Math 865, Topics in Riemannian Geometry

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Contents

Introduction

This semester will be about complex manifolds and Kähler geometry.

1 Lecture 1

1.1 Example of $\mathbb{R}^4 = \mathbb{C}^2$

We consider $\mathbb{R}^4$ and take coordinates x_1, y_1, x_2, y_2. Letting $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$, define complex one-forms

$$\begin{aligned} dz_j &= dx_j + idy_j, \\ d\bar{z}_j &= dx_j - idz_j, \end{aligned}$$

and tangent vectors

$$\begin{aligned} \partial/\partial z_j &= (1/2)\left(\partial/\partial x_j - i\partial/\partial y_j\right), \\ \partial/\partial \bar{z}_j &= (1/2)\left(\partial/\partial x_j + i\partial/\partial y_j\right). \end{aligned}$$

Note that

$$\begin{aligned} dz_j(\partial/\partial z_k) &= d\bar{z}_j(\partial/\partial \bar{z}_k) = \delta_{jk}, \\ dz_j(\partial/\partial \bar{z}_k) &= d\bar{z}_j(\partial/\partial z_k) = 0. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ denote the complexified Euclidean inner product, so that

$$\begin{aligned} \langle \partial/\partial z_j, \partial/\partial z_k \rangle &= \langle \partial/\partial \bar{z}_j, \partial/\partial \bar{z}_k \rangle = 0, \\ \langle \partial/\partial z_j, \partial/\partial \bar{z}_k \rangle &= \frac{1}{2}\delta_{jk}. \end{aligned}$$

Similarly, on 1-forms we have

$$\begin{aligned} \langle dz_j, dz_k \rangle &= \langle d\bar{z}_j, d\bar{z}_k \rangle = 0, \\ \langle dz_j, d\bar{z}_k \rangle &= 2\delta_{jk}. \end{aligned}$$

The standard complex structure $J_0 : T\mathbb{R}^4 \to T\mathbb{R}^4$ on $\mathbb{R}^4$ is given by

$$J_0(\partial/\partial x_j) = \partial/\partial y_j, \quad J_0(\partial/\partial y_j) = -\partial/\partial x_j,$$

which in matrix form is written

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.1}$$

Next, we complexify the tangent space $T \otimes \mathbb{C}$, and let

$$T^{(1,0)}(J_0) = \text{span}\{\partial/\partial z_1, \partial/\partial z_2\} = \{X - iJ_0X, X \in T_p\mathbb{R}^4\} \tag{1.2}$$

be the i-eigenspace and

$$T^{(0,1)}(J_0) = \text{span}\{\partial/\partial \bar{z}_1, \partial/\partial \bar{z}_2\} = \{X + iJ_0X, X \in T_p\mathbb{R}^4\} \tag{1.3}$$

be the $-i$-eigenspace of J_0, so that

$$T \otimes \mathbb{C} = T^{(1,0)}(J_0) \oplus T^{(0,1)}(J_0). \tag{1.4}$$

The map J_0 also induces an endomorphism of 1-forms by

$$J_0(\omega)(v_1) = \omega(J_0 v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0(dx_j) = -dy_j, \quad J_0(dy_j) = +dx_j.$$

Then complexifying the cotangent space $T^* \otimes \mathbb{C}$, we have

$$\Lambda^{1,0}(J_0) = \text{span}\{dz_1, dz_2\} = \{\alpha - iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^4\} \tag{1.5}$$

is the i-eigenspace, and

$$\Lambda^{0,1}(J_0) = \text{span}\{d\bar{z}_1, d\bar{z}_2\} = \{\alpha + iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^4\} \tag{1.6}$$

is the $-i$-eigenspace of J_0, and

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J_0) \oplus \Lambda^{0,1}(J_0). \tag{1.7}$$

We note that

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{1.8}$$

and similarly

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{1.9}$$

1.2 Complex structure in $\mathbb{R}^{2n}$

The above works in a more general setting, in any even dimension. We only need assume that $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is linear and satisfies $J^2 = -I$. In this more general setting, we have

$$T \otimes \mathbb{C} = T^{(1,0)}(J) \oplus T^{(0,1)}(J), \tag{1.10}$$

where

$$T^{(1,0)}(J) = \{X - iJX, X \in T_p\mathbb{R}^{2n}\} \tag{1.11}$$

is the i-eigenspace of J and

$$T^{(0,1)}(J) = \{X + iJX, X \in T_p\mathbb{R}^{2n}\} \tag{1.12}$$

is the $-i$-eigenspace of J.

As above, The map J also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J) \oplus \Lambda^{0,1}(J), \tag{1.13}$$

where

$$\Lambda^{1,0}(J) = \{\alpha - iJ\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.14}$$

is the i-eigenspace of J, and

$$\Lambda^{0,1}(J) = \{\alpha + iJ\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.15}$$

is the $-i$-eigenspace of J.

Again, we have the characterizations

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{1.16}$$

and

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{1.17}$$

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{1.18}$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{1.19}$$

Note that we can characterize $\Lambda^{p,q}$ as those forms satisfying

$$\alpha(v_1, \ldots, v_{p+q}) = 0, \tag{1.20}$$

if more than p if the v_j-s are in $T^{(1,0)}$ or if more than q of the v_j-s are in $T^{(0,1)}$.

Finally, we can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{1.21}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J is not a complex structure on the space $\Lambda^k_{\mathbb{C}}$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J-invariant.

1.3 Cauchy-Riemann equations

Let $f : \mathbb{C}^n \to \mathbb{C}^m$. Let the coordinates on $\mathbb{C}^n$ be given by

$$\{z^1, \dots z^n\} = \{x^1 + iy^1, \dots, x^n + iy^n\}, \tag{1.22}$$

and coordinates on $\mathbb{C}^m$ given by

$$\{w^1, \dots z^m\} = \{u^1 + iv^1, \dots, u^m + iv^m\} \tag{1.23}$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \text{span}\{\partial/\partial x^1, \dots \partial/\partial x^n, \partial/\partial y^1, \dots \partial/\partial y^n\}, \tag{1.24}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial u^1, \dots \partial/\partial u^m, \partial/\partial v^1, \dots \partial/\partial v^m\}. \tag{1.25}$$

Then the real Jacobian of

$$f = (f^1, \dots f^{2m}) = (u^1 \circ f, u^2 \circ f, \dots, v^{2m} \circ f). \tag{1.26}$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \cdots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \tag{1.27}$$

Definition 1. *A differentiable mappging f is pseudo-holomorphic if*

$$f_* \circ J_0 = J_0 \circ f_*. \tag{1.28}$$

That is, the differential of f commutes with J_0.

We have the following characterization of pseudo-holomorphic maps.

Proposition 1.1. *A mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ is pseudo-holomorphic if and only if the Cauchy-Riemann equations are satisfied, that is, writing*

$$f(z^1, \dots z^m) = (f_1, \dots, f_n) = (u_1 + iv_1, \dots u_n + iv_n), \tag{1.29}$$

and $z^j = x^j + iy^j$, for each $j = 1 \dots n$, we have

$$\frac{\partial u_j}{\partial x^k} = \frac{\partial v_j}{\partial y^k} \qquad \frac{\partial u_j}{\partial y^k} = -\frac{\partial v_j}{\partial x^k}, \tag{1.30}$$

for each $k = 1 \dots m$, and these equations are equivalent to

$$\frac{\partial}{\partial \bar{z}^k} f_j = 0, \tag{1.31}$$

for each $j = 1 \dots n$ and each $k = 1 \dots m$

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{1.32}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{1.33}$$

which is exactly the Cauchy-Riemann equations. In the general case, rearrange the coordinates so that $(x^1, \ldots, x^m, y^1, \ldots, y^m)$ are the real coordinates on $\mathbb{R}^{2m}$ and $(u^1, \ldots, u^n, v^1, \ldots, v^n)$, such that the complex structure J_0 is given by

$$J_0(\mathbb{R}^{2m}) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{1.34}$$

and similarly for $J_0(\mathbb{R}^{2n})$. Then the computation in matrix form is entirely analogous to the case of $m = n = 1$.

Finally, we compute

$$\frac{\partial}{\partial \bar{z}^k} f_j = \frac{1}{2}\Big(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\Big)(u_j + iv_j) \tag{1.35}$$

$$= \frac{1}{2}\Big\{\frac{\partial}{\partial x^k}u_j - \frac{\partial}{\partial y^k}v_j + i\Big(\frac{\partial}{\partial x^k}v_j + \frac{\partial}{\partial y^k}u_j\Big)\Big\}, \tag{1.36}$$

the vanishing of which again yields the Cauchy-Riemann equations. □

From now on, if f is a mapping satisfying the Cauchy-Riemann equations, we will just say that f is *holomorphic*.

For any differentiable f, the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^m). \tag{1.37}$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \operatorname{span}\{\partial/\partial z^1, \ldots \partial/\partial z^n, \partial/\partial \overline{z}^1, \ldots \partial/\partial \overline{z}^n\}, \tag{1.38}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \operatorname{span}\{\partial/\partial w^1, \ldots \partial/\partial w^m, \partial/\partial \overline{w}^1, \ldots \partial/\partial \overline{w}^m\}. \tag{1.39}$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \cdots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \overline{z}^1} & \cdots & \frac{\partial f^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \cdots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \overline{z}^1} & \cdots & \frac{\partial f^m}{\partial \overline{z}^n} \\ \frac{\partial \overline{f}^1}{\partial z^1} & \cdots & \frac{\partial \overline{f}^1}{\partial z^n} & \frac{\partial \overline{f}^1}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \overline{f}^m}{\partial z^1} & \cdots & \frac{\partial \overline{f}^m}{\partial z^n} & \frac{\partial \overline{f}^m}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^m}{\partial \overline{z}^n}, \end{pmatrix} \tag{1.40}$$

where $(f^1, \dots f^m) = f$ now denotes the complex components of f. This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{1.41}$$

Notice that (1.40) is of the form

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{1.42}$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.43}$$

where $\mathbb{C}$-linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \tag{1.44}$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{1.45}$$

This corresponds to the embeddings

$$Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.46}$$

where the left $\mathbb{C}$-linear is with respect to J_0, via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{1.47}$$

Note the first embedding acts as

$$(z^1, \dots, z^n)^T \mapsto (A + iB)(z^1, \dots, z^n)^T \tag{1.48}$$

Therefore if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB) \det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{1.49}$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(1,0)}. \tag{1.50}$$

A formula which will be useful later is the following for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{1.51}$$

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0, then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{1.52}$$

This corresponds to the embeddings

$$Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \tag{1.53}$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \tag{1.54}$$

The first embedding acts as

$$(z^1, \ldots, z^n)^T \mapsto (A + iB)(\overline{z}^1, \ldots, \overline{z}^n)^T. \tag{1.55}$$

Finally, note that f is anti-holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(0,1)}. \tag{1.56}$$

2 Lecture 2

2.1 Complex Manifolds

Now we can define a complex manifold

Definition 2. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\beta) \to \phi_\alpha(U_\alpha)$ satisfying the Cauchy-Riemann equations.

Definition 3. An *almost complex structure* is an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$. An almost complex structure J is said to be *integrable* if J is induced from a collection of holomorphic coordinates on M.

If M is of real dimension n, and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^n, \tag{2.1}$$

which implies that n is even. Furthermore, M is orientable and carries a natural orientation by the discussion in the previous section.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 2.1. *In any coordinate chart, define $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{2.2}$$

Then $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$ and therefore gives a globally defined almost complex structure $J : TM \to TM$ satisfying $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{2.3}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{2.4}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{2.5}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

Let (M^2, g) be any oriented Riemannian surface. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure.

In the case of surfaces, this always comes from a collection of holomorphic coordinate charts (we wil prove this later), but this is not true in higher dimensions. To understand this we proceed as follows.

2.2 The Nijenhuis tensor

Proposition 2.2. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \tag{2.6}$$

is a tensor of type $(1, 2)$ *and satisfies*

(i) $N(Y, X) = -N(X, Y)$.

(ii) $N(JX, JY) = -N(X, Y)$.

Proof. Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned}
N(fX,Y) &= 2\{[J(fX),JY] - [fX,Y] - J[fX,JY] - J[J(fX),Y]\} \\
&= 2\{[fJX,JY] - [fX,Y] - J[fX,JY] - J[fJX,Y]\} \\
&= 2\{f[JX,JY] - (JY(f))JX - f[X,Y] + (Yf)X \\
&- J(f[X,JY] - (JY(f))X) - J(f[JX,Y] - (Yf)JX)\} \\
&= fN(X,Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X.
\end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X,fY) = fN(X,Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y is obvious, and (ii) follows easily using $J^2 = -Id$. □

Notice that if M is of complex dimension 1, then there is a basis of the tangent space of the form $\{X, JX\}$, so

$$N(X,JX) = -N(JX,X) = -N(JX,J^2X) = N(JX,X), \tag{2.7}$$

which shows that the Nijenhuis tensor of any almost complex structure on a surface vanishes.

We have the following local formula for the Nijenhuis tensor.

Proposition 2.3. *In local coordinates, the Nijenhuis tensor is given by*

$$N^i_{jk} = 2\sum_{h=1}^{2n}(J^h_j\partial_h J^i_k - J^h_k\partial_h J^i_j - J^i_h\partial_j J^h_k + J^i_h\partial_k J^h_j) \tag{2.8}$$

Proof. We compute

$$\begin{aligned}
\frac{1}{2}N(\partial_j,\partial_k) &= [J\partial_j, J\partial_k] - [\partial_j,\partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j,\partial_k] \\
&= [J^l_j\partial_l, J^m_k\partial_m] - [\partial_j,\partial_k] - J[\partial_j, J^l_k\partial_l] - J[J^l_j\partial_l,\partial_k] \\
&= I + II + III + IV.
\end{aligned}$$

The first term is

$$\begin{aligned}
I &= J^l_j\partial_l(J^m_k\partial_m) - J^m_k\partial_m(J^l_j\partial_l) \\
&= J^l_j(\partial_l J^m_k)\partial_m + J^l_jJ^m_k\partial_l\partial_m - J^m_k(\partial_m J^l_j)\partial_l - J^m_kJ^l_j\partial_m\partial_l \\
&= J^l_j(\partial_l J^m_k)\partial_m - J^m_k(\partial_m J^l_j)\partial_l.
\end{aligned}$$

The second term is obviously zero. The third term is

$$III = -J(\partial_j(J^l_k)\partial_l) = -\partial_j(J^l_k)J^m_l\partial_m. \tag{2.9}$$

Finally, the fourth term is

$$III = \partial_k(J^l_j)J^m_l\partial_m. \tag{2.10}$$

Combining these, we are done. □

Next, we have

Theorem 2.1. *An almost complex structure J is integrable if and only if the Nijenhuis tensor vanishes.*

Proof. If J is integrable, then we can always find local coordinates so that $J = J_0$, and Proposition 2.3 shows that the Nijenhuis tensor vanishes. For the converse, the vanishing of the Nijenhuis tensor is the integrability condition for $T^{1,0}$ as a complex sub-distribution of $T \otimes \mathbb{C}$. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \quad (2.11)$$

But this is also a $(1,0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \quad (2.12)$$

applying J, and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \quad (2.13)$$

which is exactly the vanishing of the Nijenhuis tensor. In the analytic case, the converse then follows using a complex version of the Frobenius Theorem. The C^{∞}-case is more difficult, and is the content of the Newlander-Nirenberg Theorem, which we will discuss a bit later. □

2.3 The operators ∂ and $\overline{\partial}$

We can apply all of the linear algebra from the previous sections to almost complex manifolds, in particular we have the complex bundles $T^{(1,0)}(M)$, $T^{(0,1)}(M)$, and the bundles of (p,q)-forms, denoted by $\Lambda^{p,q}(M)$. The real operator $d : \Lambda^k_{\mathbb{R}} \to \Lambda^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \Lambda^k_{\mathbb{C}} \to \Lambda^{k+1}_{\mathbb{C}} \quad (2.14)$$

by complexification. On a complex manifold, if α is a (p,q)-form, then locally we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \quad (2.15)$$

where I and J are multi-indices of length p and q, respectively, and $\alpha_{I,J}$ are complex-valued functions. Using (1.41), we have the formula

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \Big) \wedge dz^I \wedge d\overline{z}^J. \quad (2.16)$$

Proposition 2.4. *For an almost complex structure J*

$$d(\Lambda^{p,q}) \subset \Lambda^{p+2,q-1} + \Lambda^{p+1,q} + \Lambda^{p,q+1} + \Lambda^{p-1,q+2}, \tag{2.17}$$

and J is integrable if and only if

$$d(\Lambda^{p,q}) \subset \Lambda^{p+1,q} + \Lambda^{p,q+1}. \tag{2.18}$$

(In a slight abuse of notation, by $\Lambda^{p,q}$, we mean the space of smooth sections of this bundle.)

