Math 865, Topics in Riemannian Geometry

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Fall 2016

Contents

Introduction

We will cover the following topics:

- First few lectures will be a quick review of tensor calculus and Riemannian geometry: metrics, connections, curvature tensor, Bianchi identities, commuting covariant derivatives, etc.
- Bochner-Weitzenbock formulas: various curvature conditions yield topological restrictions on a manifold.
- Decomposition of curvature tensor into irreducible summands.

Some basic references are [Bes87], [CLN06], [Lee97], [Pet06], [Poo81].

1 Lecture 1

1.1 Metrics, vectors, and one-forms

Let (M, g) be a Riemannian manifold, with metric $g \in \Gamma(S^2(T^*M))$. In coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i \otimes dx^j, \;\; g_{ij} = g_{ij}, \tag{1.1}$$

and $g_{ij} >> 0$ is a positive definite matrix. The symmetry condition is of course invariantly

$$g(X, Y) = g(Y, X). \tag{1.2}$$

A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M), \tag{1.3}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \tag{1.4}$$

is the coordinate partial. We will use the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

A 1-form is a section of the cotangent bundle, $X \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad \omega_i \in C^\infty(M). \tag{1.5}$$

Remark 1.1. Note that components of vector fields have upper indices, while components of 1-forms have lower indices. However, a collection of vector fields will be indexed by lower indices, $\{Y_1, \ldots, Y_p\}$, and a collection of 1-forms will be indexed by upper indices $\{dx^1, \ldots, dx^n\}$. This is one reason why we write the coordinates with upper indices.

1.2 The musical isomorphisms

The metric gives an isomorphism between TM and T^*M,

$$\flat : TM \to T^*M \tag{1.6}$$

defined by

$$\flat(X)(Y) = g(X,Y). \tag{1.7}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp\omega_1, \sharp\omega_2). \tag{1.8}$$

Note that if g has components g_{ij}, then $\langle \cdot, \cdot \rangle$ has components g^{ij}, the inverse matrix of g_{ij}.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{1.9}$$

where

$$X_i = g_{ij} X^j, \tag{1.10}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{1.11}$$

where

$$\omega^i = g^{ij} \omega_j, \tag{1.12}$$

thus the sharp operator "raises" an index.

1.3 Exterior algebra and wedge product

For a real vector space V, a differential form is an element of $\Lambda^p(V^*)$. The wedge product of $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ is a form in $\Lambda^{p+q}(V^*)$ defined as follows. The exterior algebra $\Lambda(V^*)$ is the tensor algebra

$$\Lambda(V^*) = \Big\{ \bigoplus_{k \geq 0} V^{\otimes^k} \Big\} / \mathcal{I} = \bigoplus_{k \geq 0} \Lambda^k(V^*) \tag{1.13}$$

where $\mathcal{I}$ is the two-sided ideal generated by elements of the form $\alpha \otimes \alpha \in V^* \otimes V^*$. The wedge product of $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ is just the multiplication induced by the tensor product in this algebra.

The space $\Lambda^k(V^*)$ satisfies the universal mapping property as follows. Let W be any vector space, and $F : (V^*)^{\otimes^k} \to W$ an alternating multilinear mapping. That is, $F(\alpha^1, \ldots, \alpha^k) = 0$ if $\alpha^i = \alpha^j$ for some i, j. Then there is a unique linear map $\tilde{F}$ which makes the following diagram

$$\begin{array}{ccc} (V^*)^{\otimes^k} & \xrightarrow{\pi} & \Lambda^k(V^*) \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle \tilde{F}} \\ & & W \end{array}$$

commutative, where π is the projection

$$\pi(\alpha^1, \ldots, \alpha^k) = \alpha^1 \wedge \cdots \wedge \alpha^k \tag{1.14}$$

We could just stick with this definition and try and prove all results using only this definition. However, for calculational purposes, it is convenient to think of differential forms as alternating linear maps from $V^{\otimes^k} \to \mathbb{R}$. For this, one has to choose a pairing

$$\Lambda^k(V^*) \cong (\Lambda^k(V))^*. \tag{1.15}$$

The pairing we will choose is as follows. If $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $v = v_1 \wedge \cdots \wedge v_k$, then

$$\alpha(v) = \det(\alpha^i(v_j)). \tag{1.16}$$

For example,

$$\alpha^1 \wedge \alpha^2(v_1 \wedge v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{1.17}$$

Then to view as a mapping from $V^{\otimes^k} \to \mathbb{R}$, we specify that if $\alpha \in (\Lambda^k(V))^*$, then

$$\alpha(v_1, \ldots, v_k) \equiv \alpha(v_1 \wedge \cdots \wedge v_k). \tag{1.18}$$

For example

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1). \tag{1.19}$$

With this convention, if $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$ then

$$\alpha \wedge \beta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}). \tag{1.20}$$

This then agrees with the definition of the wedge product given in [Spi79, Chapter 7].

Some important properties of the wedge product

- The wedge product is bilinear $(\alpha^1 + \alpha^2) \wedge \beta = \alpha^1 \wedge \beta + \alpha^2 \wedge \beta$, and $(c\alpha) \wedge \beta = c(\alpha \wedge \beta)$ for $c \in \mathbb{R}$.
- If $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$, then $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$.
- The wedge product is associative $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

It is convenient to have our 2 definitions of the wedge product because the proofs of these properties can be easier using one of the definitions, but harder using the other.

1.4 Differential forms and the d operator

A differential form is a section of $\Lambda^p(T^*M)$. I.e., a differential form is a smooth mapping $\omega : M \to \Lambda^p(T^*M)$ such that $\pi\omega = Id_M$, where $\pi : \Lambda^p(T^*M) \to M$ is the bundle projection map. We will write $\omega \in \Gamma(\Lambda^p(T^*M))$, or $\omega \in \Omega^p(M)$.

Given a coordinate system $x^i : U \to \mathbb{R}$, $i = 1 \dots n$, a local basis of T^*M is given by $dx^1, \dots, dx^n$. Then $\alpha \in \Omega^p(V^*)$ can be written as

$$\alpha = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{1.21}$$

Then we also have

$$\alpha = \frac{1}{p!} \sum_{1 \le i_1, i_2, \dots, i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \tag{1.22}$$

where the sum is over ALL indices.

However, if we want to think of α as a multilinear mapping from $TM^{\otimes^p} \to \mathbb{R}$, then we extend the coefficients $\alpha_{i_1 \dots i_p}$, which are only defined for strictly increasing sequences $i_1 < \cdots < i_p$, to ALL indices by skew-symmetry. Then we have

$$\alpha = \sum_{1 \le i_1, i_2, \dots, i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p}. \tag{1.23}$$

This convention is slightly annoying because then the projection to the exterior algebra of this is $p!$ times the original α, but has the positive feature that coefficients depending upon p do not enter into various formulas.

The exterior derivative operator [War83, Theorem 2.20],

$$d : \Omega^p(T^*M) \to \Omega^{p+1}(T^*M) \tag{1.24}$$

is the unique anti-derivation satisfying

- For $\alpha \in \Omega^p(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.
- $d^2 = 0$.
- If $f \in C^\infty(M)$ then df is the differential of f. (I.e., $f_* : TM \to \mathbb{R}$ is a element of $Hom(TM, \mathbb{R})$ which is unambiguously an element of $\Gamma(T^*M) = \Omega^1(M)$.)

Next, letting $Alt^p(TM)$ denote the alernating multilinear maps from $TM^{\otimes^p} \to \mathbb{R}$, then d can be considered as a mapping

$$d : Alt^p(TM) \to Alt^{p+1}(TM) \tag{1.25}$$

given by the formula

$$\begin{aligned} d\omega(X_0, \dots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \dots, \hat{X}_j, \dots, X_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p), \end{aligned} \tag{1.26}$$

which agrees with the formula for d given in [Spi79, Chapter 7].

Note that in a coordinate system, d is given by

$$(d\alpha)_{i_0\dots i_p} = \sum_{j=0}^{p}(-1)^j \partial_{i_j}\alpha_{i_0\dots\hat{i_j}\dots i_p}. \tag{1.27}$$

(Note this is indeed skew-symmetric in all indices.)

When we bring a Riemannian metric g into the picture, there will be an issue that comes up. If e^i is an ONB of T^*M then we would like

$$e^{i_1} \wedge \dots \wedge e^{i_p} \tag{1.28}$$

to be a unit norm element in $\Lambda^p(T^*M)$. However, when we view this as an alternating tensor, the tensor norm is given by $p!$. We will discuss this next.

1.5 Inner product on tensor bundles

The metric induces a metric on $\Lambda^k(T^*M)$. We give 3 definitions, all of which are equivalent:

- Definition 1: If

$$\begin{aligned} \omega^1 &= \alpha^1 \wedge \dots \wedge \alpha^k \\ \omega^2 &= \beta^1 \wedge \dots \wedge \beta^k, \end{aligned} \tag{1.29}$$

  then

$$\langle \omega^1, \omega^2 \rangle = \det(\langle \alpha^i, \beta^j \rangle), \tag{1.30}$$

  and extend linearly. This is well-defined.

- Definition 2: If $\{e_i\}$ is an ONB of T_pM, let $\{e^i\}$ denote the dual basis, defined by $e^i(e_j) = \delta^i_j$. Then declare that

$$e^{i_1} \wedge \dots \wedge e^{i_k}, \quad 1 \le i_1 < i_2 < \dots < i_k \le n, \tag{1.31}$$

  is an ONB of $\Lambda^k(T^*_pM)$.

- Definition 3: If $\omega \in \Lambda^k(T^*M)$, then in coordinates

$$\omega = \sum_{1\le i_1<\dots<i_k\le n} \omega_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}. \tag{1.32}$$

  Then

$$\|\omega\|^2_{\Lambda^k} = \langle \omega, \omega \rangle = \sum_{1\le i_1<\dots<i_k\le n} \omega^{i_1\dots i_k}\omega_{i_1\dots i_k}, \tag{1.33}$$

  where

$$\omega^{i_1\dots i_k} = g^{i_1 l_i} g^{i_2 l_2} \dots g^{i_k l_k} \omega_{l_1\dots l_k}. \tag{1.34}$$

To define an inner product on the full tensor bundle, we let

$$\Omega \in \Gamma\Big((TM)^{\otimes p} \otimes (T^*M)^{\otimes q}\Big). \tag{1.35}$$

We call such Ω a (p,q)-*tensor field*. As above, we can define a metric by declaring that

$$e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q} \tag{1.36}$$

to be an ONB. If in coordinates,

$$\Omega = \Omega^{i_1 \dots i_p}_{j_1 \dots j_q} \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}, \tag{1.37}$$

then

$$\|\Omega\|^2 = \langle \Omega, \Omega \rangle = \Omega^{j_1 \dots j_q}_{i_1 \dots i_p} \Omega^{i_1 \dots i_p}_{j_1 \dots j_q}, \tag{1.38}$$

where the term $\Omega^{j_1 \dots j_q}_{i_1 \dots i_p}$ is obtained by raising all of the lower indices and lowering all of the upper indices of $\Omega^{j_1 \dots j_q}_{i_1 \dots i_p}$, using the metric. By polarization, the inner product is given by

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2}\Big(\|\Omega_1 + \Omega_2\|^2 - \|\Omega_1\|^2 - \|\Omega_2\|^2\Big). \tag{1.39}$$

Remark 1.2. Recall we are using (1.16) to identify forms and alternating tensors. If $\omega \in \Lambda^p(T^*M)$, then if we view ω as an alternating p-tensor, then

$$\|\omega\|_{(T^*M)^{\otimes p}} = \sqrt{p!}\|\omega\|_{\Lambda^p}. \tag{1.40}$$

For example, as an element of $\Lambda^2(T^*M)$, $e^1 \wedge e^2$ has norm 1 if e^1, e^2 are orthonormal in T^*M. But under our identification with tensors, $e^1 \wedge e^2$ is identified with $e^1 \otimes e^2 - e^2 \otimes e^1$, which has norm $\sqrt{2}$ with respect to the tensor inner product. Thus our identification in (1.16) is *not* an isometry, but is a constant multiple of an isometry.

We remark that one may reduce a (p,q)-tensor field into a $(p-1,q-1)$-tensor field for $p \geq 1$ and $q \geq 1$. This is called a *contraction*, but one must specify which indices are contracted. For example, the contraction of Ω in the first contrvariant index and first covariant index is written invariantly as

$$Tr_{(1,1)}\Omega, \tag{1.41}$$

and in coordinates is given by

$$\delta^{j_1}_{i_1} \Omega^{i_1 \dots i_p}_{j_1 \dots j_q} = \Omega^{l i_2 \dots i_p}_{l j_2 \dots j_q}. \tag{1.42}$$

2 Lecture 2

2.1 Connections on vector bundles

A connection is a mapping $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, with the properties

- $\nabla_X s \in \Gamma(E)$,
- $\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$,
- $\nabla_X (fs) = (Xf)s + f \nabla_X s$.

In coordinates, letting $s_i, i = 1 \dots p$, be a local basis of sections of E,

$$\nabla_{\partial_i} s_j = \Gamma_{ij}^k s_k. \tag{2.1}$$

If E carries an inner product, then ∇ is *compatible* if

$$X \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle. \tag{2.2}$$

For a connection in TM, ∇ is called *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM). \tag{2.3}$$

Theorem 2.1. *(Fundamental Theorem of Riemannian Geometry) There exists a unique symmetric, compatible connection in TM.*

Invariantly, the connection is defined by

$$\begin{aligned} \langle \nabla_X Y, Z \rangle = \frac{1}{2} \Big(& X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle \\ & - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \Big). \end{aligned} \tag{2.4}$$

Letting $X = \partial_i, Y = \partial_j, Z = \partial_k$, we obtain

$$\begin{aligned} \Gamma_{ij}^l g_{lk} = \langle \Gamma_{ij}^l \partial_l, \partial_k \rangle &= \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle \\ &= \frac{1}{2} \Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \Big), \end{aligned} \tag{2.5}$$

which yields the formula

$$\boxed{\Gamma_{ij}^k = \frac{1}{2} g^{kl} \Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \Big)} \tag{2.6}$$

for the Riemannian Christoffel symbols.

2.2 Curvature in the tangent bundle

The curvature tensor is defined by

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.7}$$

for vector fields X, Y, and Z. We define

$$Rm(X,Y,Z,W) \equiv -\langle \mathcal{R}(X,Y)Z, W \rangle. \tag{2.8}$$

We will refer to $\mathcal{R}$ as the curvature tensor of type $(1,3)$ and to Rm as the curvature tensor of type $(0,4)$.

The algebraic symmetries are:

$$\mathcal{R}(X,Y)Z = -\mathcal{R}(Y,X)Z \tag{2.9}$$

$$0 = \mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y \tag{2.10}$$

$$Rm(X,Y,Z,W) = -Rm(X,Y,W,Z) \tag{2.11}$$

$$Rm(X,Y,W,Z) = Rm(W,Z,X,Y). \tag{2.12}$$

In a coordinate system we define quantities ${R_{ijk}}^l$ by

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = {R_{ijk}}^l \partial_l, \tag{2.13}$$

or equivalently,

$$\mathcal{R} = {R_{ijk}}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l. \tag{2.14}$$

Define quantities R_{ijkl} by

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l), \tag{2.15}$$

or equivalently,

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{2.16}$$

Then

$$R_{ijkl} = -\langle \mathcal{R}(\partial_i, \partial_j)\partial_k, \partial_l \rangle = -\langle {R_{ijk}}^m \partial_m, \partial_l \rangle = -{R_{ijk}}^m g_{ml}. \tag{2.17}$$

Equivalently,

$$R_{ijlk} = {R_{ijk}}^m g_{ml}, \tag{2.18}$$

that is, we lower the upper index to the *third* position.

Remark 2.2. Some authors choose to lower this index to a different position. One has to be very careful with this, or you might end up proving that S^n has negative curvature!

In coordinates, the algebraic symmetries of the curvature tensor are

$$R_{ijk}{}^{l} = -R_{jik}{}^{l} \tag{2.19}$$
$$0 = R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kij}{}^{l} \tag{2.20}$$
$$R_{ijkl} = -R_{ijlk} \tag{2.21}$$
$$R_{ijkl} = R_{klij}. \tag{2.22}$$

Of course, we can write the first 2 symmetries as a $(0,4)$ tensor,

$$R_{ijkl} = -R_{jikl} \tag{2.23}$$
$$0 = R_{ijkl} + R_{jkil} + R_{kijl}. \tag{2.24}$$

Note that using (2.22), the algebraic Bianchi identity (2.24) may be written as

$$0 = R_{ijkl} + R_{iklj} + R_{iljk}. \tag{2.25}$$

We next compute the curvature tensor in coordinates.

$$\begin{aligned}
\mathcal{R}(\partial_i, \partial_j)\partial_k &= R_{ijk}{}^{l}\partial_l \\
&= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \partial_i(\Gamma^l_{jk})\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m - \partial_j(\Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m \\
&= \Big(\partial_i(\Gamma^l_{jk}) + \Gamma^m_{jk}\Gamma^l_{im} - \partial_j(\Gamma^l_{ik}) - \Gamma^m_{ik}\Gamma^l_{jm}\Big)\partial_l,
\end{aligned} \tag{2.26}$$

which is the formula

$$\boxed{R_{ijk}{}^{l} = \partial_i(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{ik}) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}} \tag{2.27}$$

Fix a point p. Exponential coordinates around p form a normal coordinate system at p. That is $g_{ij}(p) = \delta_{ij}$, and $\partial_k g_{ij}(p) = 0$, which is equivalent to $\Gamma^k_{ij}(p) = 0$. The Christoffel symbols are

$$\Gamma^l_{jk} = \frac{1}{2}g^{lm}\Big(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\Big). \tag{2.28}$$

In normal coordinates at the point p,

$$\partial_i\Gamma^l_{jk} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} + \partial_i\partial_j g_{km} - \partial_i\partial_m g_{jk}\Big). \tag{2.29}$$

We then have at p

$$R_{ijk}{}^{l} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} - \partial_i\partial_m g_{jk} - \partial_j\partial_k g_{im} + \partial_j\partial_m g_{ik}\Big). \tag{2.30}$$

Lowering an index, we have at p

$$\begin{aligned}
R_{ijkl} &= -\frac{1}{2}\Big(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}\Big) \\
&= -\frac{1}{2}\Big(\partial^2 \owedge g\Big).
\end{aligned} \tag{2.31}$$

The $\owedge$ symbol is the Kulkarni-Nomizu product, which takes 2 symmetric $(0,2)$ tensors and gives a $(0,4)$ tensor with the same algebraic symmetries of the curvature tensor, and is defined by

$$\begin{aligned} A \owedge B(X,Y,Z,W) =& A(X,Z)B(Y,W) - A(Y,Z)B(X,W) \\ & - A(X,W)B(Y,Z) + A(Y,W)B(X,Z). \end{aligned}$$

To remember: the first term is $A(X,Z)B(Y,W)$, skew symmetrize in X and Y to get the second term. Then skew-symmetrize both of these in Z and W.

2.3 Covariant derivatives of tensor fields

Let E and E' be vector bundles over M, with covariant derivative operators ∇, and ∇', respectively. The covariant derivative operators in $E \otimes E'$ and $Hom(E,E')$ are

$$\nabla_X(s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s') \tag{2.32}$$

$$(\nabla_X L)(s) = \nabla'_X(L(s)) - L(\nabla_X s), \tag{2.33}$$

for $s \in \Gamma(E), s' \in \Gamma(E')$, and $L \in \Gamma(Hom(E,E'))$. Note also that the covariant derivative operator in $\Lambda(E)$ is given by

$$\nabla_X(s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^{r} s_1 \wedge \cdots \wedge (\nabla_X s_i) \wedge \cdots \wedge s_r, \tag{2.34}$$

for $s_i \in \Gamma(E)$.

These rules imply that if T is an (r,s) tensor, then the covariant derivative ∇T is an $(r, s+1)$ tensor given by

$$\nabla T(X, Y_1, \ldots, Y_s) = \nabla_X(T(Y_1, \ldots Y_s)) - \sum_{i=1}^{s} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s). \tag{2.35}$$

We next consider the above definitions in components for (r,s)-tensors. For the case of a vector field $X \in \Gamma(TM)$, ∇X is a $(1,1)$ tensor field. By the definition of a connection, we have

$$\nabla_m X \equiv \nabla_{\partial_m} X = \nabla_{\partial_m}(X^j \partial_j) = (\partial_m X^j)\partial_j + X^j \Gamma^l_{mj}\partial_l = (\nabla_m X^i + X^l \Gamma^i_{ml})\partial_i. \tag{2.36}$$

In other words,

$$\nabla X = \nabla_m X^i (dx^m \otimes \partial_i), \tag{2.37}$$

where

$$\nabla_m X^i = \partial_m X^i + X^l \Gamma^i_{ml}. \tag{2.38}$$

However, for a 1-form ω, (2.33) implies that

$$\nabla \omega = (\nabla_m \omega_i) dx^m \otimes dx^i, \tag{2.39}$$

with

$$\nabla_m \omega_i = \partial_m \omega_i - \omega_l \Gamma^l_{im}. \tag{2.40}$$

The definition (2.32) then implies that for a general (r,s)-tensor field S,

$$\begin{aligned}\nabla_m S^{i_1 \dots i_r}_{j_1 \dots j_s} \equiv \partial_m S^{i_1 \dots i_r}_{j_1 \dots j_s} &+ S^{l i_2 \dots i_r}_{j_1 \dots j_s} \Gamma^{i_1}_{ml} + \dots + S^{i_1 \dots i_{r-1} l}_{j_1 \dots j_s} \Gamma^{i_r}_{ml} \\ &- S^{i_1 \dots i_r}_{l j_2 \dots j_s} \Gamma^{l}_{m j_1} - \dots - S^{i_1 \dots i_r}_{j_1 \dots j_{s-1} l} \Gamma^{l}_{m j_s}.\end{aligned} \tag{2.41}$$

Remark 2.3. Some authors instead write covariant derivatives with a semi-colon

$$\nabla_m S^{i_1 \dots i_r}_{j_1 \dots j_s} = S^{i_1 \dots i_r}_{j_1 \dots j_s ; m}. \tag{2.42}$$

However, the ∇ notation fits nicer with our conventions, since the *first* index is the direction of covariant differentiation.

