik}{}^{r} R_{jqr}{}^{l} - R_{pir}{}^{l} R_{jqk}{}^{r} \\ &\qquad - \nabla_j \nabla_p R_{qik}{}^{l} + R_{pjq}{}^{r} R_{rik}{}^{l} + R_{pji}{}^{r} R_{qrk}{}^{l} + R_{pjk}{}^{r} R_{qir}{}^{l} - R_{pjr}{}^{l} R_{qik}{}^{r} \Big). \end{aligned} \tag{18.6}$$

We simplify the covariant derivative terms using again the Differential Bianchi identity

$$
\begin{aligned}
g^{pq}\Big(-\nabla_i\nabla_p R_{jqk}{}^l - \nabla_j\nabla_p R_{qik}{}^l\Big) &= -g^{pq}\Big(\nabla_i\nabla_p(g^{lm}R_{jqmk}) + \nabla_j\nabla_p(g^{lm}R_{qimk})\Big)\\
&= -g^{pq}g^{lm}\Big(\nabla_i\nabla_p R_{jqmk} + \nabla_j\nabla_p R_{qimk}\Big)\\
&= -g^{pq}g^{lm}\Big(\nabla_i\nabla_p R_{mkjq} + \nabla_j\nabla_p R_{mkqi}\Big)\\
&= g^{pq}g^{lm}\Big(\nabla_i(\nabla_m R_{kpjq} + \nabla_k R_{pmjq}) + \nabla_j(\nabla_m R_{kpqi} + \nabla_k R_{pmqi})\Big)\\
&= g^{lm}\Big(\nabla_i(\nabla_m R_{kj} - \nabla_k R_{mj}) + \nabla_j(-\nabla_m R_{ki} + \nabla_k R_{mi})\Big)\\
&= g^{lm}\Big(\nabla_i\nabla_m R_{kj} - \nabla_i\nabla_k R_{mj} - \nabla_j\nabla_m R_{ki} + \nabla_j\nabla_k R_{mi}\Big).
\end{aligned}
\tag{18.7}
$$

Substituting into the above, and noticing that two of the quadratic curvature terms simplify to have a Ricci, we have

$$
\begin{aligned}
\Delta R_{ijk}{}^l = g^{lm}&\Big(\nabla_i\nabla_m R_{kj} - \nabla_i\nabla_k R_{mj} - \nabla_j\nabla_m R_{ki} + \nabla_j\nabla_k R_{mi}\Big)\\
&+ g^{pq}\Big(R_{pij}{}^r R_{rqk}{}^l + R_{pik}{}^r R_{jqr}{}^l - R_{pir}{}^l R_{jqk}{}^r\\
&\qquad + R_{pji}{}^r R_{qrk}{}^l + R_{pjk}{}^r R_{qir}{}^l - R_{pjr}{}^l R_{qik}{}^r\Big) - R_i^r R_{jrk}{}^l - R_j^r R_{rik}{}^l.
\end{aligned}
\tag{18.8}
$$

Using the first Bianchi identity, we combine two quadratic terms

$$
\begin{aligned}
R_{pij}{}^r R_{rqk}{}^l + R_{pji}{}^r R_{qrk}{}^l &= (-R_{pij}{}^r + R_{pji}{}^r)R_{qrk}{}^l\\
&= (R_{ijp}{}^r + R_{jpi}{}^r + R_{pji}{}^r)R_{qrk}{}^l\\
&= R_{ijp}{}^r R_{qrk}{}^l.
\end{aligned}
\tag{18.9}
$$

Using this, we are done. □

Proposition 18.3. *Under the Ricci flow, $g' = -2Ric$, the evolution of the $(1,3)$ curvature tensor is given by*

$$
\begin{aligned}
\frac{\partial}{\partial t}R_{ijk}{}^l = \Delta R_{ijk}{}^l &+ R_i^p R_{jpk}{}^l + R_j^p R_{pik}{}^l + R_p^l R_{ijk}{}^p - R_k^p R_{ijp}{}^l\\
&- g^{pq}\Big(R_{ijp}{}^r R_{qrk}{}^l + R_{pik}{}^r R_{jqr}{}^l - R_{pir}{}^l R_{jqk}{}^r + R_{pjk}{}^r R_{qir}{}^l - R_{pjr}{}^l R_{qik}{}^r\Big).
\end{aligned}
\tag{18.10}
$$

Proof. Using Proposition 18.1 with $h = -2Ric$, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}R_{ijk}{}^l = g^{lm}\Big(&-\nabla_i\nabla_k R_{jm} + \nabla_i\nabla_m R_{jk} + \nabla_j\nabla_k R_{im} - \nabla_j\nabla_m R_{ik}\\
&+ R_{ijk}{}^p R_{pm} + R_{ijm}{}^p R_{kp}\Big).
\end{aligned}
\tag{18.11}
$$

Substituting the formula from Proposition 18.2,

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}{}^{l} &= \Delta R_{ijk}{}^{l} + R_i^r R_{jrk}{}^{l} + R_j^r R_{rik}{}^{l}\\ &\quad - g^{pq}\Big(R_{ijp}{}^{r}R_{qrk}{}^{l} + R_{pik}{}^{r}R_{jqr}{}^{l} - R_{pir}{}^{l}R_{jqk}{}^{r} + R_{pjk}{}^{r}R_{qir}{}^{l} - R_{pjr}{}^{l}R_{qik}{}^{r}\Big)\\ &\qquad + g^{lm}\Big(R_{ijk}{}^{p}R_{pm} + R_{ijm}{}^{p}R_{kp}\Big)\\ &= \Delta R_{ijk}{}^{l} + R_i^p R_{jpk}{}^{l} + R_j^p R_{pik}{}^{l} + R_p^l R_{ijk}{}^{p} - R_k^p R_{ijp}{}^{l}\\ &\quad - g^{pq}\Big(R_{ijp}{}^{r}R_{qrk}{}^{l} + R_{pik}{}^{r}R_{jqr}{}^{l} - R_{pir}{}^{l}R_{jqk}{}^{r} + R_{pjk}{}^{r}R_{qir}{}^{l} - R_{pjr}{}^{l}R_{qik}{}^{r}\Big)\end{aligned} \tag{18.12}$$

□

Proposition 18.4. *Under the Ricci flow, $g' = -2Ric$, the evolution of the $(0,4)$ curvature tensor is given by*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + g^{pq}(R_{iq}R_{jpkl} + R_{jq}R_{pikl} - R_{kp}R_{ijql} - R_{ql}R_{ijkp})\\ &\quad - g^{pq}\Big(R_{ijp}{}^{r}R_{qrkl} + 2R_{pikr}R_{qjl}{}^{r} + 2R_{pil}{}^{r}R_{jqkr}\Big).\end{aligned} \tag{18.13}$$

Proof. We compute

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijmk} &= \frac{\partial}{\partial t}(g_{ml}R_{ijk}{}^{l}) = -2R_{ml}R_{ijk}{}^{l} + g_{ml}\frac{\partial}{\partial t}R_{ijk}{}^{l}\\ &= -2R_{mp}R_{ijk}{}^{p} + \Delta R_{ijmk} + R_i^p R_{jpmk} + R_j^p R_{pimk} + R_{mp}R_{ijk}{}^{p} - R_k^p R_{ijmp}\\ &\quad - g^{pq}\Big(R_{ijp}{}^{r}R_{qrmk} + R_{pik}{}^{r}R_{jqmr} - R_{pimr}R_{jqk}{}^{r} + R_{pjk}{}^{r}R_{qimr} - R_{pjmr}R_{qik}{}^{r}\Big).\end{aligned} \tag{18.14}$$

Notice that

$$-R_{pimr}R_{jqk}{}^{r} + R_{pjk}{}^{r}R_{qimr} = R_{pimr}R_{qjk}{}^{r} + R_{pjk}{}^{r}R_{qimr} \tag{18.15}$$

is symmetric in p and q. Since g^{pq} is also symmetric, we can write

$$g^{pq}\Big(-R_{pimr}R_{jqk}{}^{r} + R_{pjk}{}^{r}R_{qimr}\Big) = 2g^{pq}\Big(R_{pimr}R_{qjk}{}^{r}\Big). \tag{18.16}$$

Similarly,

$$g^{pq}\Big(R_{pik}{}^{r}R_{jqmr} - R_{pjmr}R_{qik}{}^{r}\Big) = 2g^{pq}\Big(R_{pik}{}^{r}R_{jqmr}\Big). \tag{18.17}$$

Collecting all terms, and renaming indices, we are done. □

To simplify this further, Hamilton defines the quadratic curvature quantity

$$B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl} = g^{pr}g^{qs}R_{qjpi}R_{rksl} = R_{qji}{}^{r}R_{rkl}{}^{q}, \tag{18.18}$$

Proposition 18.5. *The tensor B_{ijkl} has the symmetries*

$$B_{ijkl} = B_{jilk} = B_{klij}. \tag{18.19}$$

Proof. We compute

$$\begin{aligned} B_{ijkl} &= g^{pr} g^{qs} R_{piqj} R_{rksl} \\ &= g^{pr} g^{qs} R_{qjpi} R_{slrk} \\ &= g^{qr} g^{ps} R_{pjqi} R_{slrk} \\ &= g^{qs} g^{pr} R_{pjqi} R_{rlsk} = B_{jilk}, \end{aligned} \tag{18.20}$$

and

$$\begin{aligned} B_{klij} &= {R_{qlk}}^{r} {R_{rij}}^{q} \\ &= {R_{rij}}^{q} {R_{qlk}}^{r} \\ &= {R_{qij}}^{r} {R_{rlk}}^{q} \\ &= B_{jilk} = B_{ijkl}. \end{aligned} \tag{18.21}$$

□

19 Lecture 18

19.1 Evolution of curvature tensor

Proposition 19.1. *Under the Ricci flow, $g' = -2Ric$, the evolution of the $(0,4)$ curvature tensor is given by*

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} &+ g^{pq}(R_{iq}R_{jpkl} + R_{jq}R_{pikl} - R_{kp}R_{ijql} - R_{ql}R_{ijkp}) \\ &+ 2\Big(B_{ijkl} - B_{ijlk} + B_{ljki} - B_{likj}\Big). \end{aligned} \tag{19.1}$$

Proof. We need only express the last 3 quadratic curvature terms in Proposition 18.4 in terms of the tensor B. We compute

$$\begin{aligned} &g^{pq}\Big({R_{ijp}}^{r} R_{qrkl} + 2R_{pikr}{R_{qjl}}^{r} + 2{R_{pil}}^{r} R_{jqkr}\Big) \\ &\quad = g^{pq} g^{rm} R_{ijmp} R_{qrkl} + 2g^{pq} R_{krpi} {R_{qjl}}^{r} + 2g^{pq} R_{krjq} {R_{pil}}^{r} \\ &\quad = g^{pq} g^{rm} R_{ijmp} R_{qrkl} + 2{R_{kri}}^{q} {R_{qjl}}^{r} - 2{R_{krj}}^{p} {R_{pil}}^{r} \\ &\quad = g^{pq} g^{rm} R_{ijmp} R_{qrkl} - 2{R_{rki}}^{q} {R_{qjl}}^{r} + 2{R_{rkj}}^{p} {R_{pil}}^{r} \\ &\quad = g^{pq} g^{rm} R_{ijmp} R_{qrkl} - 2B_{ljki} + 2B_{likj}. \end{aligned} \tag{19.2}$$

Apply the algebraic Bianchi identity twice to the first term

$$
\begin{aligned}
g^{pq}g^{rm}R_{ijmp}R_{qrkl} &= g^{pq}g^{rm}R_{mpij}R_{qrkl} = -g^{pq}g^{rm}R_{pmij}R_{qrkl} \\
&= -g^{pq}g^{rm}(R_{pijm} + R_{pjmi})(R_{qklr} + R_{qlrk}) \\
&= -g^{pq}g^{rm}(R_{pijm}R_{qklr} + R_{pijm}R_{qlrk} + R_{pjmi}R_{qklr} + R_{pjmi}R_{qlrk}) \\
&= -g^{pq}g^{rm}(-R_{jmpi}R_{qkrl} + R_{jmpi}R_{qlrk} - R_{mipj}R_{qkrl} + R_{mipj}R_{qlrk}) \\
&= R_{jmi}{}^{q}R_{qkl}{}^{m} - R_{jmi}{}^{q}R_{qlk}{}^{m} + R_{mij}{}^{q}R_{qkl}{}^{m} - R_{mij}{}^{q}R_{qlk}{}^{m} \\
&= -R_{rji}{}^{q}R_{qkl}{}^{r} + R_{rji}{}^{q}R_{qlk}{}^{r} + R_{rij}{}^{q}R_{qkl}{}^{r} - R_{rij}{}^{q}R_{qlk}{}^{r} \\
&= -B_{ijkl} + B_{ijlk} + B_{jikl} - B_{jilk} \\
&= -2B_{ijkl} + 2B_{ijlk},
\end{aligned}
\tag{19.3}
$$

using the symmetry (18.19). Collecting all the terms, we are done. □

Remark. Notice that B_{ijkl} does not have the symmetries of a curvature tensor, but the expression

$$
B_{ijkl} - B_{ijlk} + B_{ljki} - B_{likj} \tag{19.4}
$$

is an algebraic curvature tensor.