Proof. Let $\alpha \in \Lambda^{p,q}$, and write $p+q=r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0,\dots,X_r) = &\sum(-1)^j X_j\alpha(X_0,\dots,\hat{X}_j,\dots,X_r) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_r). \end{aligned} \tag{2.19}$$

This is easily seen to vanish if more than $p+2$ of the X_j are of type $(1,0)$ or if more than $q+2$ are of type $(0,1)$.

If J is integrable, then in a local complex coordinate system, (2.18) is easily seen to hold. For the converse we have the inclusions,

$$d(\Lambda^{1,0}) \subset \Lambda^{2,0} + \Lambda^{1,1} \text{ and } d(\Lambda^{0,1}) \subset \Lambda^{1,1} + \Lambda^{0,2}. \tag{2.20}$$

The formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{2.21}$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X,Y]$. So write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. Define $Z = [X,Y]$, then Z is also of type $(1,0)$, so

$$Z + iJZ = 0. \tag{2.22}$$

Writing this in terms of X' and Y' we see that

$$0 = 2(Z + iJZ) = -N(X',Y') - iJN(X',Y'), \tag{2.23}$$

which implies that $N \equiv 0$. □

Corollary 2.1. *On a complex manifold, $d = \partial + \overline{\partial}$ where $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $\overline{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$, and these operators satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{2.24}$$

Proof. These relations follow simply from $d^2 = 0$. □

Note that on a complex manifold, if α is a (p,q)-form written locally as

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{2.25}$$

then

$$\partial\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J. \tag{2.26}$$

$$\overline{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J. \tag{2.27}$$

Definition 4. A form $\alpha \in \Lambda^{p,0}$ is holomorphic if $\overline{\partial}\alpha = 0$.

It is easy to see that a $(p,0)$-form is holomorphic if and only if it can locally be written as

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{2.28}$$

where the α_I are holomorphic functions.

Definition 5. The (p,q) Dolbeault cohomology group is

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\{\alpha \in \Lambda^{p,q}(M) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M))}. \tag{2.29}$$

We will discuss these in more detail later, and just point out the following for now.

Definition 6. A map between almost complex manifolds $f : (M, J_M) \to (M, J_N)$ is called pseudo-holomorphic if

$$f_* \circ J_M = J_N \circ f_*. \tag{2.30}$$

If both J_M and J_N are integrable, then f is called holomorphic if it is holomorphic in coordinate charts.

Clearly, a holomorphic map is pseudo-holomorphic. If $f : M \to N$ is a holomorphic map between complex manifolds, then

$$f^*(\Lambda^{p,q}(N)) \subset \Lambda^{p,q}(M), \tag{2.31}$$

and

$$\overline{\partial} \circ f^* = f^* \circ \overline{\partial}. \tag{2.32}$$

Consequently, there is an induced mapping

$$f^* : H^{p,q}_{\overline{\partial}}(N) \to H^{p,q}_{\overline{\partial}}(M) \tag{2.33}$$

In particular, if f is a biholomorphism (one-to-one, onto, with holomorphic inverse), then the Dolbeault cohomologies of M and N are isomorphic.

3 Lecture 3

3.1 Hermitian metrics

We next consider (M, J, g) where g is a Riemannian metric, and we assume that g and J are compatible. That is,

$$g(X, Y) = g(JX, JY). \tag{3.1}$$

The metric g is called an almost-Hermitian metric. If J is also integrable, then g is called Hermitian. We extend g by complex linearity to a symmetric inner product on $T \otimes \mathbb{C}$. The following will be useful later.

Proposition 3.1. *There exist elements* $\{X_1, \dots X_n\}$ *in* $\mathbb{R}^{2n}$ *so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{3.2}$$

is an ONB for $\mathbb{R}^{2n}$ *with respect to* g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{3.3}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of $span\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. □

To a Hermitian metric $(\mathbb{R}^{2n}, J, g)$ we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{3.4}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{3.5}$$

Since

$$\omega(JX, JY) = \omega(X, Y), \tag{3.6}$$

this form is a real form of type $(1, 1)$, and is called the *Kähler form* or *fundamental 2-form*.

3.2 Complex tensor notation

Choosing any real basis of the form $\{X_1, JX_1, \ldots, X_n, JX_n\}$, let us abbreviate

$$Z_\alpha = \frac{1}{2}\Big(X_\alpha - iJX_\alpha\Big) \tag{3.7}$$

$$Z_{\overline{\alpha}} = \frac{1}{2}\Big(X_\alpha + iJX_\alpha\Big), \tag{3.8}$$

and define

$$g_{\alpha\beta} = g(Z_\alpha, Z_\beta) \tag{3.9}$$

$$g_{\overline{\alpha}\overline{\beta}} = g(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) \tag{3.10}$$

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) \tag{3.11}$$

$$g_{\overline{\alpha}\beta} = g(Z_{\overline{\alpha}}, Z_\beta). \tag{3.12}$$

Notice that

$$\begin{aligned} g_{\alpha\beta} = g(Z_\alpha, Z_\beta) &= \frac{1}{4} g(X_\alpha - iJX_\alpha, X_\beta - iJX_\beta) \\ &= \frac{1}{4}\Big(g(X_\alpha, X_\beta) - g(JX_\alpha, JX_\beta) - i(g(X_\alpha, JX_\beta) + g(JX_\alpha, X_\beta))\Big) \\ &= 0, \end{aligned}$$

since g is J-invariant, and $J^2 = -Id$. Similarly,

$$g_{\overline{\alpha}\overline{\beta}} = 0, \tag{3.13}$$

Also, from symmetry of g, we have

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) = g(Z_{\overline{\beta}}, Z_\alpha) = g_{\overline{\beta}\alpha}. \tag{3.14}$$

However, applying conjugation, since g is real we have

$$\overline{g_{\alpha\overline{\beta}}} = \overline{g(Z_\alpha, Z_{\overline{\beta}})} = g(Z_{\overline{\alpha}}, Z_\beta) = g(Z_\beta, Z_{\overline{\alpha}}) = g_{\beta\overline{\alpha}}, \tag{3.15}$$

which says that $g_{\alpha\overline{\beta}}$ is a Hermitian matrix.

We repeat the above for the fundamental 2-form ω, and define

$$\omega_{\alpha\beta} = \omega(Z_\alpha, Z_\beta) = ig_{\alpha\beta} = 0 \tag{3.16}$$

$$\omega_{\overline{\alpha}\overline{\beta}} = \omega(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = -ig_{\overline{\alpha}\overline{\beta}} = 0 \tag{3.17}$$

$$\omega_{\alpha\overline{\beta}} = \omega(Z_\alpha, Z_{\overline{\beta}}) = ig_{\alpha\overline{\beta}} \tag{3.18}$$

$$\omega_{\overline{\alpha}\beta} = \omega(Z_{\overline{\alpha}}, Z_\beta) = -ig_{\overline{\alpha}\beta}. \tag{3.19}$$

The first 2 equations are just a restatement that ω is of type $(1,1)$. Also, note that

$$\omega_{\alpha\overline{\beta}} = ig_{\alpha\overline{\beta}}, \tag{3.20}$$

defines a skew-Hermitian matrix.

On a complex manifold, the fundamental 2-form in holomorphic coordinates takes the form

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\overline{\beta}} dz^{\alpha} \wedge d\overline{z}^{\beta} \tag{3.21}$$

$$= i \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^{\alpha} \wedge d\overline{z}^{\beta}. \tag{3.22}$$

3.3 Endomorphisms

Let $End_{\mathbb{R}}(TM)$ denotes the real endomorphisms of the tangent bundle.

Proposition 3.2. *On an almost complex manifold (M, J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{\mathbb{C}}(TM) \oplus End_{\overline{\mathbb{C}}}(TM) \tag{3.23}$$

where the first factor on the left consists of endomorphisms I commuting with J,

$$IJ = JI \tag{3.24}$$

and the second factor consists of endomorphisms I anti-commuting with J,

$$IJ = -JI \tag{3.25}$$

Furthermore,

$$End_{\mathbb{C}}(TM) \cong \Big((\Lambda^{1,0} \otimes T^{1,0}) \oplus (\Lambda^{0,1} \otimes T^{0,1}) \Big)_{\mathbb{R}}, \tag{3.26}$$

and

$$End_{\overline{\mathbb{C}}}(TM) \cong \Big((\Lambda^{1,0} \otimes T^{0,1}) \oplus (\Lambda^{0,1} \otimes T^{1,0}) \Big)_{\mathbb{R}}. \tag{3.27}$$

Proof. Given J, we define

$$I^C = \frac{1}{2}(I - JIJ) \tag{3.28}$$

$$I^A = \frac{1}{2}(I + JIJ). \tag{3.29}$$

Then

$$I^C J = \frac{1}{2}(IJ - JIJ^2) = \frac{1}{2}(IJ + JI),$$

and

$$JI^C = \frac{1}{2}(JI - J^2IJ) = \frac{1}{2}(JI + IJ).$$

Next,

$$I^A J = \frac{1}{2}(IJ + JIJ^2) = \frac{1}{2}(IJ - JI),$$

and

$$JI^A = \frac{1}{2}(JI + J^2IJ) = \frac{1}{2}(JI - IJ).$$

To prove uniqueness, if

$$I = I_1^C + I_1^A = I_2^C + I_2^A, \tag{3.30}$$

then

$$I_1^C - I_2^C = I_2^A - I_1^A. \tag{3.31}$$

Denote by $\tilde{I} = I_1^C - I_2^C = I_2^A - I_1^A$. Then $\tilde{I}$ both commutes and anti commutes with I, so is then easily seen to vanish identically.

An element $\mathcal{I} \in \Lambda^{0,1} \otimes T^{1,0}$ is a complex linear mapping from $T^{0,1}$ to $T^{1,0}$, that is $\mathcal{I} \in Hom_{\mathbb{C}}(T^{0,1}, T^{1,0})$. Writing $X \in T^{0,1}$ as $X = X' + iJX'$ for real $X' \in T$ and since $\mathcal{I}$ maps to $T^{1,0}$, $\mathcal{I}$ can be written as

$$\mathcal{I} : X' + iJX' \mapsto I(X') - iJI(X'), \tag{3.32}$$

for some real endomorphism of the tangent space $I : T \to T$, by defined by

$$I(X') = Re(\mathcal{I}(X' + iJX')). \tag{3.33}$$

To show that $IJ = -JI$, we first compute

$$IJ(X') = Re\{\mathcal{I}(JX' + iJJX')\} = Re\{\mathcal{I}(J(X' + iJX')\},$$

but since $X' + iJX' \in T^{0,1}$, we have $J(X' + iJX') = -i(X' + iJX')$, so

$$IJ(X') = Re\{\mathcal{I}(-i(X' + iJX')\},$$

using complex linearity of $\mathcal{I}$,

$$IJ(X') = Re\{-i\mathcal{I}(X' + iJX')\} = Im(\mathcal{I}(X' + iJX')).$$

Next, we have

$$JI(X') = JRe(\mathcal{I}(X' + iJX')) = -Im(\mathcal{I}(X' + iJX')), \tag{3.34}$$

since $\mathcal{I}(X' + iJX')$ is a $(1,0)$ vector field, and we have shown that $IJ = -JI$.

For the converse, given a real mapping satisfying $IJ + JI = 0$, writing $X \in T^{0,1}$ as $X = X' + iJX'$ define $\mathcal{I} : T^{0,1} \to T^{1,0}$ by

$$\mathcal{I} : X' + iJX' \mapsto I(X') - iJI(X'). \tag{3.35}$$

This map is clearly real linear, and we claim that this map is moreover complex linear. To see this,

$$\begin{aligned}\mathcal{I}(i(X' + iJX')) &= \mathcal{I}(-J(X' + iJX')) \\ &= -\mathcal{I}(JX' + iJ(JX')) = -I(JX') + iJI(JX').\end{aligned}$$

Using $IJ = -JI$, this is

$$\mathcal{I}(i(X' + iJX')) = JI(X') - iIJ(JX') = JI(X') + iI(X').$$

Next,

$$i\mathcal{I}(X' + iJX') = i(I(X') - iJI(X')) = JI(X') + iI(X'),$$

so $\mathcal{I}$ is indeed complex linear.

A similar argument proves the second case, and we are done. □

In matrix terms, this proposition is equivalent to the following

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \tag{3.36}$$

Embedding this into $Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2n})$, this is

$$\begin{aligned}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2}&\begin{pmatrix} A + D + i(C - B) & 0 \\ 0 & A + D - i(C - B) \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} 0 & A - D + i(B + C) \\ A - D - i(B + C) & 0 \end{pmatrix}.\end{aligned} \tag{3.37}$$

Notice that we can refine this a bit.

Proposition 3.3. *On an almost complex manifold (M, J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{\mathbb{C},0}(TM) \oplus \mathbb{R} \oplus End_{\overline{\mathbb{C}}}(TM), \tag{3.38}$$

where the first factor consists of traceless endomorphisms, and the middle factor consists of multiples of the identity transformation.

Remark 3.1. Note that the complex anti-linear endomorphisms are necessarily traceless.

Remark 3.2. Note if we take a path of complex structures $J(t)$ with $J(0) = J$ and $J'(0) = I$, then differentiating $J^2 = -I_n$ an evaluating at $t = 0$ yields $IJ + JI = 0$. So elements of $\Lambda^{0,1} \otimes T^{1,0}$ are infinitesimal deformations of the almost complex structure.

3.4 Tensors in $T^*M \otimes T^*M$

Next, we decompose $T^*M \otimes T^*M$.

Definition 7. A tensor $B \in T^*M \otimes T^*M$ is called symmetric if

$$B(X,Y) = B(Y,X), \tag{3.39}$$

for all $X, Y \in TM$. A $(0,2)$ tensor $B \in T^*M \otimes T^*M$ is called anti-symmetric if

$$B(X,Y) = -B(Y,X), \tag{3.40}$$

for all $X, Y \in TM$.

The space of symmetric $(0,2)$ tensors will be denoted $S^2(T^*M)$, and the space of skew-symmetric tensors is $\Lambda^2(T^*M)$. Note that J acts on $T^*M \otimes T^*M$ by

$$JB(X,Y) = B(JX, JY) \tag{3.41}$$

and satisfies $J^2 = Id$, and therefore $T^*M \otimes T^*M$ splits into the $+1$ and -1 eigenspaces for J.

Definition 8. A $(0,2)$ tensor will be called Hermitian if it is in the $+1$ eigenspace for J, and skew-Hermitian if it is in the -1 eigenspace for J.

Proposition 3.4. *On an almost complex manifold (M,J) the space of tensors in $T^*M \otimes T^*M$ decomposes as*

$$\begin{aligned} T^*M \otimes T^*M &= \Lambda^2 \oplus S^2 \\ &= \Lambda^2_{+1} \oplus \Lambda^2_{-1} \oplus S^2_{+1} \oplus S^2_{-1} \\ &= \Lambda^{1,1} \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus S^2_{+1} \oplus S^2_{-1} \\ &= \Lambda^{1,1} \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1} \oplus S^2_{-1} \end{aligned} \tag{3.42}$$

Proof. The decomposition of Λ^2 has already been discussed. We need to show that

$$S^2(T^*M) \cong \Lambda^{1,1}(T^*M). \tag{3.43}$$

This is easily proved: if $T \in S^2(T^*M)$, then

$$\omega_T(X,Y) = T(JX, Y) \tag{3.44}$$

is a 2-form of type $(1,1)$. To check this, we need to show that

$$\omega_T(X,Y) = 0 \tag{3.45}$$

if either both X and Y are in $T^{(1,0)}$ or both are in $T^{(0,1)}$. For the first case,

$$\begin{aligned} \omega_T(X,Y) &= \omega_T(X' - iJX', Y' - iJY') = T(J(X' - iJX'), Y' - iJY') \\ &= T(JX' + iX', Y' - iJY') \\ &= T(JX', Y') + T(X', JY') + i(T(X', Y') - T(JX', JY')) = 0, \end{aligned}$$

since T is J-invariant. □

3.5 The musical isomorphisms

We recall the following from Riemannian geometry. The metric gives an isomorphism between TM and T^*M,

$$\flat : TM \to T^*M \tag{3.46}$$

defined by

$$\flat(X)(Y) = g(X, Y). \tag{3.47}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp\omega_1, \sharp\omega_2). \tag{3.48}$$

Note that if g has components g_{ij}, then $\langle \cdot, \cdot \rangle$ has components g^{ij}, the inverse matrix of g_{ij}.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{3.49}$$

where

$$X_i = g_{ij} X^j, \tag{3.50}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{3.51}$$

where

$$\omega^i = g^{ij} \omega_j, \tag{3.52}$$

thus the sharp operator "raises" an index.