Notice the following calculation,

$$(\nabla g)(X, Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \tag{2.43}$$

so the metric is parallel. Note that in coordinates, this says that

$$0 = \nabla_m g_{ij} = \partial_m g_{ij} - \Gamma^p_{mi} g_{pj} - \Gamma^p_{mj} g_{ip}, \tag{2.44}$$

which yield the formula

$$\partial_k g_{ij} = \Gamma^p_{ki} g_{pj} + \Gamma^p_{kj} g_{ip}. \tag{2.45}$$

This is sometimes written as

$$\partial_k g_{ij} = [ki, j] + [kj, i], \tag{2.46}$$

where $[ij;k]$ are called the Christoffel symbols of the first kind defined by

$$[ij, k] \equiv \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \Big). \tag{2.47}$$

Next, let $I : TM \to TM$ denote the identity map, which is naturally a $(1,1)$ tensor. We have

$$(\nabla I)(X, Y) = \nabla_X (I(Y)) - I(\nabla_X Y) = \nabla_X Y - \nabla_X Y = 0, \tag{2.48}$$

so the identity map is also parallel.
Note that the following statements are equivalent

- $\flat \in Hom(TM, T^*M)$ is parallel
- $\flat$ commutes with covariant differentiation.
- $\nabla_m (g_{ij} X^j) = g_{ij} \nabla_m X^j$.

Similarly, the induced metric on T^*M is parallel, and the following are equivalent.

- $\sharp \in Hom(T^*M, TM)$ is parallel
- $\sharp$ commutes with covariant differentiation.
- $\nabla_m(g^{ij}\omega_j) = g^{ij}\nabla_m\omega_j$.

Finally, note that the following are equivalent

- Tr mapping from (p,q)-tensors to $(p-1,q-1)$ tensors is parallel.
- Tr commutes with covariant differentiation.
- $\nabla_m\left(\delta^{j_1}_{i_1} X^{i_1 i_2 \dots}_{j_1 j_2 \dots}\right) = \delta^{j_1}_{i_1}\nabla_m X^{i_1 i_2 \dots}_{j_1 j_2 \dots}$.

3 Lecture 3

3.1 Double covariant derivatives

For an (r,s) tensor field T, we will write the double covariant derivative as

$$\nabla^2 T = \nabla\nabla T, \tag{3.1}$$

which is an $(r, s+2)$ tensor.

Proposition 3.1. *If T is an (r,s)-tensor field, then the double covariant derivative satisfies*

$$\nabla^2 T(X, Y, Z_1, \dots, Z_s) = \nabla_X(\nabla_Y T)(Z_1, \dots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \dots Z_s). \tag{3.2}$$

Proof. The left hand side of (3.2) is

$$\begin{aligned} \nabla^2 T(X, Y, Z_1, \dots, Z_s) &= \nabla(\nabla T)(X, Y, Z_1, \dots, Z_s) \\ &= \nabla_X(\nabla T(Y, Z_1, \dots, Z_s)) - \nabla T(\nabla_X Y, Z_1, \dots, Z_s) \\ &\quad - \sum_{i=1}^{s} \nabla T(Y, \dots, \nabla_X Z_i, \dots Z_s). \end{aligned} \tag{3.3}$$

The right hand side of (3.2) is

$$\begin{aligned} &\nabla_X(\nabla_Y T)(Z_1, \dots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \dots Z_s) \\ &= \nabla_X(\nabla_Y T(Z_1, \dots, Z_s)) - \sum_{i=1}^{s} (\nabla_Y T)(Z_1, \dots, \nabla_X Z_i, \dots, Z_s) \\ &\quad - \nabla T(\nabla_X Y, Z_1, \dots, Z_s). \end{aligned} \tag{3.4}$$

The first term on the right hand side of (3.4) is the same as first term on the right hand side of (3.3). The second term on the right hand side of (3.4) is the same as third term on the right hand side of (3.3). Finally, the last term on the right hand side of (3.4) is the same as the second term on the right hand side of (3.3). □

Remark 3.2. When we write

$$\nabla_i \nabla_j T^{j_1 \ldots j_r}_{i_i \ldots i_s} \tag{3.5}$$

we mean the components of the double covariant derivative of T as a $(r, s+2)$ tensor. This does NOT mean to take one covariant derivative ∇T, plug in ∂_j to get an (r, s) tensor, and then take a covariant derivative in the ∂_i direction; this would yield only the first term on the right hand side of (3.2).

For illustration, let's compute an example in coordinates. If $\omega \in \Omega^1(M)$, then

$$\begin{aligned} \nabla_i \nabla_j \omega_k &= \partial_i(\nabla_j \omega_k) - \Gamma^p_{ij} \nabla_p \omega_k - \Gamma^p_{ik} \nabla_j \omega_p \\ &= \partial_i(\partial_j \omega_k - \Gamma^l_{jk} \omega_l) - \Gamma^p_{ij}(\partial_p \omega_k - \Gamma^l_{pk} \omega_l) - \Gamma^p_{ik}(\partial_j \omega_p - \Gamma^l_{jp} \omega_k). \end{aligned} \tag{3.6}$$

Expanding everything out, we can write this formally as

$$\nabla^2 \omega = \partial^2 \omega_k + \Gamma * \partial \omega + (\partial \Gamma + \Gamma * \Gamma) * \omega, \tag{3.7}$$

where $*$ denotes various tensor contractions. Notice that the coefficient of ω on the right looks similar to the curvature tensor in coordinates (2.27). This is closely related to Weitzenböck formulas which we will discuss later.

3.2 Commuting covariant derivatives

Let $X, Y, Z \in \Gamma(TM)$, and compute using Proposition 3.1

$$\begin{aligned} \nabla^2 Z(X,Y) - \nabla^2 Z(Y,X) &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_X Y - \nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \\ &= \mathcal{R}(X,Y)Z, \end{aligned} \tag{3.8}$$

which is just the definition of the curvature tensor. In coordinates,

$$\boxed{\nabla_i \nabla_j Z^k = \nabla_j \nabla_i Z^k + R_{ijm}{}^k Z^m.} \tag{3.9}$$

We extend this to $(p, 0)$-tensor fields:

$$\begin{aligned} &\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X) \\ &= \nabla_X(\nabla_Y(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_X Y}(Z_1 \otimes \cdots \otimes Z_p) \\ &\quad - \nabla_Y(\nabla_X(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_Y X}(Z_1 \otimes \ldots \otimes Z_p \\ &= \nabla_X\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p\Big) - \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_X Y} Z_i \otimes \cdots \otimes Z_p \\ &- \nabla_Y\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p\Big) + \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_Y X} Z_i \otimes \cdots \otimes Z_p. \end{aligned} \tag{3.10}$$

With a slight abuse of notation, this may be rewritten as

$$
\begin{aligned}
&\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X) \\
&= \sum_{j=1}^{p} \sum_{i=1, i\neq j}^{p} Z_1 \otimes \nabla_X Z_j \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p \\
&\qquad - \sum_{j=1}^{p} \sum_{i=1, i\neq j}^{p} Z_1 \otimes \nabla_Y Z_j \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p \\
&\qquad\qquad + \sum_{i=1}^{p} Z_1 \otimes \cdots \otimes (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z_i \otimes \cdots \otimes Z_p \\
&= \sum_{i=1}^{p} Z_1 \otimes \cdots \otimes \mathcal{R}(X,Y) Z_i \otimes \cdots \otimes Z_p.
\end{aligned}
\tag{3.11}
$$

In coordinates, this is

$$
\boxed{\nabla_i \nabla_j Z^{i_1 \dots i_p} = \nabla_j \nabla_i Z^{i_i \dots i_p} + \sum_{k=1}^{p} R_{ijm}{}^{i_k} Z^{i_1 \dots i_{k-1} m i_{k+1} \dots i_p}.}
\tag{3.12}
$$

Proposition 3.3. *For a 1-form ω, we have*

$$
\nabla^2 \omega(X,Y,Z) - \nabla^2 \omega(Y,X,Z) = \omega(\mathcal{R}(Y,X)Z).
\tag{3.13}
$$

Proof. Using Proposition 3.1, we compute

$$
\begin{aligned}
&\nabla^2 \omega(X,Y,Z) - \nabla^2 \omega(Y,X,Z) \\
&= \nabla_X(\nabla_Y \omega)(Z) - (\nabla_{\nabla_X Y}\omega)(Z) - \nabla_Y(\nabla_X \omega)(Z) - (\nabla_{\nabla_Y X}\omega)(Z) \\
&= X(\nabla_Y \omega(Z)) - \nabla_Y \omega(\nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y} Z) \\
&\quad - Y(\nabla_X \omega(Z)) + \nabla_X \omega(\nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X} Z) \\
&= X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y \nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y} Z) \\
&\quad - Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) - \omega(\nabla_X \nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X} Z) \\
&= \omega\Big(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\Big) + X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z)) \\
&\quad - Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)).
\end{aligned}
\tag{3.14}
$$

The last six terms are

$$
\begin{aligned}
&X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z)) \\
&- Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)) \\
&= X\Big(Y(\omega(Z)) - \omega(\nabla_Y Z)\Big) - Y(\omega(\nabla_X Z)) - [X,Y](\omega(Z)) \\
&- Y\Big(X(\omega(Z)) - \omega(\nabla_X Z)\Big) + X(\omega(\nabla_Y Z)) \\
&= 0.
\end{aligned}
\tag{3.15}
$$

□

Remark 3.4. It would have been a lot easier to assume we were in normal coordinates, and ignore terms involving first covariant derivatives of the vector fields, but we did the above for illustration.

In coordinates, this formula becomes

$$\boxed{\nabla_i \nabla_j \omega_k = \nabla_j \nabla_i \omega_k - R_{ijk}{}^{p} \omega_p.} \tag{3.16}$$

As above, we can extend this to $(0, s)$ tensors using the tensor product, in an almost identical calculation to the $(r, 0)$ tensor case. Finally, putting everything together, the analogous formula in coordinates for a general (r, s)-tensor T is

$$\boxed{\nabla_i \nabla_j T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \nabla_j \nabla_i T^{i_1 \ldots i_r}_{j_1 \ldots j_s} + \sum_{k=1}^{r} R_{ijm}{}^{i_k} T^{i_1 \ldots i_{k-1} m i_{k+1} \ldots i_r}_{j_1 \ldots j_s} - \sum_{k=1}^{s} R_{ijj_k}{}^{m} T^{i_1 \ldots i_r}_{j_1 \ldots j_{k-1} m j_{k+1} \ldots j_s}.} \tag{3.17}$$

3.3 Gradient, Hessian, and Laplacian

As an example of the above, we consider the Hessian of a function. For $f \in C^1(M, \mathbb{R})$, the *gradient* is defined as

$$\nabla f = \sharp(df), \tag{3.18}$$

which is a vector field. This is standard notation, although in our notation above, $\nabla f = df$, where this ∇ denotes the covariant derivative. The *Hessian* is the $(0, 2)$-tensor defined by the double covariant derivative of a function, which by Proposition 3.1 is given by

$$\nabla^2 f(X, Y) = \nabla_X (\nabla_Y f) - \nabla_{\nabla_X Y} f = X(Y f) - (\nabla_X Y) f. \tag{3.19}$$

In components, this formula is

$$\nabla^2 f(\partial_i, \partial_j) = \nabla_i \nabla_j f = \partial_i \partial_j f - \Gamma^k_{ij} (\partial_k f). \tag{3.20}$$

The symmetry of the Hessian

$$\nabla^2 f(X, Y) = \nabla^2 f(Y, X), \tag{3.21}$$

then follows easily from the symmetry of the Riemannian connection. Notice that no curvature terms appear in this formula, which happens only in this special case.

The Laplacian of a function is the trace of the Hessian when considered as an endomorphism,

$$\Delta f = tr\big(X \mapsto \sharp(\nabla^2 f(X, \cdot))\big), \tag{3.22}$$

so in coordinates is given by

$$\Delta f = g^{ij} \nabla_i \nabla_j f. \tag{3.23}$$

This turns out to be equal to

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \partial_i \big(g^{ij} \partial_j f \sqrt{\det(g)} \big). \tag{3.24}$$

In a local orthonormal frame $\{e_i\}, i = 1 \dots n$, the formula for the Hessian looks like

$$\begin{aligned}(\nabla^2 f)(e_i, e_j) &= \nabla_{e_i}(\nabla_{e_j} f) - \nabla_{\nabla_{e_i} e_j} f \\ &= e_i(e_j f) - (\nabla_{e_i} e_j) f,\end{aligned} \tag{3.25}$$

and the Laplacian is given by the expression

$$\Delta f = \sum_{i=1}^{n} \nabla^2 f(e_i, e_i) = \sum_{i=1}^{n} e_i(e_i f) - \sum_{i=1}^{n} (\nabla_{e_i} e_i) f. \tag{3.26}$$

3.4 Sectional curvature, Ricci tensor, and scalar curvature

Let $\Pi \subset T_p M$ be a 2-plane, and let $X_p, Y_p \in T_p M$ span Π. Then

$$K(\Pi) = \frac{Rm(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{g(\mathcal{R}(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{3.27}$$

is independent of the particular chosen basis for Π, and is called the *sectional curvature* of the 2-plane Π. The sectional curvatures in fact determine the full curvature tensor:

Proposition 3.5. *Let Rm and Rm' be two curvature tensors of type $(0,4)$ which satisfy $K(\Pi) = K'(\Pi)$ for all 2-planes Π, then $Rm = Rm'$.*

From this proposition, if $K(\Pi) = k_0$ is constant for all 2-planes Π, then we must have

$$Rm(X,Y,Z,W) = k_0 \Big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W) \Big), \tag{3.28}$$

That is

$$Rm = \frac{k_0}{2} g \owedge g. \tag{3.29}$$

In coordinates, this is

$$R_{ijkl} = k_0 (g_{ik} g_{jl} - g_{jk} g_{il}). \tag{3.30}$$

We define the *Ricci tensor* as the $(0,2)$-tensor

$$Ric(X,Y) = tr(U \to \mathcal{R}(U,X)Y). \tag{3.31}$$

We clearly have

$$Ric(X,Y) = Ric(Y,X), \tag{3.32}$$

so $Ric \in \Gamma(S^2(T^*M))$. We let R_{ij} denote the components of the Ricci tensor,

$$Ric = R_{ij}dx^i \otimes dx^j, \tag{3.33}$$

where $R_{ij} = R_{ji}$. From the definition,

$$R_{ij} = R_{lij}{}^{l} = g^{lm}R_{limj}. \tag{3.34}$$

Notice for a space of constant curvature, we have

$$R_{jl} = g^{ik}R_{ijkl} = k_0 g^{ik}(g_{ik}g_{jl} - g_{jk}g_{il}) = (n-1)k_0 g_{jl}, \tag{3.35}$$

or invariantly

$$Ric = (n-1)k_0 g. \tag{3.36}$$

The *Ricci endomorphism* is defined by

$$Rc(X) \equiv \sharp\Big(Ric(X,\cdot)\Big). \tag{3.37}$$

The *scalar curvature* is defined as the trace of the Ricci endomorphism

$$R \equiv tr(X \to Rc(X)). \tag{3.38}$$

In coordinates,

$$R = g^{pq}R_{pq} = g^{pq}g^{lm}R_{lpmq}. \tag{3.39}$$

Note for a space of constant curvature k_0,

$$R = n(n-1)k_0. \tag{3.40}$$

3.5 Differential Bianchi Identity

Higher covariant derivatives of the curvature tensor must satisfy certain identities, the first of which is the following, which is known as the differential Bianchi identity.

Proposition 3.6. *The covariant derivative of the curvature tensor* ∇Rm *satisfies the relation*

$$\nabla Rm(X,Y,Z,V,W) + \nabla Rm(Y,Z,X,V,W) + \nabla Rm(Z,X,Y,V,W) = 0. \tag{3.41}$$

Proof. Since the equation is tensorial, we can compute in a normal coordinate system near a point p, letting the vector fields be the coordinate partials. This means we can ignore terms involving only first covariant derivatives of the vector fields. Also, Lie brackets can be ignored since they vanish identically in a neighborhood of p. We compute

$$\begin{aligned}
&\nabla Rm(X,Y,Z,V,W) + \nabla Rm(Y,Z,X,V,W) + \nabla Rm(Z,X,Y,V,W) \\
&= X(Rm(Y,Z,V,W)) + Y(Rm(Z,X,V,W)) + Z(Rm(X,Y,V,W)) \\
&= -X\langle \mathcal{R}(Y,Z)V,W\rangle - Y\langle \mathcal{R}(Z,X)V,W\rangle - Z\langle \mathcal{R}(X,Y)V,W\rangle \\
&= -\langle \nabla_X \mathcal{R}(Y,Z)V,W\rangle - \langle \nabla_Y \mathcal{R}(Z,X)V,W\rangle - \langle \nabla_Z \mathcal{R}(X,Y)V,W\rangle \\
&= -\langle \nabla_X\nabla_Y\nabla_Z V - \nabla_X\nabla_Z\nabla_Y V, W\rangle - \langle \nabla_Y\nabla_Z\nabla_X V - \nabla_Y\nabla_X\nabla_Z V, W\rangle \\
&\qquad - \langle \nabla_Z\nabla_X\nabla_Y V - \nabla_Z\nabla_Y\nabla_X V, W\rangle \\
&= -\langle \nabla_X\nabla_Y\nabla_Z V - \nabla_Y\nabla_X\nabla_Z V, W\rangle - \langle \nabla_Y\nabla_Z\nabla_X V - \nabla_Z\nabla_Y\nabla_X V, W\rangle \\
&\qquad - \langle \nabla_Z\nabla_X\nabla_Y V - \nabla_X\nabla_Z\nabla_Y V, W\rangle \\
&= Rm(X,Y,\nabla_Z V,W) + Rm(Y,Z,\nabla_X V,W) + Rm(Z,X,\nabla_Y V,W) \equiv 0.
\end{aligned} \tag{3.42}$$

□

In coordinates, this is equivalent to

$$\boxed{\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.} \tag{3.43}$$

Let us raise an index,

$$\nabla_i R_{jkm}{}^l + \nabla_j R_{kim}{}^l + \nabla_k R_{ijm}{}^l = 0. \tag{3.44}$$

Contract on the indices i and l,

$$0 = \nabla_l R_{jkm}{}^l + \nabla_j R_{klm}{}^l + \nabla_k R_{ljm}{}^l = \nabla_l R_{jkm}{}^l - \nabla_j R_{km} + \nabla_k R_{jm}. \tag{3.45}$$

This yields the Bianchi identity

$$\boxed{\nabla_l R_{jkm}{}^l = \nabla_j R_{km} - \nabla_k R_{jm}.} \tag{3.46}$$

In invariant notation, this is sometimes written as

$$\delta\mathcal{R} = d^\nabla Ric, \tag{3.47}$$

where $d^\nabla : S^2(T^*M) \to \Lambda^2(T^*M) \otimes T^*M$, is defined by

$$d^\nabla h(X,Y,Z) = \nabla h(X,Y,Z) - \nabla h(Y,Z,X), \tag{3.48}$$

and δ is the *divergence operator*.

Next, trace (3.46) on the indices k and m,

$$g^{km}\nabla_l R_{jkm}{}^l = g^{km}\nabla_j R_{km} - g^{km}\nabla_k R_{jm}. \tag{3.49}$$

Since the metric is parallel, we can move the g^{km} terms inside,

$$\nabla_l g^{km} R_{jkm}{}^l = \nabla_j g^{km} R_{km} - \nabla_k g^{km} R_{jm}. \tag{3.50}$$

The left hand side is

$$
\begin{aligned}
\nabla_l g^{km} R_{jkm}{}^{l} &= \nabla_l g^{km} g^{lp} R_{jkpm} \\
&= \nabla_l g^{lp} g^{km} R_{jkpm} \\
&= \nabla_l g^{lp} R_{jp} = \nabla_l R^l_j.
\end{aligned}
\tag{3.51}
$$

So we have the Bianchi identity

$$
\boxed{2\nabla_l R^l_j = \nabla_j R.} \tag{3.52}
$$

Invariantly, this can be written

$$
\delta Rc = \frac{1}{2} dR. \tag{3.53}
$$

Corollary 3.7. *Let (M, g) be a connected Riemannian manifold. If $n > 2$, and there exists a function $f \in C^\infty(M)$ satisfying $Ric = fg$, then $Ric = (n-1)k_0 g$, where k_0 is a constant.*

Proof. Taking a trace, we find that $R = nf$. Using (3.52), we have

$$
2\nabla_l R^l_j = 2\nabla_l \Big(\frac{R}{n} \delta^l_j \Big) = \frac{2}{n} \nabla_l R = \nabla_l R. \tag{3.54}
$$

Since $n > 2$, we must have $dR = 0$, which implies that R, and therefore f, is constant. $\square$

A metric satisfying $Ric = \Lambda g$ for a constant Λ is called an *Einstein metric*.