19.2 Uhlenbeck's method

Given a solution to the Ricci flow $g(t)$ on $[0, T)$, Let $\{e_a^0\}, a = 1 \dots n$, be a locally defined orthonormal frame field for the metric $g(0)$. Evolve the frame by the equation

$$
\frac{d}{dt}e_a(x,t) = Rc(e_a(x,t)), \quad e_a(x,0) = e_a^0(x). \tag{19.5}
$$

This is a linear ODE, so the solution also exists on $[0, T)$.

Proposition 19.2. *The frame $\{e_a(t)\}$ is an orthonormal frame field for the metric $g(t)$.*

Proof. We compute

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(g(e_a, e_b)\Big) &= \Big(\frac{\partial}{\partial t}g\Big)(e_a, e_b) + g\Big(\frac{\partial}{\partial t}e_a, e_b\Big) + g\Big(e_a, \frac{\partial}{\partial t}e_b\Big) \\
&= -2Ric(e_a, e_b) + g(Rc(e_a), e_b)) + g(e_a, Rc(e_b)) = 0.
\end{aligned}
\tag{19.6}
$$

□

In general, a manifold does not posess an globally defined frame field, so we do the following. Let V be a vector bundle over M which is bundle isomorphic to TM, and let $\iota : V \to TM$ be a fixed bundle isomorphism. Endow V the pull-back metric $h_0 = \iota^*(g_0)$. Evolve ι as follows

$$
\frac{d}{dt}\iota = Rc(\iota), \quad \iota(0) = \iota_0. \tag{19.7}
$$

The analogue of the above proposition is

Proposition 19.3. *Let $h(t) = (\iota(t))^* g(t)$. Then $h(t)$ is constant in time, so the bundle maps*

$$\iota(t) : (V, h(0)) \to (TM, g(t)) \tag{19.8}$$

are bundle isometries of $g(t)$ with the fixed metric $h(0) = \iota_0^* g(0)$.

Proof. Let $x \in M$, X, Y vectors in the fiber V_x, then

$$\begin{aligned} \frac{\partial}{\partial t} h(X,Y) &= \frac{\partial}{\partial t}\Big((\iota^* g)(X,Y)\Big) \\ &= \frac{\partial}{\partial t}\Big(g(\iota X, \iota Y)\Big) \\ &= -2Rc(\iota X, \iota Y) + g(Rc(\iota X), Y) + g(X, Rc(\iota Y)) = 0, \end{aligned} \tag{19.9}$$

which says h is independent of time, so $h = h(0) = \iota_0^* g(0)$. □

We next let $D(t) = \iota(t)^* \nabla(t)$ be the pull-backs of the Riemannian connections of $g(t)$ under $\iota(t)$. We pull-back the curvature tensor of $g(t)$: for $X, Y, Z, W \in V_x$,

$$(\iota^* Rm)(X, Y, Z, W) = Rm(\iota_* X, \iota_* Y, \iota_* Z, \iota_* W). \tag{19.10}$$

Let $\{x^k\}, k = 1 \dots n$, denote local coordinates in M, and let $\{e_a\}, a = 1 \dots n,$, be a local basis of sections of V. The components ι_a^k of $\iota(t)$ are defined

$$\iota(e_a) = \sum_{k=1}^{n} \iota_a^k \partial_k. \tag{19.11}$$

The components R_{abcd} of $\iota^* Rm$ are

$$R_{abcd} = (\iota^* Rm)(e_a, e_b, e_c, e_d) = \sum_{i,j,k,l=1}^{n} \iota_a^i \iota_b^j \iota_c^k \iota_d^l R_{ijkl}. \tag{19.12}$$

The connection $D(t)$ induces a connection on any tensor bundle, and thus we get a Laplacian on V where we trace with respect to $h(t) = h(0)$.

Proposition 19.4. *Let $g(t)$ be a solution to Ricci Flow, and $\iota(t)$ defined as in (19.7), then the evolution equation for $\iota^* Rm$ is given by*

$$\frac{\partial}{\partial t} R_{abcd} = \Delta_D R_{abcd} + 2\Big(B_{abcd} - B_{abdc} + B_{dbca} - B_{dacb}\Big), \tag{19.13}$$

where

$$B_{abcd} = h^{pq} h^{rs} R_{apbr} R_{cqds}. \tag{19.14}$$

The evolution equation for the components of ι are given by

$$\frac{\partial}{\partial t}\iota_a^k = R_l^k \iota_a^l. \tag{19.15}$$

Using this, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} R_{abcd} &= \sum_{i,j,k,l=1}^{n} \frac{\partial}{\partial t}\Big(\iota_a^i \iota_b^j \iota_c^k \iota_d^l R_{ijkl}\Big) \\
&= \frac{\partial}{\partial t}\Big(\iota_a^i\Big)\iota_b^j \iota_c^k \iota_d^l R_{ijkl} + \iota_a^i\Big(\frac{\partial}{\partial t}\iota_b^j\Big)\iota_c^k \iota_d^l R_{ijkl} + \iota_a^i \iota_b^j\Big(\frac{\partial}{\partial t}\iota_c^k\Big)\iota_d^l R_{ijkl} + \iota_a^i \iota_b^j \iota_c^k\Big(\frac{\partial}{\partial t}\iota_d^l\Big) R_{ijkl} \\
&\quad + \iota_a^i \iota_b^j \iota_c^k \iota_d^l \Big[\Delta R_{ijkl} + g^{pq}(R_{iq}R_{jpkl} + R_{jq}R_{pikl} - R_{kp}R_{ijql} - R_{ql}R_{ijkp}) \\
&\qquad + 2\Big(B_{ijkl} - B_{ijlk} + B_{ljki} - B_{likj}\Big)\Big].
\end{aligned} \tag{19.16}$$

It turns out that all of the Ricci terms cancel, and we obtain

$$\frac{\partial}{\partial t} R_{abcd} = \iota_a^i \iota_b^j \iota_c^k \iota_d^l \Big[\Delta R_{ijkl} + 2\Big(B_{ijkl} - B_{ijlk} + B_{ljki} - B_{likj}\Big)\Big]. \tag{19.17}$$

Since the maps ι are parallel, it follows that

$$\big(\iota^*(\Delta Rm)\big)_{abcd} = \Delta_D R_{abcd}. \tag{19.18}$$

Also, $B_{abcd} = (\iota^* B)(e_a, e_b, e_c, e_d)$, so we are done.

19.3 Square of curvature operator

Recall the curvature operator

$$Rm : \Lambda^2 \to \Lambda^2, \tag{19.19}$$

defined in an ONB by

$$\big(Rm(\omega)\big)_{ij} = R_{ijkl}\omega_{kl}. \tag{19.20}$$

The square of the curvature operator is given by

$$Rm^2 : \Lambda^2 \to \Lambda^2, \tag{19.21}$$

which in components is

$$\big(Rm^2(\omega)\big)_{ij} = R_{ijpq} Rm(\omega)_{pq} = R_{ijpq}R_{pqkl}\omega_{kl}. \tag{19.22}$$

In components, we have

$$(Rm^2)_{ijkl} = g^{pq} g^{rs} R_{ijpr} R_{qskl}. \tag{19.23}$$

Proposition 19.5. *The square of the curvature operator is*

$$(Rm^2)_{ijkl} = 2(B_{ijkl} - B_{ijlk}). \tag{19.24}$$

Proof. This was proved above in 19.3. □

This shows we can rewrite the curvature evolution using Uhlenbeck's trick, as

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + (Rm^2)_{abcd} + 2\Big(B_{dbca} - B_{dacb}\Big). \tag{19.25}$$

Next time we will relate the last two terms with an operation called the Lie algebra square.

20 Lecture 19

20.1 Lie algebra square

Let $\mathfrak{g}$ be any Lie algebra, and let ϕ^α be a basis of $\mathfrak{g}$. The structure constants of $\mathfrak{g}$ are defined as

$$[\phi^\alpha, \phi^\beta] = \sum_\gamma C_\gamma^{\alpha\beta} \phi^\gamma. \tag{20.1}$$

If we let ϕ^*_α denote the dual basis, and symmetric bilinear form L on $\mathfrak{g}^*$ can be viewed as an element of $S^2(\mathfrak{g})$ with components given by

$$L_{\alpha\beta} = L(\phi^*_\alpha, \phi^*_\beta). \tag{20.2}$$

The Lie algebra square of L, is $L^\# \in S^2(\mathfrak{g})$ is defined as

$$L^\#_{\alpha\beta} = C_\alpha^{\gamma\delta} C_\beta^{\epsilon\zeta} L_{\gamma\epsilon} L_{\delta\zeta}. \tag{20.3}$$

This operation is well-defined, i.e., it is independent of the basis chosen for $\mathfrak{g}$.

From (5.13) above, we know that Λ^2 is isomorphic to the Lie algebra $\mathfrak{so}(n)$, thus we can view the curvature operator as $Rm \in S^2(\mathfrak{so}(n))$.

Theorem 20.1. *Let $g(t)$ be a solution to Ricci Flow, and $\iota(t)$ defined as in (19.7), then the evolution equation for $\iota^* Rm$ is given by*

$$\frac{\partial}{\partial t} R_{abcd} = \Delta_D R_{abcd} + Rm^2 + Rm^\#. \tag{20.4}$$

Proof. From (19.25) we just need to show that

$$Rm^\#_{ijkl} = 2\Big(B_{ljki} - B_{likj}\Big). \tag{20.5}$$

This is a straightforward but tedious computation, we just give an outline. First, one writes down the explicit formula for $Rm^\#$, which is

$$(Rm^\#)_{ijkl} = R_{pquv} R_{rswx} C_{(ij)}^{(pq),(rs)} C_{(kl)}^{(uv),(wx)}, \tag{20.6}$$

where the structures constant are written with 2-form indices. Next, one explicitly calculates the structure constants for $\mathfrak{so}(n)$,

$$
\begin{aligned}
C^{(pq),(rs)}_{(ij)} &= \frac{1}{4}\Big(g^{qr}(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p) + g^{qs}(\delta_i^r\delta_j^p - \delta_i^p\delta_j^r) \\
&\qquad + g^{pr}(\delta_i^s\delta_j^q - \delta_i^q\delta_j^s) + g^{ps}(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q)\Big).
\end{aligned} \tag{20.7}
$$

We then obtain

$$
\begin{aligned}
(Rm^{\#})_{ijkl} &= R_{pquv}R_{rswx}g^{qr}(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p)g^{vw}(\delta_k^u\delta_l^x - \delta_k^x\delta_l^u) \\
&= R_{uvp}{}^{q}R_{sqx}{}^{v}(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p)(\delta_k^u\delta_l^x - \delta_k^x\delta_l^u) \\
&= R_{kvi}{}^{q}R_{jql}{}^{v} - R_{lvi}{}^{q}R_{jqk}{}^{v} - R_{kvj}{}^{q}R_{iql}{}^{v} + R_{lvj}{}^{q}R_{iqk}{}^{v}.
\end{aligned} \tag{20.8}
$$

Using the definition of $B_{ijkl} = R_{rji}{}^{q}R_{qkl}{}^{r}$, we have

$$
(Rm^{\#})_{ijkl} = B_{ikjl} - B_{iljk} - B_{jkil} + B_{jlik} = 2(B_{ljki} - B_{likj}). \tag{20.9}
$$

The proof is then finished as before using Uhlenbeck's trick. □

Corollary 20.1. *Let $g(t)$ be a solution to the Ricci flow on M^n on $[0,T)$. If $g(0)$ has positive (non-negative) curvature operator, then $g(t)$ also has positive (non-negative) curvature operator for all $t \in (0,T)$. If $g(0)$ has non-negative curvature operator and the curvature operator is strictly positive at some point $x \in M$, then then curvature operator is strictly positive for $t \in (0,T)$.*

Proof. First we show that if the curvature operator is non-negative, then $Rm^{\#}$ is also non-negative. This is a general property of the Lie algebra square operation. To see this, choose a basis for which L is diagonal, $L_{\alpha\beta} = \delta_{\alpha\beta}L_{\alpha\alpha}$ (no sum on α), and

$$
L^{\#}(v,v) = (v^{\alpha}C_{\alpha}^{\gamma\delta})(v^{\beta}C_{\beta}^{\epsilon\zeta})L_{\gamma\epsilon}L_{\delta\zeta} = (v^{\alpha}C_{\alpha}^{\gamma\delta})^2 L_{\gamma\gamma}L_{\delta\delta}, \tag{20.10}
$$

which clearly shows that $L^{\#}$ is non-negative provided that L is. Clearly Rm^2 is also non-negative provided Rm is. The result then follows from the evolution equation (20.4), and a generalization of the maximum principle, Proposition 17.1, to more general systems of tensors. □

20.2 Dimension 3

In dimension 3, let e_1, e_2, e_3 be an orthonormal frame, and e^1, e^2, e^3 be the dual orthonormal co-frame. We define the orthonormal basis $\omega^1 = e^2 \wedge e^3$, $\omega_2 = -e^1 \wedge e^3$, and $\omega^3 = e^1 \wedge e^2$ for Λ^2. With the identification of Λ^2 with $\mathfrak{so}(3)$, we have

$$
\omega^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \omega^1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \omega^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20.11}
$$

The structure constants are given by (20.7), and a computation shows that in this basis, we have

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}^{\#} = \begin{pmatrix} df - e^2 & ce - bf & be - cd \\ ce - bf & af - c^2 & bc - ae \\ be - cd & bc - ae & ad - b^2 \end{pmatrix} \tag{20.12}$$

Consider the associated ODE

$$\frac{d}{dt}\mathbb{M} = \mathbb{M}^2 + \mathbb{M}^{\#}. \tag{20.13}$$

As we saw in the maximum principle for nonlinear equations, Proposition 15.3, solutions of Ricci flow can be compared to solution of this ODE. If we choose a basis which diagonalizes the curvature operator, and label the eigenvalues

$$\lambda(0) \geq \mu(0) \geq \nu(0), \tag{20.14}$$

Then the ODE system becomes

$$\frac{d}{dt}\begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix} = \begin{pmatrix} \lambda^2 & & \\ & \mu^2 & \\ & & \nu^2 \end{pmatrix} + \begin{pmatrix} \mu\nu & & \\ & \lambda\nu & \\ & & \lambda\mu \end{pmatrix}. \tag{20.15}$$

Simple algebra shows that

$$\frac{d}{dt}(\lambda - \mu) = (\lambda - \mu)(\lambda + \mu - \nu) \tag{20.16}$$

$$\frac{d}{dt}(\mu - \nu) = (\mu - \nu)(-\lambda + \mu + \nu). \tag{20.17}$$

So the difference of the eigenvalues has a nice evolution equation. Hamilton proves that if the initial metric has positive Ricci tensor, then the Ricci eigenvalues become more and more pinched as $t \to T_{max}$, and in fact the metric, after rescaling, converges to a constant sectional curvature metric. The main tool is the maximum principle, and comparison with ODE solutions.