The $\flat$ operator extends to a complex linear mapping

$$\flat : TM \otimes \mathbb{C} \to T^*M \otimes \mathbb{C}. \tag{3.53}$$

We have the following

Proposition 3.5. *The operator $\flat$ is a complex anti-linear isomorphism*

$$\flat : T^{(1,0)} \to \Lambda^{0,1} \tag{3.54}$$

$$\flat : T^{(0,1)} \to \Lambda^{1,0}. \tag{3.55}$$

Proof. These mapping properties follow from the Hermitian property of g. Next, for any two vectors X and Y

$$\flat(JX)Y = g(JX, Y), \tag{3.56}$$

while

$$J(\flat X)(Y) = (\flat X)(JY) = g(X, JY) = -g(JX, Y). \tag{3.57}$$

□

In components, we have the following. The metric on the $T^*Z \otimes \mathbb{C}$ as components $g^{\alpha\overline{\beta}}$ where these of the component of the inverse matrix of $g_{\alpha\overline{\beta}}$. We have the identities

$$g^{\alpha\overline{\beta}} g_{\overline{\beta}\gamma} = \delta^{\alpha}_{\gamma}, \tag{3.58}$$

$$g^{\overline{\alpha}\beta} g_{\beta\overline{\gamma}} = \delta^{\overline{\alpha}}_{\overline{\gamma}}, \tag{3.59}$$

If $X = X^{\alpha} Z_{\alpha}$ is in $T^{(1,0)}$, then $\flat X$ has components

$$(\flat X)_{\overline{\alpha}} = g_{\overline{\alpha}\beta} X^{\beta}, \tag{3.60}$$

and if $X = X^{\overline{\alpha}} Z_{\overline{\alpha}}$ is in $T^{(0,1)}$, then $\flat X$ has components

$$(\flat X)_{\alpha} = g_{\alpha\overline{\beta}} X^{\overline{\beta}}, \tag{3.61}$$

Similarly, if $\omega = \omega_{\alpha} Z^{\alpha}$ is in $\Lambda^{1,0}$, then $\sharp\omega$ has components

$$(\sharp\omega)^{\overline{\alpha}} = g^{\overline{\alpha}\beta} \omega_{\beta}, \tag{3.62}$$

and if $\omega = \omega_{\overline{\alpha}} Z^{\overline{\alpha}}$ is in $\Lambda^{0,1}$, then $\sharp X$ has components

$$(\sharp\omega)^{\alpha} = g^{\alpha\overline{\beta}} \omega_{\overline{\beta}}. \tag{3.63}$$

If (M, J, g) is almost Hermitian, then the $\flat$ operator gives an identification

$$End_{\mathbb{R}}(TM) \cong T^*M \otimes TM \cong T^*M \otimes T^*M, \tag{3.64}$$

This yields a trace map defined on $T^*M \otimes T^*M$ defined as follows. If

$$h = h_{\alpha\beta} dz^{\alpha} dz^{\beta} + h_{\overline{\alpha}\overline{\beta}} d\overline{z}^{\alpha} d\overline{z}^{\beta} + h_{\alpha\overline{\beta}} dz^{\alpha} d\overline{z}^{\beta} + h_{\overline{\alpha}\beta} d\overline{z}^{\alpha} dz^{\beta}, \tag{3.65}$$

then

$$tr(h) = g^{\alpha\overline{\beta}} h_{\alpha\overline{\beta}} + g^{\overline{\alpha}\beta} h_{\overline{\alpha}\beta}. \tag{3.66}$$

Note that the components $h_{\alpha\beta}$ and $h_{\overline{\alpha}\overline{\beta}}$ do not contribute to the trace.

Remark 3.3. In Kähler geometry one sometimes sees the trace of some 2-tensor defined as just the first term in (3.66). If h is the complexification of a real tensor, then this is (1/2) of the Riemannian trace.

Using the trace map, we can put together the two decompositions from above

Proposition 3.6. *On an almost Hermitian manifold (M, J, g) we have the following decomposition*

$$\begin{aligned} T^*M \otimes T^*M &= \Lambda^2 \oplus S^2 \\ &= \Lambda^2_{+1} \oplus \Lambda^2_{-1} \oplus S^2_{+1} \oplus S^2_{-1} \\ &= \Lambda^{1,1} \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus S^2_{+1} \oplus S^2_{-1} \\ &= \Lambda^{1,1}_0 \oplus (\mathbb{R} \cdot \omega) \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}_0 \oplus (\mathbb{R} \cdot g) \oplus S^2_{-1}, \end{aligned} \tag{3.67}$$

where $\Lambda^{1,1}_0$ is the space of $(1,1)$-forms which are orthogonal to the fundamental 2-form.

4 Lecture 4

4.1 Automorphisms

Definition 9. An *infinitesimal automorphism* of a complex manifold is a real vector field X such that $\mathcal{L}_X J = 0$, where $\mathcal{L}$ denotes the Lie derivative operator.

It is straightforward to see that X is an infinitesimal automorphism if and only if its 1-parameter group of diffeomorphisms are holomorphic automorphisms, that is, $(\phi_s)_* \circ J = J \circ (\phi_s)_*$.

Proposition 4.1. *A vector field X is an infinitesimal automorphism if and only if*

$$J([X,Y]) = [X, JY], \tag{4.1}$$

for all vector fields Y.

Proof. We compute

$$[X, JY] = \mathcal{L}_X(JY) = \mathcal{L}_X(J)Y + J(\mathcal{L}_X Y) = \mathcal{L}_X(J)Y + J([X,Y]), \tag{4.2}$$

and the result follows. □

Definition 10. A *holomorphic vector field* on a complex manifold (M, J) is vector field $Z \in \Gamma(T^{1,0})$ which satisfies Zf is holomorphic for every locally defined holomorphic function f.

In complex coordinates, a holomorphic vector field can locally be written as

$$Z = \sum f_i \frac{\partial}{\partial z^i}, \tag{4.3}$$

where the f_i are locally defined holomorphic functions.

Proposition 4.2. *For $X \in \Gamma(TM)$, associate a vector field of type $(1,0)$ by mapping $X \mapsto Z = X - iJX$. Then X is an infinitesimal automorphism if and only if Z is a holomorphic vector field.*

Proof. Choose a local holomorphic coordinate system $\{z^i\}$, and for real vector fields X' and Y', write

$$X = X' - iJX' = \sum X^j \frac{\partial}{\partial z^j}, \tag{4.4}$$

$$Y = Y' - iJY' = \sum Y^j \frac{\partial}{\partial z^j}. \tag{4.5}$$

We know that X' is an infinitesimal automorphism if and only if

$$J([X', Y']) = [X', JY'], \tag{4.6}$$

for all real vector fields Y'. This condition is equivalent to

$$\sum_j \overline{Y}^j \frac{\partial X^k}{\partial \overline{z}^j} = 0, \tag{4.7}$$

for each $k = 1 \dots n$, which is equivalent to X being a holomorphic vector field.

To see this, we rewrite (4.6) in terms of complex vector fields. We have

$$X' = \frac{1}{2}(X + \overline{X}) \quad JX' = \frac{i}{2}(X - \overline{X})$$
$$Y' = \frac{1}{2}(Y + \overline{Y}) \quad JY' = \frac{i}{2}(Y - \overline{Y})$$

The left hand side of (4.6) is

$$\begin{aligned} J([X', Y']) &= J([\frac{1}{2}(X + \overline{X}), \frac{1}{2}(Y + \overline{Y})]) \\ &= \frac{1}{4} J([X, Y] + [X, \overline{Y}] + [\overline{X}, Y] + [\overline{X}, \overline{Y}]). \end{aligned}$$

But from integrability, $[X, Y]$ is also of type $(1, 0)$, and $[\overline{X}, \overline{Y}]$ is of type $(0, 1)$. So we can write this as

$$J([X', Y']) = \frac{1}{4}(i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y]). \tag{4.8}$$

Next, the right hand side of (4.6) is

$$[\frac{1}{2}(X + \overline{X}), \frac{i}{2}(Y - \overline{Y})] = \frac{i}{4}([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{4.9}$$

Then (4.8) equals (4.9) if and only if

$$J[X, \overline{Y}] + J[\overline{X}, Y] = -i[X, \overline{Y}] + i[\overline{X}, Y]. \tag{4.10}$$

This is equivalent to

$$J(Re([X, \overline{Y}])) = Im([X, \overline{Y}]). \tag{4.11}$$

This says that $[X, \overline{Y}]$ is a vector field of type $(0, 1)$. We can write the Lie bracket as

$$\begin{aligned} [X, \overline{Y}] &= \Big[\sum_j X^j \frac{\partial}{\partial z^j}, \sum_k \overline{Y}^k \frac{\partial}{\partial \overline{z}^k} \Big] \\ &= -\sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j} + \sum_k X^j (\frac{\partial}{\partial z^j} \overline{Y}^k) \frac{\partial}{\partial \overline{z}^k}, \end{aligned}$$

and the vanishing of the $(1, 0)$ component is exactly (4.7). □

4.2 The $\overline{\partial}$ operator on holomorphic vector bundles

We first illustrate this operator for the holomorphic tangent bundle $T^{1,0}$.

Proposition 4.3. *There is an first order differential operator*

$$\overline{\partial} : \Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}), \tag{4.12}$$

such that a vector field Z is holomorphic if and only if $\overline{\partial}(Z) = 0$.

Proof. Choose local holomorphic coordinates $\{z^j\}$, and write any section of Z of $T^{1,0}$, locally as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}. \tag{4.13}$$

Then define

$$\overline{\partial}(Z) = \sum_j (\overline{\partial} Z^j) \otimes \frac{\partial}{\partial z^j}. \tag{4.14}$$

This is in fact a well-defined global section of $\Lambda^{0,1} \otimes T^{1,0}$ since the transition functions of the bundle $T^{1,0}$ corresponding to a change of holomorphic coordinates are holomorphic.

To see this, if we have an overlapping coordinate system $\{w^j\}$ and

$$Z = \sum W^j \frac{\partial}{\partial w^j}. \tag{4.15}$$

Note that

$$\frac{\partial}{\partial z^j} = \frac{\partial w^k}{\partial z^j} \frac{\partial}{\partial w^j}, \tag{4.16}$$

which implies that

$$W^j = Z^p \frac{\partial w^j}{\partial z^p}. \tag{4.17}$$

We compute

$$\begin{aligned}
\overline{\partial}(Z) &= \sum \overline{\partial}(W^j) \otimes \frac{\partial}{\partial w^j} = \sum \overline{\partial}(Z^p \frac{\partial w^j}{\partial z^p}) \otimes \frac{\partial z^q}{\partial w^j} \frac{\partial}{\partial z^q} \\
&= \sum \frac{\partial w^j}{\partial z^p} \frac{\partial z^q}{\partial w^j} \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \delta_p^q \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \overline{\partial}(Z^j) \otimes \frac{\partial}{\partial z^j}.
\end{aligned}$$

□

Recall that the transition functions of a complex vector bundle are locally defined functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$, satisfying

$$\phi_{\alpha\beta} = \phi_{\alpha\gamma} \phi_{\gamma\beta}. \tag{4.18}$$

Notice the main property we used in the proof of Proposition 4.3 is that the transition functions of the bundle are holomorphic. Thus we make the following definition.

Definition 11. A vector bundle $\pi : E \to M$ is a *holomorphic vector bundle* if in complex coordinates the transitition functions $\phi_{\alpha\beta}$ are holomorphic.

Recall that a section of a vector bundle is a mapping $\sigma : M \to E$ satisfying $\pi \circ \sigma = Id_M$. In local coordinates, a section satisfies

$$\sigma_\alpha = \phi_{\alpha\beta}\sigma_\beta, \tag{4.19}$$

and conversely any locally defined collection of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}^m$ satisfying (4.19) defines a global section. A section is *holomorphic* if in complex coordinates, the σ_α are holomorphic.

We next have the generalization of Proposition 4.3.

Proposition 4.4. *If $\pi : E \to M$ is a holomorphic vector bundle, then there is an first order differential operator*

$$\overline{\partial} : \Gamma(E) \to \Gamma(\Lambda^{0,1} \otimes E), \tag{4.20}$$

such that a section σ is holomorphic if and only if $\overline{\partial}(\sigma) = 0$.

Proof. Let σ_j be a local basis of holomorphic sections in U_α, and write any section σ as

$$\sigma = \sum s_j \sigma_j. \tag{4.21}$$

Then define

$$\overline{\partial}\sigma = \sum (\overline{\partial} s_j) \otimes \sigma_j. \tag{4.22}$$

We claim this is a global section of $\Gamma(\Lambda^{0,1} \otimes E)$. If we choose a local basis σ'_j of holomorphic sections in U_β, and write σ as

$$\sigma = \sum s'_j \sigma'_j. \tag{4.23}$$

We can write

$$s'_j = A_{jl} s_l, \tag{4.24}$$

where $A : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$ is holomorphic. We also have

$$\sigma'_j = A^{-1}_{jl} \sigma_l. \tag{4.25}$$

Consequently

$$\begin{aligned} \overline{\partial}\sigma &= \sum (\overline{\partial} s'_j) \otimes \sigma'_j = \sum \overline{\partial}(A_{jk} s_k) \otimes A^{-1}_{jl} \sigma_l \\ &= \sum A_{jk} \overline{\partial}(s_k) \otimes A^{-1}_{jl} \sigma_l = \sum \delta_{kl} (\overline{\partial} s_k) \otimes \sigma_l = \sum (\overline{\partial} s_k) \otimes \sigma_k. \end{aligned}$$

□

5 Lecture 5

5.1 The Lie derivative as a $\overline{\partial}$-operator

For the special case of $T^{1,0}$ we have another operator mapping from

$$\Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}) \tag{5.1}$$

defined as follows. If X is a section of $T^{1,0}$, writing $X = X' - iJX'$ for a real vector field X' then consider $\mathcal{L}_{X'}J$. Since $J^2 = -I$, applying the Lie derivative, we have

$$(\mathcal{L}_{X'}J) \circ J + J \circ (\mathcal{L}_{X'}J) = 0, \tag{5.2}$$

that is, $\mathcal{L}_{X'}J$ anti-commutes with J, so using Proposition 3.2 we can we view $\mathcal{L}_{X'}J$ as a section of $\Lambda^{0,1} \otimes T^{1,0}$.

Proposition 5.1. *For $X \in \Gamma(T^{1,0})$,*

$$\overline{\partial} X = J \circ \mathcal{L}_{X'}J, \tag{5.3}$$

where $X' = Re(X)$.

Proof. The proof is similar to the proof of Proposition 4.2 above. For real vector fields X' and Y', we let

$$X = X' - iJX' = \sum X^j \frac{\partial}{\partial z^j},$$
$$Y = Y' - iJY' = \sum Y^j \frac{\partial}{\partial z^j},$$

and we have the formulas

$$X' = \frac{1}{2}(X + \overline{X}) \quad JX' = \frac{i}{2}(X - \overline{X})$$
$$Y' = \frac{1}{2}(Y + \overline{Y}) \quad JY' = \frac{i}{2}(Y - \overline{Y})$$

Expanding the Lie derivative,

$$(\mathcal{L}_{X'}J)(Y') = \mathcal{L}_{X'}(J(Y')) - J(\mathcal{L}_{X'}Y') = [X', JY'] - J[X', Y']. \tag{5.4}$$

In the proof of Proposition 4.2, it was shown that

$$J([X', Y']) = \frac{1}{4}(i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y]), \tag{5.5}$$

and

$$[X', JY'] = \frac{i}{4}([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{5.6}$$

So we have

$$[X', JY'] - J[X', Y'] = \frac{1}{4}(-i[X, \overline{Y}] + i[\overline{X}, Y] - J[X, \overline{Y}] - J[\overline{X}, Y])$$
$$= -\frac{1}{4}\Big(i(Z - \overline{Z}) + J(Z + \overline{Z})\Big),$$

where $Z = [X, \overline{Y}]$. We have that

$$Z = [X, \overline{Y}] = -\sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j} + \sum_k X^j (\frac{\partial}{\partial z^j} \overline{Y}^k) \frac{\partial}{\partial \overline{z}^k},$$

which we write as

$$Z = \sum Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j}. \tag{5.7}$$

We next compute

$$i(Z - \overline{Z}) + J(Z + \overline{Z}) = i\Big(Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} - \overline{W}^j \frac{\partial}{\partial z^j}\Big)$$
$$+ J\Big(\sum Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j} + \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} + \overline{W}^j \frac{\partial}{\partial z^j})$$
$$= i\Big(Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} - \overline{W}^j \frac{\partial}{\partial z^j}\Big)$$
$$+ i\Big(\sum Z^j \frac{\partial}{\partial z^j} - W^j \frac{\partial}{\partial \overline{z}^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} + \overline{W}^j \frac{\partial}{\partial z^j})$$
$$= 2i\Big(\sum Z^j \frac{\partial}{\partial z^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j}\Big).$$

We have obtained the formula

$$(\mathcal{L}_{X'} J)(Y') = \frac{-i}{2}\Big(\sum Z^j \frac{\partial}{\partial z^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j}\Big) = Im(Z^{1,0}), \tag{5.8}$$

where $Z^{1,0}$ is the $(1,0)$ part of Z, which is

$$Z^{1,0} = \sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j}. \tag{5.9}$$

Next, we need to view $\overline{\partial} X$ as a real endomorphism, and from the proof of Proposition 3.2, this is

$$(\overline{\partial} X)(Y') = Re\big((\overline{\partial} X)(Y' + iJY')\big)$$
$$= Re\Big\{\Big(\sum_j \overline{\partial} X^j \otimes \frac{\partial}{\partial z^j}\Big)(Y' + iJY')\Big\}$$
$$= Re\Big\{\Big(\sum_j \overline{\partial} X^j\Big)(Y' + iJY') \frac{\partial}{\partial z^j}\Big\}.$$

But note that

$$Y' + iJY' = \overline{Y' - iJY'} = \overline{Y} = \sum_j \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}. \tag{5.10}$$

So we have

$$\begin{aligned}(\overline{\partial} X)(Y') &= Re\Big\{\Big(\sum_j \overline{\partial} X^j\Big)(\overline{Y})\frac{\partial}{\partial z^j}\Big\} \\ &= Re\Big\{\sum_{p,j} \overline{Y}^p\Big(\frac{\partial}{\partial \overline{z}^p} X^j\Big)\frac{\partial}{\partial z^j}\Big\} = Re(Z^{1,0}).\end{aligned}$$

But since $Z^{1,0}$ is of type $(1,0)$,

$$Im(Z^{1,0}) = -J(Re(Z^{1,0}). \tag{5.11}$$

Finally, we have

$$(\overline{\partial} X)(Y') = Re(Z^{1,0}) = J(Im(Z^{1,0})) = J((\mathcal{L}_{X'} J)(Y')), \tag{5.12}$$

and we are done. □

Letting Θ denote $T^{1,0}$, there is moreover an entire complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,3} \otimes \Theta) \xrightarrow{\overline{\partial}} \cdots . \tag{5.13}$$

We have that the holomorphic vector fields (equivalently, the automorphisms of the complex structure) are $H^0(M, \Theta)$. The higher cohomology groups $H^1(M, \Theta)$ and $H^2(M, \Theta)$ of this complex play a central role in the theory of deformations of complex structures.