4 Lecture 4

4.1 The divergence of a tensor

If T is an (r, s)-tensor, we define the *divergence* of T, div T to be the $(r, s-1)$ tensor

$$
(\text{div } T)(Y_1, \dots, Y_{s-1}) = tr\Big(X \to \sharp(\nabla T)(X, \cdot, Y_1, \dots, Y_{s-1}) \Big), \tag{4.1}
$$

that is, we trace the covariant derivative on the *first* two covariant indices. In coordinates, this is

$$
(\text{div } T)^{i_1 \dots i_r}_{j_1 \dots j_{s-1}} = g^{ij} \nabla_i T^{i_1 \dots i_r}_{j j_1 \dots j_{s-1}}. \tag{4.2}
$$

Using an local orthonormal frame $\{e_i\}$, $i = 1 \dots n$, the divergence can also be written as

$$
(\text{div } T)(Y_1, \dots Y_{s-1}) = \sum_{i=1}^{n} (\nabla_{e_i} T)(e_i, Y_1, \dots, Y_{s-1}). \tag{4.3}
$$

If X is a vector field, define

$$(\text{div } X) = tr(\nabla X), \tag{4.4}$$

which is in coordinates

$$\text{div } X = \delta^i_j \nabla_i X^j = \nabla_j X^j. \tag{4.5}$$

For vector fields and 1-forms, these two are of course closely related:

Proposition 4.1. *For a vector field X,*

$$\text{div } X = \text{div } (\flat X). \tag{4.6}$$

Proof. We compute

$$\text{div } X = \delta^i_j \nabla_i X^j = \delta^i_j \nabla_i g^{jl} X_l = \delta^i_j g^{jl} \nabla_i X_l = g^{il} \nabla_i X_l = \text{div } (\flat X). \tag{4.7}$$

□

In a local orthonormal frame $\{e_i\}, i = 1 \dots n$, the divergence of a 1-form is given by

$$\begin{aligned} \text{div } \omega &= \sum_{i=1}^n (\nabla_{e_i} \omega)(e_i) \\ &= \sum_{i=1}^n e_i(\omega(e_i)) - \omega\Big(\sum_{i=1}^n \nabla_{e_i} e_i \Big), \end{aligned} \tag{4.8}$$

whereas the divergence of a vector field is given by

$$\text{div } X = \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle. \tag{4.9}$$

4.2 Volume element and Hodge star

If M is oriented, we define the Riemannian volume element dV to be the oriented unit norm element of $\Lambda^n(T^*M_x)$. Equivalently, if $\omega^1, \dots, \omega^n$ is a positively oriented ONB of T^*M_x, then

$$dV = \omega^1 \wedge \cdots \wedge \omega^n. \tag{4.10}$$

In coordinates,

$$dV = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n. \tag{4.11}$$

Recall the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$ defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV_x, \tag{4.12}$$

where $\alpha, \beta \in \Lambda^p$.

Remark 4.2. The inner product in (4.12) is the inner product on p-forms, *not* the tensor inner product.

The folowing proposition summarizes the main properties of the Hodge star operator that we will require.

Proposition 4.3. *The Hodge star operator satisfies the following.*

1. *The Hodge star is an isometry from Λ^p to Λ^{n-p}.*

2. $*(\omega^1 \wedge \cdots \wedge \omega^p) = \omega^{p+1} \wedge \cdots \wedge \omega^n$ *if* $\omega^1, \ldots, \omega^n$ *is a positively oriented ONB of* T^*M_x*. In particular,* $*1 = dV$*, and* $*dV = 1$.

3. *On* Λ^p, $*^2 = (-1)^{p(n-p)}$.

4. *For* $\alpha, \beta \in \Lambda^p$,

$$\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta) = *(\beta \wedge *\alpha). \tag{4.13}$$

5. *If* $\{e_i\}$ *and* $\{\omega^i\}$ *are dual ONB of* T_xM*, and* T_x^*M*, respectively, and* $\alpha \in \Lambda^p$*, then*

$$*(\omega^j \wedge \alpha) = (-1)^p i_{e_j}(*\alpha), \tag{4.14}$$

where $i_X : \Lambda^p \to \Lambda^{p-1}$ *is* interior multiplication *defined by*

$$i_X\alpha(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}). \tag{4.15}$$

6. *For* $\alpha \in \Omega^p(M)$*, in a coordinate system,*

$$(*\alpha)_{i_1 \ldots i_{n-p}} = \frac{1}{p!}\alpha^{j_1 \ldots j_p}\sqrt{\det(g)}\epsilon_{j_1 \ldots j_p i_1 \ldots i_{n-p}}, \tag{4.16}$$

where the ϵ *symbol is equal to* 1 *if* $(j_1, \ldots, j_p, i_1, \ldots, i_{n-p})$ *is an even permutation of* $(1, \ldots, n)$*, equal to* -1 *if it is an odd permutation, and zero otherwise.*

Proof. The proof is left to the reader. □

Remark 4.4. Note that interior multiplication is not canonically defined – it depends upon our identification of p-forms with alternating tensors of type $(0, p)$.

Remark 4.5. In general, locally there will be two different Hodge star operators, depending upon the two different choices of local orientation. Each will extend to a *global* Hodge star operator if and only if M is orientable. However, one can still construct *global* operators using the Hodge star, even if M is non-orientable, an example of which will be the Laplacian.

4.3 Exterior derivative and covariant differentiation

We next give a formula relating the exterior derivative and covariant differentiation.

Proposition 4.6. *The exterior derivative $d : \Omega^p \to \Omega^{p+1}$ is given by*

$$d\omega(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{4.17}$$

(recall the notation means that the $\hat{X}_j$ term is omitted). If $\{e_i\}$ and $\{\omega^i\}$ are dual ONB of T_xM, and T_x^*M, then this may be written

$$d\omega = \sum_{i=1}^{n} \omega^i \wedge \nabla_{e_i}\omega. \tag{4.18}$$

In coordinates, this is

$$(d\omega)_{i_0\ldots i_p} = \sum_{j=0}^{p} (-1)^j \nabla_{i_j}\omega_{i_0\ldots\hat{i_j}\ldots i_p}. \tag{4.19}$$

Proof. Recall the formula for the exterior derivative [War83, Proposition 2.25],

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{4.20}$$

Since both sides of the equation (4.17) are tensors, we may assume that $[X_i, X_j]_x = 0$, at a fixed point x. Since the connection is Riemannian, we also have $\nabla_{X_i}X_j(x) = 0$. We then compute at the point x.

$$\begin{aligned} d\omega(X_0, \ldots, X_p)(x) &= \sum_{j=0}^{p} (-1)^j X_j \Big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \Big)(x) \\ &= \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p)(x), \end{aligned} \tag{4.21}$$

using the definition of the covariant derivative. This proves the first formula (4.17). The formula (4.19) is just (4.17) in a coordinate system.

For (4.18), note that

$$\nabla_{X_j}\omega = \nabla_{(X_j)^i e_i}\omega = \sum_{i=1}^{n} \omega^i(X_j) \cdot (\nabla_{e_i}\omega), \tag{4.22}$$

so we have

$$\begin{aligned} d\omega(X_0,\ldots,X_p) &= \sum_{j=0}^{p}(-1)^j \sum_{i=1}^{n} \omega^i(X_j)\cdot(\nabla_{e_i}\omega)(X_0,\ldots,\hat{X}_j,\ldots,X_p) \\ &= \sum_i (\omega^i \wedge \nabla_{e_i}\omega)(X_0,\ldots,X_p), \end{aligned} \tag{4.23}$$

where we used (1.20) to obtain the last equality. □

4.4 The divergence theorem for a Riemannian manifold

We begin with a useful formula for the divergence of a vector field.

Proposition 4.7. *For a vector field X,*

$$*(\mathrm{div}\ X) = (\mathrm{div}\ X)dV = d(i_X dV) = \mathcal{L}_X(dV). \tag{4.24}$$

In a coordinate system, we have

$$\mathrm{div}\ X = \frac{1}{\sqrt{\det(g)}}\partial_i\Big(X^i\sqrt{\det(g)}\Big). \tag{4.25}$$

Proof. Fix a point $x \in M$, and let $\{e_i\}$ be an orthonormal basis of T_xM. In a small neighborhood of x, parallel translate this frame along radial geodesics. For such a frame, we have $\nabla_{e_i}e_j(x) = 0$. Such a frame is called an *adapted* moving frame field at x. Let $\{\omega^i\}$ denote the dual frame field. We have

$$\begin{aligned} \mathcal{L}_X(dV) &= (di_X + i_X d)dV = d(i_X dV) \\ &= \sum_i \omega^i \wedge \nabla_{e_i}\big(i_X(\omega^1\wedge\cdots\wedge\omega^n)\big) \\ &= \sum_i \omega^i \wedge \nabla_{e_i}\Big((-1)^{j-1}\sum_{j=1}^{n}\omega^j(X)\omega^1\wedge\cdots\wedge\hat{\omega}^j\wedge\cdots\wedge\omega^n\Big) \\ &= \sum_{i,j}(-1)^{j-1}e_i\big(\omega^j(X)\big)\omega^i\wedge\omega^1\wedge\cdots\wedge\hat{\omega}^j\wedge\cdots\wedge\omega^n \\ &= \sum_i \omega^i(\nabla_{e_i}X)dV \\ &= (\mathrm{div}\ X)dV = *(\mathrm{div}\ X). \end{aligned} \tag{4.26}$$

Applying $*$ to this formula, we have

$$\begin{aligned}
\text{div } X &= *d(i_X dV) \\
&= *d(i_X \sqrt{\det(g)} dx^1 \wedge \cdots \wedge dx^n) \\
&= *d\Big(\sum_{j=1}^{n} (-1)^{j-1} X^j \sqrt{\det(g)} dx^1 \wedge \ldots \hat{dx^j} \cdots \wedge dx^n \Big) \\
&= *\Big(\partial_i (X^i \sqrt{\det(g)}) dx^1 \wedge \cdots \wedge dx^n \Big) \\
&= *\Big(\partial_i (X^i \sqrt{\det(g)}) \frac{1}{\sqrt{\det(g)}} dV \Big) \\
&= \frac{1}{\sqrt{\det(g)}} \partial_i \Big(X^i \sqrt{\det(g)} \Big).
\end{aligned} \tag{4.27}$$

□

Corollary 4.8. *Let (M, g) be compact, orientable and with boundary ∂M. If X is a vector field of class C^1, and f is a function of class C^1, then*

$$\int_M (\text{div } X) f dV = -\int_M df(X) dV + \int_{\partial M} \langle X, \hat{n} \rangle f dS, \tag{4.28}$$

where $\hat{n}$ is the outer unit normal. If ω is a one-form of class C^1, then

$$\int_M (\text{div } \omega) f dV = -\int_M \langle \omega, df \rangle dV + \int_{\partial M} \omega(\hat{n}) f dS. \tag{4.29}$$

If u and v are functions of class C^2, then

$$\int_M (\Delta u) v dV = -\int_M \langle \nabla u, \nabla v \rangle dV + \int_{\partial M} \langle \nabla u, \hat{n} \rangle v dS, \tag{4.30}$$

and

$$\int_M (\Delta u) v dV - \int_M u (\Delta v) dV = \int_{\partial M} \langle \nabla u, \hat{n} \rangle v dS - \int_{\partial M} v \langle \nabla u, \hat{n} \rangle dS. \tag{4.31}$$

Consequently, if M is compact without boundary, then Δ is a self-adjoint operator.

Proof. We compute

$$d(f i_X dV) = df \wedge (i_X dV) + f d(i_X dV). \tag{4.32}$$

Using Stokes' Theorem and Proposition 4.7,

$$\int_M f (\text{div } X) dV + \int_M df \wedge (i_X dV) = \int_{\partial M} f i_X dV. \tag{4.33}$$

A computation like above shows that

$$df \wedge (i_X dV) = df(X) dV. \tag{4.34}$$

Next, on ∂M, decompose $X = X_T + X_N$ into its tangential and normal components. Then

$$\begin{aligned} i_X dV &= dV(X_T + X_N, \dots) \\ &= dV(\langle X, \hat{n}\rangle \hat{n}, \dots) \\ &= \langle X, \hat{n}\rangle dS, \end{aligned} \tag{4.35}$$

since the volume element on the boundary is $dS = i_{\hat{n}} dV$. The proof for 1-forms is the dual argument. Green's first formula (4.30) follows using $\Delta u = \mathrm{div}(\nabla u)$, and Green's second formula (4.31) follows from (4.30). □

We point out the following. The formula (4.28), gives a nice way to derive the coordinate formula for the divergence as follows. Fix a coordinate system, and assume that X and f have compact support in these coordinates. Then

$$\begin{aligned} \int_M f(\mathrm{div}\ X) dV &= -\int_M df(X) dV \\ &= -\int_M \partial_i f dx^i (X^j \partial_j) \sqrt{\det(g)} dx \\ &= -\int_{\mathbb{R}^n} \partial_i f X^i \sqrt{\det(g)} dx \\ &= \int_{\mathbb{R}^n} f \partial_i \Big(X^i \sqrt{\det(g)} \Big) dx \\ &= \int_M f \frac{1}{\sqrt{\det(g)}} \partial_i \Big(X^i \sqrt{\det(g)} \Big) dV. \end{aligned} \tag{4.36}$$

Since this is true for any f, we must have

$$\mathrm{div}\ X = \frac{1}{\sqrt{\det(g)}} \partial_i \Big(X^i \sqrt{\det(g)} \Big). \tag{4.37}$$

This formula yields a slightly non-obvious formula for the contraction of the Christoffel symbols on the upper and one lower index.

Corollary 4.9. *The Christoffel symbols satisfy*

$$\sum_{i=1}^n \Gamma^i_{ij} = \frac{1}{2} g^{pq} \partial_j g_{pq} = \frac{1}{\sqrt{\det(g)}} \partial_j \sqrt{\det(g)} = \frac{1}{2} \partial_j \log \det(g). \tag{4.38}$$

Proof. The first equality follows easily from the coordinate formula for the Christoffel symbols (2.6). Next, on one hand, we have the formula

$$\mathrm{div}\ X = \nabla_i X^i = \partial_i X^i + \Gamma^i_{ip} X^p. \tag{4.39}$$

On the other hand we have

$$\mathrm{div}\ X = \partial_i X^i + \frac{1}{\sqrt{\det(g)}} \Big(\partial_p \sqrt{\det(g)} \Big) X^p. \tag{4.40}$$

Since this is true for an arbitrary vector field X, the coefficient of X^p must be the same. □

Exercise 4.10. Prove the middle equality in (4.38) directly. (Hint: use Jacobi's formula for the derivative of a determinant.)

This also yields a useful formula for the Laplacian of a function.

Corollary 4.11. *In a coordinate system, the Laplacian of a function is given by*

$$\Delta f = \frac{1}{\sqrt{\det(g)}} \partial_i \Big(g^{ij} \partial_j f \sqrt{\det(g)} \Big). \tag{4.41}$$

Proof. Since $\Delta f = \mathrm{div}(\nabla f)$, just let $X^i = g^{ij}\partial_i f$ in (4.37). □

5 Lecture 5

5.1 Integration and adjoints

We begin with an integration-by-parts formula for (r,s)-tensor fields.

Proposition 5.1. *Let (M,g) be compact and without boundary, T be an (r,s)-tensor field, and S be a $(r,s+1)$ tensor field. Then*

$$\int_M \langle \nabla T, S \rangle dV = - \int_M \langle T, \mathrm{div}\ S \rangle dV. \tag{5.1}$$

Proof. Let us view the inner product $\langle T, S\rangle$ as a 1-form ω. In coordinates

$$\omega = \langle T, S \rangle = T^{j_1 \dots j_s}_{i_1 \dots i_r} S^{i_1 \dots i_r}_{j j_1 \dots j_s} dx^j. \tag{5.2}$$

Note the indices on T are reversed, since we are taking an inner product. Taking the divergence, since g is parallel we compute

$$\begin{aligned} \mathrm{div}\ (\langle T, S \rangle) &= \nabla^j (T^{j_1 \dots j_s}_{i_1 \dots i_r} S^{i_1 \dots i_r}_{j j_1 \dots j_s}) \\ &= \nabla^j (T^{j_1 \dots j_s}_{i_1 \dots i_r}) S^{i_1 \dots i_r}_{j j_1 \dots j_s} + T^{j_1 \dots j_s}_{i_1 \dots i_r} \nabla^j S^{i_1 \dots i_r}_{j j_1 \dots j_s} \\ &= \langle \nabla T, S \rangle + \langle T, \mathrm{div}\ S \rangle. \end{aligned} \tag{5.3}$$

The result then follows from Proposition 4.1 and Corollary 4.8. □

Remark 5.2. Some authors define $\nabla^* = -\mathrm{div}$. Then

$$\int_M \langle \nabla T, S \rangle dV = \int_M \langle T, \nabla^* S \rangle dV, \tag{5.4}$$

so that ∇^* is the formal L^2-adjoint of ∇, for example [Pet06].

Recall the adjoint of d, $\delta : \Omega^p \to \Omega^{p-1}$, is defined by

$$\delta \omega = (-1)^{n(p+1)+1} * d * \omega. \tag{5.5}$$

Proposition 5.3. *For (M, g) compact without boundary, the operator δ is the L^2 adjoint of d,*

$$\int_M \langle \delta\alpha, \beta \rangle dV = \int_M \langle \alpha, d\beta \rangle dV, \tag{5.6}$$

where $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$.

Proof. We compute

$$\begin{aligned}
\int_M \langle \alpha, d\beta \rangle dV &= \int_M d\beta \wedge *\alpha \\
&= \int_M \Big(d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big) \\
&= \int_M (-1)^{p+(n-p+1)(p-1)} \beta \wedge * * d * \alpha \\
&= \int_M \langle \beta, (-1)^{n(p+1)+1} * d * \alpha \rangle dV \\
&= \int_M \langle \beta, \delta\alpha \rangle dV.
\end{aligned} \tag{5.7}$$

□

We note the following. If $\alpha \in \Omega^p(T^*M)$, then we can define the divergence operator $\mathrm{div} : \Omega^p(M) \to \Omega^{p-1}(M)$ as follows.

$$\mathrm{div}\, \alpha = \sum_{j=1}^{n} i_{e_j} \nabla_{e_j} \alpha. \tag{5.8}$$

This is a well-defined global operator $\mathrm{div} : \Omega^p(M) \to \Omega^{p-1}(M)$, and agrees with our previous definition of div under our identification of p-forms with alternating tensors. To see this, fix a point $x \in M$, and let $\{e_i\}$ and $\{\omega^i\}$ denote an adapted orthonormal frame field at x. Recall that p-form is written as

$$\alpha = \frac{1}{p!} \sum_{1 \leq i_1, i_2, \ldots, i_p \leq n} \alpha_{i_1 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}. \tag{5.9}$$

So then (5.8), evaluated at x, is

$$
\begin{aligned}
\text{div } \alpha &= \frac{1}{p!} \sum_{1 \le i_1, i_2, \ldots, i_p \le n} \sum_{j=1}^{n} e_j(\alpha_{i_1 \ldots i_p}) i_{e_j}(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) \\
&= \frac{1}{p!} \sum_{1 \le i_1, i_2, \ldots, i_p \le n} \sum_{j=1}^{n} e_j(\alpha_{i_1 \ldots i_p}) \sum_{k=1}^{p} (-1)^{k-1} \delta_j^{i_k} (\omega^{i_1} \wedge \cdots \wedge \widehat{\omega^{i_k}} \wedge \cdots \wedge \omega^{i_p}) \\
&= \frac{1}{p!} \sum_{1 \le i_1, i_2, \ldots, i_p \le n} \sum_{k=1}^{p} e_{i_k}(\alpha_{i_1 \ldots i_p}) (-1)^{k-1} (\omega^{i_1} \wedge \cdots \wedge \widehat{\omega^{i_k}} \wedge \cdots \wedge \omega^{i_p}) \\
&= \frac{1}{(p-1)!} \sum_{1 \le i_1, i_2, \ldots, i_{p-1} \le n} \sum_{k=1}^{n} e_k(\alpha_{k i_1 \ldots i_{p-1}}) (\omega^{i_1} \wedge \cdots \wedge \omega^{i_{p-1}}).
\end{aligned}
\tag{5.10}
$$

So the components of div α at x are

$$
(\text{div } \alpha)_{i_1 \ldots i_{p-1}} = \sum_{k=1}^{n} e_k(\alpha_{k i_1 \ldots i_{p-1}}). \tag{5.11}
$$

On the other hand, the alternating $(0, p)$-tensor corresponding to α is

$$
\tilde{\alpha} = \sum_{1 \le i_1, i_2, \ldots, i_p \le n} \alpha_{i_1 \ldots i_p} \omega^{i_1} \otimes \cdots \otimes \omega^{i_p}, \tag{5.12}
$$

and the definition of div $\tilde{\alpha}$ from (4.2), evaluated at x, is

$$
(\text{div } \tilde{\alpha})_{i_1 \ldots i_{p-1}} = \sum_{j=1}^{n} \nabla_{e_j} \alpha_{j i_1 \ldots i_{p-1}} = \sum_{j=1}^{n} e_j(\alpha_{j i_1 \ldots i_{p-1}}). \tag{5.13}
$$

Consequently, our definitions agree. The next proposition says that our divergence operator agrees with the Hodge δ operator, up to a sign, a fact which is not at all obvious.