Theorem 20.2 (Hamilton [Ham82])**.** *If (M^3, g) is a compact three-manifold with positive Ricci tensor, then the normalized Ricci flow convergences exponentially fast to a constant positive curvature metric as $t \to \infty$.*

The remaining details can be found in [CK04, Chapter 6].

21 Lecture 20

21.1 Conformal geometry

Let $u : M \to \mathbb{R}$. Then $\tilde{g} = e^{-2u} g$, is said to be *conformal* to g.

Proposition 21.1. *The Christoffel symbols transform as*

$$\tilde{\Gamma}^i_{jk} = g^{il}\Big(- (\partial_j u) g_{lk} - (\partial_k u) g_{lj} + (\partial_l u) g_{jk}\Big) + \Gamma^i_{jk}. \tag{21.1}$$

Invariantly,

$$\tilde{\nabla}_X Y = \nabla_X Y - du(X) Y - du(Y) X + g(X,Y)\nabla u. \tag{21.2}$$

Proof. Using (2.31), we compute

$$\begin{aligned}
\tilde{\Gamma}^i_{jk} &= \frac{1}{2}\tilde{g}^{il}\Big(\partial_j \tilde{g}_{kl} + \partial_k \tilde{g}_{jl} - \partial_l \tilde{g}_{jk}\Big)\\
&= \frac{1}{2} e^{2u} g^{il}\Big(\partial_j (e^{-2u} g_{kl}) + \partial_k (e^{-2u} g_{jl}) - \partial_l (e^{-2u} g_{jk})\Big)\\
&= \frac{1}{2} e^{2u} g^{il}\Big(- 2e^{-2u}(\partial_j u) g_{kl} - 2e^{-2u}(\partial_k u) e^{-2u} g_{jl} + 2e^{-2u}(\partial_l u) g_{jk}\\
&\qquad + e^{-2u}\partial_j (g_{kl}) + e^{-2u}\partial_k (g_{jl}) - e^{-2u}\partial_l (g_{jk})\Big)\\
&= g^{il}\Big(- (\partial_j u) g_{kl} - (\partial_k u) g_{jl} + (\partial_l u) g_{jk}\Big) + \Gamma^i_{jk}.
\end{aligned} \tag{21.3}$$

This is easily seen to be equivalent to the invariant expression. □

Proposition 21.2. *The* $(0,4)$*-curvature tensor transforms as*

$$\tilde{Rm} = e^{-2u}\Big[Rm + \Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big) \owedge g\Big]. \tag{21.4}$$

Proof. Recall the formula (3.28) for the $(1,3)$ curvature tensor

$$\tilde{R}_{ijk}{}^{l} = \partial_i(\tilde{\Gamma}^l_{jk}) - \partial_j(\tilde{\Gamma}^l_{ik}) + \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik}. \tag{21.5}$$

Take a normal coordinate system for the metric g at a point $x \in M$. All computations below will be evaluated at x. Let us first consider the terms with derivatives of Christoffel symbols, we have

$$\begin{aligned}
\partial_i(\tilde{\Gamma}^l_{jk}) - \partial_j(\tilde{\Gamma}^l_{ik}) &= \partial_i\Big[g^{lp}\Big(- (\partial_j u) g_{pk} - (\partial_k u) g_{pj} + (\partial_p u) g_{jk}\Big) + \Gamma^l_{jk}\Big]\\
&- \partial_j\Big[g^{lp}\Big(- (\partial_i u) g_{kp} - (\partial_k u) g_{ip} + (\partial_p u) g_{ik}\Big) + \Gamma^l_{ik}\Big]\\
&= g^{lp}\Big(- (\partial_i\partial_j u) g_{pk} - (\partial_i\partial_k u) g_{pj} + (\partial_i\partial_p u) g_{jk}\Big) + \partial_i(\Gamma^l_{jk})\\
&- g^{lp}\Big(- (\partial_j\partial_i u) g_{kp} - (\partial_j\partial_k u) g_{ip} + (\partial_j\partial_p u) g_{ik}\Big) - \partial_j(\Gamma^l_{ik})\\
&= g^{lp}\Big(- (\partial_i\partial_k u) g_{pj} + (\partial_i\partial_p u) g_{jk} + (\partial_j\partial_k u) g_{ip} - (\partial_j\partial_p u) g_{ik}\Big) + R_{ijk}{}^{l}.
\end{aligned} \tag{21.6}$$

A simple computation shows this is

$$\partial_i(\tilde{\Gamma}^l_{jk}) - \partial_j(\tilde{\Gamma}^l_{ik}) = g^{lp}(\nabla^2 u \owedge g)_{ijpk} + R_{ijk}{}^l. \tag{21.7}$$

Next, we consider the terms that are quadratic Christoffel terms.

$$\begin{aligned}\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik} = g^{lp}\Big(&- (\partial_i u)g_{mp} - (\partial_m u)g_{ip} + (\partial_p u)g_{im}\Big)\\ \times g^{mr}\Big(&- (\partial_j u)g_{kr} - (\partial_k u)g_{jr} + (\partial_r u)g_{jk}\Big)\\ -g^{lp}\Big(&- (\partial_j u)g_{mp} - (\partial_m u)g_{jp} + (\partial_p u)g_{jm}\Big)\\ \times g^{mr}\Big(&- (\partial_i u)g_{kr} - (\partial_k u)g_{ir} + (\partial_r u)g_{ik}\Big).\end{aligned} \tag{21.8}$$

Terms in the first product which are symmetric in i and j will cancel with the corresponding terms of the second product, so this simplifies to

$$\begin{aligned}&\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik}\\ &= g^{lp}g^{mr}\Big((\partial_i u)g_{mp}(\partial_j u)g_{kr} + (\partial_m u)g_{ip}(\partial_k u)g_{jr} + (\partial_p u)g_{im}(\partial_r u)g_{jk}\\ &\quad +(\partial_i u)g_{mp}(\partial_k u)g_{jr} - (\partial_i u)g_{mp}(\partial_r u)g_{jk} + (\partial_m u)g_{ip}(\partial_j u)g_{kr}\\ &\quad -(\partial_m u)g_{ip}(\partial_r u)g_{jk} - (\partial_p u)g_{im}(\partial_j u)g_{kr} - (\partial_p u)g_{im}(\partial_k u)g_{jr}\\ &\quad - \text{same 9 terms with i and j exchanged}\Big)\\ &= g^{lp}\Big((\partial_i u)(\partial_j u)g_{kp} + (\partial_j u)(\partial_k u)g_{ip} + (\partial_p u)(\partial_i u)g_{jk}\\ &\quad +(\partial_i u)(\partial_k u)g_{jp} - (\partial_i u)(\partial_p u)g_{jk} + (\partial_k u)(\partial_j u)g_{ip}\\ &\quad -g^{mr}(\partial_m u)(\partial_r u)g_{ip}g_{jk} - (\partial_p u)(\partial_j u)g_{ik} - (\partial_p u)(\partial_k u)g_{ij}\\ &\quad - \text{same 9 terms with i and j exchanged}\Big)\end{aligned} \tag{21.9}$$

The first and ninth terms are symmetric in i and j. The fourth and sixth terms, taken together, are symmetric in i and j. The third and fifth terms cancel, so we have

$$\begin{aligned}\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik} = g^{lp}\Big(&(\partial_j u)(\partial_k u)g_{ip} - (\partial_p u)(\partial_j u)g_{ik} - |\nabla u|^2 g_{ip}g_{jk}\\ &- \text{same 3 terms with i and j exchanged}\Big).\end{aligned} \tag{21.10}$$

Writing out the last term, we have

$$\begin{aligned}\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik} = g^{lp}\Big(&(\partial_j u)(\partial_k u)g_{ip} - (\partial_i u)(\partial_k u)g_{jp} - (\partial_p u)(\partial_j u)g_{ik} + (\partial_p u)(\partial_i u)g_{jk}\\ &- |\nabla u|^2 g_{ip}g_{jk} + |\nabla u|^2 g_{jp}g_{ik}\Big).\end{aligned} \tag{21.11}$$

Another simple computation shows this is

$$\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik} = g^{lp}\Big[\Big(du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk}. \tag{21.12}$$

Adding together (21.7) and (21.12), we have

$$\tilde{R}_{ijk}{}^{l} = g^{lp}\Big[\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk} + R_{ijk}{}^{l}. \tag{21.13}$$

We lower the the index on the right using the metric $\tilde{g}_{lp}$, to obtain

$$\tilde{R}_{ijpk} = e^{-2u}\Big[\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk} = e^{-2u}R_{ijpk}, \tag{21.14}$$

and we are done. □

Proposition 21.3. *Let $\tilde{g} = e^{-2u}g$. The $(1,3)$ Weyl tensor is conformally invariant. The $(0,4)$ Weyl tensor transforms as*

$$\tilde{W}_{ijkl} = e^{-2u}W_{ijkl}. \tag{21.15}$$

The Schouten $(0,2)$ tensor transforms as

$$\tilde{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A. \tag{21.16}$$

The Ricci $(0,2)$ tensor transforms as

$$\tilde{Ric} = (n-2)\Big(\nabla^2 u + \frac{1}{n-2}(\Delta u)g + du \otimes du - |\nabla u|^2 g\Big) + Ric. \tag{21.17}$$

The scalar curvature transforms as

$$\tilde{R} = e^{2u}\Big(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R\Big). \tag{21.18}$$

Proof. We expand (21.13) in terms of Weyl,

$$\tilde{W}_{ijk}{}^{l} + (\tilde{A} \owedge \tilde{g})_{ijk}{}^{l} = g^{lp}\Big[\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk} + W_{ijk}{}^{l} + (A \owedge g)_{ijk}{}^{l}. \tag{21.19}$$

Note that

$$\begin{aligned}(\tilde{A} \owedge \tilde{g})_{ijk}{}^{l} &= \tilde{g}^{lp}(\tilde{A} \owedge e^{-2u}g)_{ijpk} \\ &= g^{lp}(\tilde{A} \owedge g)_{ijpk}.\end{aligned} \tag{21.20}$$

We can therefore rewrite (21.19) as

$$\tilde{W}_{ijk}{}^{l} - W_{ijk}{}^{l} = g^{lp}\Big[\Big(-\tilde{A} + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 + A\Big) \owedge g\Big]_{ijpk}. \tag{21.21}$$

In dimension 2 and 3 the right hand side is zero, so the left hand side is also. In any dimension, recall from Section 4.2, that the left hand side is in $Ker(c)$, and the right hand side is in $Im(\psi)$ (with respect to either g or $\tilde{g}$). This implies that both sides must vanish. To see this, assume $R \in Ker(c) \cap Im(\psi)$. Then $R = h \owedge g$, so

$c(R) = (n-2)h + tr(h)g = 0$, which implies that $h = 0$ for $n \neq 2$. This implies conformal invariance of $Weyl$, and also the formula for the conformal transformation of the Schouten tensor. We lower an index of the Weyl,

$$\tilde{W}_{ijkl} = \tilde{g}_{pk}\tilde{W}_{ijl}{}^{p} = e^{-2u} g_{pk} W_{ijl}{}^{p} = e^{-2u} W_{ijkl}, \tag{21.22}$$

which proves (21.15). We have the formula

$$\Big(-\tilde{A} + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 + A \Big) \owedge g = 0. \tag{21.23}$$