5.2 The space of almost complex structures

We define

$$\mathcal{J}(\mathbb{R}^{2n}) \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n, \mathbb{R}), J^2 = -I_{2n}\} \tag{5.14}$$

In this subsection, we give some alternative descriptions of this space.

Proposition 5.2. *The space $\mathcal{J}(\mathbb{R}^{2n})$ is the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.*

Proof. We note that $GL(2n, \mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n, \mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$\Phi_A : J \mapsto AJA^{-1}. \tag{5.15}$$

Obviously,

$$(AJA^{-1})^2 = AJA^{-1}AJA^{-1} = AJ^2A^{-1} = -I, \tag{5.16}$$

and

$$\Phi_{AB}(J) = (AB)J(AB)^{-1} = ABJB^{-1}A^{-1} = \Phi_A\Phi_B(J), \tag{5.17}$$

so is indeed a group action. Given J and J', there exists bases

$$\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \text{ and } \{e'_1, \ldots, e'_n, J'e'_1, \ldots, J'e'_n\}. \tag{5.18}$$

Define $S \in GL(2n, \mathbb{R})$ by $Se_k = e'_k$ and $S(Je_k) = J'e'_k$. Then $J' = SJS^{-1}$, and the action is therefore transitive. The stabilizer subgroup of J_0 is

$$Stab(J_0) = \{A \in GL(2n, \mathbb{R}) : AJ_0A^{-1} = J_0\}, \tag{5.19}$$

that is, A commutes with J_0. We have seen above in (**??**) that this can be identified with $GL(n, \mathbb{C})$. □

We next give yet another description of this space. Define

$$\begin{aligned}\mathcal{C}(\mathbb{R}^{2n}) = \{P \subset \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n} \mid dim_{\mathbb{C}}(P) = n,\\ P \text{ is a complex subspace satisfying } P \cap \overline{P} = \{0\}\}.\end{aligned}$$

If we consider $\mathbb{R}^{2n} \otimes \mathbb{C}$, we note that complex conjugation is a well defined complex anti-linear map $\mathbb{R}^{2n} \otimes \mathbb{C} \to \mathbb{R}^{2n} \otimes \mathbb{C}$.

Proposition 5.3. *The space $\mathcal{C}(\mathbb{R}^{2n})$ can be explicitly identified with $\mathcal{J}(\mathbb{R}^{2n})$ by the following. If $J \in \mathcal{J}(\mathbb{R}^{2n})$ then let*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = T^{1,0}(J) \oplus T^{0,1}(J), \tag{5.20}$$

where

$$T^{0,1}(J) = \{X + iJX, X \in \mathbb{R}^{2n}\} = \{-i\}\text{-eigenspace of } J. \tag{5.21}$$

This an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and letting $T^{1,0}(J) = \overline{T^{0,1}(J)}$, we have $T^{1,0} \cap T^{0,1} = \{0\}$.

For the converse, given $P \in \mathcal{C}(\mathbb{R}^{2n})$, then P may be written as a graph over $\mathbb{R}^{2n} \otimes 1$, that is

$$P = \{X' + iJX' \mid X' \in \mathbb{R}^{2n} \subset \mathbb{C}^{2n}\}, \tag{5.22}$$

with $J \in \mathcal{J}(\mathbb{R}^{2n})$, and

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \overline{P} \oplus P = T^{1,0}(J) \oplus T^{0,1}(J). \tag{5.23}$$

Proof. For the forward direction, we already know this. To see the other direction, consider the projection map Re restricted to P

$$\pi = Re : P \to \mathbb{R}^{2n}. \tag{5.24}$$

We claim this is a real linear isomorphism. Obviously, it is linear over the reals. Let $X \in P$ satisfy $\pi(X) = 0$. Then $Re(X) = 0$, so $X = iX'$ for some real $X' \in \mathbb{R}^{2n}$. But $\overline{X} = -iX' \in P \cap \overline{P}$, so by assumption $X = 0$. Since these spaces are of the same real dimension, π has an inverse, which we denote by J. Clearly then, (5.22) is satisfied. Since P is a complex subspace, given any $X = X' + iJX' \in P$, the vector $iX' = (-JX') + iX'$ must also lie in P, so

$$(-JX') + iX' = X'' + iJX'', \tag{5.25}$$

for some real X'', which yields the two equations

$$JX' = -X'' \tag{5.26}$$

$$X' = JX''. \tag{5.27}$$

applying J to the first equation yields

$$J^2X' = -JX'' = -X'. \tag{5.28}$$

Since this is true for any X', we have $J^2 = -I_{2n}$. □

Remark 5.1. We note that $J \mapsto -J$ corresponds to interchanging $T^{0,1}$ and $T^{1,0}$.

Remark 5.2. The above propositions embed $\mathcal{J}(\mathbb{R}^{2n})$ as a subset of the complex Grassmannian $G(n, 2n, \mathbb{C})$. These spaces have the same dimension, so it is an *open* subset. Furthermore, the condition that the projection to the real part is an isomorphism is generic, so it is also dense.

6 Lecture 6

6.1 Deformations of complex structure

We next let $J(t)$ be a path of complex structures through $J = J(0)$. Such a $J(t)$ is equivalent to a decomposition

$$TM \otimes \mathbb{C} = T^{1,0}(J_t) \oplus T^{0,1}(J_t). \tag{6.1}$$

Note that, for t sufficiently small, this determines an element $\phi(t) \in \Lambda^{0,1}(J) \otimes T^{1,0}(J)$ which we view as a mapping

$$\phi(t) : T^{0,1}(J) \to T^{1,0}(J), \tag{6.2}$$

by writing

$$T^{0,1}(J_t) = \{v + \phi(t)v \mid v \in T^{0,1}(J_0)\}. \tag{6.3}$$

That is, we write $T^{0,1}(J_t)$ as a graph over $T^{0,1}(J_0)$. Conversely, a path $\phi(t)$ in (6.2), corresponds to a path $J(t)$ of almost complex structures.

Corollary 6.1. *Let M be compact, and J an almost complex structure. The there is a canonical correspondence between paths of almost complex structures $J(t)$ through J and paths $\phi(t)$ of sections of $\Lambda^{0,1}(J) \otimes T^{1,0}(J)$ for t small.*

Proof. Call the base complex structure J_0. Given

$$\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0) = Hom_{\mathbb{C}}(T^{0,1}(J_0), T^{1,0}(J_0)), \tag{6.4}$$

then

$$T^{0,1}(J) = \{v + \phi v, v \in T^{0,1}(J_0)\}. \tag{6.5}$$

This is always an n-dimensional complex subspace. If $X \in T^{0,1}(J) \cap \overline{T^{0,1}(J)}$, then

$$v + \phi v = w + \overline{\phi} w, \tag{6.6}$$

where $v \in T^{0,1}(J_0)$ and $w \in T^{1,0}(J_0)$. This yields the equations

$$\overline{\phi} w = v \tag{6.7}$$

$$\phi v = w. \tag{6.8}$$

This says that $\phi\overline{\phi}$ has 1 as an eigenvalue. But if ϕ is sufficiently small, this cannot happen.

Conversely, given a path $J(t)$, we obtain a path $\phi(t)$ by choosing $P(t) = T^{0,1}(J(t))$ and writing $P(t)$ as a graph over $T^{0,1}(J_0)$. But projection from $P(t)$ to $P(0)$ is an isomorphism if $P(t)$ is sufficiently close to $P(0)$ in the Grassmanian.

□

6.2 The Nijenhuis tensor

We next return to the Nijenhuis tensor, which we recall is defined by

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\}. \tag{6.9}$$

Proposition 6.1. *For any almost complex structure, the Nijenhuis tensor is a section of $\Lambda^{0,2} \otimes T^{1,0}$ as follows. Let X and Y be in $T^{0,1}$, and write $X = X' + iJX'$ and $Y = Y' + iJY'$ for real vectors X' and Y'. Then*

$$\Pi_{T^{1,0}}[X,Y] = -\frac{1}{4}(N(X',Y') - iJN(X',Y')). \tag{6.10}$$

Proof. To see this, we compute

$$[X,Y] = [X' + iJX', Y' + iJY'] = [X',Y'] - [JX',JY'] + i([X',JY'] + [JX',Y']). \tag{6.11}$$

Notice that $\Pi_{T^{1,0}}(Z) = \frac{1}{2}(Z - iJZ)$, so

$$\begin{aligned}\Pi_{T^{1,0}}([X,Y]) =& \frac{1}{2}([X',Y'] - [JX',JY'] + i([X',JY'] + [JX',Y'])\\ &- \frac{1}{2}(i(J[X',Y'] - J[JX',JY']) - J[X',JY'] - J[JX',Y'])\\ &= \frac{1}{2}([X',Y'] - [JX',JY'] + J[X',JY'] + J[JX',Y'])\\ &+ \frac{1}{2}iJ([X',Y'] - [JX',JY'] + J[X',JY'] + J[JX',Y'])\\ &= -\frac{1}{4}(N(X',Y') - iJN(X',Y')).\end{aligned}$$

□

Next, we write the integrability condition for a path of almost complex structures $J(t) = J^C(t) + I(t)$ with corresponding $\phi(t) \in \Lambda^{0,1} \otimes T^{1,0}$.

Proposition 6.2. *The complex structure $J(t) = J^C(t) + I(t)$ is integrable if and only if*

$$\overline{\partial}\phi(t) + [\phi(t), \phi(t)] = 0, \tag{6.12}$$

where $[\phi(t), \phi(t)] \in \Lambda^{0,2} \otimes T^{1,0}$ is a term which is quadratic in the $\phi(t)$ and its first derivatives, that is,

$$\|[\phi(t), \phi(t)]\| \leq \|\phi\| \cdot \|\nabla\phi\|, \tag{6.13}$$

in any local coordinate system.

Proof. By Proposition 6.1, the integrability equation is equivalent to $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$. Writing

$$\phi = \sum \phi_{ij} d\overline{z}_i \otimes \frac{\partial}{\partial z^j}, \tag{6.14}$$

if $J(t)$ is integrable, then we must have

$$\Big[\frac{\partial}{\partial \overline{z}^i} + \phi\Big(\frac{\partial}{\partial \overline{z}^i}\Big), \frac{\partial}{\partial \overline{z}^k} + \phi\Big(\frac{\partial}{\partial \overline{z}^k}\Big)\Big] \in T_t^{0,1}. \tag{6.15}$$

This yields

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] \in T_t^{0,1} \tag{6.16}$$

The first two terms are

$$\begin{aligned}\Big[\frac{\partial}{\partial \overline{z}^i}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] &= \sum_j \Big(\frac{\partial \phi_{kj}}{\partial \overline{z}^i} - \frac{\partial \phi_{ij}}{\partial \overline{z}^k}\Big)\frac{\partial}{\partial z^j}\\ &= (\overline{\partial}\phi)\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial z^j}\Big).\end{aligned}$$

The third term is

$$\begin{aligned}\Big[\phi_{ij}\frac{\partial}{\partial z^j},\phi_{kl}\frac{\partial}{\partial z^l}\Big] &= \phi_{ij}\Big(\frac{\partial}{\partial z^j}\phi_{kl}\Big)\frac{\partial}{\partial z^l} - \phi_{kl}\Big(\frac{\partial}{\partial z^l}\phi_{ij}\Big)\frac{\partial}{\partial z^j}\\ &= [\phi,\phi]\Big(\frac{\partial}{\partial \overline{z}^i},\frac{\partial}{\partial \overline{z}^k}\Big),\end{aligned}$$

where $[\phi,\phi]$ is defined by

$$[\phi,\phi] = \sum (d\overline{z}^i \wedge d\overline{z}^k)\Big[\phi_{ij}\frac{\partial}{\partial z^j},\phi_{kl}\frac{\partial}{\partial z^l}\Big], \tag{6.17}$$

and is easily seen to be a well-defined global section of $\Lambda^{0,2}\otimes T^{1,0}$. We have shown that

$$(\overline{\partial}\phi(t) + [\phi(t),\phi(t)])\Big(\frac{\partial}{\partial \overline{z}^i},\frac{\partial}{\partial \overline{z}^k}\Big) \in T^{0,1}_t. \tag{6.18}$$

But the left hand side is also in $T^{1,0}$. For sufficiently small t however, $T^{0,1}_t \cap T^{1,0} = \{0\}$, and therefore (6.12) holds.

For the converse, if (6.12) is satisfied, then the above argument in reverse shows that the integrability of $T^{0,1}_t$ holds as a distribution, which by Proposition 6.1 is equivalent to integrability of the complex structure $J(t)$. □

Using the above we can identify the $\overline{\partial}$ in the second term of the complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1}\otimes\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2}\otimes\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,3}\otimes\Theta) \xrightarrow{\overline{\partial}} \cdots \tag{6.19}$$

with the linearized Nijenhuis tensor at $t = 0$:

Proposition 6.3. *Let $J(t)$ be a path of almost complex structures with $J'(0) = I$, corresponding to $\phi \in \Lambda^{0,1}\otimes T^{1,0}$. Then*

$$\overline{\partial}\phi = -\frac{1}{4}(N'_J(I) - iJN'_J(I)). \tag{6.20}$$

Note we are using the following identification: since the Nijenhuis tensor is J anti-invariant, and skew-symmetric, it is a skew-hermitian 2-form, so by Proposition **??**, *$N + iJN$ is a section of $\Lambda^{0,2}\otimes T^{1,0}$.*

Proof. This follows from the above, using the fact that the quadratic term $[\phi,\phi]$ does not contribute to the linearization. □

7 Lecture 7

7.1 A fixed point theorem

The following is a cruial tool in the analytic study of moduli spaces and gluing theorems, see for example [Biq13, Lemma 7.3].

Lemma 7.1. *Let $H : E \to F$ be a differentiable mapping between Banach spaces. Define $Q = H - H(0) - H'(0)$. Assume that there are positive constants C_1, s_0, C_2 so that the following are satisfied:*

- (1) *The nonlinear term Q satisfies*

$$\|Q(x) - Q(y)\|_F \leq C_1(\|x\|_E + \|y\|_E)\|x - y\|_E$$

for every $x, y \in B_E(0, s_0)$.

- (2) *The linearized operator at 0, $H'(0) : E \to F$ is an isomorphism with inverse bounded by C_2.*

If s and $\|H(0)\|_F$ are sufficiently small (depending upon C_1, s_0, C_2), then there is a unique solution $x \in B_E(0, s)$ of the equation $H(x) = 0$.