Proposition 5.4. *On Ω^p, $\delta = -\text{div}$.*

Proof. Let $\omega \in \Omega^p$. Choose locally defined dual ONB $\{e_i\}$ and $\{\omega^i\}$. We compute

$$
\begin{aligned}
(\text{div } \omega) &= \sum_j i_{e_j} \nabla_{e_j} \omega \\
&= \sum_j (-1)^{p(n-p)} \Big(i_{e_j} \big(* * (\nabla_{e_j} \omega) \big) \Big) \\
&= (-1)^{p(n-p)} \sum_j (-1)^{n-p} * (\omega^j \wedge * \nabla_{e_j} \omega) \\
&= (-1)^{(p+1)(n-p)} \sum_j * \big(\omega^j \wedge \nabla_{e_j} (*\omega) \big) \\
&= (-1)^{n(p+1)} (*d * \omega).
\end{aligned}
\tag{5.14}
$$

□

An alternative proof of the proposition is as follows. Assume that $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$ are supported in a coordinate system. Then using Proposition 5.3, formula (4.19), and Proposition 5.1, we have

$$\begin{aligned}
\int_M \langle \alpha, \delta\beta \rangle dV &= \int_M \langle d\alpha, \beta \rangle dV \\
&= \frac{1}{p!} \int_M (d\alpha)_{i_0 \dots i_{p-1}} \beta^{i_0 \dots i_{p-1}} dV \\
&= \frac{1}{p!} \int_M \sum_{j=0}^{p-1} (-1)^j \nabla_{i_j} \alpha_{i_0 \dots \hat{i_j} \dots i_{p-1}} \beta^{i_0 \dots i_{p-1}} dV \\
&= \frac{1}{(p-1)!} \int_M \nabla_{i_0} \alpha_{i_1 \dots i_{p-1}} \beta^{i_0 \dots i_{p-1}} dV \\
&= \frac{1}{(p-1)!} \int_M \langle \nabla \alpha, \beta \rangle_{ten} dV \\
&= \frac{1}{(p-1)!} \int_M \langle \alpha, -\text{div } \beta \rangle_{ten} dV \\
&= \int_M \langle \alpha, -\text{div } \beta \rangle dV.
\end{aligned} \tag{5.15}$$

where $\langle \cdot, \cdot \rangle_{ten}$ denotes the tensor inner product. Thus both δ and $-\text{div}$ are L^2 adjoints of d. The result then follows from uniqueness of the L^2 adjoint.

Exercise 5.5. Try and prove Proposition 5.4 directly in coordinates, using the coordinate formulas (4.2), (4.16), and (4.19).

5.2 The Hodge Laplacian and the rough Laplacian

For T an (r, s)-tensor, the rough Laplacian is an (r, s) tensor given by

$$\Delta T = \text{div } \nabla T, \tag{5.16}$$

and is given in coordinates by

$$(\Delta T)^{i_1 \dots i_r}_{j_1 \dots j_s} = g^{ij} \nabla_i \nabla_j T^{i_1 \dots i_r}_{j_1 \dots j_s}. \tag{5.17}$$

If $\omega \in \Omega^p(M)$, the rough Laplacian is defined by

$$\Delta \omega = \sum_{j=1}^{n} \nabla_{e_j} \nabla_{e_j} \omega, \tag{5.18}$$

and this agrees with the rough Laplacian above under our identification of p-forms with alternating tensors.

For $\omega \in \Omega^p$ we define the *Hodge laplacian* $\Delta_H : \Omega^p \to \Omega^p$ by

$$\Delta_H \omega = (d\delta + \delta d)\omega. \tag{5.19}$$

We say a p-form is *harmonic* if it is in the kernel of Δ_H.

Proposition 5.6. *If M is compact without boundary, then for T and S both (r,s)-tensors,*

$$\int_M \langle \Delta T, S\rangle dV = -\int_M \langle \nabla T, \nabla S\rangle dV = \int_M \langle T, \Delta S\rangle dV. \tag{5.20}$$

For $\alpha, \beta \in \Omega^p$,

$$\int_M \langle \Delta_H \alpha, \beta\rangle dV = \int_M \langle d\alpha, d\beta\rangle dV + \int_M \langle \delta\alpha, \delta\beta\rangle dV = \int_M \langle \alpha, \Delta_H \beta\rangle dV. \tag{5.21}$$

Consequently, a p-form is harmonic ($\Delta_H \alpha = 0$) if and only if it is both closed and co-closed ($d\alpha = 0$ and $\delta\alpha = 0$).

Proof. Formula (5.20) is an application of (5.16) and Proposition 5.1. For the second, from Proposition 5.3,

$$\begin{aligned}
\int_M \langle \Delta_H \alpha, \beta\rangle dV &= \int_M \langle (d\delta + \delta d)\alpha, \beta\rangle dV \\
&= \int_M \langle d\delta\alpha, \beta\rangle dV + \int_M \langle \delta d\alpha, \beta\rangle dV \\
&= \int_M \langle \delta\alpha, \delta\beta\rangle dV + \int_M \langle d\alpha, d\beta\rangle dV \\
&= \int_M \langle \alpha, d\delta\beta\rangle dV + \int_M \langle \alpha, \delta d\beta\rangle dV \\
&= \int_M \langle \alpha, \Delta_H \beta\rangle dV.
\end{aligned} \tag{5.22}$$

The last statement follows easily by letting $\alpha = \beta$ in (5.21). □

Note that Δ maps alternating $(0,p)$ tensors to alternating $(0,p)$ tensors, therefore it induces a map $\Delta : \Omega^p \to \Omega^p$ (note that on [Poo81, page 159] it is stated that the rough Laplacian of an r-form is in general not an r-form, but this seems to be incorrect). On p-forms, Δ and Δ_H are two self-adjoint linear second order differential operators. How are they related? Next, we will look at the simplest case of 1-forms.

5.3 Harmonic 1-forms

Consider the case of 1-forms.

Proposition 5.7. *Let $\omega \in \Omega^1(M)$.*

$$\Delta\omega = -\Delta_H(\omega) + Ric(\sharp\omega, \cdot). \tag{5.23}$$

Proof. We compute

$$\begin{aligned}
(\delta d\omega)_j &= \delta(\nabla_i \omega_j - \nabla_j \omega_i) \\
&= -g^{pq}\nabla_p \nabla_q \omega_j + g^{pq}\nabla_p \nabla_j \omega_q \\
&= -\Delta\omega_j + g^{pq}\nabla_p \nabla_j \omega_q.
\end{aligned} \tag{5.24}$$

Next,

$$
\begin{aligned}
(d\delta\omega)_j &= d(-g^{pq}\nabla_p\omega_q)_j \\
&= -\nabla_j(g^{pq}\nabla_p\omega_q) \\
&= -g^{pq}\nabla_j\nabla_p\omega_q.
\end{aligned} \tag{5.25}
$$

Adding these together,

$$
\begin{aligned}
(\Delta_H\omega)_j &= -\Delta\omega_j + g^{pq}(\nabla_p\nabla_j - \nabla_j\nabla_p)\omega_q \\
&= -\Delta\omega_j + g^{pq}(-R_{pjq}{}^{i}\omega_i) \\
&= -\Delta\omega_j - g^{pq}(R_{pjiq}\omega^i) \\
&= -\Delta\omega_j + R_{ij}\omega^i,
\end{aligned} \tag{5.26}
$$

recalling that our convention is to lower the upper index of the $(1,3)$ curvature tensor to the third position. □

Theorem 5.8 (Bochner)**.** *If (M, g) has non-negative Ricci curvature, then any harmonic 1-form is parallel. In this case $b_1(M) \leq n$. If, in addition, Ric is positive definite at some point, then any harmonic 1-form is trivial. In this case $b_1(M) = 0$.*

Proof. If ω satisfies $\Delta_H\omega = 0$, then formula (5.23) is

$$
\Delta\omega = Ric(\sharp\omega, \cdot). \tag{5.27}
$$

Take the inner product with ω, and integrate

$$
\int_M \langle\Delta\omega, \omega\rangle dV = -\int_M |\nabla\omega|^2 dV = \int_M Ric(\sharp\omega, \sharp\omega) dV. \tag{5.28}
$$

This clearly implies that $\nabla\omega \equiv 0$, thus ω is parallel, so is determined everywhere by its value at any point. If in addition *Ric* is strictly positive somewhere, ω must vanish identically. The conclusion on the first Betti number follows from the Hodge Theorem. □

6 Lecture 6

6.1 Eigenvalue estimates

The above argument actually proves slightly more than was stated.

Proposition 6.1. *If (M^n, g) is compact and satisfies*

$$
Ric \geq (n-1)a \cdot g, \tag{6.1}
$$

where $a > 0$ is a constant. Then the first eigenvalue of the Hodge Laplacian on 1-forms satisfies

$$
\lambda_1 \geq (n-1)a. \tag{6.2}
$$

Furthermore, equality in (6.2) *is never attained.*

Proof. An eigenform is a nontrivial solution of

$$\Delta_H \omega = \lambda_1 \omega, \tag{6.3}$$

so using (5.23), we have

$$-\Delta\omega + Ric(\sharp\omega, \cdot) = \lambda_1 \omega. \tag{6.4}$$

Pairing with ω and integrating,

$$-\int_M \langle \Delta\omega, \omega \rangle dV + \int_M Ric(\sharp\omega, \sharp\omega) dV = \lambda_1 \int_M |\omega|^2 dV. \tag{6.5}$$

Integrating by parts, and using the inequality (6.7), we obtain

$$\int_M |\nabla\omega|^2 dV \leq (\lambda_1 - (n-1)a) \int_M |\omega|^2 dV. \tag{6.6}$$

If ω is non-trivial, then the L^2 norm on the right hand side is non-zero, and inequality (6.2) follows. If we have equality in (6.2), then (6.6) implies that ω is parallel. Propositions 4.6 and 5.4 then imply that $\Delta_H \omega = 0$, which contradicts (6.3). □

The inequality (6.2) is not sharp, so we have to work a little harder. Next, we have a theorem about the lowest eigenvalue of the Laplacian on functions.

Theorem 6.2 (Lichnerowicz-Obata)**.** *If (M^n, g) is compact and satisfies*

$$Ric \geq (n-1)a, \tag{6.7}$$

where $a > 0$ is a constant. Then the first non-zero eigenvalue of the Hodge Laplacian on functions satisfies

$$\lambda_0 \geq na. \tag{6.8}$$

Furthermore, equality is attained in (6.8) if and only if (M, g) is isometric to the sphere of radius $\frac{1}{\sqrt{a}}$ in $\mathbb{R}^{n+1}$.

Proof. Assuming that

$$\Delta_H f = \lambda_0 f, \tag{6.9}$$

then since d commutes with the Hodge Laplacian, we have

$$\Delta_H \omega = \lambda_0 \omega \tag{6.10}$$

where ω is the 1-form df. From the proof of Proposition 6.1, we have

$$\int_M |\nabla\omega|^2 dV \leq (\lambda_0 - (n-1)a) \int_M |\omega|^2 dV. \tag{6.11}$$

Using the matrix inequality

$$|A|^2 \geq \frac{1}{n}(trA)^2, \tag{6.12}$$

the left hand side of (6.11) is estimated

$$\int_M |\nabla\omega|^2 dV = \int_M |\nabla^2 f|^2 dV \geq \frac{1}{n}\int_M (\Delta_H f)^2 dV. \tag{6.13}$$

The integral on the right hand side of (6.11) is

$$\int_M |\omega|^2 dV = \int_M |df|^2 dV = \int_M f(\Delta_H f) dV = \frac{1}{\lambda_0}\int_M (\Delta_H f)^2 dV. \tag{6.14}$$

Combining the above, we obtain the inequality

$$\frac{1}{n}\int_M (\Delta_H f)^2 dV \leq \frac{\lambda_0 - (n-1)a}{\lambda_0}\int_M (\Delta_H f)^2 dV. \tag{6.15}$$

The integral is a multiple of the L^2-norm of f, so is strictly positive if f is non-trivial. Consequently,

$$\frac{1}{n} \leq \frac{\lambda_0 - (n-1)a}{\lambda_0}, \tag{6.16}$$

which simplifies to (6.8).

Next, if we have equality in this inequality, we must have equality in all of the inequalities we used so far. In particular, equality in (6.12) implies that we have a non-trivial solution of

$$\nabla^2 f = \frac{1}{n}\Delta f \cdot g. \tag{6.17}$$

with $Ric = (n-1)ag$ whenever $df \neq 0$. Since f is an eigenfunction, this is rewritten as

$$\nabla^2 f = -af \cdot g. \tag{6.18}$$

This implies that along any unit-speed geodesic,

$$f(s) = A\cos(\sqrt{a}s) + B\sin(\sqrt{a}s), \tag{6.19}$$

where s is the arc-length from a fixed point P_+. If we choose the point P_+ to be a maximum of f, then $f(s) = A\cos(\sqrt{a}s)$ along any geodesic through P_+. This then implies that level sets of f must have constant curvature, and then one can construct an isometry with a round sphere, but we will omit the details. For more details, see [?] and also [?] for an excellent exposition. □

Let us return to 1-forms. We have the following decomposition.

Proposition 6.3. *For (M, g) compact, the space of 1-forms admits the orthogonal decomposition*

$$\Omega^1(M) = d\{\Omega^0_0(M)\} \oplus \{Ker(\delta)\}, \tag{6.20}$$

where $\Omega^0_0(M)$ denotes the space of functions with mean value zero.

Proof. Given a 1-form α, we can find a function f solving

$$\delta\alpha = \Delta_H f. \tag{6.21}$$

This is true because the left hand side has zero mean value from the divergence theorem. So then

$$\delta\alpha = \delta df, \tag{6.22}$$

which says that

$$\delta(\alpha - df) = 0, \tag{6.23}$$

which proves the decomposition

$$\Omega^1(M) = d\{\Omega^0_0(M)\} + \{Ker(\delta)\}. \tag{6.24}$$

These spaces have trivial intersection since if df is divergence-free, then $\delta df = \Delta_H f = 0$, which implies that f is constant, and (6.20) follows. By Proposition (5.3), this is clearly an orthogonal decomposition. □

Let α be an eigenform of Δ_H, with $\Delta_H \alpha = \lambda_1 \alpha$. If $\alpha = df$, with $f \in \Omega^0_0(M)$, then

$$\Delta_H(df) = \lambda_1 df \tag{6.25}$$

implies that

$$d(\Delta_H f - \lambda_1 f) = 0, \tag{6.26}$$

so if M is connected,

$$\Delta_H f - \lambda_1 f = c, \tag{6.27}$$

for some constant c. Integrating this, from the divergence theorem, and the condition on f, we conclude that $c = 0$.

Consequently, f is an eigenfunction of Δ_H with eigenvalue λ_1, so if $Ric \geq (n-1)a$, by Theorem 6.2, we conclude that

$$\lambda_1 \geq na. \tag{6.28}$$

What about co-closed 1-forms? Assume that $\Delta_H \alpha = \lambda_1 \alpha$ and $\delta\alpha = 0$. Consider the 2-form $d\alpha$. If $d\alpha = 0$, then α is a harmonic 1-form. If $a > 0$, then $\alpha \equiv 0$ by Theorem 5.8. So then we have that $d\alpha$ is a non-trivial solution of

$$\Delta_H d\alpha = \lambda_1 d\alpha. \tag{6.29}$$

So to get anywhere, it looks like we would need to estimate the least eigenvalue of the Hodge Laplacian on closed 2-forms. We will return to this later, but next we will show how to estimate this eigenvalue without having to go to 2-forms.

6.2 Killing 1-forms

We next define an operator on 1-forms mapping

$$\mathcal{L} : \Omega^1(M) \to \Gamma(S^2(T^*M)) \tag{6.30}$$

by

$$(\mathcal{L}(\omega))_{ij} = \nabla_i \omega_j + \nabla_j \omega_i. \tag{6.31}$$

Definition 6.4. We say that ω is *Killing* if $\mathcal{L}\omega = 0$. A vector field X is a *Killing field* if the 1-parameter group of local diffeomorphisms generated by X consists of local isometries of g.

Proposition 6.5. *A vector field is a Killing field if and only if $\mathcal{L}_X g = 0$, which is equivalent to the skew-symmetry of ∇X, viewed as an endomorphism.*

A one-form is ω Killing if and only if the vector field $\sharp\omega$ is Killing.

Proof. Let ϕ_t denote the 1-parameter group of X,

$$\begin{aligned} \frac{d}{ds}(\phi_s^* g)\Big|_t &= \frac{d}{ds}(\phi_{s+t}^* g)\Big|_0 \\ &= \phi_t^* \frac{d}{ds}(\phi_s^* g)\Big|_0 \\ &= \phi_t^* \mathcal{L}_X g. \end{aligned} \tag{6.32}$$

It follows that $\phi_t^* g = g$ for every t if and only if $\mathcal{L}_X g = 0$.

For a vector field X, the covariant derivative ∇X is a $(1,1)$ tensor. Equivlently, $\nabla X \in \Gamma(End(TM))$. Any endomorphism T of an inner product space can be decomposed into its symmetric and skew-symmetric parts via

$$\begin{aligned} \langle Tu, v\rangle &= \frac{1}{2}\left(\langle Tu, v\rangle + \langle u, Tv\rangle\right) + \left(\langle Tu, v\rangle - \langle u, Tv\rangle\right) \\ &= \langle T_{sym} u, v\rangle + \langle T_{sk} u, v\rangle. \end{aligned} \tag{6.33}$$

Next, recalling the formula for the Lie derivative of a $(0,2)$ tensor,

$$\begin{aligned} \mathcal{L}_X g(Y,Z) &= X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\ &= 2g((\nabla X)_{sym} Y, Z), \end{aligned} \tag{6.34}$$

so $\mathcal{L}_X g = 0$ if and only if ∇X is skew-symmetric. Finally, since g is parallel, $\mathcal{L}\omega = 0$ if and only if $\mathcal{L}_{\sharp\omega} g = 0$. $\square$

Next, define another Laplacian $\square : \Omega^1(M) \to \Omega^1(M)$ by

$$\square = \text{div } \mathcal{L} \tag{6.35}$$

The Weitzenböck formula for $\square$ is

Proposition 6.6. *Let $\omega \in \Omega^1(M)$. Then*

$$\Box\omega = \Delta\omega - d\delta\omega + Ric(\sharp\omega, \cdot). \tag{6.36}$$

This may also be written as

$$\Box\omega = -2d\delta - \delta d + 2Ric(\sharp\omega, \cdot). \tag{6.37}$$

Proof. We compute

$$\begin{aligned}(\text{div } \mathcal{L}\omega)_j &= \text{div } (\nabla_i\omega_j + \nabla_j\omega_i)\\ &= g^{pq}\nabla_p\nabla_q\omega_j + g^{pq}\nabla_p\nabla_j\omega_q\\ &= \Delta\omega_j + g^{pq}\nabla_p\nabla_j\omega_q\\ &= \Delta\omega_j + g^{pq}(\nabla_j\nabla_p\omega_q - R_{pjq}{}^l\omega_l)\\ &= \Delta\omega_j - d\delta\omega + R^l_j\omega_l.\end{aligned} \tag{6.38}$$

Finally, (6.37) follows from this and (5.23). □

We recall that for (M, g) compact, the isometry group $\text{Iso}(M, g)$ is a compact Lie group, with Lie algebra the space of Killing vector fields. Furthermore,

$$\dim(\text{Iso}(M, g)) \leq \frac{n(n+1)}{2}. \tag{6.39}$$

See [Kob95] for a proof of these facts. We have the following corollary of the above.

Theorem 6.7 (Bochner)**.** *Let (M, g) be compact. If (M, g) has negative semi-definite Ricci tensor, then $dim(Iso(M, g)) \leq n$. If, in addition, the Ricci tensor is negative definite at some point, then $Iso(M, g)$ is finite.*

Proof. First, note that taking the trace of $\mathcal{L}\omega = 0$ implies that $\delta\omega = 0$. Given a Killing form, from Proposition 6.6 it follows that

$$0 = \Delta\omega + Ric(\sharp\omega, \cdot). \tag{6.40}$$

Pairing with ω and integrating by parts,

$$0 = -\int_M |\nabla\omega|^2 dV + \int_M Ric(\omega, \omega) dV. \tag{6.41}$$

This implies that any Killing form is parallel. If the isometry group is not finite, then there exists a non-trivial 1-parameter group $\{\phi_t\}$ of isometries. By Proposition 6.5, this generates a non-trivial Killing vector field X which is parallel and satisfies $Ric(X, X) = 0$. Since X is parallel, it is determined by its value at a single point, so the dimension of the space of Killing vector fields is less than n, which implies that $dim(\text{Iso}(M, g)) \leq n$, since the space of Killing fields is the Lie algebra of Lie group $\text{Iso}(M, g)$. If Ric is negative definite at some point x, then $Ric(X_x, X_x) = 0$ implies that $X_x = 0$, and thus $X \equiv 0$ since it is parallel. Consequently, there are no nontrival 1-parameter groups of isometries, so $\text{Iso}(M, g)$ must be finite, since it is compact. □

Note that a flat metric on an n-dimensional torus $S^1 \times \cdots \times S^1$ attains equality in the above inequality. Note also that by the Gauss-Bonnet Theorem, any metric on a surface of genus $g \geq 2$ must have a point of negative curvature, so any non-positively curved metric on a surface of genus $g \geq 2$ must have finite isometry group.

Let us return the the problem at the end of the previous section. Recall from Proposition (6.3), that

$$\Omega^1(M) = d\{\Omega^0_0(M)\} \oplus \{Ker(\delta)\}. \tag{6.42}$$

We can now prove the following.