Recall that $c(A \owedge g) = (n-2)A + tr(A)g = Ric$, so we obtain

$$-\tilde{Ric} + (n-2)(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2) + (\Delta u)g + (1 - \frac{n}{2})|\nabla u|^2 + Ric = 0, \tag{21.24}$$

which implies (21.17). Finally,

$$\begin{aligned} \tilde{R} &= \tilde{g}^{-1}\tilde{Ric} = e^{2u} g^{-1}\tilde{Ric} \\ &= (n-2)e^{2u}\Big(\Delta u + \frac{n}{n-2}\Delta u + (1-n)|\nabla u|^2 + R\Big) \\ &= e^{2u}\Big(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R\Big), \end{aligned} \tag{21.25}$$

which is (21.18). □

By writing the conformal factor differently, the scalar curvature equation takes a nice semilinear form, which is the famous Yamabe equation:

Proposition 21.4. *If $n \neq 2$, and $\tilde{g} = v^{\frac{4}{n-2}} g$, then*

$$-4\frac{n-1}{n-2}\Delta v + Rv = \tilde{R} v^{\frac{n+2}{n-2}}. \tag{21.26}$$

Proof. We have $e^{-2u} = v^{\frac{4}{n-2}}$, which is

$$u = -\frac{2}{n-2}\ln v. \tag{21.27}$$

Using the chain rule,

$$\nabla u = -\frac{2}{n-2}\frac{\nabla v}{v}, \tag{21.28}$$

$$\nabla^2 u = -\frac{2}{n-2}\Big(\frac{\nabla^2 v}{v} - \frac{\nabla v \otimes \nabla v}{v^2}\Big). \tag{21.29}$$

Substituting these into (21.18), we obtain

$$\begin{aligned} \tilde{R} &= v^{\frac{-4}{n-2}}\Big(-4\frac{n-1}{n-2}\Big(\frac{\Delta v}{v} - \frac{|\nabla v|^2}{v^2}\Big) - 4\frac{n-1}{n-2}\frac{|\nabla v|^2}{v^2} + R\Big) \\ &= v^{\frac{-n+2}{n-2}}\Big(-4\frac{n-1}{n-2}\Delta v + Rv\Big). \end{aligned} \tag{21.30}$$

□

Proposition 21.5. *If $n = 2$, and $\tilde{g} = e^{-2u}g$, the conformal Gauss curvature equation is*

$$\Delta u + K = \tilde{K}e^{-2u}. \tag{21.31}$$

Proof. This follows from (21.18), and the fact that in dimension 2, $R = 2K$. □

21.2 Negative scalar curvature

Proposition 21.6. *If (M, g) is compact, and $R < 0$, then there exists conformal metric $\tilde{g} = e^{-2u}g$ with $\tilde{R} = -1$.*

Proof. If $n > 2$, we would like to solve the equation

$$-4\frac{n-1}{n-2}\Delta v + Rv = -v^{\frac{n+2}{n-2}}. \tag{21.32}$$

If $n > 2$, let $p \in M$ be a point where v attains a its global maximum. Then (21.26) evaluated at p becomes

$$R(p)v(p) \leq -(v(p))^{\frac{n+2}{n-2}}. \tag{21.33}$$

Dividing, we obtain

$$(v(p))^{\frac{4}{n-2}} \leq -R(p), \tag{21.34}$$

which gives an *a priori* upper bound on v. Similarly, by evaluating a a global minimum point q, we obtain

$$(v(p))^{\frac{4}{n-2}} \geq -R(q), \tag{21.35}$$

which gives an a priori strictly positive lower bound on v. We have shown there exists a constant C_0 so that $\|v\|_{C^0} < C_0$. The standard elliptic estimate says that there exists a constant C, depending only on the background metric, such that (see [GT01, Chapter 4])

$$\begin{aligned} \|v\|_{C^{1,\alpha}} &\leq C(\|\Delta v\|_{C^0} + \|v\|_{C^0}) \\ &\leq C(\|Rv + v^{\frac{n+2}{n-2}}\|_{C^0} + CC_0 \leq C_1, \end{aligned} \tag{21.36}$$

where C_1 depends only upon the background metric. Applying elliptic estimates again,

$$\|v\|_{C^{3,\alpha}} \leq C(\|\Delta v\|_{C^{1,\alpha}} + \|v\|_{C^{1,\alpha}}) \leq C_3, \tag{21.37}$$

where C_3 depends only upon the background metric.

In terms of u, the equation is

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R = -e^{-2u}. \tag{21.38}$$

Let $t \in [0,1]$, and consider the family of equations

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R = \big((1-t)R - 1\big)e^{-2u}. \tag{21.39}$$

Define an operator $F_t : C^{2,\alpha} \to C^{\alpha}$ by

$$F_t(u) = 2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R - \big((1-t)R - 1\big)e^{-2u}. \tag{21.40}$$

Let $u_t \in C^{2,\alpha}$ satisfy $F_t(u_t) = 0$. The linearized operator at u_t, $L_t : C^{2,\alpha} \to C^{\alpha}$, is given by

$$L_t(h) = 2(n-1)\Delta h - (n-1)(n-2)2\langle \nabla u, \nabla h\rangle + 2\big((1-t)R - 1\big)e^{-2u}h. \tag{21.41}$$

Notice that the coefficient h is strictly negative. The maximum principle and linear theory imply that the linearized operator is invertible. Next, define

$$S = \{t \in [0,1] \mid \text{there exists a solution } u_t \in C^{2,\alpha} \text{ of } F_t(u_t) = 0\}. \tag{21.42}$$

Since the linearized operator is invertible, the implicit function theorem implies that S is open. Assume u_{t_i} is a sequence of solutions with $t_i \to t_0$ as $i \to \infty$. The above elliptic estimates imply there exist a constant C_4, independent of t, such that $\|u_{t_i}\|_{C^{3,\alpha}} < C_4$. By Arzela-Ascoli, there exists $u_{t_0} \in C^{2,\alpha}$ and a subsequence $\{j\} \subset \{i\}$ such that $u_{t_j} \to u_{t_0}$ strongly in $C^{2,\alpha}$. The limit u_{t_0} is a solution at time t_0. This shows that S is closed. Since the interval $[0,1]$ is connected, this implies that $S = [0,1]$, and consequently there must exist a solution at $t = 1$. In the case $n = 2$, the same arugment applied to (21.31) yields a similar a priori estimate, and the proof remains valid. □

For $n = 2$, the Gauss-Bonnet formula says that

$$\int_M K_g dV_g = 2\pi\chi(M). \tag{21.43}$$

So the case of negative Gauss curvature in the above theorem can only occur in the case of genus $g \geq 2$.

22 Lecture 21

22.1 The Yamabe Problem

We just saw that in the strictly negative scalar curvature case, it is easy to conformally deform to constant negative scalar curvature. It turns out that on any compact manifold, one can always deform to a constant scalar curvature metric, without any conditions. For $n = 2$, this is implied by the uniformization theorem (however, this can be proved directly using PDE alone). For $n \geq 3$, the Yamabe equation takes the form

$$-4\frac{n-1}{n-2}\Delta v + R \cdot v = \lambda \cdot v^{\frac{n+2}{n-2}}, \tag{22.1}$$

where λ is a constant. These are the Euler-Lagrange equations of the *Yamabe functional*,

$$\mathcal{Y}(\tilde{g}) = Vol(\tilde{g})^{-\frac{n-2}{n}} \int_M R_{\tilde{g}} dvol_{\tilde{g}}, \tag{22.2}$$

for $\tilde{g} \in [g]$, where $[g]$ denotes the conformal class of g. An important related conformal invariant is the *Yamabe invariant* of the conformal class $[g]$:

$$Y([g]) \equiv \inf_{\tilde{g} \in [g]} \mathcal{Y}(\tilde{g}). \tag{22.3}$$

The Yamabe problem has been completely solved through the results of many mathematicians, over a period of approximately thirty years. Initially, Yamabe claimed to have a proof in [Yam60]. The basic strategy was to prove the existence of a minimizer of the Yamabe functional through a sub-critical regularization technique. Subsequently, an error was found by N. Trudinger, who then gave a solution with a smallness assumption on the Yamabe invariant [Tru68]. Later, Aubin showed that the problem is solvable provided that

$$Y([g]) < Y([g_{round}]), \tag{22.4}$$

where $[g_{round}]$ denotes the conformal class of the round metric on the n-sphere, and verified this inequality for $n \geq 6$ and g not locally conformally flat [Aub76b], [Aub76a], [Aub98]. Schoen solved the remaining cases [Sch84]. It is remarkable that Schoen employed the positive mass conjecture from general relativity to solve these remaining most difficult cases. A great reference for the solution of the Yamabe problem is Lee and Parker [LP87].

22.2 Constant curvature

Let g denote the Euclidean metric on $\mathbb{R}^n, n \geq 3$, and consider conformal metrics $\tilde{g} = e^{-2u} g$.

Proposition 22.1. *If $\tilde{g}$ is Einstein, then there exists constant a, b_i, c, such that*

$$\tilde{g} = \left(a|x|^2 + b_i x^i + c\right)^{-2} g. \tag{22.5}$$

Proof. For the Schouten tensor, we must have

$$\tilde{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g. \tag{22.6}$$

Let us rewrite the conformal factor as $\tilde{g} = v^{-2} g$, that is $u = \ln v$. The equation is then written

$$v^2 \tilde{A} = v \nabla^2 v - \frac{1}{2}|\nabla v|^2 g. \tag{22.7}$$

Let us assume that $\tilde{g}$ is Einstein, which is equivalent to $\tilde{g}$ having constant curvature. In this case, we have

$$\tilde{A} = \frac{tr(A)}{n}\tilde{g} = \frac{R}{2n(n-1)}v^{-2}g, \tag{22.8}$$

so we obtain

$$\frac{K}{2}g = v\nabla^2 v - \frac{1}{2}|\nabla v|^2 g, \tag{22.9}$$

where $R = n(n-1)K$. The off-diagonal equation is

$$v_{ij} = 0, \ i \neq j, \tag{22.10}$$

implies that we may write $v_i = h_i(x_i)$ for some function h_i. The diagonal entries say that

$$\frac{K}{2} = vv_{ii} - \frac{1}{2}|\nabla v|^2. \tag{22.11}$$

Differentiate this in the x^j direction,

$$0 = v_j v_{ii} + vv_{iij} - v_l v_{lj}. \tag{22.12}$$

If $j = i$, then we obtain

$$v_{iii} = 0. \tag{22.13}$$

In terms of h,

$$(h_i)_{ii} = 0. \tag{22.14}$$

This implies that

$$h_i = a_i x_i + b_i, \tag{22.15}$$

for some constants a_i, b_i. If $j \neq i$, then (22.12) is

$$0 = v_j(v_{ii} - v_{jj}). \tag{22.16}$$

This says that $a_i = a_j$ for $i \neq j$. This forces v to be of the form

$$v = a|x|^2 + b_i x^i + c. \tag{22.17}$$

$\square$

From conformal invariance of the Weyl, we know that $\tilde{W} = 0$, so $\tilde{g}$ being Einstein is equivalent to having constant sectional curvature. The sectional curvature of such a metric is

$$\begin{aligned} K &= 2vv_{ii} - |\nabla v|^2 \\ &= 2(a|x|^2 + b_i x^i + c)2a - |2ax_i + b_i|^2 \\ &= 4ac - |b|^2. \end{aligned} \tag{22.18}$$

If $K > 0$, then the discriminant is negative, so there are no real roots, and v is defined on all of $\mathbb{R}^n$. The metric

$$\tilde{g} = \frac{4}{(1+|x|^2)^2} g \tag{22.19}$$

represents the round metric with $K = 1$ on S^n under stereographic projection. If $K < 0$ then the solution is defined on a ball, or the complement of a ball, or a half space. The metric

$$\tilde{g} = \frac{4}{(1-|x|^2)^2} g \tag{22.20}$$

is the usual ball model of hyperbolic space, and

$$\tilde{g} = \frac{1}{x_n^2} g \tag{22.21}$$

is the upper half space model of hyperbolic space. If $K = 0$ and $|b| \neq 0$, the solution is defined on all of $\mathbb{R}^n$.

22.3 Conformal transformations

The case $K = 0$ of this proposition implies the follow theorem of Liouville.

Theorem 22.1 (Liouville)**.** *For $n \geq 3$, then group of conformal transformations of $\mathbb{R}^n$ is generated by rotations, scalings, translations, and inversions.*

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a conformal transformation. Then $T^* g = v^{-2} g$ for some positive function v, which says v is a flat metric which is conformal to the Euclidean metric. By above, we must have $v = a|x|^2 + b_i x^i + c$, with $|b|^2 = 4ac$. If $a = 0$, then $v = c$, so T is a scaling composed with an isometry. If $a \neq 0$, then

$$v = \frac{1}{a} \sum_i (a x_i + \frac{1}{2} b_i)^2. \tag{22.22}$$

From this it follows that T is a scaling and inversion composed with an isometry. □

We note the following fact: the group of conformal transformations of the round S^n is isomorphic to the group of isometric of hyperbolic space H^{n+1}. This is proved by showing that in the ball model of hyperbolic space, isometries of H^{n+1} restrict to conformal automorphisms of the boundary n-sphere. By identifying H^{n+1} with a component of the unit sphere in $\mathbb{R}^{n,1}$, one shows that $Iso(H^n) = O(n,1)$. We have some special isomorphisms in low dimensions. For $n = 1$,

$$\begin{aligned} SO(2,1) &= PSL(2,\mathbb{R}), \\ SO(3,1) &= PSL(2,\mathbb{C}) \\ SO(5,1) &= PSL(2,\mathbb{H}). \end{aligned} \tag{22.23}$$

For the first case,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R}) \tag{22.24}$$

acts upon H^2 in the upper half space model by fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}, \tag{22.25}$$

where z satisfies $Im(z) > 0$. The boundary of H^2 is S^1, which is identified with 1-dimensional real projective space $\mathbb{RP}^1$. The conformal transformations of S^1 are

$$[r_1, r_2] \mapsto [ar_1 + br_2, cr_1 + dr_2]. \tag{22.26}$$

It is left as an exercise to find explicit maps from the groups on the right to the isometries of hyperbolic space, and conformal transformations of the sphere in the other two cases.