Outline of Proof. The equation $H(x) = 0$ expands to

$$H(0) + H'(0)(x) + Q(x) = 0. \tag{7.1}$$

If we let $x = Gy$, where G is the inverse of $H'(0)$, then we have

$$H(0) + y + Q(Gy) = 0, \tag{7.2}$$

or

$$y = -H(0) - Q(Gy). \tag{7.3}$$

In other words, y is a fixed point of the mapping

$$T : y \mapsto -H(0) - Q(Gy). \tag{7.4}$$

With the assumptions in the lemma, it follows that T is a contraction mapping, so a fixed point exists by the standard fixed point theorem ($T^n y_0$ converges to a unique fixed point for any y_0 sufficiently small). □

Next, we have

Proposition 7.1. *If $H'(0)$ is Fredholm, (finite-dimensional kernel and cokernel and closed range), and there exists a complement of the cokernel on which $H'(0)$ has a bounded right inverse, then there exists a map*

$$\Psi : Ker(H'(0)) \to Coker(H'(0)), \tag{7.5}$$

whose zero set is locally isomorphic to the zero set of H.

Proof. Consider $P = \Pi \circ H$, where Π is projection to a complement of $Coker(H'(0))$. The differential of the map P, $P'(0)$ is now surjective. Choose any complement K to the space $Ker(H'(0))$, and restrict the mapping to this complement. Equivalently, let G be any right inverse, i.e., $H'(0)G = Id$, and let K be the image of G. Given a kernel element $x_0 \in H^1_E$, the equation $H(x_0 + Gy) = 0$ expands to

$$H(0) + H'(0)(x_0 + Gy) + Q(x_0 + Gy) = 0. \tag{7.6}$$

We therefore need to find a fixed point of the map

$$T_{x_0} : y \mapsto -H(0) - Q(x_0 + Gy), \tag{7.7}$$

and the proof is the same as before. □

7.2 Infinitesimal slice theorem for the moduli space of almost complex structures

We want a local model for the space of almost complex structures near a complex structure J modulo diffeomorphism. The main tool for this is the following infinitesimal version of a "slice" theorem due to Ebin-Palais, adapted to the complex case by Koiso [Koi83]. The notation $C^{k,\alpha}$ will denote the space of Hölder continuous mappings (or tensors) with $0 < \alpha < 1$. Fix a hermitian metric g compatible with J.

Theorem 7.1. *For each ACS J_1 in a sufficiently small $C^{\ell+1,\alpha}$-neighborhood of J ($\ell \geq 1$), there is a $C^{\ell+2,\alpha}$-diffeomorphism $\varphi : M \to M$ such that*

$$\tilde{\theta} \equiv \varphi^* J_1 - J \tag{7.8}$$

satisfies

$$\delta_g(\tilde{\theta}) = 0. \tag{7.9}$$

Proof. Let $\{\omega_1, \ldots, \omega_\kappa\}$ denote a basis of the space of real holomorphic vector fields. Consider the map

$$\mathcal{N} : C^{\ell+2,\alpha}(TM) \times \mathbb{R}^\kappa \times C^{\ell+1,\alpha}(\Lambda^{0,1} \otimes T^{1,0}) \to C^{\ell,\alpha}(T^*M) \tag{7.10}$$

given by

$$\mathcal{N}(X, v, \theta) = \mathcal{N}_\theta(X, v) = \big(\delta_g\big[\varphi^*_{X,1}(J+\theta)\big] + \sum_i v_i\omega_i\big), \tag{7.11}$$

where $\varphi_{X,1}$ denotes the diffeomorphism obtained by following the flow generated by the vector field X for unit time. Linearizing in (X, v) at $(X, v, \theta) = (0, 0, 0)$, we find

$$\begin{aligned}
\mathcal{N}_0'(Y, a) &= \frac{d}{d\epsilon}\big(\delta_g\big[\varphi^*_{\epsilon Y,1}(J)\big] + \sum_i (\epsilon a_i)\omega_i\big)\Big|_{\epsilon=0} \\
&= \big(\delta_g[\mathcal{L}_Y J] + \sum_i a_i\omega_i\big) \\
&= \big(\Box Y + \sum_i a_i\omega_i\big),
\end{aligned}$$

where $\Box = \delta_g\mathcal{L}_Y(J)$. Notice that from above, we can identify

$$\Box = \overline{\partial}^{\#}\overline{\partial}, \tag{7.12}$$

so $\Box$ is a self-adjoint operator.

The adjoint map $(\mathcal{N}_0')^* : C^{m+2,\alpha}(T^*M) \to C^{m,\alpha}(TM) \times \mathbb{R}^\kappa$ is given by

$$(\mathcal{N}_0')^*(\eta) = \Big((\Box\eta)^\sharp, \int_M \langle \eta, \omega_i\rangle \, dV_g\Big), \tag{7.13}$$

where $(\Box\eta)^\sharp$ is the vector field dual to $\Box\eta$.

If η is in the kernel of the adjoint, the first equation implies that η is a holomorphic vector field, while the second implies that η is orthogonal (in L^2) to the space of holomorphic vector fields. It follows that $\eta = 0$, so the map $\mathcal{N}_0'$ is surjective.

Omitting a few technical details for simplicity, applying the fixed point theorem from above, , given $\theta_1 \in C^{\ell+1,\alpha}(\Lambda^{0,1} \otimes T^{1,0})$ small enough, we can solve the equation $\mathcal{N}_{\theta_1} = 0$; i.e., there is a vector field $X \in C^{\ell+2,\alpha}(TM)$, and a $v \in \mathbb{R}^\kappa$, such that

$$\delta_g[\varphi^* J_1] + \sum_i v_i \omega_i = 0, \tag{7.14}$$

where $\varphi = \varphi_{X,1}$. Letting $\tilde{\theta} = \varphi^* J_1 - J$, then $\tilde{\theta}$ satisfies

$$\delta_g[\tilde{\theta}] + \sum_i v_i \omega_i = 0, \tag{7.15}$$

Pairing with ω_j, for $j = 1 \dots \kappa$, and integrating by parts, we see that $v_j = 0$, and we are done. □

Remark 7.1. The above is ust an "infinitesimal" version of the Slice Theorem. The full Ebin-Palais Slice Theorem for Riemannian metrics constructs a local slice for the action of the diffeomorphism group, see [Ebi68]. The main difficulty is that the natural action of the diffeomorphism group on the space of Riemannian metrics is not differentiable as a mapping of Banach spaces (with say Sobolev or Hölder norms). It is however differentiable as a mapping of ILH spaces, see [Omo70, Koi78]. Koiso proved an adaption of the slice theorem for Riemannian metric to the space of complex structures in [Koi83]. For the purposes of these lectures, we will content ourselves with the infinitesimal version, and will not go into details about the full slice theorem

8 Lecture 8

8.1 The Kuranishi map

We now have the following theorem.

Theorem 8.1. *Let (M, J) be a complex surface. The space $H^1(M, \Theta)$ is identified with*

$$H^1(M, \Theta) \simeq \frac{Ker(N_J)'}{Im(X \to \mathcal{L}_X J)}, \tag{8.1}$$

and therefore consists of essential infinitesimal deformations of the complex structure.

Furthermore, there is a map

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta) \tag{8.2}$$

called the Kuranishi map *such that the moduli space of complex structures near J is given by the orbit space*

$$\Psi^{-1}(0)/H^0(M, \Theta). \tag{8.3}$$

Proof. The identification (8.1) follows from the computations in the previous lecture. The remaining part takes a lot of machinery, so we will only give an outline here.

We consider the three term complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta), \tag{8.4}$$

and we will abbreviate this as

$$\Gamma(A) \xrightarrow{\overline{\partial}_A} \Gamma(B) \xrightarrow{\overline{\partial}_B} \Gamma(C) \xrightarrow{\overline{\partial}_C} \cdots \tag{8.5}$$

It is not hard to show that this complex is elliptic. We define a map

$$F : \Gamma(B) \to \Gamma(C) \oplus \Gamma(A) \tag{8.6}$$

by

$$F(\phi) = (\Pi_{\Gamma(C)} N_{J_\phi}, \overline{\partial}_A^* \phi). \tag{8.7}$$

where we have fixed a hermitian metric g compatible with J, and the adjoint is taken with respect to g.

Claim 8.1. *For ϕ sufficiently small, zeroes of F correspond to integrable complex structures near ϕ, modulo diffeomorphism.*

For the forward direction, if $\Pi_{\Gamma(C)(J)} N_{J_\phi} = 0$, then $N_{J_\phi} = 0$ if ϕ is sufficiently small. For the converse, we have that given any J_ϕ near J, there exists a diffeomorphism $f : M \to M$ such that $f^* J_\phi = J_{\phi'}$ with $\overline{\partial}_A^* \phi' = 0$. This follows since $\overline{\partial}_A^*$ is the divergence operator with respect to g, and then this follows from a version of the Ebin slice theorem. This finishes the claim.

Next, the linearization of F at $\phi = 0$, defined by

$$P(h) = \frac{d}{dt} F(\phi(t))\Big|_{t=0}, \tag{8.8}$$

where $\phi(t)$ is any path satisfying $\phi(0) = 0$, and $\phi'(0) = h$, is given by

$$P(h) = (\overline{\partial}_B(h), \overline{\partial}_A^*(h)). \tag{8.9}$$

This is an elliptic operator, since the above complex is elliptic. We also know that

$$N_{J_\phi} = \overline{\partial}\phi + [\phi, \phi], \tag{8.10}$$

and the nonlinear term satisfies

$$\|[\phi_1, \phi_1] - [\phi_2, \phi_2]\| \leq C(\|\phi_1\| + \|\phi_2\|) \cdot \|\phi_1 - \phi_2\|. \tag{8.11}$$

Consequently, one can use elliptic theory and this estimate on the nonlinear term together with an infinite-dimensional fixed point theorem to show that the zero set of F is equivalent to the zero set of a map

$$\Psi : Ker(P) \to Coker(P) = Ker(P^*), \tag{8.12}$$

defined between finite-dimensional spaces. Since M is compact, basic Hodge theory shows that

$$Ker(P) \simeq Ker(\overline{\partial}_B) \cap Ker(\overline{\partial}_A^*) \simeq \frac{Ker(\overline{\partial}_B)}{Im(\overline{\partial}_A)} \simeq H^1(M, \Theta), \tag{8.13}$$

and

$$Coker(P) \simeq Ker(\overline{\partial}_B^*) \oplus Ker((\overline{\partial}_A) \simeq \frac{Ker(\overline{\partial}_C)}{Im(\overline{\partial}_B)} \oplus H^0(M, \Theta) \tag{8.14}$$

$$\simeq H^2(M, \Theta) \oplus H^0(M, \Theta). \tag{8.15}$$

So we have

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta) \oplus H^0(M, \Theta) \tag{8.16}$$

Finally, the map Ψ is equivariant with respect to the holomorphic automorphsim group $H^0(M, \Theta)$, so we only need to consider Ψ as a mapping from

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta), \tag{8.17}$$

and we then obtain the actual moduli space as the orbit space of the action of $H^0(M, \Theta)$ on $\Psi^{-1}(0)$. □

Corollary 8.1. *If $H^2(M, \Theta) = 0$, then any such infinitesimal deformation I is* integrable*, that is, $I = J'(0)$ for an actual path of complex structures $J(t)$. If both $H^2(M, \Theta) = 0$ and $H^0(M, \Theta) = 0$ then the moduli space of complex structures near J is smooth of dimension $H^1(M, \Theta)$.*

8.2 The higher dimensional case

The argument in the previous section does not work as given in higher dimensions because the complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta), \tag{8.18}$$

is not elliptic. One case attempt to modify the above argument by projection the Nijenhuis tensor to the image of $\overline{\partial}$, but this introduces some technical difficulties which we do not want to get into. Instead, we will outline Kuranishi's method following Kodaira-Morrow [MK71]. Let $\Box$ denote the Laplacian

$$\Box = \overline{\partial}\overline{\partial}^{\#} + \overline{\partial}^{\#}\overline{\partial}, \tag{8.19}$$

where $\overline{\partial}^{\#}$ is the L^2-adjoint of $\overline{\partial}$ (we have fixed a Hermitian metric compatible with J).

Let $\mathbb{H}^k$ denote the space of harmonic forms in $\Lambda^{0,k} \otimes \Theta$, that is

$$\mathbb{H}^k = \{\phi \in \Gamma(\Lambda^{0,k} \otimes \Theta) | \Box\phi = 0\}. \tag{8.20}$$

Hodge theory tells us that $H^{0,k}(M, \Theta) \cong \mathbb{H}^k$ and that

$$\Gamma(\Lambda^{0,k} \otimes \Theta) = \mathbb{H}^k \oplus Im(\Box) \tag{8.21}$$
$$= \mathbb{H}^k \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^{\#}), \tag{8.22}$$

where these are orthogonal direct sums in L^2. Given any $\phi \in \Gamma(\Lambda^{0,k} \otimes \Theta)$, we have

$$\phi = h + \Box\psi \tag{8.23}$$

where h is harmonic. Applying the same decomposition to ψ

$$\psi = h_1 + \psi_1 \tag{8.24}$$

where h_1 is harmonic and $\psi_1 \in Im(\Box)$, enables us to write

$$\phi = h + \Box\psi_1, \tag{8.25}$$

where ψ_1 is orthogonal to $\mathbb{H}^k$. It is straightforward to show that ψ_1 is unique.

Definition 12. The Green's operator is defined as

$$G\phi = \psi_1, \tag{8.26}$$

so that any ϕ can be written as

$$\phi = \mathbf{H}\phi + \Box G\phi, \tag{8.27}$$

where $\mathbf{H}$ is harmonic projection onto $\mathbb{H}^k$.

Define a mapping

$$\Psi : \mathbb{H}^1 \to \Gamma((\Lambda^{0,1} \otimes \Theta)) \tag{8.28}$$

as follows. Given $\phi_1 \in \mathbb{H}^1$, solve the following equation for ϕ:

$$\phi = \phi_1 + \overline{\partial}^{\#} G[\phi, \phi] \tag{8.29}$$

This admits a unique solution using an iteration procedure similar to the above process using Hölder norms (details omitted). Next,

Proposition 8.1. *If ϕ_1 is in a sufficiently small ball around the origin in $\mathbb{H}^1$, then the solution ϕ of* (8.29) *solves*

$$\overline{\partial}\phi + [\phi, \phi] = 0 \tag{8.30}$$

if and only if

$$\mathbf{H}[\phi, \phi] = 0. \tag{8.31}$$

We will omit the proof, and just point out that the Kuranishi map is the corresponding mapping

$$\Phi : \mathbb{H}^1 \to \mathbb{H}^2, \tag{8.32}$$

and the zeroes of Φ parametrize the integrable complex structures near J. Note that elements $\phi \in \mathbb{H}^1$ necessarily satisfy $\overline{\partial}^{\#}\phi = 0$, which reflects the fact that this parametrizes complex structues near J modulo diffeomorphism if there are no automorphisms. Kuranishi also shows that that Φ is holomorphic, so that the moduli space is an analytic subset.

In the case of non-trivial automorphisms, note that Φ is equivariant under automorphisms of J, so if (M, J) admits non-trivial holomorphic vector fields, then the moduli space of complex structure modulo diffeomorphic is isomorphic to

$$\Psi^{-1}(0)/\mathbb{H}^0, \tag{8.33}$$

but the full proof of this identification requires a more elaborate slice theorem.

A special case where this mapping has been computed is the case of Calabi-Yau metrics. In this case, the following is known (we will not discuss the proof, and refer the reader to [Huy05]):

Theorem 8.2 (Tian-Todorov)**.** *For a Calabi-Yau metric* (X, g)*, the Kuranishi map* $\Psi \equiv 0$*. That is, every infintesimal Einstein deformation integrates to an actual deformation.*

9 Lecture 9

9.1 Serre duality

For a real oriented Riemannian manifold of dimension n, the Hodge star operator is a mapping

$$* : \Lambda^p \to \Lambda^{n-p} \tag{9.1}$$

defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV_g, \tag{9.2}$$

for $\alpha, \beta \in \Lambda^p$, where dV_g is the oriented Riemannian volume element.

If M is a complex manifold of complex dimension $m = n/2$, and g is a Hermitian metric, then the Hodge star extends to the complexification

$$* : \Lambda^p \otimes \mathbb{C} \to \Lambda^{2m-p} \otimes \mathbb{C}, \tag{9.3}$$

and it is not hard to see that

$$* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}. \tag{9.4}$$

Therefore the operator

$$\overline{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \tag{9.5}$$

is a $\mathbb{C}$-antilinear mapping and satisfies

$$\alpha \wedge \overline{*}\beta = \langle \alpha, \overline{\beta} \rangle dV_g. \tag{9.6}$$

for $\alpha, \beta \in \Lambda^p \otimes \mathbb{C}$.