Proposition 6.8. *If (M^n, g) is compact and satisfies*

$$Ric \geq (n-1)a, \tag{6.43}$$

where $a > 0$ is a constant.

The first eigenvalue of Δ_H restricted to the first factor in the decomposition (6.42) *satisfies*

$$\lambda_1 \geq na, \tag{6.44}$$

with equality if and only if (M, g) is isometric to the sphere of radius $\frac{1}{\sqrt{a}}$ in $\mathbb{R}^{n+1}$.

The first eigenvalue of the Hodge Laplacian on divergence-free 1*-forms satisfies*

$$\lambda_1 \geq 2(n-1)a, \tag{6.45}$$

with equality if and only if the corresponding eigenspace consists of Killing 1*-forms.*

Proof. The first part has already been proved above. For the second part, let α be a 1-form satisfying $\Delta_H \alpha = \lambda_1 \alpha$, and $\delta\alpha = 0$. Then (6.37) says

$$\begin{aligned} \square\alpha &= -\delta d\alpha + 2Ric(\sharp\alpha, \cdot) \\ &= -\Delta_H \alpha + 2Ric(\sharp\alpha, \cdot) \\ &= -\lambda_1 \alpha + 2Ric(\sharp\alpha, \cdot). \end{aligned} \tag{6.46}$$

Pairing with α and integrating,

$$\int_M \langle \square\alpha, \alpha \rangle dV \geq (-\lambda_1 + 2(n-1)a) \int_M |\alpha|^2 dV. \tag{6.47}$$

The left hand side is

$$\begin{aligned} \int_M \langle \square\alpha, \alpha \rangle dV &= \int_M \langle -\delta \mathcal{L}\alpha, \alpha \rangle dV \\ &= -\frac{1}{2} \int_M \langle \mathcal{L}\alpha, \mathcal{L}\alpha \rangle dV. \end{aligned} \tag{6.48}$$

So we have the inequality

$$-\frac{1}{2} \int_M |\mathcal{L}\alpha|^2 dV \geq (-\lambda_1 + 2(n-1)a) \int_M |\alpha|^2 dV, \tag{6.49}$$

which clearly yields the inequality (6.45), with equality if and only if $\mathcal{L}\alpha = 0$. $\square$

7 Lecture 7

7.1 Conformal Killing 1-forms

We next define an operator on 1-forms mapping

$$\mathcal{K} : \Omega^1(M) \to \Gamma(S^2_0(T^*M)) \tag{7.1}$$

by

$$(\mathcal{K}(\omega))_{ij} = \nabla_i \omega_j + \nabla_j \omega_i - \frac{2}{n}(\text{div }\omega) g_{ij}. \tag{7.2}$$

Definition 7.1. We say that ω is *conformal Killing* if $\mathcal{K}\omega = 0$. A vector field X is a *conformal Killing field* if the 1-parameter group of local diffeomorphisms generated by X consists of local conformal transformations of g.

The analogue of Proposition 6.5 is the following.

Proposition 7.2. *A vector field is a conformal Killing field if and only if $\mathcal{L}_X g = 2(\delta X)g$, which is equivalent to the skew-symmetry of the trace-free part of ∇X, viewed as an endomorphism.*

A one-form is ω conformal Killing if and only if the vector field $\sharp\omega$ is conformal Killing.

Proof. The proof is similar to the proof of Proposition 6.5 and is omitted. □

Next, define another Laplacian $\Box_{\mathcal{K}} : \Omega^1(M) \to \Omega^1(M)$ by

$$\Box_{\mathcal{K}} = \text{div }\mathcal{K} \tag{7.3}$$

The Weitzenböck formula for $\Box_{\mathcal{K}}$ is

Proposition 7.3. *Let $\omega \in \Omega^1(M)$. Then*

$$\Box_{\mathcal{K}}\omega = -2\Big(\frac{n-1}{n}\Big)d\delta - \delta d + 2Ric(\sharp\omega, \cdot). \tag{7.4}$$

Proof. We have that

$$\begin{aligned} \Box_{\mathcal{K}}\omega &= \text{div }\Big(\mathcal{L}\omega + \frac{2}{n}(\delta\omega)\Big)g \\ \Box_{\mathcal{K}}\omega &= \Box\omega + \frac{2}{n}d\delta\omega, \end{aligned} \tag{7.5}$$

and (7.4) then follows from (6.37). □

We recall that for (M, g) compact, the conformal automorphism group $\text{Conf}(M, g)$ is a Lie group, with Lie algebra the space of conformal Killing vector fields. Furthermore,

$$\dim(\text{Conf}(M, g)) \leq \frac{(n+1)(n+2)}{2}. \tag{7.6}$$

Theorem 7.4. *Let (M,g) be compact. If (M,g) has negative semi-definite Ricci tensor, then Conf(M,g) = Iso(M,g). Furthermore, dim(Conf(M,g)) $\leq n$. If, in addition, the Ricci tensor is negative definite at some point, then Conf(M,g) is finite.*

Proof. Given a conformal Killing form ω, from Proposition 7.3 it follows that

$$0 = -2\Big(\frac{n-1}{n}\Big)d\delta\omega - \delta d\omega + 2Ric(\sharp\omega,\cdot). \tag{7.7}$$

Pairing with ω and integrating by parts,

$$0 = -2\Big(\frac{n-1}{n}\Big)\int_M |\delta\omega|^2 dV - \int_M |d\omega|^2 dV + 2\int_M Ric(\omega,\omega)dV. \tag{7.8}$$

This implies that any conformal Killing form is both closed and co-closed. In particular, it is a Killing form. Since the Lie algebras agree, we must have Conf(M,g) = Iso(M,g), and the result follows from Theorem (6.7). □

Using the formula (7.4), we can also give an alternative proof of Theorem 6.2.

Proof of Theorem 6.2. If f satisfies $\Delta_H f = \lambda_0 f$, then plug $\omega = df$ into (7.4) to obtain

$$\begin{aligned}\text{div } \mathcal{K}\omega &= -2\Big(\frac{n-1}{n}\Big)d\delta df + 2Ric(\sharp\omega,\cdot)\\ &= -2\Big(\frac{n-1}{n}\Big)\lambda_0\omega + 2Ric(\sharp\omega,\cdot).\end{aligned} \tag{7.9}$$

Pairing with ω and integrating,

$$\int_M \langle \text{div } \mathcal{K}\omega,\omega\rangle dV = -2\Big(\frac{n-1}{n}\Big)\lambda_0\int_M |\omega|^2 dV + \int_M 2Ric(\sharp\omega,\sharp\omega)dV. \tag{7.10}$$

Integrating by parts, and using $Ric \geq (n-1)a\cdot g$, we obtain

$$-\frac{1}{2}\int_M |\mathcal{K}\omega|^2 dV \geq \Big\{-2\Big(\frac{n-1}{n}\Big)\lambda_0 + 2a(n-1)\Big\}\int_M |\omega|^2 dV. \tag{7.11}$$

If f is non-constant, then $\omega = df$ is nontrivial, and we obtain the inequality

$$\lambda_0 \geq na. \tag{7.12}$$

Furthermore, we have equality if and only if there is a non-trivial conformal Killing form ω of the form $\omega = df$. This implies that f satisfies (6.17), and the rest of the proof is the same as above. □

7.2 Conformal Killing forms on S^n

On the sphere, we can completely describe the conformal Killing 1-forms.

Proposition 7.5. *On S^n with the round metric, any conformal Killing form may be decomposed as*

$$\omega = df + \omega_0, \tag{7.13}$$

where f is a first order spherical harmonic, and ω_0 is a Killing form.

Consequently, the eigenspace corresponding to the eigenvalue n of Δ_H is of dimension $(n+1)$, and eigenforms are conformal Killing Fields.

The eigenspace corresponding to the eigenvalue $2(n-1)$ of Δ_H consists of Killing 1-forms, and is therefore of dimension $\frac{n(n+1)}{2}$.

Proof. From Proposition 6.3, we know that

$$\omega = df + \omega_0, \tag{7.14}$$

where f is a function with mean value zero, and ω_0 is divergence-free. Since ω is conformal Killing, we have

$$\begin{aligned} 0 = \mathcal{K}\omega &= \mathcal{K}df + \mathcal{K}\omega_0 \\ &= 2\mathring{\nabla}^2 f + \mathcal{K}\omega_0. \end{aligned} \tag{7.15}$$

which is

$$\mathring{\nabla}^2 f = -\frac{1}{2}\mathcal{K}\omega_0, \tag{7.16}$$

where $\mathring{\nabla}^2$ denotes the traceless Hessian operator. Pair both sides with of (7.16) with $\mathring{\nabla}^2 f$, and integrate by parts

$$\begin{aligned} \int_{S^n} |\mathring{\nabla}^2 f|^2 dV &= -\frac{1}{2}\int_{S^n} \langle \mathring{\nabla}^2 f, \mathcal{K}\omega_0\rangle dV \\ &= -\frac{1}{2}\int_{S^n} \langle \mathcal{K}^*\mathring{\nabla}^2 f, \omega_0\rangle dV. \end{aligned} \tag{7.17}$$

Since the L^2 adjoint of $\mathcal{K}$ is 2δ, this is

$$\int_{S^n} |\mathring{\nabla}^2 f|^2 dV = -\int_{S^n} \langle \delta\mathring{\nabla}^2 f, \omega_0\rangle dV. \tag{7.18}$$

Next, let us compute the divergence of the trace-free Hessian of f,

$$\begin{aligned} (\delta\mathring{\nabla}^2 f)_j &= -g^{pq}\nabla_p\Big(\nabla_q\nabla_j f - \frac{1}{n}(\Delta f)g_{qj}\Big) \\ &= -g^{pq}\nabla_p\nabla_j\nabla_q f - \frac{1}{n}\nabla_j(\Delta f) \\ &= -g^{pq}(\nabla_j\nabla_p\nabla_q f - R_{pjq}{}^l\nabla_l f) - \frac{1}{n}\nabla_j(\Delta f) \\ &= -\nabla_j(\Delta f) - g^{pq}Ric_{jp}\nabla_q f - \frac{1}{n}\nabla_j(\Delta f) \\ &= -(n-1)\nabla_j\Big(\frac{1}{n}\Delta f + f\Big). \end{aligned} \tag{7.19}$$

Plugging this into (7.18),

$$\begin{aligned}\int_{S^n} |\mathring{\nabla}^2 f|^2 dV &= (n-1)\int_{S^n} \Big\langle \nabla\Big(\frac{1}{n}\Delta f + f\Big), \omega_0 \Big\rangle dV \\ &= (n-1)\int_{S^n} \Big\langle \Big(\frac{1}{n}\Delta f + f\Big), \delta\omega_0 \Big\rangle dV = 0.\end{aligned} \tag{7.20}$$

We conclude that $\mathring{\nabla}^2 f = 0$. Then (7.19) implies that

$$\Delta f = -nf + c, \tag{7.21}$$

where c is a constant. Integrating, implies that $c = 0$. □

8 Lecture 8

8.1 Eigenvalues of elliptic operators

Let $P : \Gamma(E) \to \Gamma(E)$ be self-ajoint elliptic operator for a Riemannian vector bundle E over a compact Riemannian manifold. Then the following holds.

- There exists a countable sequence of real numbers

  $$\lambda_0 < \lambda_1 < \lambda_2 < \dots, \tag{8.1}$$

  with no accumulation points such that the space

  $$V_{\lambda_i} = \{\sigma \in \Gamma(E) \mid P\sigma = \lambda_i \sigma\} \tag{8.2}$$

  is finite-dimensional, and consists entirely of smooth sections.

- If $\lambda \in \mathbb{R} \setminus \{\lambda_0, \lambda_1, \dots\}$ then the operator $P - \lambda I$ is invertible with bounded inverse.

- Furthermore $L^2(E)$ admits the orthogonal decompostion

  $$L^2(E) = \bigoplus_{i=0}^{\infty} V_{\lambda_i}. \tag{8.3}$$

  That is, the space of all eigenfunctions is dense in $L^2(E)$.

The proof was outlined in the lecture, to be completed later.

9 Lecture 9

9.1 Eigenfunctions of Δ_H on functions on S^{n-1}

Proposition 9.1. *Consider $S^{n-1} \subset \mathbb{R}^n$ with the round metric g_S. The eigenvalues of Δ_H acting on functions are*

$$\lambda_k = k(k+n-2), \quad k \geq 0. \tag{9.1}$$

Furthermore, the correponding eigenspace V_{λ_k} is exactly the space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^n$, restricted to S^{n-1}.

The proof was given in the lecture, to be added here later.

10 Lecture 10

10.1 Some computations

In the following, we will consider metrics of the form

$$g = dt^2 + g_S(t,x), \tag{10.1}$$

where $g_S(t,x)$ is a time-dependent metric on S^3. The index 0 will correspond to the t coordinate, while the indices $1,2,3$ will correspond to a coordinate system on S^3.

The Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = \Gamma^k_{ij}(g_S),\ i,j,k \geq 1 \tag{10.2}$$

$$\Gamma^0_{ij} = \frac{1}{2}g^{00}(\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij}) = -\frac{1}{2}(\dot{g}_S)_{ij},\ i,j \geq 1 \tag{10.3}$$

$$\Gamma^0_{i0} = \frac{1}{2}g^{00}(\partial_i g_{00} - \partial_0 g_{i0} - \partial_0 g_{i0}) = 0,\ i \geq 1 \tag{10.4}$$

$$\Gamma^p_{00} = \frac{1}{2}g^{pl}(\partial_0 g_{l0} + \partial_0 g_{l0} - \partial_l g_{00}) = 0,\ p \geq 0 \tag{10.5}$$

$$\Gamma^p_{0j} = \frac{1}{2}g^{pl}(\partial_0 g_{lj} + \partial_j g_{l0} - \partial_l g_{0j}) = \frac{1}{2}g^{pl}(\partial_0 g_{lj}) = \frac{1}{2}g^{pl}_S(\dot{g}_S)_{lj},\ p,j \geq 1. \tag{10.6}$$

We will write $\tilde{\omega} = f dr + \omega$, where $\omega \in \Omega^1(S^3)$.

10.2 Divergence operator on 1-forms

We compute the divergence on 1-forms.

$$\begin{aligned}
\delta\tilde{\omega} &= g^{ij}\nabla_i\tilde{\omega}_j,\ i,j \geq 0, && (10.7)\\
&= g^{00}\nabla_0\tilde{\omega}_0 + g^{ij}\nabla_i\tilde{\omega}_j,\ i,j \geq 1, && (10.8)\\
&= \dot{f} + g^{ij}(\partial_i\omega_j - \Gamma^p_{ij}\tilde{\omega}_p),\ i,j \geq 1, p \geq 0 && (10.9)\\
&= \dot{f} + g^{ij}(\partial_i\omega_j - \Gamma^0_{ij}f - \Gamma^k_{ij}\omega_k),\ i,j,k \geq 1 && (10.10)\\
&= \dot{f} + \delta_S\omega + \frac{1}{2}f tr_{g_S}(\dot{g}_S). && (10.11)
\end{aligned}$$

Using this, the Laplacian on functions is:

$$\Delta_H\phi = -\ddot{\phi} - \frac{1}{2}tr_{g_S}(\dot{g}_S)\dot{\phi} + \delta_S d_S\phi \tag{10.12}$$

Special case of $g = dt^2 + a(t)g_S$, where g_S is the round metric:

$$\Delta_H\phi = -\ddot{\phi} - \frac{1}{2}(n-1)\frac{\dot{a}}{a}\dot{\phi} + \frac{1}{a}\Delta_{g_S}\phi \tag{10.13}$$

10.3 The cylindrical metric

This is $g = dt^2 + g_S$, so $a \equiv 1$. Therefore

$$\Delta_H \phi = -\ddot{\phi} + \Delta_{g_S} \phi. \tag{10.14}$$

Let $\phi = f(t)B(\theta)$, where B is an eigenfunction, then

$$\Delta_H fB = (-\ddot{f} + k(k+n-2)f)B. \tag{10.15}$$

Solutions are given by

$$f = c_1 + c_2 t, \tag{10.16}$$

if $k = 0$, or

$$f = c_1 \cosh(t\sqrt{k(k+n-2)}) + c_2 \sinh(t\sqrt{k(k+n-2)}), \tag{10.17}$$

if $k > 0$.

Properties of harmonic functions on the cylinder:

- Any bounded harmonic function is constant.
- (Liouville Theorem) If $\Delta_H \phi = 0$ and $\phi = O(e^{C_2|t|})$ as $t \to \pm\infty$, then ϕ is a finite linear combination of harmonic functions of the above form.

10.4 Euclidean space

This is $g = dr^2 + r^2 g_S$, $r > 0$, so $a \equiv r^2$. Therefore

$$\Delta_H \phi = -\ddot{\phi} - \frac{n-1}{r}\dot{\phi} + \frac{1}{r^2}\Delta_{g_S}\phi. \tag{10.18}$$

Let $\phi = f(t)B(\theta)$, where B is an eigenfunction, then

$$\Delta_H fB = (-\ddot{f} - \frac{n-1}{r}\dot{f} + \frac{k(k+n-2)}{r^2} f)B. \tag{10.19}$$

For $n = 2$, and $k = 0$, the solution is $f(r) = c_1 + c_2 \log r$. Otherwise, for each k, there are 2 solutions given by

$$f = r^p \tag{10.20}$$

for $p = k, 2 - n - k$.

Properties of harmonic functions on Euclidean space:

- The two solutions above are related by the Kelvin transform.
- Any bounded harmonic function on $\mathbb{R}^n$ is constant.
- (Liouville Theorem) If $\Delta_H \phi = 0$ and $\phi = O(r^N)$ as $r \to \infty$, then ϕ is a harmonic polynomial.

10.5 The sphere

This is $g = dr^2 + sin^2(r) g_S$, $0 < r < \pi$, so $a \equiv sin^2(r)$. This is a compact manifold, and the only global harmonic functions are constant. But let us consider eigenfunctions. This is an n-dimensional sphere, so the eigenvalues are $k(k+n-1)$. So let ψ_k be an eigenfunction such that

$$\Delta_{S^n} \psi_k = k(k+n-1)\psi_k. \tag{10.21}$$

Let us separate variables so that

$$\psi_{k,l} = f_k(r) B_l \tag{10.22}$$

where B_l is an eigenfunction on S^{n-1} satisfying

$$\Delta_{S^{n-1}} B_l = l(l+n-2) B_l. \tag{10.23}$$

Then the eigenvalue equation is

$$\Delta_H \psi_{k,l} = (-\ddot{f}_k - (n-1)\cot(r)\dot{f}_k + \frac{l(l+n-2)}{\sin(r)^2} f_k) B_l = k(k+n-1) f_k B_l. \tag{10.24}$$

This yields an ODE

$$-\ddot{f}_k - (n-1)\cot(r)\dot{f}_k + \Big(\frac{l(l+n-2)}{\sin(r)^2} - k(k+n-1)\Big) f_k = 0. \tag{10.25}$$

For $n = 2$, this is

$$-\ddot{f}_k - \cot(r)\dot{f}_k + \Big(\frac{l^2}{\sin(r)^2} - k(k+1)\Big) f_k = 0. \tag{10.26}$$

For each k, there 2 solutions for each $0 \leq l \leq k$, which are Legendre polynomials in $\sin(r), \cos(r)$. So each k gives $2k+1$ solutions.

What we have done here is follows. The space of harmonic polynomials of degree k on $\mathbb{R}^3$, call it $\mathcal{H}_k$, is of dimension $2k+1$. This is a irreducible representation space of SO(3). However, if we restrict the action to SO(2) $\subset$ SO(3), by rotations fixing the north and south poles, then the representation $\mathcal{H}_k$ decomposes a 1-dimensional representations (which is the zonal harmonic), and k 2-dimensional representations.

11 Lecture 11

11.1 Mapping properties of the Laplacian

Let us define the set

$$\mathcal{I} = \begin{cases} \mathbb{Z} & n = 2,3 \\ \mathbb{Z} \setminus \{3-n, \dots, -1\} & n \geq 4 \end{cases} \tag{11.1}$$

Proposition 11.1. *In $\mathbb{R}^n$, consider*

$$\Delta : r^k H^2(S^{n-1}) \to r^{k-2} L^2(S^{n-1}). \tag{11.2}$$

- *If $k \in \mathbb{Z} \setminus \mathcal{I}$, then this mapping is an isomorphism.*
- *If $k \in \mathbb{Z}_{\geq 0}$, unless $k = 0$ and $n = 2$, then*

  $$\Delta : \log(r) r^k V_k \oplus r^k H^2(S^{n-1}) \to r^{k-2} L^2(S^{n-1}). \tag{11.3}$$

  is surjective, where V_k is the kth eigenspace of the Laplacian on S^{n-1}.
- *If $k = 0$ and $n = 2$, then*

  $$\Delta : \{\log(r)^2\} \oplus H^2(S^1) \to r^{-2} L^2(S^1). \tag{11.4}$$

  is surjective, where V_k is the kth eigenspace of the Laplacian on S^{n-1}.
- *If $k \in \mathbb{Z}$ and $k \leq 2 - n$, unless $k = 0$ and $n = 2$, then*

  $$\Delta : \log(r) r^k V_{2-n-k} \oplus r^k H^2(S^{n-1}) \to r^{k-2} L^2(S^{n-1}). \tag{11.5}$$

  is surjective.