22.4 Obata Theorem

The metrics in the previous section with $K = 1$ are none other than the spherical metric. We following characteriztion of the round metric on S^n due to Obata.

Theorem 22.2 (Obata [Oba72]). *Let $\tilde{g}$ be a constant scalar curvature metric on S^n conformal to the standard round metric g. Then $(S^n, \tilde{g})$ is isometric to (S^n, g), plus a scaling.*

Proof. We write $g = v^{-2}\tilde{g}$. The transformation of the Schouten tensor is

$$A_g = v^{-1}\tilde{\nabla}^2 v - \frac{1}{2v^2}|\tilde{\nabla} v|^2 \tilde{g} + A_{\tilde{g}}. \tag{22.27}$$

Let $E = Ric - (R/n)g$ denote the traceless Ricci tensor. In terms of E, we have

$$0 = E_g = (n-2)v^{-1}\Big(\tilde{\nabla}^2 v - \frac{1}{n}(\tilde{\Delta} v)\tilde{g}\Big) + E_{\tilde{g}}. \tag{22.28}$$

We integrate

$$\begin{aligned} \int_{S^n} v|E_{\tilde{g}}|^2 dV_{\tilde{g}} &= -(n-2)\int_{S^n} \Big\langle E_{\tilde{g}}, \tilde{\nabla}^2 v - \frac{1}{n}(\tilde{\Delta} v)\tilde{g}\Big\rangle dV_{\tilde{g}} \\ &= -(n-2)\int_{S^n} \langle E_{\tilde{g}}, \tilde{\nabla}^2 v\rangle dV_{\tilde{g}} \\ &= (n-2)\int_{S^n} \langle \tilde{\delta} E_{\tilde{g}}, \tilde{\nabla} v\rangle dV_{\tilde{g}} = 0, \end{aligned} \tag{22.29}$$

since $\delta \tilde{Rc} = \frac{1}{2}d\tilde{R} = 0$, by assumption the $\tilde{g}$ has constant scalar curvature. We conclude that $\tilde{g}$ is Einstein. We have shown in (22.20) that the round metric on S^n is conformal to the Euclidean metric. From conformal invariance of the Weyl, the round metric therefore has vanishing Weyl tensor. Since $\tilde{g}$ in conformal to g, it also has vanishing Weyl. This plus the Einstein condition implies $\tilde{g}$ has constant sectional curvature. Thus $\tilde{g}$ is isometric to the round metric by [Lee97, Theorem ?], and a scaling fixes the curvature. □

We have a further characterization:

Theorem 22.3 (Obata [Oba72]). *Let g be a an Einstein metric. The g is the unique constant scalar curvature metric in its conformal class, unless (M, g) is conformally equivalent to (S^n, g_{round}), in which case there is a $(n+1)$ parameter family of solutions, all of which is isometric to the round metric, up to scaling.*

Proof. From the preceeding proof, we know the any constant scalar metric conformal to an Einstein metric is also Einstein. From (22.28) we thus have a non-zero solution of the equation

$$\nabla^2 v = \frac{1}{n}(\Delta v)g. \tag{22.30}$$

This implies (M, g) is conformal to the round metric, as proved in [?]. □

22.5 Differential Bianchi for Weyl

Proposition 22.2. *The divergence of the Weyl is given by*

$$\delta W = (n-3)d^{\nabla}A, \tag{22.31}$$

where A is the Schouten tensor. In coordinates,

$$\nabla_l W_{jkm}{}^l = (n-3)(\nabla_j A_{km} - \nabla_k A_{jm}). \tag{22.32}$$

Proof. The divergence of the curvature tensor was given in (4.5)

$$\nabla_l R_{jkm}{}^l = \nabla_j R_{km} - \nabla_k R_{jm}. \tag{22.33}$$

Decomposing the curvature tensor,

$$\nabla_l W_{jkm}{}^l + g^{lp}(A \owedge g)_{jkpm}) = \nabla_j R_{km} - \nabla_k R_{jm}, \tag{22.34}$$

which yields the formula

$$\begin{aligned}\nabla_l W_{jkm}{}^l &= -g^{lp}\nabla_l(A \owedge g)_{jkpm} + \nabla_j R_{km} - \nabla_k R_{jm}\\ &= -g^{lp}\nabla_l\big(A_{jp}g_{km} - A_{kp}g_{jm} - A_{jm}g_{kp} + A_{km}g_{jp}\big)\\ &\quad + \nabla_j\Big((n-2)A_{km} + \frac{R}{2(n-1)}g_{km}\Big) - \nabla_k\Big((n-2)A_{jm} + \frac{R}{2(n-1)}g_{jm}\Big).\end{aligned} \tag{22.35}$$

The Bianchi identity in terms of A is

$$\nabla_j A_i^j = \frac{1}{2(n-1)}\nabla_i R. \tag{22.36}$$

Substituting this in the above expression, we find that all of the scalar curvature terms vanish, and we are left with

$$\nabla_l W_{jkm}{}^l = (n-3)(\nabla_j A_{km} - \nabla_k A_{jm}). \tag{22.37}$$

□

23 Lecture 22

23.1 Local conformal flatness

We define the *Cotton tensor* as

$$C = d^{\nabla} A, \tag{23.1}$$

or in coordinates,

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}. \tag{23.2}$$

We say a manifold (M, g) is locally conformal flat if for each point $p \in M$, there is a function $u : U \to \mathbb{R}$ defined in a neighborhood of p such the that metric $\tilde{g} = e^{-2u} g$ is a flat metric.

Proposition 23.1. *Let (M, g) be an n-dimensional Riemannian manifold. If $n = 2$, then g is locally conformally flat. If $n = 3$, then g is locally conformally flat if and only if the Cotton tensor vanishes. If $n \geq 4$, g is locally conformally flat if and only if the Weyl tensor vanishes identically.*

Proof. For $n = 2$, the equation for local conformal flatness is

$$\Delta u + K = 0. \tag{23.3}$$

This is just Laplace's equation, which can always be solved locally.

If (M, g) is locally conformally flat, then for $n \geq 4$, the Weyl tensor must vanish from conformal invariance. For $n = 3$, local conformal flatness implies we have a solution the equation

$$\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g = 0. \tag{23.4}$$

We apply d^{∇} to this equation,

$$\begin{aligned}
(d^{\nabla} A_g)_{ijk} &= \nabla_i A_{jk} - \nabla_j A_{ik} \\
&= \nabla_i \Big(\nabla_j \nabla_k u + \nabla_j u \nabla_k u - \frac{1}{2} |\nabla u|^2 g_{jk} \Big) - \text{ skew in } i \text{ and } j \\
&= \nabla_i \nabla_j \nabla_k u + (\nabla_i \nabla_j u)(\nabla_k u) + (\nabla_j u)(\nabla_i \nabla_k u) - (\nabla^l u)(\nabla_l \nabla_i u) g_{jk} \\
&\qquad - \text{ skew in } i \text{ and } j \\
&= \nabla_i \nabla_j \nabla_k u + (\nabla_j u)(\nabla_i \nabla_k u) - (\nabla^l u)(\nabla_l \nabla_i u) g_{jk} \\
&\qquad - \text{ skew in } i \text{ and } j \\
&= -R_{ijk}{}^p \nabla_p u + (\nabla_j u)(\nabla_i \nabla_k u) - (\nabla^l u)(\nabla_l \nabla_i u) g_{jk} \\
&\quad - (\nabla_i u)(\nabla_j \nabla_k u) + (\nabla^l u)(\nabla_l \nabla_j u) g_{ik}
\end{aligned} \tag{23.5}$$

The first term is

$$\begin{aligned}-R_{ijk}{}^{p}\nabla_p u &= -g^{pl}R_{ijlk}\nabla_p u \\ &= -g^{pl}(A_{il}g_{jk} - A_{jl}g_{ik} - A_{ik}g_{jl} + A_{jk}g_{il})\nabla_p u \\ &= -A_{il}g_{jk}\nabla^l u + A_{jl}g_{ik}\nabla^l u + A_{ik}\nabla_j u - A_{jk}\nabla_i u.\end{aligned} \tag{23.6}$$

Substituting (23.4), we find that all term cancel, and thus

$$d^{\nabla}A_g = 0 \tag{23.7}$$

is a necessary condition in dimension 3.

Finally, we deal with the sufficiency. By Proposition 22.2, the Cotton tensor vanishes also in case $n \geq 4$. We must find a solution of the equation

$$\nabla^2 u = -du \otimes du + \frac{1}{2}|\nabla u|^2 g - A_g. \tag{23.8}$$

From classical tensor calculus, the integrability condition for this overdetermined system is exactly the vanishing of the Cotton tensor [**?**].

Another way to see this is to think of $du = \alpha$ as a 1-form. The equation is then

$$\nabla\alpha = -\alpha \otimes \alpha + \frac{1}{2}|\alpha|^2 g - A_g. \tag{23.9}$$

In local coordinates,

$$(\nabla\alpha)_{ij} = \partial_i\alpha_j - \Gamma_{ij}^k\alpha_k, \tag{23.10}$$

so this overdetermined system looks like

$$\partial_i\alpha_j = f_{ij}(\alpha_1, \ldots, \alpha_n) + h_{ij}. \tag{23.11}$$

The vanishing of the Cotton tensor is exactly the integrability condition required in the Frobenius Theorem [**?**, Chapter 6]. □

23.2 Examples

Besides constant sectional curvature metrics, there are two other commonly found examples of locally conformally flat metrics. First, the product of two constant sectional curvature metrics, on manifolds M and N with $K_M = 1$, and $K_N = -1$, respectively, is locally conformally flat. To see this, note we can write the product metric

$$g_{M\times N} = g_M + g_N. \tag{23.12}$$

Since the sectional curvatures are constant, we have

$$g_M = \frac{1}{2}g_M \owedge g_M, \text{ and } g_N = -\frac{1}{2}g_N \owedge g_N, \tag{23.13}$$

so

$$
\begin{aligned}
R_{M\times N} = R_M + R_N &= \frac{1}{2}\Big(g_M \owedge g_M - g_N \owedge g_N\Big)\\
&= \frac{1}{2}(g_M + g_N) \owedge (g_M - g_N)\\
&= \frac{1}{2} g_{M\times N} \owedge (g_M - g_N)\\
&= \frac{1}{2}\Psi(g_M - g_N),
\end{aligned}
\tag{23.14}
$$

and therefore the Weyl tensor vanishes.

We can actually exhibit such metrics directly as follows. Topologically,

$$
\begin{aligned}
\mathbb{R}^{p+q} \setminus \mathbb{R}^{q-1} &= \{\mathbb{R}^{p+1} \setminus \{0\}\} \times \mathbb{R}^{q-1}\\
&= S^p \times \mathbb{R}^+ \times \mathbb{R}^{q-1}\\
&= S^p \times H^q,
\end{aligned}
\tag{23.15}
$$

Let us endow this space with the metric $g = r^{-2} g_{\mathbb{R}^{p+q}}$, where

$$
r^2 = \sum_{i=1}^{p+1} x_i^2, \tag{23.16}
$$

and $g_{\mathbb{R}^{p+q}}$ is the Euclidean metric on $\mathbb{R}^{p+q}$. This metric is conformal to the Euclidean metric, so by definition is locally conformally flat. Let g_{S^p} denote the standard metric on the unit sphere $S^p \subset \mathbb{R}^{p+1}$. We can rewrite our metric as

$$
g = \frac{1}{r^2}\Big(dr^2 + r^2 g_{S^p} + \sum_{i=p+2}^{p+q} dx_i^2\Big) = g_{S^p} + \frac{1}{r^2}\Big(dr^2 + \sum_{i=p+2}^{p+q} dx_i^2\Big), \tag{23.17}
$$

and we see the left hand side is the product metric of g_{S^p} with the hyperbolic upper half space H^q.

The second example is the product of a manifold of constant sectional curvature with S^1 or $\mathbb{R}$. To see that this is locally conformally flat, we again write

$$
\begin{aligned}
R_{M\times\mathbb{R}} &= \frac{K}{2} g_M \owedge g_M\\
&= \frac{K}{2}(g_M + dx^2) \owedge (g_M - dx^2),
\end{aligned}
\tag{23.18}
$$

since $dx^2 \owedge dx^2 = 0$, thus the Weyl tensor vanishes.