The L^2-adjoint of $\overline{\partial}$ is given by

$$\overline{\partial}^* = - * \overline{\partial} *, \tag{9.7}$$

and the $\overline{\partial}$-Laplacian is defined by

$$\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*. \tag{9.8}$$

Letting

$$\mathbb{H}^{p,q}(M, g) = \{ \alpha \in \Lambda^{p,q} | \Delta_{\overline{\partial}} \alpha = 0 \}, \tag{9.9}$$

Hodge theory tells us that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathbb{H}^{p,q}(M, g), \tag{9.10}$$

is finite-dimensional, and that

$$\Lambda^{p,q} = \mathbb{H}^{p,q}(M, g) \oplus Im(\Delta_{\overline{\partial}}) \tag{9.11}$$

$$= \mathbb{H}^{p,q}(M, g) \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^*), \tag{9.12}$$

with this being an orthogonal direct sum in L^2.

Corollary 9.1. *Let (M, J) be a compact complex manifold of real dimension $n = 2m$. Then*

$$H^{p,q}_{\overline{\partial}}(M) \cong (H^{n-p,n-q}_{\overline{\partial}}(M))^*, \tag{9.13}$$

and therefore

$$b^{p,q}(M) = b^{n-p,n-q}(M) \tag{9.14}$$

Proof. One verifies that

$$\overline{*} \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}} \overline{*}, \tag{9.15}$$

so the mapping $\overline{*}$ preserves the space of harmonic forms, and is invertible. The result then follows from Hodge theory. □

This same argument works with form taking values in a holomorphic bundle, and the conclusion of Serre duality is that

$$H^p(M, E) \cong (H^{n-p}(M, K \otimes E^*), \tag{9.16}$$

where $K = \Lambda^{n,0}$ is the canonical bundle. Note that

$$H^p(M, \Omega^q(E)) \cong \mathbb{H}^{q,p}(M, E). \tag{9.17}$$

Proposition 9.1. *If (M, J) is a compact complex manifold then*

$$b^k(M) \leq \sum_{p+q=k} b^{p,q}(M), \tag{9.18}$$

and

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b^k(M) = \sum_{p,q=0}^{m} (-1)^{p+q} b^{p,q}(M). \tag{9.19}$$

Proof. This requires some machinery; it follows from the Frölicher spectral sequence [?]. □

9.2 Hodge numbers of a Kähler manifold

Now let us assume that (M, J, g) is Kähler. That is, the fundamental 2-form ω is closed. Consider the 3 Laplacians

$$\Delta_H = d^*d + dd^*, \tag{9.20}$$
$$\Delta_\partial = \partial^*\partial + \partial\partial^* \tag{9.21}$$
$$\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*, \tag{9.22}$$

where $\cdot^*$ denotes the L^2-adjoint. The key is the following

Proposition 9.2. *If (M, J, g) is Kähler, then*

$$\Delta_H = 2\Delta_\partial = 2\Delta_{\overline{\partial}}. \tag{9.23}$$

Proof. Let L denote the mapping

$$L : \Lambda^p \to \Lambda^{p+2} \tag{9.24}$$

given by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form. Then we have the identities

$$[\overline{\partial}^*, L] = i\partial \tag{9.25}$$
$$[\partial^*, L] = -i\overline{\partial}. \tag{9.26}$$

These are proved proved first in $\mathbb{C}^n$ and then on a Kähler manifold using Kähler normal coordinates. The proposition then follows from these identities (proof omitted). □

Proposition 9.3. *If (M, J, g) is a compact Kähler manifold, then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M), \tag{9.27}$$

and

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{q,p}_{\overline{\partial}}(M)^*. \tag{9.28}$$

Consequently,

$$b^k(M) = \sum_{p+q=k} b^{p,q}(M) \tag{9.29}$$

$$b^{p,q}(M) = b^{q,p}(M). \tag{9.30}$$

Proof. This follows because if a harmonic k-form is decomposed as

$$\phi = \phi^{p,0} + \phi^{p-1,1} + \cdots + \phi^{1,p-1} + \phi^{0,p}, \tag{9.31}$$

then

$$0 = \Delta_H \phi = 2\Delta_{\overline{\partial}}\phi^{p,0} + 2\Delta_{\overline{\partial}}\phi^{p-1,1} + \cdots + 2\Delta_{\overline{\partial}}\phi^{1,p-1} + 2\Delta_{\overline{\partial}}\phi^{0,p}, \tag{9.32}$$

therefore

$$\Delta_{\overline{\partial}}\phi^{p-k,k} = 0, \tag{9.33}$$

for $k = 0 \dots p$.

Next,

$$\overline{\Delta_{\overline{\partial}}\phi} = \Delta_{\partial}\overline{\phi}, \tag{9.34}$$

so conjugation sends harmonic forms to harmonic forms. □

This yields a topologicial obstruction for a complex manifold to admit a Kähler metric:

Corollary 9.2. *If (M, J, g) is a compact Kähler manifold, then the odd Betti numbers of M are even.*

Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$

$$(z_1, z_2) \to 2^k(z_1, z_2). \tag{9.35}$$

This is a free and properly discontinuous action, so the quotient $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ is a manifold, which is called a primary Hopf surface. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$, which has $b^1 = 1$, therefore it does not admit any Kähler metric.

9.3 The Hodge diamond

The following picture is called the Hodge diamond:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & & \vdots & & & & \\
h^{n,0} & & \cdots & & \vdots & & \cdots & & h^{0,n}. \\
 & & & & \vdots & & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}
\tag{9.36}
$$

Reflection about the center vertical is conjugation. Reflection about the center horizontal is Hodge star. The composition of these two operations, or rotation by π, is Serre duality.

For a surface, the Hodge diamond is

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
. \tag{9.37}
$$

10 Lecture 10

10.1 Complex projective space

Complex projective spaces is defined to be the space of lines through the origin in $\mathbb{C}^{n+1}$. This is equivalent to $\mathbb{C}^{n+1}/\sim$, where $\sim$ is the equivalence relation

$$(z^0, \dots, z^n) \sim (w^0, \dots, w^n) \tag{10.1}$$

if there exists $\lambda \in \mathbb{C}^*$ so that $z^j = \lambda w^j$ for $j = 1 \dots n$. The equivalence class of $(z^0, \dots, z^n)$ will be denoted by $[z^0 : \dots : z^n]$. Letting $U_j = \{[z^0 : \dots : z^n] | z^j \neq 0\}$, $\mathbb{CP}^n$ is covered by $(n+1)$ coordinate charts $\phi_j : U_j \to \mathbb{C}^n$ defined by

$$\phi_j : [z^0 : \dots : z^n] \mapsto \Big(\frac{z^0}{z_j}, \dots, \frac{z^{j-1}}{z_j}, \frac{z^{j+1}}{z_j}, \dots, \frac{z^n}{z_j}\Big), \tag{10.2}$$

with inverse given by

$$\phi_j^{-1} : (w^1, \dots, w^n) \mapsto [w^1 : \dots w^{j-1} : 1 : w^j : \dots : w^n]. \tag{10.3}$$

The overlap maps are holomorphic, which gives $\mathbb{CP}^n$ the structure of a complex manifold.

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\}$ denote the projection map, and consider the form

$$\tilde{\Phi} = -4i\partial\overline{\partial}\log\Big(\sum_{j=0}^{n}|z^j|^2\Big) \tag{10.4}$$

It is easy to see that this form is the pull-back of a $(1,1)$-form on $\mathbb{CP}^n$, that is $\tilde{\Phi} = \pi^*\Phi$. Furthermore, Φ is positive-definite, which implies that Φ is the fundamental 2-form of a Hermitian metric g_{FS}. Furthermore, since $d = \partial + \overline{\partial}$, it follows that $d\Phi = 0$, so g_{FS} is a Kähler metric. Note that $\tilde{\Phi}$ is invariant under the action of $\mathrm{U}(n+1)$, which implies that the isometry group of g_{FS} contains $\mathrm{PU}(n+1)$, the projective Unitary group. Moreover, these isometries are holomorphic. The full isometry group has 2 components; the non-identity component consists of anti-holomoprhic isometries (the $\mathrm{U}(n+1)$-action composed with conjugation of the coordinates).

Remark 10.1. This metric seems to just come from nowhere, but we will see in a bit that is a very natural definition (but we need to discuss line bundles first to understand this). Also, the normalization in (10.4) is to arrange that the holomorphic sectional curvature of g_{FS} is equal to 1, we will discuss this later.

The only non-trivial integral cohomology of $\mathbb{CP}^n$ is in even degrees

$$H^{2j}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z} \tag{10.5}$$

for $j = 1 \dots n$. Using Proposition 9.3, it follows that the Hodge numbers are given by

$$b^{p,q}(\mathbb{CP}^n) = \begin{cases} 1 & p = q \\ 0 & p \neq q. \end{cases} \tag{10.6}$$

For example, the Hodge diamond of $\mathbb{CP}^1$ is given by

$$\begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix} \quad , \tag{10.7}$$

and the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{10.8}$$

10.2 Line bundles and divisors

A line bundle over a complex manifold M is a rank 1 complex vector bundle $\pi : E \to M$. The transition functions are defined as follows. A trivialization is a mapping

$$\Phi_\alpha : U_\alpha \times \mathbb{C} \to E \tag{10.9}$$

which maps $x \times \mathbb{C}$ linearly onto a fiber. The transition functions are

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*, \tag{10.10}$$

defined by

$$\varphi_{\alpha\beta}(x) = \frac{1}{v}\pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{10.11}$$

for $v \neq 0$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{10.12}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satifsying (10.12), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$, and this bundle is uniquely defined up to bundle equivalence, which we will define below. If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle, while if they are holomorphic, we say that E is a holomorphic vector bundle. Note that total space of a holomorphic vector bundle over a complex manifold is a complex manifold.

A vector bundle mapping is a mapping $f : E_1 \to E_2$ which is linear on fibers, and covers the identity map. Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions $f_\alpha : U_\alpha \to \mathbb{C}$ defined by

$$f_\alpha(x) = \frac{1}{v}\pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)) \tag{10.13}$$

for $v \neq 0$. It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi_{\alpha\beta}^{E_2} f_\beta \varphi_{\beta\alpha}^{E_1}, \tag{10.14}$$

equivalently,

$$\varphi_{\beta\alpha}^{E_2} f_\alpha = f_\beta \varphi_{\beta\alpha}^{E_1}. \tag{10.15}$$

We say that two bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $f : E_1 \to E_2$. This is equivalent to non-vanishing of the local representatives, that is, $f_\alpha : U_\alpha \to \mathbb{C}^*$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$\phi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{10.16}$$

The tensor product $E_1 \otimes E_2$ of two line bundles E_1 and E_2 is again a line bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E_1 \otimes E_2} = \varphi_{\alpha\beta}^{E_1} \varphi_{\alpha\beta}^{E_2}. \tag{10.17}$$

The dual E^* of a line bundle E, is again a line bundle, and has transition functions

$$\varphi^{E^*}_{\alpha\beta} = (\varphi^E_{\beta\alpha})^{-1}. \tag{10.18}$$

Note that for any line bundle,

$$E \otimes E^* \cong \mathbb{C}, \tag{10.19}$$

is the trivial line bundle.

For our purpose, a divisor is defined to be the zero set of a holomorphic section of a nontrivial line bundle. Conversely, an irreducible holomorphic subvariety of codimension 1 defines a line bundle by taking local defining functions to be the transition functions, that is,

$$\varphi_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \tag{10.20}$$

10.3 Line bundles on complex projective space

If M is any smooth manifold, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^* \to 1. \tag{10.21}$$

where $\mathcal{E}$ is the sheaf of germs of C^∞ functions, and $\mathcal{E}^*$ is the sheaf of germs of non-vanishing C^∞ functions. The associated long exact sequence in cohomology is

$$\begin{aligned} \ldots \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{E}) \to H^1(M,\mathcal{E}^*) \\ \to H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{E}) \to H^2(M,\mathcal{E}^*) \to \ldots. \end{aligned} \tag{10.22}$$

But $\mathcal{E}$ is a flabby sheaf due to existence of partitions of unity in the smooth category, so $H^k(M,\mathcal{E}) = \{0\}$ for $k \geq 1$. This implies that

$$H^1(M,\mathcal{E}^*) \cong H^2(M,\mathbb{Z}). \tag{10.23}$$

Using Čech cohomology, the left hand side is easily seen to be the set of smooth line bundles on M up to equivalence.

Next, if M is a complex manifold, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1. \tag{10.24}$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions, and $\mathcal{O}^*$ is the sheaf of germs of non-vanishing holomorphic functions. The associated long exact sequence in cohomology is

$$\begin{aligned} \ldots \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \\ \xrightarrow{c_1} H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{O}) \to H^2(M,\mathcal{O}^*) \to \ldots. \end{aligned} \tag{10.25}$$

Now $\mathcal{O}$ is not flabby (there are no nontrivial holomorphic partitions of unity!). However

$$\dim(H^k(M, \mathcal{O})) = b^{0,k}. \tag{10.26}$$

Since $b^{0,1} = b^{0,2} = 0$ for $\mathbb{CP}^n$, we have

$$H^1(\mathbb{CP}^n, \mathcal{O}^*) \cong H^2(M, \mathbb{Z}) \cong \mathbb{Z}. \tag{10.27}$$

Again, using Čech cohomology, the left hand side is easily seen to be the set of holomorphic line bundles on M up to equivalence. Consequently, on $\mathbb{CP}^n$ the smooth line bundles are the same as holomorphic line bundles up to equivalence:

Corollary 10.1. *The set of holomorphic line bundles on $\mathbb{CP}^n$ up to equivalence is isomorphic to $\mathbb{Z}$, with the tensor product corresponding to addition.*

The line bundles on $\mathbb{CP}^n$ are denoted by $\mathcal{O}(k)$, where k is the integer obtained under the above isomorphism, which is the first Chern class. Of course, every line bundle must be a tensor power of a generator. If $H \subset \mathbb{CP}^n$ is a hyperplane, then the line bundle corresponding to H, denoted by $[H]$ is $\mathcal{O}(1)$. The dual of this bundle, $\mathcal{O}(-1)$ has a nice description, it is called the tautological bundle. This is

$$\mathcal{O}(-1) = \{([x], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v \in [x]\}. \tag{10.28}$$

To see that $[H]$ corresponds to $\mathcal{O}(1)$, use the following:

Proposition 10.1 ([GH78, page 141])**.** *The first Chern class of a complex line bundle L is equal to the Euler class of the underlying oriented real rank 2 bundle, and is the Poincaré dual to the zero locus of a transverse section. Furthermore, if g is a Hermitian metric on L, then the curvature form of the Chern connection on L is given by*

$$\Theta = 2\pi i \partial\overline{\partial}|\sigma|^2, \tag{10.29}$$

where σ is is any locally defined holomorphic section. Finally,

$$c_1(L) = \Big[\frac{i}{2\pi}\Theta\Big]. \tag{10.30}$$

Returning to the Fubini-Study metric: note the $\mathcal{O}(-1)$ admits a Hermitian metric h by restricting the inner product in $\mathbb{C}^{n+1}$ to a fiber. Thus we see that

Proposition 10.2. *The Kähler form of the Fubini-Study metric is $(-i/2\pi)$ times the curvature form of h.*

10.4 Adjunction formula

Let $V \subset M^n$ be a smooth complex hypersurface. The exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}M\big|_V \to N_V \to 0, \tag{10.31}$$

defines the holomorphic normal bundle. The adjunction formula says that

$$N_V = [V]|_V. \tag{10.32}$$

To see this, let f_α be local defining functions for V, so that the transition functions of $[V]$ are $g_{\alpha\beta} = f_\alpha f_\beta^{-1}$. Apply d to the equation

$$f_\alpha = g_{\alpha\beta} f_\beta \tag{10.33}$$

to get

$$df_\alpha = d(g_{\alpha\beta}) f_\beta + g_{\alpha\beta} df_\beta. \tag{10.34}$$

Restricting to V, since $f_\beta = 0$ defines V, we have

$$df_\alpha = g_{\alpha\beta} df_\beta. \tag{10.35}$$

Note that df_α is a section of N_V^*. For a smooth hypersurface, the differential of a local defining function is nonzero on normal vectors. Consequently, $N_V^* \otimes [V]$ is the trivial bundle when restricted to V since it has a non-vanishing section.