Proof. TO BE COMPLETED. □

We can actually prove a stronger result

Proposition 11.2. *In $\mathbb{R}^n$, for $q \geq 1$, consider*

$$\Delta : P_q(\log(r)) r^k H^2(S^{n-1}) \to P_q(\log(r)) r^{k-2} L^2(S^{n-1}), \tag{11.6}$$

where P_q is a polynomial of degree $q \geq 1$.

- *If $k \in \mathbb{Z} \setminus \mathcal{I}$, then this mapping is an isomorphism.*
- *If $k \in \mathbb{Z}_{\geq 0}$, then*

  $$\Delta : \log(r)^{q+1} r^k V_k \oplus P_q(\log(r)) r^k H^2(S^{n-1}) \to P_q(\log(r)) r^{k-2} L^2(S^{n-1}). \tag{11.7}$$

  is surjective, where V_k is the kth eigenspace of the Laplacian on S^{n-1}.
- *If $k \in \mathbb{Z}$ and $k \leq 2 - n$, then*

  $$\Delta : \log(r)^{q+1} r^k V_{2-n-k} \oplus P_q(\log(r)) r^k H^2(S^{n-1}) \to P_q(\log(r)) r^{k-2} L^2(S^{n-1}). \tag{11.8}$$

  is surjective.

Proof. TO BE COMPLETED. □

12 Lecture 12

12.1 Harmonic functions blowing up at a point

Using the above result, we will prove the following.

Theorem 12.1. *Let (M, g) be a compact manifold and let h_k be any homogeneous harmonic polynomial h_k of degree k in $\mathbb{R}^n$. For $k \geq 1$, and any $p \in M$ there exists a harmonic function $\phi_k : M \setminus \{p\} \to \mathbb{R}$ such that*

$$\phi_k = r^{2-n-k}(r^{-k}h_k) + O(r^{3-n-k}), \tag{12.1}$$

as $r \to 0$, where r is the distance from p.

If $n > 2$, then there does not exist a harmonic function on $M \setminus \{p\}$, satisfying

$$G = r^{2-n} + O(r^{3-n}), \tag{12.2}$$

as $r \to 0$.

If $n = 2$, then there does not exists a harmonic function on $M^2 \setminus \{p\}$, satisfying

$$G = \log(r) + O(1), \tag{12.3}$$

as $r \to 0$.

Proof. TO BE COMPLETED. □

Let us return to the case of harmonic functions on S^n, for which the ODE is

$$-\ddot{f} - (n-1)\cot(r)\dot{f} + \frac{l(l+n-2)}{\sin(r)^2} f = 0. \tag{12.4}$$

For $n = 2$, this is

$$-\ddot{f} - \cot(r)\dot{f} + \frac{l^2}{\sin(r)^2} f = 0. \tag{12.5}$$

The general solution of this is for $l \geq 1$ is

$$c_1 \cot(r/2)^l + c_2 \cot(r/2)^{-l}. \tag{12.6}$$

We see that for each $l > 0$, there are harmonic functions h_l which are smooth on $S^2 \setminus \{N\}$ (N is the north pole) satisfying $h_l = c_1 r^{-l} + O(r^{-k+1})$ as $r \to 0$, which illustrates the previous theorem.

However, for $l = 0$, the solution is

$$c_1 + c_2 \log\Big(\frac{1-\cos(r)}{1+\cos(r)}\Big). \tag{12.7}$$

Of course, we do not find any solution which is asymptotic to $\log(r)$ near N, but which is smooth on $S^2 \setminus \{N\}$, since the second solution in (12.7) also has a singularity at the south pole. Note that

$$\log\left(\frac{1-\cos(r)}{1+\cos(r)}\right) \sim 2\log(r) \tag{12.8}$$

as $r \to 0$, but

$$\log\left(\frac{1-\cos(r)}{1+\cos(r)}\right) \sim -2\log(\pi - r), \tag{12.9}$$

as $r \to \pi$. This is a special case of the following.

Theorem 12.2. *Consider N distinct points $\{p_1, \ldots p_N\} \subset M$. If $n \geq 3$, then there exists a harmonic function on $M \setminus \{p_1, \ldots, p_N\}$ satisfying*

$$G = c_i r_i^{2-n} + O(r_i^{3-n}), \tag{12.10}$$

as $r_i \to 0$ where $r_i(x) = d(p_i, x)$ and c_i are constant, for each $i = 1 \ldots N$, if and only if the "balancing condition"

$$\sum_{i=1}^{N} c_i = 0 \tag{12.11}$$

is satisfied.

A similar statement holds for $n = 2$ if r_i^{2-n} is replaced by $\log(r_i)$.

Proof. TO BE COMPLETED. □

For a smooth function $f : M \to \mathbb{R}$, consider the operator

$$\Box\phi = \Delta_H \phi + f\phi. \tag{12.12}$$

Note that if $f > 0$, and M is compact, then $Ker(\Box) = \{0\}$. To see this, if $\Box\phi = 0$, then

$$0 = \int_M \phi\Box\phi = \int_M |\nabla\phi|^2 dV + \int_M f\phi^2 dV, \tag{12.13}$$

which implies that $\phi = 0$.

Theorem 12.3. *Let (M, g) be a compact manifold, and let h_k be any homogeneous harmonic polynomial h_k of degree k in $\mathbb{R}^n$. Assume that $Ker(\Box) = \{0\}$. For $k \geq 0$, and any $p \in M$ there exists a function $\phi_k : M \setminus \{p\} \to \mathbb{R}$ satisfying $\Box\phi_k = 0$, and such that*

$$\phi_k = r^{2-n-k}(r^{-k}h_k) + O(r^{3-n-k}), \tag{12.14}$$

as $r \to 0$, where r is the distance from p.

If $n > 2$, then there exists a function on $M \setminus \{p\}$ satisfying $\square G = 0$, and such that

$$G = r^{2-n} + O(r^{3-n}), \tag{12.15}$$

as $r \to 0$.

If $n = 2$, then there exists a function on $M^2 \setminus \{p\}$ satisfying $\square G = 0$, and such that

$$G = \log(r) + O(1), \tag{12.16}$$

as $r \to 0$.

Proof. TO BE COMPLETED. $\square$

What if there is some nontrivial kernel of $\square$? In this case, the answer for which leading terms extend is quite complicated. It depends upon the order of vanishing of the kernel elements at the point. The relative index theorem will say something about this, but this will be discussed much later.

12.2 Remarks

For construction the Green's function of $\square$, we have an expansion like

$$\square(r^{2-n} + G_{3-n} + \cdots + G_{-1} + G_0) = O(r^{-1}). \tag{12.17}$$

Note that all of these terms are completely determined, but when we get to the zeroth order term G_0, we are free to add a constant to the expansion. However, the actual Green's function is unique, so this constant should be determined! This is true, but for global reasons. In our construction, we need to apply the inverse operator to our "approximate" Green's function, and this will then give the correct constant. It is really a globally determined quantity, and simply cannot be determined from local considerations alone.

12.3 Polyhomogeneous expansions

For a harmonic function near a point, we have an expansion

$$h \sim \sum_{k=-N}^{\infty} r^k \Big(\sum_{j=0}^{N_k} c_{jk} (\log(r))^j \Big) \phi_k, \tag{12.18}$$

where $\phi_k : S^{n-1} \to \mathbb{R}$.

13 Lecture 13

13.1 The conformal Laplacian

Let's consider a different operator on S^n. The following operator is known as the conformal Laplacian:

$$\Box = 4\frac{n-1}{n-2}\Delta_H + R = 4\frac{n-1}{n-2}\Delta_H + n(n-1) = 4\frac{n-1}{n-2}(\Delta_H + \frac{n(n-2)}{4}) \tag{13.1}$$

Let's try and find a solution which only depends on r. The resulting ODE is

$$-\ddot{f} - (n-1)\cot(r)\dot{f} + \frac{n(n-2)}{4}f = 0. \tag{13.2}$$

Let's try $n = 4$, the general solution is

$$f(r) = \frac{c_1}{\sin^2(r)} + c_2\frac{\cos(r)}{\sin^2(r)}. \tag{13.3}$$

It is not hard to see that when $c_1 = c_2$, this function is smooth at the south pole, so let

$$G(r) = \frac{1+\cos(r)}{\sin^2(r)}. \tag{13.4}$$

Here is a trick: consider the metric

$$G(r)^2 g_{S^4} = \frac{(1+\cos(r))^2}{\sin^4(r)}(dr^2 + \sin^2(r)g_{S^3}). \tag{13.5}$$

If we make the change of variables $s = \cot(r) + \csc(r)$, then

$$G(r)^2 g_{S^4} = ds^2 + \frac{(1+\cos(r))^2}{\sin^2(r)}g_{S^3} = ds^2 + s^2 g_{S^3}, \tag{13.6}$$

which is the flat metric!

13.2 Conformal geometry

Let $u : M \to \mathbb{R}$. Then $\tilde{g} = e^{-2u}g$, is said to be *conformal* to g.

Proposition 13.1. *The Christoffel symbols transform as*

$$\tilde{\Gamma}^i_{jk} = g^{il}\Big(- (\partial_j u)g_{lk} - (\partial_k u)g_{lj} + (\partial_l u)g_{jk}\Big) + \Gamma^i_{jk}. \tag{13.7}$$

Invariantly,

$$\tilde{\nabla}_X Y = \nabla_X Y - du(X)Y - du(Y)X + g(X,Y)\nabla u. \tag{13.8}$$

Proof. Using (2.6), we compute

$$
\begin{aligned}
\tilde{\Gamma}^i_{jk} &= \frac{1}{2}\tilde{g}^{il}\Big(\partial_j\tilde{g}_{kl} + \partial_k\tilde{g}_{jl} - \partial_l\tilde{g}_{jk}\Big) \\
&= \frac{1}{2}e^{2u}g^{il}\Big(\partial_j(e^{-2u}g_{kl}) + \partial_k(e^{-2u}g_{jl}) - \partial_l(e^{-2u}g_{jk})\Big) \\
&= \frac{1}{2}e^{2u}g^{il}\Big(- 2e^{-2u}(\partial_j u)g_{kl} - 2e^{-2u}(\partial_k u)e^{-2u}g_{jl} + 2e^{-2u}(\partial_l u)g_{jk} \\
&\qquad + e^{-2u}\partial_j(g_{kl}) + e^{-2u}\partial_k(g_{jl}) - e^{-2u}\partial_l(g_{jk})\Big) \\
&= g^{il}\Big(- (\partial_j u)g_{kl} - (\partial_k u)g_{jl} + (\partial_l u)g_{jk}\Big) + \Gamma^i_{jk}.
\end{aligned}
\tag{13.9}
$$

This is easily seen to be equivalent to the invariant expression. □

Proposition 13.2. *Let $\tilde{g} = e^{-2u}g$. The scalar curvature transforms as*

$$\tilde{R} = e^{2u}\Big(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R\Big). \tag{13.10}$$

Proof. TBC. □

By a little more computation, we have the following:

Proposition 13.3. *The $(0,4)$-curvature tensor transforms as*

$$\tilde{Rm} = e^{-2u}\Big[Rm + \Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big) \owedge g\Big]. \tag{13.11}$$

Proof. TBC. □

By writing the conformal factor differently, the scalar curvature equation takes a nice semilinear form, which is the famous Yamabe equation:

Proposition 13.4. *If $n \neq 2$, and $\tilde{g} = v^{\frac{4}{n-2}}g$, then*

$$-4\frac{n-1}{n-2}\Delta v + Rv = \tilde{R}v^{\frac{n+2}{n-2}}. \tag{13.12}$$

If $n = 2$, and $\tilde{g} = e^{-2u}g$, the conformal Gauss curvature equation is

$$\Delta u + K = \tilde{K}e^{-2u}. \tag{13.13}$$

Proof. We have $e^{-2u} = v^{\frac{4}{n-2}}$, which is

$$u = -\frac{2}{n-2}\ln v. \tag{13.14}$$

Using the chain rule,

$$\nabla u = -\frac{2}{n-2}\frac{\nabla v}{v}, \tag{13.15}$$

$$\nabla^2 u = -\frac{2}{n-2}\Big(\frac{\nabla^2 v}{v} - \frac{\nabla v \otimes \nabla v}{v^2}\Big). \tag{13.16}$$

Substituting these into (13.10), we obtain

$$\begin{aligned}\tilde{R} &= v^{\frac{-4}{n-2}}\Big(-4\frac{n-1}{n-2}\Big(\frac{\Delta v}{v} - \frac{|\nabla v|^2}{v^2}\Big) - 4\frac{n-1}{n-2}\frac{|\nabla v|^2}{v^2} + R\Big) \\ &= v^{\frac{-n+2}{n-2}}\Big(-4\frac{n-1}{n-2}\Delta v + Rv\Big).\end{aligned} \tag{13.17}$$

Finally, (13.13) follows from (13.10), and the fact that in dimension 2, $R = 2K$. □

13.3 Uniformization on S^2

Since the conformal group of (S^2, g_S), where g_S is the round metric, is noncompact, we cannot hope to prove existence of a constant curvature metric by a compactness argument as in the $k \geq 1$ case. However, there is a trick to solve this case using only linear theory.

Theorem 13.5. *If (M, g) is a Riemann surface of genus 0, then g is conformal to (S^2, g_S).*

Proof. We remove a point p from M, and consider the manifold $(M \setminus \{p\}, g)$. We want to find a conformal factor $u : M \setminus \{p\} \to \mathbb{R}$ such that $\tilde{g} = e^{-2u}g$ is flat. The equation for this is

$$\Delta u = -K. \tag{13.18}$$

However, by the Gauss-Bonnet theorem, the right hand side has integral 4π, so this equation has no smooth solution. But we will find a solution u on $M \setminus \{p\}$ so that $u = O(\log(r))$ and $r \to 0$, where $r(x) = d(p, x)$. Let ϕ be a smooth cutoff function satisfying

$$\phi = \begin{cases} 1 & r \leq r_0 \\ 0 & r \geq 2r_0 \end{cases}, \tag{13.19}$$

and $0 \leq \phi \leq 1$, for r_0 very small. Consider the function $f = \Delta(\phi \log(r))$. Computing in normal coordinates, near p we have

$$\begin{aligned}\Delta f &= \frac{1}{\sqrt{\det(g)}}\partial_i(g^{ij}u_j\sqrt{\det(g)}) = \frac{1}{\sqrt{\det(g)}}\partial_r(u_r\sqrt{\det(g)}) \\ &= (\log(r))'' + (\log(r))'\frac{(\sqrt{\det(g)})'}{\sqrt{\det(g)}}.\end{aligned}$$

Expanding the volume element in radial coordinates, $\sqrt{\det(g)} = r + O(r^3)$ as $r \to 0$, so we have

$$\Delta f = -\frac{1}{r^2} + \frac{1}{r}\Big(\frac{1 + O(r^2)}{r + O(r^3)}\Big) = -\frac{1}{r^2} + \frac{1}{r^2}\Big(\frac{1 + O(r^2)}{1 + O(r^2)}\Big) = O(1) \tag{13.20}$$

as $r \to 0$.

Next, we compute

$$\int_M f dV = \lim_{\epsilon \to 0} \int_{M \setminus B(p,r)} \Delta(\phi \log(r)) dV = -\lim_{\epsilon \to 0} \int_{S(p,r)} \partial_r(\log(r)) d\sigma = -2\pi.$$

Note the minus sign is due to using the *outward* normal of the domain $M \setminus B(p,r)$. Consequently, we can solve the equation

$$\Delta(u) = -2\Delta(\phi \log(r)) - K, \tag{13.21}$$

by the Gauss-Bonnet Theorem and Fredholm Theory in L^2. Rewriting this as

$$\Delta \tilde{u} = \Delta(u + 2\phi \log(r)) = -K. \tag{13.22}$$

The space $(M \setminus \{p\}, e^{-2\tilde{u}} g)$ is therefore isometric to Euclidean space, since it is clearly complete and simply connected. Using stereographic projection, the spherical metric is conformal to the flat metric, and we can therefore write

$$g_S = \frac{4}{(1+|x|^2)^2} e^{-2\tilde{u}} g = e^{-2v} g. \tag{13.23}$$

It is easy to see that v is a bounded solution of

$$\Delta v + K = e^{-2v} \tag{13.24}$$

on $M \setminus \{p\}$ and extends to a smooth solution on all of M by elliptic regularity. □

Corollary 13.6. *If (M, J) is a Riemann surface homeomorphic to S^2 then it is biholomorphic to the Riemann sphere (S^2, J_S).*

14 Lecture 14

14.1 The Green's function metric

Let (M, g) be compact, and assume that $R > 0$, of dimension $n \geq 3$ Given $p \in M$, by Theorem 12.3, there exists a unique Green's function satisfying

$$\Box G = \Big(-4\frac{n-1}{n-2}\Delta + R \Big) G = 0, \tag{14.1}$$

and admiting an expansion of the form

$$G = r^{2-n} + G_{3-n} + \cdots + G_{-1} + A \log r + G_0 + o(1) \tag{14.2}$$

in normal coordinates around p as $r \to 0$, where G_j is homogeneous of degree j for $3 - n \leq j \leq 0$. Since $G \to +\infty$ as $r \to 0$, there exists a point $x \in M$ at which G obtains its minimum. At this point, we have

$$0 = -4\frac{n-1}{n-2}(\Delta G)(x) + R(x) G(x). \tag{14.3}$$

If $G(x) < 0$, then the second term is strictly negative, which implies that $(\Delta G)(x) < 0$. However, in any coordindate system around x, the Laplacian of G at x is just the Euclidean Laplacian at x, and this contradicts the fact that x is a minimum of G. If $G(x) = 0$, then this argument does not lead to a contradiction. In this case, we can appeal to the Hopf strong maximum principle to say this cannot happen [GT01, Section 3.2].

Since $G > 0$, we can consider the metric $(X, \tilde{g} = G^{\frac{4}{n-2}} g)$, where $X = M \setminus \{p\}$. By (13.12), $R_{\tilde{g}} = 0$. Consider the coordinate system defined on $\mathbb{R}^n \setminus B(0, R)$ for some $R > 0$ large defined by

$$y^i = s^{-2} x^i, \ i = 1 \dots n, \tag{14.4}$$

and $s^2 = (y^1)^2 + \cdots + (y^n)^2$. Let us expand the metric $\tilde{g}$ in these coordinates. First, since the coordinates x^i are normal, for any N, we can write

$$g = (\delta_{ij} + (H_2)_{ij} + \cdots + (H_N) + O(|x|^{N+1}))dx^i \otimes dx^j \tag{14.5}$$

as $|x| \to 0$, where H_j is homogeneous of degree j for $2 \leq j \leq N$. Let Φ denote the mapping defined by $x^i = s^{-2} y^i$. Then

$$\Phi^* g = \Big(\delta_{ij} + (H_{-2})_{ij} + \cdots + (H_{-N})_{ij} + O(|y|^{-N-1}) \Big) \Phi^* dx^i \otimes \Phi^* dx^j, \tag{14.6}$$

where H_j is now homogeneous of degree $-j$ in the y coordinates. We compute

$$\Phi^* dx^i = d(s^{-2} y^i) = (-2 s^{-4} y^i y^l + s^{-2} \delta_{il}) dy^l \tag{14.7}$$

This implies that

$$\Phi^* g = s^{-4} \Big(\delta_{ij} + (H_{-2})_{ij} + \cdots + (H_{-N})_{ij} + O(|y|^{-N-1}) \Big) dy^i \otimes dy^j, \tag{14.8}$$

where we have re-defined the H_j.

Next, expanding the Green's function in the y^i coordinates,

$$\Phi^* G = s^{n-2} + G_{n-4} + \cdots + G_1 - A \log s + G_0 + o(1) \tag{14.9}$$

as $s \to \infty$, where the G_j are homogeneous of degree j. So then

$$(\Phi^* G)^{\frac{4}{n-2}} = s^4 + G_2 + G_1 + G_0 + \dots, \tag{14.10}$$

where again we have redefined the G_j.

We then have

$$\begin{aligned} \Phi^* \tilde{g} = (\Phi^* G)^{\frac{4}{n-2}} \Phi^* g &= \Big(s^4 + G_2 + G_1 + G_0 + \dots \Big) \\ &\quad \cdot s^{-4} \Big(\delta_{ij} + (H_{-2})_{ij} + \cdots + (H_{-N})_{ij} + O(|y|^{-N-1}) \Big) dy^i \otimes dy^j \\ &= \Big(\delta_{ij} + (J_{-2})_{ij} + \dots \Big) dy^i \otimes dy^j. \end{aligned} \tag{14.11}$$

So the Green's function metric in inverted normal coordinates looks like the Euclidean metric plus decaying terms. This is a special case of an *asymptotically flat metric*, which we will define more generally later.

14.2 Transformation law for the conformal Laplacian

The following is a transformation formula for the conformal Laplacian.