24 Lecture 23

24.1 Conformal invariants

Proposition 24.1. *For $n = 3$, the Cotton tensor is conformally invariant. That is, under the transformation $\tilde{g} = e^{-2u} g$,*

$$
\tilde{C} = C. \tag{24.1}
$$

Proof. This is a calculation, very similar to the calulation performed in the proof of Proposition 23.1. In that proof we assumed that

$$\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g = 0, \tag{24.2}$$

but here we have that

$$\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g = \tilde{A}. \tag{24.3}$$

That computation shows that

$$\begin{aligned} d^{\nabla}\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big) = \\ d^{\nabla} A_g - \tilde{A}_{il} g_{jk} \nabla^l u + \tilde{A}_{jl} g_{ik} \nabla^l u + \tilde{A}_{ik} \nabla_j u - \tilde{A}_{jk} \nabla_i u, \end{aligned} \tag{24.4}$$

since $W = 0$ when $n = 3$. So we have that

$$(d^{\nabla} \tilde{A})_{ijk} = (d^{\nabla} A_g)_{ijk} - \tilde{A}_{il} g_{jk} \nabla^l u + \tilde{A}_{jl} g_{ik} \nabla^l u + \tilde{A}_{ik} \nabla_j u - \tilde{A}_{jk} \nabla_i u. \tag{24.5}$$

Next, we compute

$$\begin{aligned} (d^{\tilde{\nabla}} \tilde{A})_{ijk} &= \tilde{\nabla}_i \tilde{A}_{jk} - \tilde{\nabla}_j \tilde{A}_{ik} \\ &= \partial_i \tilde{A}_{jk} - \tilde{\Gamma}^p_{ij} \tilde{A}_{pk} - \tilde{\Gamma}^p_{ik} \tilde{A}_{jp} - \text{skew in } i \text{ and } j \\ &= \partial_i \tilde{A}_{jk} - \tilde{\Gamma}^p_{ik} \tilde{A}_{jp} - \text{skew in } i \text{ and } j. \end{aligned} \tag{24.6}$$

Recall the formula (21.1) for the conformal change of Christoffel symbol:

$$\tilde{\Gamma}^i_{jk} = g^{il}\Big(- (\partial_j u) g_{lk} - (\partial_k u) g_{lj} + (\partial_l u) g_{jk}\Big) + \Gamma^i_{jk}. \tag{24.7}$$

Substituting this, we obtain

$$\begin{aligned} (d^{\tilde{\nabla}} \tilde{A})_{ijk} &= (d^{\nabla} \tilde{A})_{ijk} - g^{pl}\Big(- (\partial_i u) g_{lk} - (\partial_k u) g_{li} + (\partial_l u) g_{ik}\Big) \tilde{A}_{jp} - \text{skew in } i \text{ and } j \\ &= (d^{\nabla} \tilde{A})_{ijk} + (\partial_i u) \tilde{A}_{jk} + (\partial_k u) \tilde{A}_{ji} - (\nabla^l u) g_{ik} \tilde{A}_{jl} - \text{skew in } i \text{ and } j \\ &= (d^{\nabla} \tilde{A})_{ijk} + (\partial_i u) \tilde{A}_{jk} - (\nabla^l u) g_{ik} \tilde{A}_{jl} - (\partial_j u) \tilde{A}_{ik} + (\nabla^l u) g_{jk} \tilde{A}_{il}. \end{aligned} \tag{24.8}$$

Comparing terms, we are done. □

Proposition 24.2. *For $n \geq 4$, the quantity*

$$\int_M |Weyl_g|^{\frac{n}{2}} dV_g, \tag{24.9}$$

is conformally invariant. For $n = 3$, the quantity

$$\int_M |C_g| dV_g = \int_M \sqrt{\langle C_g, C_g \rangle_g} dV_g, \tag{24.10}$$

is a conformal invariant.

Proof. We know that the $(1,3)$ Weyl tensor is a conformal invariant. Let $\tilde{g} = \lambda g$, and compute

$$\begin{aligned}\int_M |\tilde{W}|_{\tilde{g}}^{\frac{n}{2}} dV_{\tilde{g}} &= \int_M \left(\tilde{g}^{ip}\tilde{g}^{jq}\tilde{g}^{kr}\tilde{g}_{sl}\tilde{W}_{pqr}{}^{s}\tilde{W}_{ijk}{}^{l} \right)^{\frac{n}{4}} dV_{\tilde{g}} \\ &\int_M \left(\lambda^{-2} g^{ip}g^{jq}g^{kr}g_{sl}W_{pqr}{}^{s}W_{ijk}{}^{l} \right)^{\frac{n}{4}} \lambda^{n/2} dV_g \\ &= \int_M |W|^{\frac{n}{2}} dV_g.\end{aligned} \tag{24.11}$$

For the second, we know that the $(0,3)$ Cotton tensor is a conformal invariant. For $\tilde{g} = \lambda g$, we compute

$$\begin{aligned}\int_M |\tilde{C}|_{\tilde{g}} dV_{\tilde{g}} &= \int_M \left(\tilde{g}^{ip}\tilde{g}^{jq}\tilde{g}^{kr}\tilde{C}_{pqr}\tilde{C}_{ijk} \right)^{\frac{1}{2}} dV_{\tilde{g}} \\ &= \int_M \left(\lambda^{-3} g^{ip}g^{jq}g^{kr}C_{pqr}C_{ijk} \right)^{\frac{1}{2}} \lambda^{\frac{3}{2}} dV_g \\ &= \int_M |C_g|_g dV_g.\end{aligned} \tag{24.12}$$

□

24.2 Weitzenböck formula revisited

Recall from Section 9.1 that for $\omega \in \Gamma(\Lambda^p)$,

$$\Delta_H \omega = -\Delta\omega + \rho_\omega, \tag{24.13}$$

where

$$\begin{aligned}\rho_\omega = &-\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} R_{lmi_j i_k}\omega_{i_1\dots i_{j-1} l i_{j+1}\dots i_{k-1} m i_{k+1}\dots i_p} \\ &+ \sum_{m=1}^{n}\sum_{j=1}^{p} R_{i_j m}\omega_{i_1\dots i_{j-1} m i_{j+1}\dots i_p}.\end{aligned} \tag{24.14}$$

If (M,g) is locally conformally flat, for the first term, we have

$$\begin{aligned}&-\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} R_{lmi_j i_k}\omega_{i_1\dots i_{j-1} l i_{j+1}\dots i_{k-1} m i_{k+1}\dots i_p} \\ &= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} \Big(A_{li_j}g_{mi_k} - A_{mi_j}g_{li_k} - A_{li_k}g_{mi_j} \\ &\qquad + A_{mi_k}g_{li_j} \Big)\omega_{i_1\dots i_{j-1} l i_{j+1}\dots i_{k-1} m i_{k+1}\dots i_p}.\end{aligned} \tag{24.15}$$

It is easy to see that the four terms terms are the same, so we have

$$\begin{aligned}
\rho_\omega &= -2\sum_{l=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p} A_{li_j} g_{mi_k}\omega_{i_1\dots i_{j-1}li_{j+1}\dots i_{k-1}mi_{k+1}\dots i_p} \\
&\quad + \sum_{m=1}^{n}\sum_{j=1}^{p} R_{i_jm}\omega_{i_1\dots i_{j-1}mi_{j+1}\dots i_p} \\
&= -2(p-1)\sum_{l=1}^{n}\sum_{j=1}^{p} A_{li_j}\omega_{i_1\dots i_{j-1}li_{j+1}\dots i_p} \qquad (24.16)\\
&\quad + \sum_{m=1}^{n}\sum_{j=1}^{p} R_{i_jm}\omega_{i_1\dots i_{j-1}mi_{j+1}\dots i_p} \\
&= \frac{n-2p}{n-2}\sum_{l=1}^{n}\sum_{j=1}^{p} R_{li_j}\omega_{i_1\dots i_{j-1}li_{j+1}\dots i_p} + \frac{p(p-1)}{(n-1)(n-2)}R\omega.
\end{aligned}$$

Remark. Note that if (M, g) has constant sectional curvature K, then this becomes

$$\begin{aligned}
\rho_\omega &= \frac{n-2p}{n-2}\sum_{l=1}^{n}\sum_{j=1}^{p} R_{li_j}\omega_{i_1\dots i_{j-1}li_{j+1}\dots i_p} + \frac{p(p-1)}{(n-1)(n-2)}R\omega \\
&= Kp(n-p)\omega.
\end{aligned} \qquad (24.17)$$

Proposition 24.3. *If (M, g) is compact, of dimension $n = 2m$, locally conformally flat, and $R > 0$, then $b_m(M) = 0$.*

Proof. Let ω be a harmonic m-form. From the Bochner formula (9.14), we have

$$\begin{aligned}
0 &= \int_M |\nabla\omega|^2 + \int_M \langle\rho_\omega, \omega\rangle dV_g \\
&= \int_M |\nabla\omega|^2 + \frac{m}{2(2m-1)}\int_M R_g|\omega|^2 dV_g.
\end{aligned} \qquad (24.18)$$

If $R > 0$, then clearly we must have $\omega \equiv 0$. □

Proposition 24.4. *If (M^n, g) is compact, locally conformally flat, and $Ric > 0$, then $b_i(M) = 0$ for $i = 1 \dots n-1$.*

Proof. Again, this follows easily from the Bochner formula and Poincaré duality. □

Remark. This is not surprising, since Kuiper has shown that any compact simply connected locally conformally flat manifold is conformally diffeomorphic to the round n-sphere [?]. In this case, since $Ric > 0$, Myers' Theorem implies that the universal cover in compact, and is therefore conformally equivalent to the round S^n. Then (M, g) is a compact conformal quotient of S^n. But any co-compact subgroup of the conformal group $SO(n+1, 1)$ is conjugate to a subgroup of isometries, so (M, g) is conformal to a positive space form.

We can also write ρ_ω in terms of the Schouten tensor,

$$\rho_\omega = (n-2p)\sum_{l=1}^{n}\sum_{j=1}^{p} A_{li_j}\omega_{i_1\ldots i_{j-1}li_{j+1}\ldots i_p} + \frac{p}{2(n-1)}R\omega. \tag{24.19}$$

Under various positivity assumptions on the Schouten tensor, some Betti number vanishing theorems were shown in [?]. We mention also that Schoen and Yau have shown vanishing theorems for certain homotopy groups in the locally conformally flat positive scalar curvature case [?]. Chern and Simons, and Kulkarni have shown that the Pontrjagin forms depend only upon the Weyl tensor. Therefore all the Pontryagin classes of any compact locally conformally flat manifold vanish [?], [?].

We look again at the case of 2-forms. On 2-forms, the Weitzenböck formula is

$$(\Delta_H\omega)_{ij} = -(\Delta w)_{ij} - \sum_{l,m} R_{lmij}\omega_{lm} + \sum_m R_{im}\omega_{mj} + \sum_m R_{jm}\omega_{im}. \tag{24.20}$$

In dimension 4, the Ricci terms vanish, and we obtain

$$(\rho_\omega)_{ij} = -\sum_{l,m=1}^{n} W_{lmij}\omega_{lm} + \frac{1}{3}R\omega_{ij}. \tag{24.21}$$

Let λ denote the minimum eigenvalue of this Weyl curvature operator.

Proposition 24.5. *Let (M^4, g) be a compact 4-manifold. If $\lambda < \frac{R}{3}$, then $b_2(M) = 0$.*

Proof. Again, an easy application of the Bochner formula. □

25 Lecture 24

25.1 Laplacian of Schouten

Let (M, g) be a Riemannian manifold, and recall the Schouten tensor

$$A = \frac{1}{n-2}\left(Ric - \frac{R}{2(n-1)}g\right), \tag{25.1}$$

and the Cotton tensor

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}. \tag{25.2}$$

We differentiate the formula

$$\nabla_i A_{jk} = \nabla_j A_{ik} + C_{ijk}, \tag{25.3}$$

to get

$$(\Delta A)_{jk} = \nabla^i\nabla_j A_{ik} + \nabla^i C_{ijk}. \tag{25.4}$$

Commuting covariant derivatives,

$$\begin{aligned}
\nabla^i \nabla_j A_{ik} &= g^{il} \nabla_l \nabla_j A_{ik} \\
&= g^{il} \left(\nabla_j \nabla_l A_{ik} - R_{lji}{}^{m} A_{mk} - R_{ljk}{}^{m} A_{im} \right) \\
&= \nabla_j \nabla^i A_{ik} - g^{il} g^{mp} \left(R_{ljpi} A_{mk} + R_{ljpk} A_{im} \right)
\end{aligned} \tag{25.5}$$

Using the Bianchi identity (4.11), we compute

$$\begin{aligned}
\nabla^i A_{ik} &= \frac{1}{n-2} \left(\nabla^i R_{ik} - \frac{1}{2(n-1)} \nabla^i (R g_{ik}) \right) \\
&= \frac{1}{n-2} \left(\frac{1}{2} \nabla_k R - \frac{1}{2(n-1)} \nabla_k R \right) \\
&= \frac{\nabla_k R}{2(n-1)} \\
&= \nabla_k (tr(A)).
\end{aligned} \tag{25.6}$$