For any short exact sequence

$$0 \to A \to B \to C \to 0, \tag{10.36}$$

it holds that

$$\Lambda^{\dim(B)}(B) \cong \Lambda^{\dim(A)}(A) \otimes \Lambda^{\dim(C)}(C), \tag{10.37}$$

so the adjunction formula can be rephrased as

$$K_V = (K_M \otimes [V])|_V. \tag{10.38}$$

11 Lecture 11

11.1 Characteristic numbers of hypersurfaces

Let $V \subset \mathbb{P}^n$ be a smooth complex hypersurface. We know that the line bundle $[V] = \mathcal{O}(d)$ for some $d \geq 1$. We have the exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}\mathbb{P}^n\big|_V \to N_V \to 0. \tag{11.1}$$

The adjunction formula says that

$$N_V = \mathcal{O}(d)\big|_V. \tag{11.2}$$

We have the smooth splitting of (11.1),

$$T^{(1,0)}\mathbb{P}^n\big|_V = T^{(1,0)}(V) \oplus \mathcal{O}(d)\big|_V. \tag{11.3}$$

Taking Chern classes,

$$c(T^{(1,0)}\mathbb{P}^n\big|_V) = c(T^{(1,0)}(V)) \cdot c(\mathcal{O}(d)\big|_V). \tag{11.4}$$

From the Euler sequence [GH78, page 409],

$$0 \to \mathbb{C} \to \mathcal{O}(1)^{\oplus(n+1)} \to T^{(1,0)}\mathbb{P}^n \to 0, \tag{11.5}$$

it follows that

$$c(T^{(1,0)}\mathbb{P}^n) = (1 + c_1(\mathcal{O}(1)))^{n+1}. \tag{11.6}$$

Note that for any divisor D,

$$c_1([D]) = \eta_D, \tag{11.7}$$

where η_D is the Poincaré dual to D. That is

$$\int_D \xi = \int_{\mathbb{P}^n} \xi \wedge \eta_D, \tag{11.8}$$

for all $\xi \in H^{2n-2}(\mathbb{P})$, see [GH78, page 141]. So in particular $c_1(\mathcal{O}(1)) = \omega$, where ω is the Poincaré dual of a hyperplane in $\mathbb{P}^n$ (note that ω is integral, and is some multiple of the Fubini-Study metric). Therefore

$$c(T^{(1,0)}\mathbb{P}^n) = (1 + \omega)^{n+1}. \tag{11.9}$$

Also $c_1(\mathcal{O}(d)) = d \cdot \omega$, since $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. The formula (11.4) is then

$$(1 + \omega)^{n+1}\big|_V = (1 + c_1 + c_2 + \dots)(1 + d \cdot \omega\big|_V). \tag{11.10}$$

A crucial tool in the following is the Lefschetz hyperplane theorem:

Theorem 11.1 ([GH78, page 156])**.** *Let $M \subset \mathbb{P}^n$ be a hypersurface of dimension $n-1$, and H be a hyperplane, and let $V = M \cap H$. Then the inclusion $\iota : V \to M$ induces a mapping*

$$\iota^* : H^q(M, \mathbb{Q}) \to H^q(V, \mathbb{Q}) \tag{11.11}$$

which is an isomorphism for $q \leq n-3$ and injective for $q = n-2$.

Furthermore, the mapping

$$\iota_* : \pi_q(V) \to \pi_q(M) \tag{11.12}$$

is an isomorphism for $q \leq n-3$ and surjective for $q = n-2$.

11.2 Dimension $n = 2$

Consider a curve in $\mathbb{P}^2$. The formula (11.10) is

$$(1 + 3\omega)\big|_V = (1 + c_1)(1 + d \cdot \omega\big|_V), \tag{11.13}$$

which yields

$$c_1 = (3 - d)\omega\big|_V. \tag{11.14}$$

The top Chern class is the Euler class, so we have

$$\chi(V) = \int_V (3 - d)\omega \tag{11.15}$$

$$= (3 - d)\int_{\mathbb{P}^2} \omega \wedge (d \cdot \omega) \tag{11.16}$$

$$= d(3 - d)\int_{\mathbb{P}^2} \omega^2 = d(3 - d). \tag{11.17}$$

Here we used the fact that $d \cdot \omega$ is Poincaré dual to V, and ω^2 is a positive generator of $H^4(\mathbb{P}^2, \mathbb{Z})$. Equivalently, we can write

$$\int_V \omega = \int_{\mathbb{P}^2} \omega \wedge \eta_V = \int_{\mathbb{P}^2} \eta_H \wedge \eta_V. \tag{11.18}$$

Since cup product is dual to intersection under Poincaré duality, the integral simply counts the number of intersection points of V with a generic hyperplane.

In term of the genus g,

$$g = \frac{(d - 1)(d - 2)}{2}. \tag{11.19}$$

11.3 Dimension $n = 3$

We consider a hypersurface in $\mathbb{P}^3$, which is topologically a 4-manifold. The formula (11.10) is

$$(1 + 4\omega + 6\omega^2)\big|_V = (1 + c_1 + c_2) \cdot (1 + d \cdot \omega\big|_V), \tag{11.20}$$

so that

$$c_1 = (4 - d)\omega\big|_V, \tag{11.21}$$

and then

$$c_2 = (6 - d(4 - d))\omega^2\big|_V. \tag{11.22}$$

Since the top Chern class is the Euler class,

$$\chi(V) = \int_V (6 - d(4-d))\omega^2 \tag{11.23}$$

$$= (6 - d(4-d)) \int_{\mathbb{P}^3} \omega^2 \wedge (d \cdot \omega) \tag{11.24}$$

$$= d(6 - d(4-d)), \tag{11.25}$$

again using the fact that $d \cdot \omega$ is Poincaré dual to V, and that ω^3 is a positive generator of $H^6(\mathbb{P}^3, \mathbb{Z})$.

It follows from the Lefschetz hyperplane theorem that M has $b_1 = 0$, therefore $b^{1,0} = b^{0,1} = 0$.

11.4 Hirzebruch Signature Theorem

We think of V as a real 4-manifold, with complex structure given by J. Then the kth Pontrjagin Class is defined to be

$$p_k(V) = (-1)^k c_{2k}(TV \otimes \mathbb{C}) \tag{11.26}$$

Since (V, J) is complex, we have that

$$TV \otimes \mathbb{C} = TV \oplus \overline{TV}, \tag{11.27}$$

so

$$c(TV \otimes \mathbb{C}) = c(TV) \cdot c(\overline{TV}) \tag{11.28}$$

$$= (1 + c_1 + c_2) \cdot (1 - c_1 + c_2) \tag{11.29}$$

$$= 1 + 2c_2 - c_1^2, \tag{11.30}$$

which yields

$$p_1(V) = c_1^2 - 2c_2. \tag{11.31}$$

Consider next the intersection pairing $H^2(V) \times H^2(V) \to \mathbb{R}$, given by

$$(\alpha, \beta) \to \int \alpha \wedge \beta \in \mathbb{R}. \tag{11.32}$$

Let b_2^+ denote the number of positive eigenvalues, and b_2^- denote the number of negative eigenvalues. By Poincaré duality the intersection pairing is non-degenerate, so

$$b_2 = b_2^+ + b_2^-. \tag{11.33}$$

The *signature* of V is defined to be

$$\tau = b_2^+ - b_2^-. \tag{11.34}$$

The Hirzebruch Signature Theorem [MS74, page 224] states that

$$\tau = \frac{1}{3}\int_V p_1(V) \tag{11.35}$$

$$= \frac{1}{3}\int_V (c_1^2 - 2c_2). \tag{11.36}$$

Rewriting this,

$$2\chi + 3\tau = \int_V c_1^2. \tag{11.37}$$

Remark 11.1. This implies that S^4 does not admit any almost complex structure, since the left hand side is 4, but the right hand side trivially vanishes.

11.5 Representations of $U(2)$

As discussed above, some representations which are irreducible for $SO(4)$ become reducible when restricted to $U(2)$. Under $SO(4)$, we have

$$\Lambda^2 T^* = \Lambda^2_+ \oplus \Lambda^2_-, \tag{11.38}$$

where

$$\Lambda_2^+ = \{\alpha \in \Lambda^2(M, \mathbb{R}) : *\alpha = \alpha\} \tag{11.39}$$

$$\Lambda_2^- = \{\alpha \in \Lambda^2(M, \mathbb{R}) : *\alpha = -\alpha\}. \tag{11.40}$$

But under U(2), we have the decomposition

$$\Lambda^2 T^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}. \tag{11.41}$$

Notice that these are the complexifications of real vector spaces. The first is of dimension 2, the second is of dimension 4. Let ω denote the 2-form $\omega(X, Y) = g(JX, Y)$. This yields the orthogonal decomposition

$$\Lambda^2 T^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R} \cdot \omega \oplus \Lambda_0^{1,1}, \tag{11.42}$$

where $\Lambda_0^{1,1} \subset \Lambda^{1,1}$ is the orthogonal complement of the span of ω, and is therefore 2-dimensional (the complexification of which is the space of *primitive* $(1,1)$-forms).

Proposition 11.1. *Under* U(2)*, we have the decomposition*

$$\Lambda^2_+ = \mathbb{R} \cdot \omega \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \tag{11.43}$$

$$\Lambda^2_- = \Lambda_0^{1,1}. \tag{11.44}$$

Proof. We can choose an oriented orthonormal basis of the form

$$\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}. \tag{11.45}$$

Let $\{e^1, e^2, e^3, e^4\}$ denote the dual basis. The space of $(1,0)$ forms, $\Lambda^{1,0}$ has generators

$$\theta^1 = e^1 + ie^2, \quad \theta^2 = e^3 + ie^4. \tag{11.46}$$

We have

$$\begin{aligned}\omega &= \frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 + \theta^2 \wedge \overline{\theta}^2)\\ &= \frac{i}{2}\Big((e^1 + ie^2) \wedge (e^1 - ie^2) + (e^3 + ie^4) \wedge (e^3 - ie^4)\Big)\\ &= e^1 \wedge e^2 + e^3 \wedge e^4 = \omega^1_+.\end{aligned} \tag{11.47}$$

Similarly, we have

$$\frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 - \theta^2 \wedge \overline{\theta}^2) = e^1 \wedge e^2 - e^3 \wedge e^4 = \omega^1_-, \tag{11.48}$$

so ω^1_- is of type $(1,1)$, so lies in $\Lambda^{1,1}_0$. Next,

$$\begin{aligned}\theta^1 \wedge \theta^2 &= (e^1 + ie^2) \wedge (e^3 + ie^4)\\ &= (e^1 \wedge e^3 - e^2 \wedge e^4) + i(e^1 \wedge e^4 + e^2 \wedge e^3)\\ &= \omega^2_+ + i\omega^3_+.\end{aligned} \tag{11.49}$$

Solving, we obtain

$$\omega^2_+ = \frac{1}{2}(\theta^1 \wedge \theta^2 + \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{11.50}$$

$$\omega^3_+ = \frac{1}{2i}(\theta^1 \wedge \theta^2 - \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{11.51}$$

which shows that ω^2_+ and ω^3_+ are in the space $\Lambda^{2,0} \oplus \Lambda^{0,2}$. Finally,

$$\begin{aligned}\theta^1 \wedge \overline{\theta}^2 &= (e^1 + ie^2) \wedge (e^3 - ie^4)\\ &= (e^1 \wedge e^3 + e^2 \wedge e^4) + i(-e^1 \wedge e^4 + e^2 \wedge e^3)\\ &= \omega^2_- - i\omega^3_-,\end{aligned} \tag{11.52}$$

which shows that ω^2_- and ω^3_- are in the space $\Lambda^{1,1}_0$. □

This decomposition also follows from the proof of the Hodge-Riemann bilinear relations [GH78, page 123].

Corollary 11.1. *If (M^4, g) is Kähler, then*

$$b_2^+ = 1 + 2b^{2,0}, \tag{11.53}$$

$$b_2^- = b^{1,1} - 1, \tag{11.54}$$

$$\tau = b_2^+ - b_2^- = 2 + 2b^{2,0} - b^{1,1}. \tag{11.55}$$

Proof. This follows from Proposition 11.1, and Hodge theory on Kähler manifolds, see [GH78]. □

So we have that

$$\chi = 2 + b_2 = 2 + b^{1,1} + 2b^{2,0} \tag{11.56}$$
$$\tau = 2 + 2b^{2,0} - b^{1,1}. \tag{11.57}$$

Remark 11.2. Notice that

$$\chi + \tau = 4(1 + b^{2,0}), \tag{11.58}$$

so in particular, the integer $\chi + \tau$ is divisible by 4 on a Kähler manifold with $b_1 = 0$. This is in fact true for *any* almost complex manifold of real dimension 4, this follows from a version of Riemann-Roch Theorem which holds for almost complex manifolds, see [Gil95, Lemma 3.5.3]. This implies that there is no almost complex structure on $\overline{\mathbb{P}}^2$, that is, there is no almost complex structure on $\mathbb{P}^2$ which induces the reversed orientation to that induced by the usual complex structure on $\mathbb{P}^2$.

Applying these formulas to our example, we find that

$$2\chi + 3\tau = (4-d)^2 \int_V \omega^2 = d(4-d)^2. \tag{11.59}$$

Using the formula for the Euler characteristic from above,

$$\chi = d(6 - d(4-d)), \tag{11.60}$$

we find that

$$\tau = -\frac{1}{3} d(d+2)(d-2). \tag{11.61}$$

Some arithmetic shows that

$$b_2 = d^3 - 4d^2 + 6d - 2 \tag{11.62}$$
$$b_2^+ = \frac{1}{3}(d^3 - 6d^2 + 11d - 3) \tag{11.63}$$
$$b_2^- = \frac{1}{3}(d-1)(2d^2 - 4d + 3) \tag{11.64}$$
$$b^{2,0} = b^{0,2} = \frac{1}{6}(d-3)(d-2)(d-1) \tag{11.65}$$
$$b^{1,1} = \frac{1}{3} d(2d^2 - 6d + 7) \tag{11.66}$$
$$b^{1,0} = b^{0,1} = 0. \tag{11.67}$$

For $d = 2$, we find that $b_2^+ = 1$, $b_2^- = 1$. This is not surprising, as any non-degenerate quadric is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ [GH78, page 478]. The Hodge numbers are $b^{1,1} = 2$, $b^{0,2} = b^{2,0} = 0$, so the Hodge diamond is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{11.68}$$

For $d = 3$, we find $b_2^+ = 1$, $b_2^- = 6$. This is expected, since any non-degenerate cubic is biholomorphic to $\mathbb{P}^2$ blown up at 6 points, and is therefore diffeomorphic to $\mathbb{P}^2\#6\overline{\mathbb{P}^2}$ [GH78, page 489]. The Hodge numbers in this case are $b^{1,1} = 7$, $b^{0,2} = b^{2,0} = 0$, so the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{11.69}$$

For $d = 4$, this is a $K3$ surface [GH78, page 590]. We find $b_2^+ = 3$, $b_2^- = 19$, so $\chi = 24$, and $\tau = -16$. The intersection form is given by

$$2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11.70}$$

Since $c_1 = 0$, the canonical bundle is trivial. The Hodge numbers in this case are $b^{1,1} = 20$, $b^{0,2} = b^{2,0} = 1$, so the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{11.71}$$

For $d = 5$, we find $b_2^+ = 9$, $b_2^- = 44$, so $\chi = 55$, and $\tau = -35$. From Freedman's topological classification of simply-connected 4-manifolds, V must be homeomorphic to $9\mathbb{P}^2\#44\overline{\mathbb{P}^2}$, see [FQ90]. By the work of Gromov-Lawson [GL80], this latter smooth manifold admits a metric of positive scalar curvature, and therefore all of its Seiberg-Witten invariants vanish [Wit94]. But V is Kähler, so it has some non-zero Seiberg-Witten invariant [Mor96, Theorem 7.4.4]. We conclude that V is homeomorphic to $9\mathbb{P}^2\#44\overline{\mathbb{P}^2}$, but not diffeomorphic.