Proposition 14.1. *If $n \neq 2$, and $\tilde{g} = v^{\frac{4}{n-2}} g$, and*

$$\Box_g \phi = -\frac{4(n-1)}{n-2} \Delta_g \phi + R_g \phi, \tag{14.12}$$

then

$$\Box_{\tilde{g}} (v^{-1}\phi) = v^{-\frac{n+2}{n-2}} \Box_g \phi. \tag{14.13}$$

For $n = 2$, if $\tilde{g} = e^{2w} g$, then

$$\Delta_{\tilde{g}} \phi = e^{-2w} \Delta_g \phi. \tag{14.14}$$

Proof. First compute that

$$\Delta_{\tilde{g}} \phi = v^{-\frac{4}{n-2}} \Delta_g \phi + 2 v^{-\frac{n+2}{n-2}} \langle \nabla v, \nabla \phi \rangle. \tag{14.15}$$

From this, it follows that

$$\Delta_{\tilde{g}} (v^{-1}\phi) = v^{-\frac{n+2}{n-2}} \Big(\Delta_g \phi - \frac{\Delta v}{v} \phi \Big). \tag{14.16}$$

□

14.3 Harmonic functions on the Green's function metric

Since the Green's function metric is scalar-flat, the transformation formula (14.13) says that

$$-\frac{4(n-1)}{n-2} \Delta_{\tilde{g}} (G^{-1}\phi) = G^{-\frac{n+2}{n-2}} \Box_g \phi. \tag{14.17}$$

So if ϕ is a function which is $\Box_g$-harmonic, then $G^{-1}\phi$ is a harmonic function for $\tilde{g}$. Therefore, from Theorem 12.3, we have the following.

Theorem 14.2. *Let $(X, \tilde{g})$ be a Green's function metric, $n \geq 3$, and let h_k be any homogeneous harmonic polynomial h_k of degree k in $\mathbb{R}^n$. For $k \geq 0$, there exists a harmonic function $\phi_k : X \to \mathbb{R}$ such that*

$$\phi_k = |y|^k (|y|^{-k} h_k) + O(|y|^{k-1}), \tag{14.18}$$

as $|y| \to \infty$.

Proof. This follows directly from Theorem 12.3. □

The same result is true for *any* asymptotically flat metric, but this will be more difficult to prove, since we cannot use the same trick in general. This trick converted the difficult step of inverting the Laplacian on the ALE metric to the much easier step of inverting the conformal Laplacian on the conformal compactification.

15 Lecture 15

15.1 Analysis on the cylinder

For $\delta \in \mathbb{R}$, define $L^2_\delta(C) = e^{\delta t} L^2(C)$, with norm

$$\|u\|_{L^2_\delta(C)} = \|e^{-\delta t} u\|_{L^2(C)}. \tag{15.1}$$

More generally, for any integer $k \geq 0$, defined $W^{k,2}_\delta(C) = e^{\delta t} W^{k,2}(C)$, with norm

$$\|u\|_{W^{k,2}_\delta(C)} = \|e^{-\delta t} u\|_{W^{k,2}(C)}. \tag{15.2}$$

The Laplacian on the cylinder is given by

$$\Delta u = \ddot{u} + \Delta_{S^{n-1}} u. \tag{15.3}$$

Then

$$\Delta : W^{k,2}_\delta \to W^{k-2,2}_\delta \tag{15.4}$$

is a bounded linear mapping.

Another imporant operator on the cylinder is the following. In $\mathbb{R}^n \setminus \{0\}$ consider $|x|^2 \Delta_0$, where Δ_0 is the Euclidean Laplacian. Recall that

$$\Delta_0 f = \ddot{f} + \frac{n-1}{r} \dot{f} + \frac{1}{r^2} \Delta_{S^{n-1}} f. \tag{15.5}$$

Let us make the change of variable $r = e^t$, or $t = \log r$. Writing $f(r) = h(\log r)$, we have

$$f'(r) = e^{-t} h'(t) \tag{15.6}$$

$$f''(r) = e^{-2t}(h''(t) - h'(t)), \tag{15.7}$$

so that

$$|x|^2 \Delta_0 f = \ddot{h} + (n-2)\dot{h} + \Delta_{S^{n-1}} h. \tag{15.8}$$

Clearly then

$$|x|^2 \Delta_0 : W^{k,2}_\delta \to W^{k-2,2}_\delta \tag{15.9}$$

is a bounded linear mapping.

15.2 Basic elliptic estimate

We have the following.

Proposition 15.1. *Let L be either Δ or $|x|^2\Delta_0$, and fix any $\delta \in \mathbb{R}$. Then there exists a constant C so that*

$$\|u\|_{W^{k,2}_\delta(C)} \leq C(\|Lu\|_{W^{k-2,2}_\delta(C)} + \|u\|_{W^{k-2,2}_\delta(C)}) \tag{15.10}$$

Proof. Use the usual elliptic estimate for $\Omega' \Subset \Omega$, where $\Omega' = (i-1, i+1) \times S^{n-1}$ and $\Omega = (i-2, i+2) \times S^{n-1}$ to get a constant C so that

$$\|u\|^2_{W^{k,2}(\Omega')} \leq C(\|Lu\|^2_{W^{k-2,2}(\Omega)} + \|u\|^2_{W^{k-2,2}(\Omega)}. \tag{15.11}$$

Multiply this by $e^{-2\delta i}$, and then sum over $i \in \mathbb{Z}$. ∎

15.3 Indicial roots

Let $u = a(t)\phi(\theta)$ then

$$\Delta u = \ddot{a}\phi + a\Delta_{S^{n-1}}\phi. \tag{15.12}$$

If ϕ is an eigenfunction, so that

$$\Delta_{S^{n-1}}\phi = -k(k+n-2)\phi, \tag{15.13}$$

then

$$\Delta u = (\ddot{a} - k(k+n-2)a)\phi. \tag{15.14}$$

The solutions are

$$a(t) = e^{\pm t\sqrt{k(k+n-2)}}, \tag{15.15}$$

if $k \neq 0$, or

$$a(t) = c_0 + c_1 t \tag{15.16}$$

if $k = 0$.

Definition 15.2. The set of indicial roots of Δ on the cylinder is the set of real numbers $\pm\sqrt{k(k+n-2)}$ for $k \in \mathbb{Z}_{\geq 0}$.

For the operator $|x|^2\Delta_0$, we have

$$|x|^2\Delta_0 u = (\ddot{a} + (n-2)\dot{a} - k(k+n-2)a)\phi. \tag{15.17}$$

The solutions are

$$a(t) = e^{kt}, e^{(2-n-k)t} \tag{15.18}$$

unless $n = 2$ and $k = 0$, in which case the solutions are

$$a(t) = c_0 + c_1 t. \tag{15.19}$$

Definition 15.3. If $n > 3$, the set of indicial roots of $|x|^2\Delta_0$ on the cylinder is the set of real numbers $\mathbb{Z} \setminus \{3-n, \dots, -1\}$. If $n = 2, 3$, then the set of indicial roots is $\mathbb{Z}$.

15.4 The key estimate

The main result is the following.

Theorem 15.4. *Let $L = \Delta$ or $|x|^2\Delta_0$ on the cylinder, and assume that δ is not an indicial root. Then there exists a constant C such that*

$$\|u\|_{L^2_\delta(C)} \leq C\|Lu\|_{L^2_\delta(C)}. \tag{15.20}$$

which implies that there exists C_k so that

$$\|u\|_{W^{k,2}_\delta(C)} \leq C_k \|Lu\|_{W^{k-2,2}_\delta(C)}. \tag{15.21}$$

Moreover, the mapping

$$L : W^{k,2}_\delta(C) \to W^{k-2,2}_\delta(C) \tag{15.22}$$

is an isomorphism.

Proof. The basic idea is the following. Consider the equation $\Delta u = f$. Rewrite this as

$$B_\delta U = (e^{-\delta t} \Delta e^{\delta t}) U = F, \tag{15.23}$$

where $U = e^{-\delta t} u$ and $F = e^{-\delta t} f$ are in L^2. The operator

$$B_\delta(u) = \ddot{u} + 2\delta \dot{u} + (\Delta_{S^{n-1}} + \delta^2) u. \tag{15.24}$$

Taking the Fourier transform in the t variable yields

$$(\Delta_{S^{n-1}} + (\delta + i\xi)^2) \hat{U} = \hat{F}. \tag{15.25}$$

If δ is not an incidial root, then $(\delta + i\xi)^2$ is not an eigenvalue, so the operator $\hat{B}_\delta$ has an inverse $R_\delta : L^2(S^{n-1}) \to L^2(S^{n-1})$, which is bounded. Furthermore, using Plancheral's formula, it is easy to see that the bound of the inverse depends only on δ and not on ξ. To complete the proof, take the inverse Fourier transform, and use that the Fourier transform is an isometry in L^2. □

15.5 Analysis of Δ_0 on $\mathbb{R}^n \setminus \{0\}$

Letting $\mathbb{R}^n_*$ denote $\mathbb{R}^n \setminus \{0\}$, define $L^2_\delta(\mathbb{R}^n_*)$ by the following

$$\|u\|_{L^2_\delta(\mathbb{R}^n_*)} = \left\{ \int_{\mathbb{R}^n_*} |u|^2 r^{-2\delta - n} dx \right\}^{\frac{1}{2}}. \tag{15.26}$$

Also, define $W^{k,2}_\delta(\mathbb{R}^n_*)$ by

$$\|u\|_{W^{k,2}_\delta(\mathbb{R}^n_*)} = \sum_{j=0}^{k} \|D^j u\|_{L^2_{\delta-j}(\mathbb{R}^n_*)}. \tag{15.27}$$

Proposition 15.5. *Under the change of variables $r = e^t$, the $W^{k,2}_\delta(\mathbb{R}^n_*)$ norm is equivalent to the norm $W^{k,2}_\delta(C)$.*

Proof. This is done by direct calculation. □

Using the results from the previous lecture, we have the following corollaries.

Corollary 15.6. *Fix any $\delta \in \mathbb{R}$. Then there exists a constant C so that*

$$\|u\|_{W^{k,2}_{\delta}(\mathbb{R}^n_*)} \leq C(\|\Delta_0 u\|_{W^{k-2,2}_{\delta-2}(\mathbb{R}^n_*)} + \|u\|_{W^{k-2,2}_{\delta}(\mathbb{R}^n_*)}) \tag{15.28}$$

Proof. This can be proved directly by a scaling argument in dyadic annuli. However, it follows directly from the previous lecture, noting that the weight is shifted in the first term on the right hand side. This is because we apply the result on the cylinder to the operator $|x|^2\Delta_0$. □

The key result is then the following.

Theorem 15.7. *Consider Δ_0 on $\mathbb{R}^n_*$, and assume that δ is not an indicial root of $|x|^2\Delta_0$. Then there exists a constant C such that*

$$\|u\|_{L^2_{\delta}(\mathbb{R}^n_*)} \leq C\|\Delta u\|_{L^2_{\delta-2}(\mathbb{R}^n_*)}, \tag{15.29}$$

which implies that there exists C_k so that

$$\|u\|_{W^{k,2}_{\delta}(\mathbb{R}^n_*)} \leq C_k\|\Delta u\|_{W^{k-2,2}_{\delta-2}(\mathbb{R}^n_*)}. \tag{15.30}$$

Moreover, the mapping

$$\Delta_0 : W^{k,2}_{\delta}(\mathbb{R}^n_*) \to W^{k-2,2}_{\delta-2}(\mathbb{R}^n_*) \tag{15.31}$$

is an isomorphism.

A great reference for this section is Frank Pacard's lecture notes.

16 Lecture 16

16.1 Asymptotically flat metrics

The Green's function metric introduced above is a special case of the following class of metrics.

Definition 16.1. A complete Riemannian manifold (X^n, g) is called *asymptotically flat* or *AF* of order $\tau > 0$ if there exists a diffeomorphism $\psi : X \setminus K \to (\mathbb{R}^n \setminus B(0, R))$ where K is a compact subset of X, and such that under this identification,

$$(\psi_* g)_{ij} = \delta_{ij} + O(\rho^{-\tau}), \tag{16.1}$$
$$\partial^{|k|}(\psi_* g)_{ij} = O(\rho^{-\tau-k}), \tag{16.2}$$

for any partial derivative of order k, as $r \to \infty$, where ρ is the distance to some fixed basepoint.

Let $w > 0$ be a smooth positive weight function $w : X \to \mathbb{R}$ defined as follows

$$w = \begin{cases} |x| \circ \Psi & |x| \circ \Psi \geq R \\ 1 & \rho \leq 1 \end{cases}, \tag{16.3}$$

and $1 < w < R$ otherwise. Define $L^2_\delta(X)$ by the following

$$\|u\|_{L^2_\delta(X)} = \left\{ \int_X |u|^2 w^{-2\delta - n} dV_g \right\}^{\frac{1}{2}}. \tag{16.4}$$

Also, define $W^{k,2}_\delta(X)$ by

$$\|u\|_{W^{k,2}_\delta(X)} = \sum_{j=0}^{k} \|D^j u\|_{L^2_{\delta-j}(X)}. \tag{16.5}$$

16.2 Fredholm Properties of Δ

Next, we have the following expansion of the Laplacian of g.

Proposition 16.2. *In the AF coordinate system,*

$$\Delta u = a^{ij}(x)\partial_i \partial_j u + b^i(x)\partial_i u, \tag{16.6}$$

where

$$a^{ij} = \delta^{ij} + O(r^{-\tau}), \tag{16.7}$$

$$b^i = O(r^{-\tau-1}) \tag{16.8}$$

as $r \to \infty$.

Proof. We compute

$$\begin{aligned} \Delta u &= g^{ij}\nabla_i \nabla_j u \\ &= g^{ij}(\partial_i \partial_j u - \Gamma^k_{ij}\partial_k u) \\ &= (\delta^{ij} + O(r^{-\tau})^{ij})(\partial_i \partial_j u + O(r^{-\tau-1})^k_{ij}\partial_k u) \\ &= (\delta^{ij} + O(r^{-\tau})^{ij})\partial_i \partial_j u + O(r^{-\tau-1})^k_{ij}\partial_k u. \end{aligned} \tag{16.9}$$

□

This has the following corollary.

Corollary 16.3. *For any $\delta \in \mathbb{R}$, the Laplacian is a bounded linear mapping*

$$\Delta : W^{k,2}_\delta(X) \to W^{k-2,2}_{\delta-2}(X). \tag{16.10}$$

Furthermore, there exists a constant C_k such that

$$\|u\|_{W^{k,2}_\delta(X)} \leq C_k(\|\Delta u\|_{W^{k-2,2}_{\delta-2}(X)} + \|u\|_{W^{k-2,2}_\delta(X)}). \tag{16.11}$$

The "key result" from above is crucial to proving the following.

Theorem 16.4. *If δ is not an indicial root, then the mapping*

$$\Delta : W^{k,2}_{\delta}(X) \to W^{k-2,2}_{\delta-2}(X) \tag{16.12}$$

is semi-Fredholm (has finite-dimensional kernel and closed range)

Proof. We claim that there exists an $R \gg 0$ and a constant C such that any $u \in W^{2,2}_{\delta}(X)$ satisfies

$$\|u\|_{W^{2,2}_{\delta}(X)} \leq C(\|\Delta u\|_{L^2_{\delta-2}(X)} + \|u\|_{L^2(B_R)}), \tag{16.13}$$

where B_R is a ball of radius R centered at the basepoint.

Let χ_R be a smooth cutoff function defined by

$$\chi_R = \begin{cases} 1 & \rho < R \\ 0 & \rho > 2R \end{cases}, \tag{16.14}$$

and $0 \leq \chi_R \leq 1$. Write $u = u_0 + u_\infty$ where $u_0 = \chi_R u$, and $u_\infty = (1 - \chi_R)u$. Then

$$\begin{aligned} \|u\|_{W^{2,2}_{\delta}(X)} &\leq \|u_0\|_{W^{2,2}_{\delta}(X)} + \|u_\infty\|_{W^{2,2}_{\delta}(X)} \\ &\leq \|u_0\|_{W^{2,2}(B_R)} + \|u_\infty\|_{W^{2,2}_{\delta}(X)} \\ &\leq C(\|\Delta u_0\|_{L^2(B_{2R})} + \|u_0\|_{L^2(B_{2R})}) + \|u_\infty\|_{W^{2,2}_{\delta}(X)} \\ &\leq C(\|\Delta u_0\|_{L^2_{\delta-2}(B_{2R})} + \|u_0\|_{L^2_{\delta}(B_{2R})}) + \|u_\infty\|_{W^{2,2}_{\delta}(X)}. \end{aligned} \tag{16.15}$$

Next, since u_∞ is supported near infinity, and δ is not an indicial root, we can use Theorem 15.7 to estimate

$$\begin{aligned} \|u_\infty\|_{W^{2,2}_{\delta}(X)} &\leq C\|u_\infty\|_{W^{2,2}_{\delta}(\mathbb{R}^n_*)} \\ &\leq C\|\Delta_0 u_\infty\|_{L^2_{\delta-2}(\mathbb{R}^n_*)} \\ &\leq C\|(\Delta_0 - \Delta + \Delta)u_\infty\|_{L^2_{\delta-2}(\mathbb{R}^n_*)} \\ &\leq C\|(\Delta_0 - \Delta)u_\infty\|_{L^2_{\delta-2}(\mathbb{R}^n_*)} + C\|\Delta u_\infty\|_{L^2_{\delta-2}(\mathbb{R}^n_*)} \\ &\leq C\|O(r^{-\tau})D^2 u_\infty + O(r^{\tau-1})Du_\infty\|_{L^2_{\delta-2}(X)} + C\|\Delta u_\infty\|_{L^2_{\delta-2}(X)} \\ &\leq o(1)\|u_\infty\|_{W^{2,2}_{\delta}(X)} + C\|\Delta u_\infty\|_{L^2_{\delta-2}(X)}, \end{aligned} \tag{16.16}$$

as $R \to \infty$. Therefore, for R sufficiently large, we can absorb the first term on the right hand side into the left, and obtain

$$\|u\|_{W^{2,2}_{\delta}(X)} \leq C(\|\Delta u_0\|_{L^2_{\delta-2}(X)} + \|u_0\|_{L^2_{\delta}(B_{2R})}) + C\|\Delta u_\infty\|_{L^2_{\delta-2}(X)}. \tag{16.17}$$

Finally, the terms involving derivatives of the cutoff function only contain derivatives of u up to order 1, so they can be absorbed into the left hand side using interpolation, which completes the proof of the claimed estimate (16.13).

The Fredholm property then follows from this using the standard argument involving the Rellich Lemma, similar to what we did in the compact case. For any sequence of kernel elements $u_i \in W^{2,2}_\delta$ of unit norm, then by the Rellich Lemma, some subsequence converges strongly in $L^2(B_R)$. The above estimate (16.13) then shows that u_i is a Cauchy sequence in $W^{2,2}_\delta$, so has a convergent subsequence. This implies that the kernel is finite-dimensional.

To get closed range, first there exists a closed subspace Z so that $W^{2,2}_\delta = Z \oplus Ker(\Delta)$. We claim there exists a constant C so that

$$\|u\|_{W^{2,2}_\delta(X)} \leq C\|\Delta u\|_{L^2_{\delta-2}(X)} \tag{16.18}$$

for all $u \in Z$. If not, then there exists a sequence u_i with $\|u_i\|_{W^{2,2}_\delta(X)} = 1$ and $\|\Delta u_i\|_{L^2_{\delta-2}(X)} \to 0$ as $i \to \infty$. The above estimate and the Rellich Lemma show that there is a subsequence of the u_i which is Cauchy in Z. But the limit of such a sequence is non-zero element in $Z \cap Ker(\Delta)$, which is a contradiction. Finally, let $f_i \in L^2_{\delta-2}(X)$ be a Cauchy sequence with $\Delta u_i = f_i$ and $f_i \to f_\infty$. We can assume that $u_i \in Z$. The estimate (16.18) then shows that u_i is Cauchy in $W^{2,2}_\delta(X)$, so converges strongly to u_∞, which then satisfies $\Delta u_\infty = f_\infty$.

□

Remark 16.5. If δ is an indicial root, the kernel is still finite-dimensional. The problem is that the mapping does not have closed range. Example involving $\log(r)$ TO BE ADDED.

17 Lecture 17

17.1 Existence of expansions at infinity

We will prove the following Liouville-type theorem.

Theorem 17.1. *Let (X, g) be AF, $u \in L^2_\delta(X)$ satisfy $\Delta u = 0$, and let $k = k^-(\delta)$ (the largest indicial root less than or equal to δ).*

- *If $k < 0$, then $u \equiv 0$.*
- *If $k \geq 0$ then u admits an expansion $u = h_k + O(r^{k-1})$ as $r \to \infty$, where h_k is a harmonic polynomial in the AF coordinates.*
- *If $(X, g) = (\mathbb{R}^n, g_0)$, then u is exactly a harmonic polynomial, in the standard coordinate system.*

Proof. TBC. □

Definition 17.2. Let $\mathcal{H}_\delta = \{u \in L^2_\delta(X) \mid \Delta u = 0\}$.

A corollary of the above expansion is the following.

Corollary 17.3. *Let $\delta \in \mathbb{R}$. If $k^-(\delta) < 0$, then* $\dim(\mathcal{H}_\delta) = 0$. *If $k^-(\delta) \geq 0$, then*

$$\dim(\mathcal{H}_\delta) \leq \sum_{j=0}^{k^-(\delta)} N_k, \tag{17.1}$$

where N_k is the dimension of the space of homogeneous harmonic polynomials of degree k in $\mathbb{R}^n$, which is given by

$$N_k = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}. \tag{17.2}$$

For any integer $k \geq 0$, and $0 < \epsilon < 1$ then

$$\dim(\mathcal{H}_{k+\epsilon}) - \dim(\mathcal{H}_{k-\epsilon}) \leq N_k. \tag{17.3}$$

Proof. TBC. □

17.2 Existence of harmonic functions

First, we have the dual of weighted L^2 spaces.