Substituting into (25.5) we obtain

$$\begin{aligned}
\nabla^i \nabla_j A_{ik} &= \nabla_j \nabla_k (tr(A)) + g^{mp} R_{jp} A_{mk} - g^{il} g^{mp} R_{ljpk} A_{im} \\
&= \nabla_j \nabla_k (tr(A)) + g^{mp} \big((n-2) A_{jp} + tr(A) g_{jp} \big) A_{mk} \\
&\quad - g^{il} g^{mp} \Big(W_{ljpk} + A_{lp} g_{jk} - A_{jp} g_{lk} - A_{lk} g_{jp} + A_{jk} g_{lp} \Big) A_{im} \\
&= \nabla_j \nabla_k (tr(A)) + (n-2) A_j^m A_{mk} + tr(A) A_{jk} \\
&\quad - g^{il} g^{mp} W_{ljpk} A_{im} - |A|^2 g_{jk} + 2 A_j^m A_{km} - tr(A) A_{jk} \\
&= \nabla_j \nabla_k (tr(A)) - g^{il} g^{mp} W_{ljpk} A_{im} + n (A^2)_{jk} - |A|^2 g_{jk}.
\end{aligned} \tag{25.7}$$

Substituting into the above, we arrive at the general formula

$$(\Delta A)_{jk} = \nabla^i C_{ijk} + \nabla_j \nabla_k (tr(A)) - g^{il} g^{mp} W_{ljpk} A_{im} + n (A^2)_{jk} - |A|^2 g_{jk}. \tag{25.8}$$

Proposition 25.1. *If (M, g) is locally conformally flat, and has constant scalar curvature, then*

$$\Delta A = n A^2 - |A|^2 g. \tag{25.9}$$

Proof. From Proposition 23.1, the Weyl and Cotton tensor terms vanish, and also the scalar term vanishes in (25.8). □

25.2 The Yamabe Flow

The Yamabe flow, introduced by Hamilton, is

$$\frac{d}{dt} g = -(R_g - r_g) g, \tag{25.10}$$

where

$$r_g = \frac{1}{Vol(g)} \int_M R dV_g, \tag{25.11}$$

is the average scalar curvature. Note this flow remains in the conformal class of the initial metric. Also, this flow preserves volume. To see this,

$$\begin{aligned} \frac{d}{dt} Vol(g(t)) &= \frac{1}{2} \int_M tr_g(h) dV_g \\ &= -\frac{n}{2} \int_M (R_g - r_g) dV_g = 0 \end{aligned} \tag{25.12}$$

We recall the formula for the linearization of the Ricci tensor

$$\begin{aligned} (Ric')_{ij} = \frac{1}{2}\Big(& -\Delta h_{ij} + \nabla_i (div\ h)_j + \nabla_j (div\ h)_i - \nabla_i \nabla_j (trh) \\ & - 2R_{iljp} h^{lp} + R_i^p h_{jp} + R_j^p h_{ip} \Big). \end{aligned} \tag{25.13}$$

For the Yamabe flow, we have

$$\frac{d}{dt}(Ric)_{ij} = \frac{1}{2}\Big((\Delta R) g_{ij} + (n-2) \nabla_i \nabla_j R \Big). \tag{25.14}$$

The evolution of the scalar curvature is given by

$$\begin{aligned} \delta R = \delta(g^{-1} Ric) &= g^{-1}(\delta Ric) + (\delta g^{-1}) Ric \\ &= \frac{1}{2} g^{-1} \Big((\Delta R) g_{ij} + (n-2) \nabla_i \nabla_j R \Big) + g^{-1}(R - r) g g^{-1} Ric \\ &= (n-1)\Delta R + R(R - r). \end{aligned} \tag{25.15}$$

Proposition 25.2. *The Yamabe flow preserves positive scalar curvature.*

Proof. The change of scalar curvature for the unnormalized Yamabe flow is

$$\frac{d}{dt} R = (n-1)\Delta R + R^2, \tag{25.16}$$

so the maximum principle applies. □

We next restrict to the case that (M, g) is locally conformally flat. In this case, (25.8) is

$$\Delta A = \nabla^2 (tr(A)) + nA^2 - |A|^2 g. \tag{25.17}$$

Rewrite this in terms of the Ricci tensor

$$\frac{1}{n-2} \Delta \left(Ric - \frac{R}{2(n-1)} g \right) = \frac{1}{2(n-1)} \nabla^2 R + nA^2 - |A|^2 g. \tag{25.18}$$

which is

$$\Delta Ric = \frac{1}{2(n-1)}\Big((\Delta R)g + (n-2)\nabla^2 R\Big) + (n-2)\Big(nA^2 - |A|^2 g\Big). \qquad (25.19)$$

Substituting into (25.14) we obtain

$$\frac{d}{dt}Ric = (n-1)\Delta Ric - (n-2)(n-1)\Big(nA^2 - |A|^2 g\Big). \qquad (25.20)$$

In an ONB, we compute that

$$nA_{im}A_{jm} - |A|^2 g_{ij} = \frac{1}{(n-2)^2}\left(nR_{im}R_{jm} - \frac{n}{n-1}RR_{ij} - |Ric|^2 g_{ij} + \frac{R^2}{n-1}g_{ij}\right). \qquad (25.21)$$

We have show that in the locally conformally flat under the Yamabe flow the evolution of the Ricci tensor is given by

$$\frac{d}{dt}Ric = (n-1)\Delta Ric - \frac{n-1}{(n-2)}\left(nRic^2 - \frac{n}{n-1}RRic - |Ric|^2 g + \frac{R^2}{n-1}g\right). \qquad (25.22)$$

Proposition 25.3. *If (M, g) is locally conformally flat, then the Yamabe flow preserves positive Ricci.*

Proof. As in the Ricci flow case, we just need to verify the null eigenvector assumption. If Ricci in non-negative and has a zero eigenvector, then we require that

$$|Ric|^2 \leq \frac{R^2}{n-1}, \qquad (25.23)$$

which is true, using (16.8) in dimension $n-1$. □

As we discussed in the previous section, Kuiper's Theorem implies that if (M, g) is compact, locally conformally flat and has positive Ricci, then (M, g) is conformal to a constant positive curvature metric. One may ask: if one starts the Yamambe flow with such a metric, does the flow converge to the constant curvature metric. Ben Chow showed that this is indeed the case with proof similar to that of Hamilton [Cho92]. Subsequently, Rugang Ye proved the Yamabe flow converges on a locally conformally flat manifold with any initial data [Ye94].

In the general case (not necessarily locally conformally flat), Hamilton proved existence of the flow for all time and proved convergence in the case of negative scalar curvature. The case of positive scalar curvature however is highly non-trivial. Schwetlick and Struwe [SS03] proved convergence for $3 \leq n \leq 5$ provided an certain energy bound on the initial metric is satsified. In the beautiful paper [Bre05], Simon Brendle proved convergence for $3 \leq n \leq 5$ for arbitrary initial data.

26 Lecture 25

26.1 Curvature in dimension 4

Recall the Hodge star operator on Λ^p in dimension n satisfies

$$*^2 = (-1)^{p(n-p)} I. \tag{26.1}$$

In the case of Λ^2 in dimension 4, $*^2 = I$. The space of 2-form decomposes into

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{26.2}$$

the $+1$ and -1 eigenspaces of the Hodge star operator, respectively. Note that $dim_{\mathbb{R}}(\Lambda^2) = 6$, whilst $dim_{\mathbb{R}}(\Lambda^2_\pm) = 3$. Elements of Λ^2_+ are called *self-dual* 2-forms, and elements of Λ^2_- are called *anti-self-dual* 2-forms

We fix an oriented orthonormal basis e_1, e_2, e_3, e_4 and let

$$\begin{aligned}
\omega_1^\pm &= e^1 \wedge e^2 \pm e^3 \wedge e^4, \\
\omega_2^\pm &= e^1 \wedge e^3 \pm e^4 \wedge e^2, \\
\omega_3^\pm &= e^1 \wedge e^4 \pm e^2 \wedge e^3,
\end{aligned}$$

note that $*\omega_i^\pm = \pm\omega_i^\pm$, and $\frac{1}{\sqrt{2}}\omega_i^\pm$ is an orthonormal basis of $\Lambda^2_\pm$.

Remark. In dimension 6, on Λ^3, we have $*^2 = -1$, so $\Lambda^3 \otimes \mathbb{C} = \Lambda^3_+ \oplus \Lambda^3_-$, the $+i$ and $-i$ eigenspaces of the Hodge star. That is, $*$ gives a complex structure on Λ^3 in dimension 6. In general, in dimensions $n = 4m$, $\Lambda^{2m} = \Lambda^{2m}_+ \oplus \Lambda^{2m}_-$, the ± 1 eigenspaces of Hodge star, whilst in dimensions $n = 4m + 2$, the Hodge star gives a complex structure on Λ^{2m+1}.

In the remainder of this section, we will perform computations in a local oriented ONB. For a 2-form ω, the components of ω are defined by

$$\omega_{ij} = \omega(e_i, e_j), \tag{26.3}$$

so that the 2-form can be written

$$\omega = \frac{1}{2}\sum_{i,j} \omega_{ij} e^i \wedge e^j. \tag{26.4}$$

What are the components of $e^p \wedge e^q$? We write

$$\begin{aligned}
e^p \wedge e^q &= \frac{1}{2}\sum_{i,j} (e^p \wedge e^q)_{ij} e^i \wedge e^j \\
&= \frac{1}{2}\sum_{i,j} \delta^{pq}_{ij} e^i \wedge e^j,
\end{aligned} \tag{26.5}$$

so the components of $e^p \wedge e^q$ are given by $(e^p \wedge e^q)_{ij} = \delta^{pq}_{ij}$, the generalized Kronecker delta symbol, which is defined to be $+1$ if $(p, q) = (i, j)$, -1 if $(p, q) = (j, i)$, and 0 otherwise.

In Section 5.2, we defined the curvature operator

$$\mathcal{R} \in \Gamma\Big(S^2\big(\Lambda^2(T^*M)\big)\Big). \tag{26.6}$$

In an local ONB, we write

$$(\mathcal{R}\omega)_{ij} = \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_{kl}. \tag{26.7}$$

Notice we have changed previous convention, and introduced a factor of 1/2, the reason for this will be seen below. Therefore the curvature operator is

$$\begin{aligned} \mathcal{R}\omega &= \frac{1}{2}\sum_{i,j}(\mathcal{R}\omega)_{ij} e^i \wedge e^j \\ &= \frac{1}{4}\sum_{i,j,k,l} R_{ijkl}\omega_{kl} e^i \wedge e^j. \end{aligned} \tag{26.8}$$

Note that any tensor with components P_{ijkl} satisfying $P_{ijkl} = -P_{jikl} = -P_{ijlk}$ and $P_{ijkl} = P_{klij}$ yields a symmetric operator $\mathcal{P} : \Lambda^2 \to \Lambda^2$ defined by the same formula

$$\mathcal{P}\omega = \frac{1}{4}\sum_{i,j,k,l} P_{ijkl}\omega_{kl} e^i \wedge e^j. \tag{26.9}$$

Conversely, any symmetric operator $\mathcal{P} : \Lambda^2 \to \Lambda^2$ is equivalent to a $(0,4)$ tensor, by

$$P_{pqrs} = \langle \mathcal{P}(e^p \wedge e^q), e^r \wedge e^s\rangle. \tag{26.10}$$

For the operator $\mathcal{P}$, we have

$$\begin{aligned} P_{pqrs} &= \langle \mathcal{P}(e^p \wedge e^q), e^r \wedge e^s\rangle \\ &= \langle \frac{1}{4}\sum_{i,j,k,l} P_{ijkl}(e^p \wedge e^q)_{kl} e^i \wedge e^j, e^r \wedge e^s\rangle \\ &= \frac{1}{4}\langle \sum_{i,j,k,l} P_{ijkl}\delta^{pq}_{kl} e^i \wedge e^j, e^r \wedge e^s\rangle \\ &= \frac{1}{4}\langle \sum_{i,j}(P_{ijpq} - P_{ijqp}) e^i \wedge e^j, e^r \wedge e^s\rangle \\ &= \frac{1}{2}\langle \sum_{i,j} P_{ijpq} e^i \wedge e^j, e^r \wedge e^s\rangle \\ &= \frac{1}{2}(P_{rspq} - P_{srpq}) = P_{pqrs}. \end{aligned} \tag{26.11}$$

This computation was just to verify that with our convention of 1/4 in (26.9), we get back the same tensor we started with.

In dimension 4 there is the special coincidence that the curvature operator acts on 2-forms, and the space of 2-forms decomposes as above. Recall from Section 5.1, the full curvature tensor decomposes as

$$Rm = W + \frac{1}{2} E \owedge g + \frac{R}{24} g \owedge g, \tag{26.12}$$

where

$$E = Ric - \frac{R}{4} g \tag{26.13}$$

is the *traceless Ricci tensor*.