Proposition 17.4. *Let (X, g) be AE. The dual space of $L^2_\delta(X)$ is $L^2_{-n-\delta}(X)$.*

Proof. If $u \in L^2_\delta(X)$, and $v \in L^2_{-n-\delta}(X)$, then

$$\begin{aligned}
\int_X uv dV_g &= \int_X uw^{-\delta-n/2} vw^{\delta+n/2} dV_g \\
&\leq \Big\{ \int_X |u|^2 w^{-2\delta-n} dV_g \Big\}^{\frac{1}{2}} \Big\{ \int_X |v|^2 w^{2\delta+n} dV_g \Big\}^{\frac{1}{2}} \\
&= \Big\{ \int_X |u|^2 w^{-2\delta-n} dV_g \Big\}^{\frac{1}{2}} \Big\{ \int_X |v|^2 w^{2(\delta+n)-n} dV_g \Big\}^{\frac{1}{2}} \\
&= \|u\|_{L^2_\delta(X)} \|v\|_{L^2_{-n-\delta}(X)}.
\end{aligned} \tag{17.4}$$

□

Next we have the following, which is the generalization of Theorem 14.2.

Proposition 17.5. *Let (X, g) be AF, and let h_k be a homogeneous harmonic polynomial of degree k. Then there exists a harmonic function $\phi_k : X \to \mathbb{R}$ admitting the expansion*

$$\phi_k = h_k + O(r^{k-1}) \tag{17.5}$$

as $r \to \infty$.

Proof. TBC. □

As a corollary of this, we have

Corollary 17.6. *For any integer $k \geq 0$,*

$$\dim(\mathcal{H}_{k+\epsilon}) - \dim(\mathcal{H}_{k-\epsilon}) = N_k. \tag{17.6}$$

For $\delta \in \mathbb{R}$,

$$\dim(\mathcal{H}_\delta) = \sum_{j=0}^{k^-(\delta)} N_k, \tag{17.7}$$

Proof. TBC. □

A great reference for this section is Bartnik's article on the mass of an ALE space [?].

18 Lecture 18

18.1 Operators asymptotic to Δ_0

We next consider a more general class of operators.

Definition 18.1. A self-adjoint second order elliptic differential operator P is asymptotic to Δ_0 at rate $\sigma > 0$ if in AF coordinates, we have

$$Pu = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u, \tag{18.1}$$

where the coefficients admit expansions

$$a^{ij} = \delta^{ij} + O(r^{-\sigma}) \tag{18.2}$$

$$b^i = O(r^{-\sigma-1}) \tag{18.3}$$

$$c = O(r^{-\sigma-2}), \tag{18.4}$$

as $r \to \infty$.

The same argument given above to prove Theorem 16.4 proves the following.

Proposition 18.2. *If δ is not an indicial root then*

$$P : W^{2,k}_\delta(X) \to W^{k-2,2}_\delta(X) \tag{18.5}$$

is Fredholm.

We also have the analogue of existence of harmonic expansions.

Theorem 18.3. *Let (X, g) be AF, $u \in L^2_\delta(X)$ satisfy $Pu = 0$. If u is not identically zero, then there exists an exceptional value $k \leq k^-(\delta)$ such that u admits an expansion $u = h_k + O(r^{k-1})$ as $r \to \infty$, where h_k is harmonic and homogeneous of degree k in the AF coordinates.*

Proof. The proof is almost the same as the proof of Theorem 17.1 above, with the following extra step. If $u = O(r^{-k})$ as $r \to \infty$, for any k, then consider the step in the proof using the Kelvin transform. In this step, we find a harmonic function with derivative of any order vanishing at the origin. By a unique continutation theorem for harmonic functions, this harmonic function must vanish. This implies that u vanishes in a neighborhood of infinity, and must be identically zero again by local unique continuation for elliptic operators. □

Analogous to above, define the following.

Definition 18.4. Let $\mathcal{H}_\delta = \{u \in L^2_\delta(X) \mid Pu = 0\}$.

Since we are allowing a zero order term in P, we can no longer say that there is no decaying kernel of P. But we can say the following.

Proposition 18.5. *There exists $N > 0$ such that if $\delta < -N$ then* $\dim(\mathcal{H}_\delta) = 0$.

Proof. Note that for $\delta' < \delta$, we have $\mathcal{H}_{\delta'} \subset \mathcal{H}_\delta$. Since the operator is Fredholm, these spaces are finite-dimensional. From Theorem 18.3, we are done. □

Using this, we can only conclude the following.

Proposition 18.6. *Let (X, g) be AF, and P as above. The there exists $N > 0$ so that if $k \geq N$ and h_k is a homogeneous harmonic polynomial of degree k. Then there exists a harmonic function $\phi_k : X \to \mathbb{R}$ admitting the expansion*

$$\phi_k = h_k + O(r^{k-1}) \tag{18.6}$$

as $r \to \infty$.

Consequently, for $k \geq N$,

$$\dim(\mathcal{H}_{k+\epsilon}) - \dim(\mathcal{H}_{k-\epsilon}) = N_k. \tag{18.7}$$

Proof. TBC. □

Another useful result is the following.

Proposition 18.7. *Let δ be a non-indicial root, and $u \in W^{2,2}_\delta(X)$ satisfy $Pu = f$, where $f \in L^2_{\delta'}(X)$ where $k^-(\delta) < \delta' \leq \delta$. Then $u \in W^{2,2}_{\delta'}(X)$.*

Proof. TBC. □

18.2 The relative index theorem

Definition 18.8. The index of P at a non-indicial root $\delta \in \mathbb{R}$ is

$$I_\delta = \dim \mathcal{H}_\delta - \dim \mathcal{H}_{2-n-\delta}. \tag{18.8}$$

The main theorem is the following.

Theorem 18.9. *The index satisfies the following properties.*

- *If* $\delta < \delta'$, *are nonindicial and there is no indicial root in between* δ *and* δ' *then* $I_\delta = I_{\delta'}$.

- *If* $0 < \epsilon < 1$, *and* k *is an indicial root, then*

$$I_{k+\epsilon} - I_{k-\epsilon} = \begin{cases} N_k & k \geq 0 \\ N_{2-n-k} & k \leq 2-n \end{cases}. \tag{18.9}$$

Proof. This is proved by a duality argument, integration-by-parts, etc. TBC. □

18.3 The case of $M \setminus \{p\}$, M compact

We next consider the following class of operators.

Definition 18.10. A self-adjoint second order elliptic differential operator is asymptotic to Δ_0 at rate $\sigma > 0$ if in coordinates near p, we have

$$Pu = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u, \tag{18.10}$$

where the coefficients admit expansions

$$a^{ij} = \delta^{ij} + O(r^\sigma) \tag{18.11}$$

$$b^i = O(r^{\sigma-1}) \tag{18.12}$$

$$c = O(r^{\sigma-2}), \tag{18.13}$$

as $r \to 0$.

In this setting, the relative index theorem takes the following form (there is a sign change from the AF version).

Theorem 18.11. *The index satisfies the following properties.*

- *If* $\delta < \delta'$, *are nonindicial and there is no indicial root in between* δ *and* δ' *then* $I_\delta = I_{\delta'}$.

- *If* $0 < \epsilon < 1$, *and* k *is an indicial root, then*

$$I_{k+\epsilon} - I_{k-\epsilon} = \begin{cases} -N_k & k \geq 0 \\ -N_{2-n-k} & k \leq 2-n \end{cases}. \tag{18.14}$$

Proof. TBC. □

19 Lection 19

19.1 Zonal harmonics

Recall that on the sphere S^{n-1}, in normal coordinates, based at the north pole, $g = dr^2 + sin^2(r)g_S$, $0 < r < \pi$, We already know that the eigenvalues are $k(k+n-2)$. So let ψ_k be an eigenfunction such that

$$\Delta_{S^{n-1}}\psi_k = k(k+n-2)\psi_k. \tag{19.1}$$

Let us consider eigenfunctions depending only upon r, so that $\Psi_k = f_k(r)$. Then the eigenvalue equation is

$$\Delta_H\psi_k = -\ddot{f}_k - (n-2)\cot(r)\dot{f}_k = k(k+n-2)f_k. \tag{19.2}$$

This yields an ODE

$$-\ddot{f}_k - (n-2)\cot(r)\dot{f}_k - k(k+n-2)f_k = 0. \tag{19.3}$$

Making the change of variable $f_k(r) = P_k(\cos r)$, this becomes the differential equation

$$(1-x^2)\ddot{P}_k - (n-1)x\dot{P}_k + k(k+n-2)P_k = 0 \tag{19.4}$$

For $n = 3$, this is

$$(1-x^2)\ddot{P}_k - 2x\dot{P}_k + k(k+1)P_k = 0, \tag{19.5}$$

a solution of which is a Legendre Polynomial of degree k.

The solution space of the ODE (19.4) is 2-dimensional, and the solutions are not necessarily polynomial. However, it turns out that there is always a polynomial solution, which is not so easy to see directly from the ODE. We will prove this shortly.

Define the following. Given $x \in S^{n-1}$, then the mapping

$$\Phi_k : \mathcal{H}_k \to \mathbb{R}, \tag{19.6}$$

given by $\Phi_x^{(k)}(h_k) = h_k(x)$ is a linear functional on $\mathcal{H}_k$. By the Riesz representation theorem, there exists a unique element $Z_x^{(k)}(y) \in \mathcal{H}_k$ so that

$$\int_{S^{n-1}} h_k(y)Z_x^{(k)}(y)d\sigma = h_k(x). \tag{19.7}$$

Clearly, if $Z_x^{(k)}(y)$ satisfies this property, then any rotation of $Z_x^{(k)}(y)$ around an axis containing x also satisfies this property, so by uniqueness, $Z_x^{(k)}(y)$ is an eigenfunction which is invariant under rotations around x-axis. Therefore, it must be a solution of the ODE above, where the north pole is replaced by the point x. These eigenfunctions are called zonal harmonics, and have the following special properties.

Proposition 19.1. *The zonal harmonics $Z_x^{(k)}(y)$ satisfy the following.*

- *Let $e_1, \dots e_{N_k}$ be an L^2 ONB for $\mathcal{H}_k$. Then*

$$Z_x^{(k)}(y) = \sum_{i=1}^{N_k} e_i(x)e_i(y). \tag{19.8}$$

- $Z_x^{(k)}(y) = Z_y^{(k)}(x)$.
- $Z_x^{(k}(x) = \frac{1}{n\omega_n}\dim(\mathcal{H}_k) = \frac{1}{n\omega_n}N_k$.
- $|Z_x^{(k}(y)| \leq \frac{1}{n\omega_n}N_k$.
- *Extending to $\mathbb{R}^n \setminus \{0\}$ by homogeneity, there exists a constant C such that*

$$|Z_x^{(k)}(y)| \leq Ck^{n-2}|y|^k \tag{19.9}$$

Proof. This is left as an exercise. □

There is a nice corollary of the above.

Corollary 19.2. *If u is harmonic in a ball $B(p, r)$ in $\mathbb{R}^n$, then u is real analytic in $B(p, r)$, and the Taylor series at p converges in any compact subset of the ball.*

Proof. Recall the Poisson kernel of the ball $B(0, R)$ is given by

$$P(x, y) = \frac{1}{n\omega_n R}\frac{R^2 - |x|^2}{|x - y|^n}, \tag{19.10}$$

so that any harmonic u satisfies

$$u(x) = \int_{S^{n-1}(R)} u(y)P(x, y)d\sigma. \tag{19.11}$$

Consider the case that u is harmonic on $\overline{B(0, 1)}$. Proposition 19.1 implies that the sum

$$\sum_{k=0}^{\infty} Z_y^{(k)}(x) \tag{19.12}$$

converges absolutely and uniformly on any compact subset of $B(0, 1)$. Fix any point $x \in B(0, 1)$, $f \in L^2(S^{n-1})$. Then

$$\int_{S^{n-1}} f(y)P(x, y)d\sigma = \int_{S^{n-1}} f(y)\sum_{k=0}^{\infty} Z_y^{(k)}(x)d\sigma \tag{19.13}$$

holds since it is clearly true if f is a harmonic polynomial, and such polynomials are dense in $L^2(S^{n-1})$. Therefore we have the pointwise formula

$$P(x, y) = \frac{1 - |x|^2}{|x - y|^n} = \sum_{k=0}^{\infty} Z_y^{(k)}(x), \tag{19.14}$$

which implies the corollary, after plugging this into (19.11). The general case follows by translating and dilating $B(0, r)$. □

19.2 Expansion of the Newton kernel

The above was to illustrate that the zonal harmonics are very important, since they give an expansion of the Poission kernel. Next, we will see that they also give an expansion of the Newton kernel.

First, consider

$$\begin{aligned}\frac{1}{|x-y|^{n-2}} &= \frac{1}{|x|^{n-2}}\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|^{n-2}}\\ &= \frac{1}{|x|^{n-2}}\frac{1}{\left|1-2\frac{x\cdot y}{|x|^2}+\frac{|y|^2}{|x|^2}\right|^{\frac{n-2}{2}}}\\ &= \frac{1}{|x|^{n-2}}\frac{1}{\left|1-2\frac{|y|}{|x|}(\frac{x}{|x|}\cdot\frac{y}{|y})+\left(\frac{|y|}{|x|}\right)^2\right|^{\frac{n-2}{2}}}\\ &= \frac{1}{|x|^{n-2}}\frac{1}{(1-2st+s^2)^{\frac{n-2}{2}}},\end{aligned} \tag{19.15}$$

where $s = |y|/|x|$ and $t = (x\cdot y)/(|x||y|)$. Now, if $|y| < |x|$, then

$$\begin{aligned}\frac{1}{|x-y|^{n-2}} &= \frac{1}{|x|^{n-2}}\frac{1}{(1-2st+s^2)^{\frac{n-2}{2}}}\\ &= \frac{1}{|x|^{n-2}}\sum_{k=0}^{\infty}P_k(t)s^k.\end{aligned} \tag{19.16}$$

Then $P_k(t)$ are polynomials of degree k and are known as ultraspherical polynomials, which are a special case of Gegenbauer polynomials.

Proposition 19.3. *We have*

$$P_k(t) = c_{k,n}Z^{(k)}_{y/|y|}(x/|x|), \tag{19.17}$$

where $c_{k,n}$ is a constant given by

$$c_{k,n} = n\omega_n\frac{n-2}{n-2+2k}. \tag{19.18}$$

Proof. These coefficients must be harmonic, and they are invariant by rotations in the x-axis, so they must be a multiple of the zonal harmonic. For determination of the constant, assume x lies along one axis, and then evaluate at another point along the same axis. So assume that $x/|x| = y/|y|$, with $|y| < |x|$. Since $t = 1$,

$$\begin{aligned}\sum_{k=0}^{\infty}P_k(t)s^k &= \frac{1}{(1-2s+s^2)^{\frac{n-2}{2}}}\\ &= (1-s)^{-(n-2)}\\ &= \sum_{k=0}^{\infty}\frac{r_k}{k!}s^k,\end{aligned} \tag{19.19}$$

where $r_0 = 1$, $r_1 = n - 2$, and for $k \geq 1$,

$$r_k = (n-2)\big((n-2)+1\big)\dots\big((n-2)+k-1\big) \tag{19.20}$$

Then from (19.16), we see that

$$P_k(1) = \frac{r_k}{k!}. \tag{19.21}$$

Next, from Proposition 19.1, we have that

$$Z_p^{(k)}(p) = \frac{1}{n\omega_n} N_k, \tag{19.22}$$

which implies that

$$c_{k,n} = \frac{n\omega_n r_k}{k! N_k}. \tag{19.23}$$

Some algebra shows that this simplifies to (19.18). □

Remark 19.4. For $n = 3$, we have

$$c_{k,n} = 3\omega_3 \frac{1}{2k+1}. \tag{19.24}$$

Also, in this case, $Z_{y/|y|}^{(k)}(x/|x|) = \frac{2k+1}{3\omega_3} P^{(k)}(\cos(\theta))$ where θ is the angle between x and y, and $P^{(k)}$ is the Legendre polynomial of degree k, normalized so that $P^{(k)}(1) = 1$. In other words, $P_k(t) = P^{(k)}(t)$ in dimension 3.

We have proved the following.

Proposition 19.5. *The Newton kernel has the following expansions.*

- *If $|y| < |x|$, then*

$$\frac{1}{|x-y|^{n-2}} = \frac{1}{|x|^{n-2}} \sum_{k=0}^{\infty} c_{k,n} Z_{y/|y|}^{(k)}(x/|x|) \Big(\frac{|y|}{|x|}\Big)^k. \tag{19.25}$$

- *If $|y| > |x|$, then*

$$\frac{1}{|x-y|^{n-2}} = \frac{1}{|y|^{n-2}} \sum_{k=0}^{\infty} c_{k,n} Z_{y/|y|}^{(k)}(x/|x|) \Big(\frac{|x|}{|y|}\Big)^k. \tag{19.26}$$

Note the following. Using Proposition 19.1, we can rewrite for example the first expansion as

$$\begin{aligned} \frac{1}{|x-y|^{n-2}} &= \frac{1}{|x|^{n-2}} \sum_{k=0}^{\infty} c_{k,n} \sum_{j=1}^{N_k} \phi_{k,j}(x/|x|)\phi_{k,j}(y/|y|) \Big(\frac{|y|}{|x|}\Big)^k \\ &= \sum_{k=0}^{\infty} c_{k,n} \sum_{j=1}^{N_k} |x|^{2-n-k} \phi_{k,j}(x/|x|) |y|^k \phi_{k,j}(y/|y|). \end{aligned} \tag{19.27}$$

where for $k \geq 0$, the functions $\phi_{k,j}$ for $j = 1 \dots N_k$ are an L^2 orthonormal basis of $\mathcal{H}^k$ on S^{n-1}. The term $|x|^{2-n-k}\phi_{k,j}(x/|x|)$ is the Kelvin transform of a degree k harmonic polynomial in x, and $|y|^k \phi_{k,j}(y/|y|)$ is a degree k harmonic polynomial in y. So from this expansion, it is clearly harmonic in both x and y, although convergence is not very clear from this expression.

Remark 19.6. Note that for $n = 3$, the constants work out so that if $|y| < |x|$, then

$$\frac{1}{|x-y|} = \frac{1}{|x|} \sum_{k=0}^{\infty} P^{(k)}(\cos(\theta)) \Big(\frac{|y|}{|x|} \Big)^k, \tag{19.28}$$

where, as mentioned above, $P^{(k)}$ is the usual Legendre polynomial of degree k.

19.3 Weighted Hölder spaces

20 Lecture 20

20.1 Weighted L^p spaces

21 Lecture 21

21.1 Manifold with 2 ends

This is $g = dr^2 + (r^2 + m) g_S$, $m > 0$, and $t \in \mathbb{R}$, so $a = r^2 + m$. Therefore

$$\Delta_H \phi = -\ddot{\phi} - \frac{(n-1)r}{r^2+m} \dot{\phi} + \frac{1}{r^2+m} \Delta_{g_S} \phi. \tag{21.1}$$

Let $\phi = f(t) B(\theta)$, where B is an eigenfunction, then

$$\Delta_H f B = (-\ddot{f} - \frac{(n-1)r}{r^2+m} \dot{f} + \frac{k(k+n-2)}{r^2+m} f) B. \tag{21.2}$$

Note, if $n = 3$, and $k = 0$, this has the solution

$$f(t) = c_1 + c_2 \frac{1}{\sqrt{m}} \arctan(\frac{t}{\sqrt{m}}). \tag{21.3}$$

Properties of harmonic functions on this manifold:

- There exists a nonconstant bounded harmonic function.
- If $\Delta_H \phi = 0$ and $\phi = O(|t|^N)$ as $t \to \pm\infty$, then ϕ admits an expansion

  $$\phi = h_k + O(|t|^k), \tag{21.4}$$

  as $t \to \pm\infty$. where h_k is a harmonic polynomial in $\mathbb{R}^n$.

22 Lecture 22

22.1 Hyperbolic space

This is $g = dr^2 + \sinh^2(r) g_S$, $r > 0$, so $a \equiv \sinh^2(r)$. Therefore

$$\Delta_H \phi = -\ddot{\phi} - (n-1)\coth(r)\dot{\phi} + \frac{1}{\sinh^2(r)} \Delta_{g_S} \phi. \tag{22.1}$$

Let $\phi = f(t)B(\theta)$, where B is an eigenfunction, then

$$\Delta_H f B = (-\ddot{f} - (n-1)\coth(r)\dot{f} + \frac{k(k+n-2)}{\sinh^2(r)} f)B. \tag{22.2}$$

Let's look at this for $n = 2$. In this case, the ODE is

$$-\ddot{f} - \coth(r)\dot{f} + \frac{k^2}{\sinh^2(r)} f = 0. \tag{22.3}$$

This has solutions

$$f = c_1 \cosh(k \log(\coth(r/2))) + c_2 \sinh(k \log(\coth(r/2))). \tag{22.4}$$

Properties of harmonic functions on hyperbolic space:

- For any $k \geq 0$, there exist harmonic functions which satisfy $\phi(r) = O(r^{2-n-k})$ as $r \to 0$, but satisfy $\phi(r) = e^{-(n-1)r}$ as $\to \infty$.
- For any $k \geq 0$, there exists harmonic functions which satisfy $\phi(r) = O(r^k)$ as $r \to 0$, but satisfy $\phi(r) = O(1)$ as $\to \infty$.
- Any bounded harmonic function on $\mathbb{R}^n$ is constant.
- (Solvability of the Dirichlet problem) Given $f \in C^0(S^{n-1})$, there exists a global harmonic function $\phi : \mathbb{H}^n \to r$ such that $\phi|_{S^{n-1}} = f$.

23 Lecture 23

23.1 Weitzenböck formula for 2-forms

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