Corresponding to this decomposition, we define the *Weyl curvature operator*, $\mathcal{W} : \Lambda^2 \to \Lambda^2$ as

$$(\mathcal{W}\omega)_{ij} = \frac{1}{2} \sum_{k,l} W_{ijkl} \omega_{kl}. \tag{26.14}$$

We also define $\mathcal{W}^{\pm} : \Lambda^2 \to \Lambda^2_{\pm}$ as

$$\mathcal{W}^{\pm}\omega = \pi_{\pm} \mathcal{W} \omega, \tag{26.15}$$

where $\pi_{\pm} : \Lambda^2 \to \Lambda^2_{\pm}$ is the projection $\frac{1}{2}(I \pm *)$. The operator $\mathcal{W}^+$ is a symmetric operator, so by the above procedure, it corresponds to a curvature-like tensor W^+, the components of which are defined by

$$\begin{aligned} W^+_{pqrs} &= \langle \mathcal{W}^+(e^p \wedge e^q), e^r \wedge e^s \rangle \\ &= \frac{1}{2} \langle \mathcal{W}\big(e^p \wedge e^q + *(e^p \wedge e^q)\big), e^r \wedge e^s \rangle. \end{aligned} \tag{26.16}$$

For example,

$$\begin{aligned} W^+_{1234} &= \frac{1}{2} \langle \mathcal{W}\big(e^1 \wedge e^2 + e^3 \wedge e^4\big), e^3 \wedge e^4 \rangle \\ &= \frac{1}{2}(W_{1234} + W_{3434}). \end{aligned} \tag{26.17}$$

Correspondingly, we can decompose the Weyl curvature tensor as

$$W = W^+ + W^-, \tag{26.18}$$

the *self-dual* and *anti-self-dual* components of the Weyl curvature, respectively. Therefore in dimension 4 we have the further orthogonal decomposition of the curvature tensor

$$Rm = W^+ + W^- + \frac{1}{2} E \owedge g + \frac{R}{24} g \owedge g. \tag{26.19}$$

The *traceless Ricci curvature operator* $\mathcal{E}$ is the operator associated to the curvature-like tensor $E \owedge g$, and the *scalar curvature operator* $\mathcal{S}$ is the operator associated to $Rg \owedge g$.

Proposition 26.1. *The Weyl curvature operator preserves the type of forms,*

$$\mathcal{W} : \Lambda^2_{\pm} \to \Lambda^2_{\pm}, \tag{26.20}$$

and therefore

$$\mathcal{W}_{\pm} : \Lambda^2_{\pm} \to \Lambda^2_{\pm}. \tag{26.21}$$

The scalar curvature operator acts as a multiple of the identity

$$\mathcal{S}\omega = 2R\omega. \tag{26.22}$$

The traceless Ricci operator reverses types,

$$\mathcal{E} : \Lambda^2_{\pm} \to \Lambda^2_{\mp}. \tag{26.23}$$

In block form corresponding to the decomposition (26.2), the full curvature operator is

$$\mathcal{R} = \left(\begin{array}{c|c} \mathcal{W}^+ + \frac{R}{12} I & \mathcal{E} \\ \hline \mathcal{E} & \mathcal{W}^- + \frac{R}{12} I \end{array} \right). \tag{26.24}$$

Proof. We first consider the traceless Ricci operator. We compute

$$\begin{aligned}
((E \owedge g)\omega)_{ij} &= \frac{1}{2}(E \owedge g)_{ijkl}\omega_{kl} \\
&= \frac{1}{2}(E_{ik}g_{jl}\omega_{kl} - E_{jk}g_{il}\omega_{kl} - E_{il}g_{jk}\omega_{kl} + E_{jl}g_{ik}\omega_{kl}) \\
&= \frac{1}{2}(E_{ik}\omega_{kj} - E_{jk}\omega_{ki} - E_{il}\omega_{jl} + E_{jl}\omega_{il}) \\
&= E_{ik}\omega_{kj} - E_{jk}\omega_{ki},
\end{aligned} \tag{26.25}$$

since ω is skew-symmetric. Next assume that E_{ij} is diagonal, so that $E_{ij} = \lambda_i g_{ij}$, and we have

$$\begin{aligned}
\frac{1}{2}(E \owedge g)_{ijkl}\omega_{kl} &= \lambda_i \delta_{ik}\omega_{kj} - \lambda_j \delta_{jk}\omega_{ki} \\
&= \lambda_i \omega_{ij} - \lambda_j \omega_{ji} \\
&= (\lambda_i + \lambda_j)\omega_{ij}.
\end{aligned} \tag{26.26}$$

Next compute

$$\begin{aligned}
\frac{1}{2}(E \owedge g)_{ijkl}(\omega_1^+)_{kl} &= \frac{1}{2}(E \owedge g)_{ijkl}(\delta^{kl}_{12} + \delta^{kl}_{34}) \\
&= (\lambda_i + \lambda_j)(\delta^{ij}_{12} + \delta^{ij}_{34}) \\
&= (\lambda_1 + \lambda_2)\delta^{ij}_{12} + (\lambda_3 + \lambda_4)\delta^{ij}_{34}.
\end{aligned} \tag{26.27}$$

Since E is traceless, $\lambda_1 + \lambda_2 = -\lambda_3 - \lambda_4$, so we have

$$\frac{1}{2}(E \owedge g)_{ijkl}(\omega_1^+)_{kl} = (\lambda_1 + \lambda_2)(\delta_{12}^{ij} - \delta_{34}^{ij}), \tag{26.28}$$

which equivalently is

$$(E \owedge g)(\omega_1^+) = (\lambda_1 + \lambda_2)\omega_1^-. \tag{26.29}$$

A similar computation shows that

$$(E \owedge g) : \Lambda^2_{\pm} \to \Lambda^2_{\mp}. \tag{26.30}$$

The dimension of the space $\{M : \Lambda^2 \to \Lambda^2, M \text{ symmetric}, M* = - * M\}$ is 9. The dimension of the maps of the form $E \owedge g$, where is a traceless symmetric tensor is also 9, since the map $E \to E \owedge g$ is injective for $n > 2$. We conclude that the remaining part of the curvature tensor

$$\mathcal{W}^{\pm} + \frac{1}{24}\mathcal{S} : \Lambda^2_{\pm} \to \Lambda^2_{\pm}. \tag{26.31}$$

The proposition follows, noting that $g \owedge g = 2I$, twice the identity. To see this, we have

$$\begin{aligned}
\big((g \owedge g)\omega\big)_{ij} &= \frac{1}{2}(g \owedge g)_{ijkl}\omega_{kl} \\
&= \frac{1}{2}(g_{ik}g_{jl} - g_{jk}g_{il} - g_{il}g_{jk} + g_{jl}g_{ik})\omega_{kl} \\
&= (g_{ik}g_{jl} - g_{jk}g_{il})\omega_{kl} \\
&= (\omega_{ij} - \omega_{ji}) = 2\omega_{ij}.
\end{aligned} \tag{26.32}$$

□

Of course, instead of appealing to the dimension argument, one can show directly that (26.31) is true, using the fact that the Weyl is in the kernel of Ricci contraction, that is, the Weyl tensor satisfies $W_{iljl} = 0$. For example,

$$\begin{aligned}
(\mathcal{W}\omega_1^+)_{ij} &= \frac{1}{2}W_{ijkl}(\delta_{kl}^{12} + \delta_{kl}^{34}) \\
&= W_{ij12} + W_{ij34},
\end{aligned} \tag{26.33}$$

taking an inner product,

$$\begin{aligned}
\langle \mathcal{W}\omega_1^+, \omega_1^- \rangle &= \frac{1}{2}(W_{ij12} + W_{ij34})(\delta_{ij}^{12} - \delta_{ij}^{34}) \\
&= W_{1212} - W_{3412} + W_{1234} - W_{3434} \\
&= W_{1212} - W_{3434}.
\end{aligned} \tag{26.34}$$

But we have

$$\begin{aligned}
W_{1212} + W_{1313} + W_{1414} &= 0 \\
W_{1212} + W_{3232} + W_{4242} &= 0,
\end{aligned} \tag{26.35}$$

adding these,

$$2W_{1212} = -W_{1313} - W_{1414} - W_{3232} - W_{4242}. \tag{26.36}$$

We also have

$$\begin{aligned} W_{1414} + W_{2424} + W_{3434} = 0 \\ W_{3131} + W_{3232} + W_{3434} = 0, \end{aligned} \tag{26.37}$$

adding,

$$2W_{3434} = -W_{1414} - W_{2424} - W_{3131} - W_{3232} = 2W_{1212}, \tag{26.38}$$

which shows that

$$\langle \mathcal{W}\omega_1^+, \omega_1^- \rangle = 0. \tag{26.39}$$

Next

$$\begin{aligned} \langle \mathcal{W}\omega_1^+, \omega_2^- \rangle &= \frac{1}{2}(W_{ij12} + W_{ij34})(\delta_{ij}^{13} - \delta_{ij}^{42}) \\ &= W_{1312} - W_{4212} + W_{1334} - W_{4234} \\ &= -W_{1231} - W_{4212} - W_{4313} - W_{4234}. \end{aligned} \tag{26.40}$$

But from vanishing Ricci contraction, we have

$$\begin{aligned} W_{4212} + W_{4313} &= 0, \\ W_{1231} + W_{4234} &= 0, \end{aligned} \tag{26.41}$$

which shows that

$$\langle \mathcal{W}\omega_1^+, \omega_2^- \rangle = 0.$$

A similar computation can be done for the other components.

27 Lecture 26

27.1 Some representation theory

As $SO(4)$ modules, we have the decomposition

$$\begin{aligned} S^2(\Lambda^2) &= S^2(\Lambda^2_+ \oplus \Lambda^2_-) \\ &= S^2(\Lambda^2_+) \oplus (\Lambda^2_+ \otimes \Lambda^2_-) \oplus S^2(\Lambda^2_-), \end{aligned} \tag{27.1}$$

which is just the "block form" decomposition in (26.24).

Proposition 27.1. *We have the following isomorphisms of Lie groups*

$$Spin(3) = Sp(1) = SU(2), \tag{27.2}$$

and

$$Spin(4) = Sp(1) \times Sp(1) = SU(2) \times SU(2). \tag{27.3}$$

Proof. Recall that $Sp(1)$ is the group of unit quaternions,

$$Sp(1) = \{q \in \mathbb{H} : q\overline{q} = |q|^2 = 1\}, \tag{27.4}$$

where for $q = x_0 + x_1 i + x_2 j + x_3 k$, the conjugate is $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. The first isomorphism is, for $q_1 \in Sp(1)$, and $q \in Im(\mathbb{H}) = \{x_1 i + x_2 j + x_3 k\}$,

$$q_1 \mapsto q_1 q \overline{q}_1 \in SO(Im(\mathbb{H})) = SO(3), \tag{27.5}$$

is a double covering of $SO(3)$. For the isomorphism $Sp(1) = SU(2)$, we send

$$q = \alpha + j\beta \mapsto \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \tag{27.6}$$

where $\alpha, \beta \in \mathbb{C}$.

To see that $Sp(1) \times Sp(1) = Spin(4)$, take $(q_1, q_2) \in Sp(1) \times Sp(1)$, and define $\phi : \mathbb{H} \to \mathbb{H}$ by

$$\phi(q) = q_1 q \overline{q}_2. \tag{27.7}$$

This define a homomorpishm $f : Sp(1) \times Sp(1) \to SO(4)$, with

$$ker(f) = \{(1,1), (-1,-1)\}, \tag{27.8}$$

and f is a non-trivial double covering. $\square$

We let V denote the standard 2-dimensional complex representation of $SU(2)$, which is just matrix multiplication of (27.6) on column vectors. Let $S^r(V)$ denote the space of complex totally symmetric r-tensors. This is the same as homogeneous polynomials of degree r in 2 variable, so $\dim_{\mathbb{C}}(S^r(V)) = r + 1$. The following proposition can be found in [**?**].

Proposition 27.2. *If W is an irreducible complex representation of $Spin(3) = SU(2)$ then W is equivalent to $S^r(V)$ for some $r \geq 0$. Such a representation descends to $SO(3)$ if and only if r is even, in which case are complexifications of real representations of $SO(3)$. Furthermore,*

$$S^p(V) \otimes S^q(V) = \bigoplus_{r=0}^{\min(p,q)} S^{p+q-2r} V. \tag{27.9}$$

For G_1 and G_2 compact Lie groups, it is well-known that the irreducible representations of $G_1 \times G_2$ are exactly those of the form $V_1 \otimes V_2$ for irreducible representations V_1 and V_2 of G_1 and G_2, respectively [**?**]. For $Spin(4) = SU(2) \times SU(2)$, let V_+ and V_- denote the standard irreducible complex 2-dimensional representations of the first and second factors, respectively.

Proposition 27.3. *If W is an irreducible complex representation of $Spin(4) = SU(2) \times SU(2)$ then W is equivalent to*

$$S^{p,q} = S^p(V_+) \otimes S^q(V_-), \tag{27.10}$$

for some non-negative integers p, q. Such a representation descends to $SO(4)$ if and only if $p+q$ is even, in which case these are complexifications of real representations of $SO(4)$.

Note that

$$dim_{\mathbb{C}}(S^{p,q}) = (p+1)(q+1), \tag{27.11}$$

which yields that $dim_{\mathbb{C}}(S^{1,1}) = 4$. Since $p + q = 2$ is even, this corresponds to an irreducible real representation of $SO(4)$. It it not hard to show that the standard real 4-dimensional representation of $SO(4)$, call it T, is irreducible. Therefore, we must have

$$T \otimes \mathbb{C} = V_+ \otimes_{\mathbb{C}} V_-. \tag{27.12}$$

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