12 Lecture 12

12.1 Complete Intersections

Let $V^k \subset \mathbb{P}^n$ be a smooth complete intersection of $n - k$ homogeneous polynomials of degree $d_1, \ldots, d_{n-k}$. Consider again the exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}\mathbb{P}^n\big|_V \to N_V \to 0, \tag{12.1}$$

where N_V is now a bundle of rank $n - k$ bundle. The adjunction formula says that

$$N_V = \mathcal{O}(d_1)\big|_V \oplus \cdots \oplus \mathcal{O}(d_{n-k})\big|_V. \tag{12.2}$$

We have the smooth splitting of (11.1),

$$T^{(1,0)}\mathbb{P}^n\big|_V = T^{(1,0)}(V) \oplus N_V. \tag{12.3}$$

Taking Chern classes,

$$c(T^{(1,0)}\mathbb{P}^n\big|_V) = c(T^{(1,0)}(V)) \cdot c\left(\mathcal{O}(d_1)\big|_V \oplus \cdots \oplus \mathcal{O}(d_{n-k})\big|_V\right), \tag{12.4}$$

which is

$$(1+\omega)^{n+1}\big|_V = (1 + c_1 + \cdots + c_k)(1 + d_1 \cdot \omega\big|_V) \cdots (1 + d_{n-k} \cdot \omega\big|_V). \tag{12.5}$$

12.2 Calabi-Yau complete intersections

Note that if

$$n + 1 = d_1 + \cdots + d_{n-k} \tag{12.6}$$

then V has vanishing first Chern class, and therefore carries a Ricci-flat metric by Yau's theorem. If any of the degrees d_j is equal to 1, then this reduces to a complete intersection in a lower dimensional projective space. So without loss of generality, assume that $d_j \geq 2$. Then (12.6) implies the inequality

$$n \leq 2k + 1. \tag{12.7}$$

A Calabi-Yau manifold admits a non-zero holomorphic $(n, 0)$-form, which is denoted by Ω. This form yield an isomorphism of bundles

$$\Theta \cong \Lambda^{n-1,0} \tag{12.8}$$

by the mapping $X \mapsto \iota_X \Omega$, where ι is interior multiplication. Consequently, the lowest cohomologies of the holomorphic tangent sheaf are given by

$$H^0(V, \Theta) = H^0(V, \Lambda^{n-1}) \tag{12.9}$$
$$H^1(V, \Theta) = H^1(V, \Lambda^{n-1}) \tag{12.10}$$
$$H^2(V, \Theta) = H^2(V, \Lambda^{n-1}), \tag{12.11}$$

so that

$$\dim(H^0(V, \Theta)) = h^{n-1,0} \tag{12.12}$$
$$\dim(H^1(V, \Theta)) = h^{n-1,1} \tag{12.13}$$
$$\dim(H^2(V, \Theta)) = h^{n-2,2}. \tag{12.14}$$

Consider the case of Calabi-Yau surfaces, $k = 2$, so that (12.7) implies that $n \leq 5$. The possibilites are in Table 1. The computation of the Euler characteristic is straightforward from the above formulas, using that the integral of the top Chern class is the Euler class. Notice that all of these have the same Euler characteristic. This is not an accident, it turns out that all of these are in fact diffeomorphic [?]. By

Degrees	$\subset \mathbb{P}^n$	$\chi(V)$	$\dim(H^1(V,\Theta)) = h^{1,1}$
(4)	$\mathbb{P}^3$	24	20
$(2,3)$	$\mathbb{P}^4$	24	20
$(2,2,2)$	$\mathbb{P}^5$	24	20

Table 1: Complete intersection Calabi-Yau surfaces.

the Tian-Todorov theorem, the moduli space of complex structures is of dimension 20. A computation shows that the dimension of the space of quartics in $\mathbb{P}^3$ modulo the action of the automorphism group of $\mathbb{P}^3$, which is $\mathrm{PGL}(4,\mathbb{C})$, is equal to 19. Therefore "most" $K3$ surfaces are not algebraic.

Consider the case of Calabi-Yau threefolds, $k = 3$, so that (12.7) implies that the number $n \leq 7$. The possibilites are in Table 2. This shows that, in contrast to surfaces, Calabi-Yau threefolds are not necesssarily diffeomorphic, and their Hodge numbers are not always the same. In fact, this leads to the big subject of mirror symmetry, which we will not discuss.

Degrees	$\subset \mathbb{P}^n$	$\chi(V)$	$\dim(H^1(V,\Theta)) = h^{2,1}$	$h^{1,1}$
(5)	$\mathbb{P}^4$	-200	101	1
$(4,2)$	$\mathbb{P}^5$	-176	89	1
$(3,3)$	$\mathbb{P}^5$	-144	73	1
$(3,2,2)$	$\mathbb{P}^6$	-144	73	1
$(2,2,2,2)$	$\mathbb{P}^7$	-128	65	1

Table 2: Complete intersection Calabi-Yau threefolds.

Next, we make some remarks on how to compute the numbers appearing in Table 2. Again, the computation of the Euler characteristic is straightforward from the above formulas, using that the integral of the top Chern class is the Euler class.Next, note that

$$H^{2,0}(V) \cong H^{3,1}(V) \cong H^1(V,\Omega^3) \cong H^1(V,\mathcal{O}) \cong H^{0,1}(V). \tag{12.15}$$

Consequently, the Hodge diamond of a simply-connected Calabi-Yau threefold is given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}, \tag{12.16}$$

and one has

$$\chi(V) = 2(h^{1,1} - h^{2,1}). \tag{12.17}$$

Note that $h^{1,1}(V) = 1$ for the above examples by the Lefschetz hyperplane theorem. However, there are many Calabi-Yau threefolds which have $h^{1,1}(V) > 1$ [**?**].

The space of quintics in $\mathbb{P}^4$ modulo the automorphism group $\mathrm{PGL}(5, \mathbb{C})$ has dimension 101, and therefore all deformations of the quintic are still quintics, in contrast to what happens in the $K3$ case. It turns out that all Calabi-Yau manifolds in dimensions 3 and greater are algebraic [**?**]. To see this, we use Bochner's vanishing theorem:

Theorem 12.1 (Bochner)**.** *If (M, g, J) is Kähler and has non-negative Ricci tensor, and not identically zero, then there are no nontrivial holomorphic $(p, 0)$-forms. Furthermore, if $Ric \equiv 0$, then holomorphic $(p, 0)$-forms are parallel.*

If (M, g, J) is Kähler and has non-positive Ricci tensor, and not identically zero, then there are no non-trivial holomorphic vector fields. Furthermore, if $Ric \equiv 0$, then any holomorphic vector field is parallel.

Proof. One just goes through the usual Weitzenbock argument on p-forms, and show that for $(p, 0)$-forms, the curvature term is given by the Ricci tensor (but need to check the sign of this term). Note that if a $(p, 0)$-form is harmonic for the Hodge Laplacian, then it is harmonic for the $\overline{\partial}$-Laplacian, and thus $\overline{\partial}$-closed and $\overline{\partial}$-co-closed, but a $(p, 0)$-form is automatically $\overline{\partial}$-co-closed, so the harmonic $(p, 0)$-forms are exactly the holomorphic $(p, 0)$-forms.

The statment on holomorphic vector fields is the dual to the statement on holomorphic $(1, 0)$-forms, and the sign of the curvature term in the Weitzenbock formula is opposite. If time, we will go through the details later. □

So assume we have a Calabi-Yau manifold (V^n, J, g) with holonomy exactly $\mathrm{SU}(n)$. This implies that the canonical bundle is flat, and since the curvature form of the canonical bundle is a multiple of the Ricci form, the metric g must be Ricci-flat. Then Bochner's Theorem implies that all harmonic $(p, 0)$-forms are parallel. We already know that the canonical bundle admit a parallel section. For $0 < p < n$, existence of such a parallel form would imply reduction of the holonomy group to a proper subgroup of $\mathrm{SU}(n)$. So if $n \geq 3$, we have that $h^{2,0} = 0$. By the Kähler identities, we also have that $h^{0,2} = 0$, and therefore

$$H^2(V, \mathbb{C}) = H^{1,1}(V). \tag{12.18}$$

The Kähler cone is therefore an open cone in $H^2(V, \mathbb{C})$, so it must contain an integral class in $H^2(V, \mathbb{Z})$. Consequently, by Kodaira's embedding theorem, V is projective, and by Chow's Theorem, it is algebraic.

Note that a flat torus cross a $K3$ surface is not algebraic, but this does not contradict the above because the holonomy in this case is a proper subgroup of $\mathrm{SU}(3)$. For threefolds, above we proved that is V^3 is simply connected with trivial canonical bundle, then $h^{2,0} = 0$. Finally, note that the second part of Bochner's Theorem implies that Calabi-Yau metrics have discrete automorphism group (in fact, it must be finite).

12.3 Riemann surface complete intersections

Let us now just consider the simple case of a complete intersection of $n-1$ hypersurfaces in $\mathbb{P}^n$, of degrees $d_1, \ldots, d_{n-1}$. We have

$$1 + (n+1)\omega\big|_V = 1 + c_1 + (d_1 + \cdots + d_{n-1})\omega\big|_V, \tag{12.19}$$

which yields

$$c_1 = (n + 1 - d_1 - \cdots - d_{n-1})\omega\big|_V. \tag{12.20}$$

The Euler characteristic is

$$\chi(V) = (n + 1 - d_1 - \cdots - d_{n-1}) \int_V \omega. \tag{12.21}$$

By definition of the Poincaré dual,

$$\int_V \omega = \int_{\mathbb{P}^n} \omega \wedge \eta_V = \int_{\mathbb{P}^n} \eta_H \wedge \eta_V. \tag{12.22}$$

We use some intersection theory to understand the integral. Intersecting cycles is Poincaré dual to the cup product, thus the integral counts the number of intersection points of V with a generic hyperplane. Consequently,

$$\chi(V) = (n + 1 - d_1 - \cdots - d_{n-1}) d_1 d_2 \cdots d_{n-1}. \tag{12.23}$$

The genus g is given by

$$g = 1 - \frac{1}{2}(n + 1 - d_1 - \cdots - d_{n-1}) d_1 d_2 \cdots d_{n-1}. \tag{12.24}$$

Proposition 12.1. *For Riemann surface complete intersections, we have the following:*

- *A curve of genus zero arises as a nontrivial complete intersection only if it is a quadric in $\mathbb{P}^2$.*
- *A curve of genus 1 arises as a complete intersection only if it is a cubic in $\mathbb{P}^2$ or the intersection of two quadrics in $\mathbb{P}^3$.*
- *A curve of genus 2 does not arise as a complete intersection.*

Proof. The first two cases are an easy computation. For the last case, assume by contradiction that it does. If any of the $d_i = 1$, then it is a complete intersection in a lower dimensional projective space. So without loss of generality, assume that $d_i \geq 2$. We would then have

$$2 = -(n + 1 - d_1 - \cdots - d_{n-1}) d_1 d_2 \cdots d_{n-1}. \tag{12.25}$$

The right hand side is a product of integers. Since 2 is prime, the only possibility is that $n = 2$, and $d_1 = 2$, in which case the above equation reads

$$2 = -2, \tag{12.26}$$

which is a contradiction. □

12.4 The twisted cubic

Here is an example of a surface which is not a complete intersection, called the twisted cubic. Consider

$$\phi : \mathbb{P}^1 \to \mathbb{P}^3, \tag{12.27}$$

by

$$\phi([u,v]) = [u^3, u^2 v, uv^2, v^3]. \tag{12.28}$$

Using coordinates $[z_0, z_1, z_2, z_3]$, the image of ϕ lies on the intersection of 3 quadrics,

$$z_0 z_2 = z_1^2 \tag{12.29}$$

$$z_1 z_3 = z_2^2 \tag{12.30}$$

$$z_0 z_3 = z_1 z_2. \tag{12.31}$$

The intersection of any 2 of these equations vanishes on the twisted cubic, but has another zero component, and the third equation then picks out the correct component.

13 Lecture 13

13.1 Riemann-Roch Theorem

Instead of using the Hirzebruch signature Theorem to compute these characteristic numbers, we can use the Riemann-Roch formula for complex manifolds.

Let $\mathcal{E}$ be a complex vector bundle over V of rank k. Assume that $\mathcal{E}$ splits into a sum of line bundles

$$\mathcal{E} = L_1 \oplus \cdots \oplus L_k. \tag{13.1}$$

Let $a_i = c_1(L_i)$. Then

$$c(\mathcal{E}) = (1 + a_1) \cdots (1 + a_k), \tag{13.2}$$

which shows that $c_j(\mathcal{E})$ is given by the elementary symmetric functions of the a_i, that is

$$c_j(\mathcal{E}) = \sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k}. \tag{13.3}$$

Any other symmetric polynomial can always be expressed as a polynomial in the elementary symmetric functions. We define the *Chern character* as

$$ch(\mathcal{E}) = e^{a_i} + \cdots + e^{a_k}. \tag{13.4}$$

Re-expressing in terms of the Chern classes, we have the first few terms of the Chern character:

$$ch(\mathcal{E}) = rank(E) + c_1(\mathcal{E}) + \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right) + \ldots. \tag{13.5}$$

The Todd Class is associated to

$$Td(\mathcal{E}) = \frac{a_1}{1-e^{-a_1}} \cdots \frac{a_k}{1-e^{-a_k}} \tag{13.6}$$

Re-expressing in terms of the Chern classes, we have the first few terms of the Todd class:

$$Td(\mathcal{E}) = 1 + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}\left(c_1(\mathcal{E})^2 + c_2(\mathcal{E})\right) + \frac{1}{24}c_1(\mathcal{E})c_2(\mathcal{E}) + \dots. \tag{13.7}$$

For an almost complex manifold V, let $Td(V) = Td(T^{(1,0)}V)$.

Note the following fact: except for ch_0, all of the Chern character and Todd polynomials are independent of the rank of the bundle.

Recall the Dolbeault complex with coefficients in a holomorphic vector bundle,

$$\Omega^p(\mathcal{E}) \xrightarrow{\bar{\partial}} \Omega^{p+1}(\mathcal{E}). \tag{13.8}$$

Let $H^p(V, \mathcal{E})$ denote the pth cohomology group of this complex, and define the *holomorphic Euler characteristic* as

$$\chi(V, \mathcal{E}) = \sum_{p=0}^{k} (-1)^p \dim_{\mathbb{C}}(H^p(V, \mathcal{E})). \tag{13.9}$$

Theorem 13.1. *(Riemann-Roch) Let $\mathcal{E}$ be a holomorphic vector bundle over a complex manifold V. Then*

$$\chi(V, \mathcal{E}) = \int_V ch(\mathcal{E}) \wedge Td(V). \tag{13.10}$$

We look at a few special cases. Let V be a curve, and let $\mathcal{E}$ be a line bundle over V, then we have

$$\dim H^0(V, \mathcal{E}) - \dim H^1(V, \mathcal{E}) = \int_V c_1(\mathcal{E}) + \frac{1}{2}c_1(V). \tag{13.11}$$

Recall that $c_1(V)$ is the Euler class, and $\int_V c_1(\mathcal{E})$ is the degree d of the line bundle. Using Serre duality, this is equivalent to

$$\dim H^0(V, \mathcal{E}) - \dim H^0(V, K \otimes \mathcal{E}^*) = d + 1 - g, \tag{13.12}$$

which is the classical Riemann-Roch Theorem for curves (g is the genus of V).

Next, let V be of dimension 2, and $\mathcal{E}$ be a line bundle, then

$$\begin{aligned} &\dim H^0(V, \mathcal{E}) - \dim H^1(V, \mathcal{E}) + \dim H^2(V, \mathcal{E}) \\ &\qquad = \int_V \frac{1}{2}c_1(\mathcal{E})c_1(V) + \frac{1}{12}\left(c_1(V)^2 + c_2(V)\right). \end{aligned} \tag{13.13}$$

If $\mathcal{E}$ is the trivial line bundle, then this is

$$1 - b^{0,1} + b^{0,2} = \frac{1}{12}\int_V \big(c_1(V)^2 + c_2(V)\big) \tag{13.14}$$

If V is a hypersurface of degree d in $\mathbb{P}^3$, then this gives

$$b^{0,2} = \frac{1}{6}(d-3)(d-2)(d-1), \tag{13.15}$$

which is of course in agreement with (11.65) above. All of the other characteristic numbers follow from this.

If $\mathcal{E}$ is a rank 2 bundle, then

$$\begin{aligned}&\dim H^0(V,\mathcal{E}) - \dim H^1(V,\mathcal{E}) + \dim H^2(V,\mathcal{E})\\ &= \int_V \frac{1}{2}c_1(\mathcal{E})c_1(V) + \frac{1}{6}\big(c_1(V)^2 + c_2(V)\big) + \frac{1}{2}\big(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\big).\end{aligned} \tag{13.16}$$

For fun, again we let V be a complex hypersurface in $\mathbb{P}^3$, and let $\mathcal{E}$ be $\Omega^1 = \Lambda^{(1,0)} = (T^{(1,0)})^*$, so $c_1(\Omega^1) = -c_1(V)$, and $c_2(\Omega^1) = c_2(V)$. We have $b^{0,1} = b^{1,0} = 0$, and by Serre duality $b^{1,2} = b^{1,0} = 0$. So Riemann-Roch gives

$$\begin{aligned}-b^{1,1} &= \int_V \frac{1}{2}c_1(\Omega^1)c_1(V) + \frac{1}{6}\big(c_1(V)^2 + c_2(V)\big) + \frac{1}{2}\big(c_1(\Omega^1)^2 - 2c_2(\Omega^1)\big)\\ &= \int_V \frac{-1}{2}c_1(V)^2 + \frac{1}{6}\big(c_1(V)^2 + c_2(V)\big) + \frac{1}{2}\big(c_1(V)^2 - 2c_2(V)\big)\\ &= \int_V \left(\frac{1}{6}c_1(V)^2 - \frac{5}{6}c_2(V)\right)\\ &= -\frac{1}{3}d(2d^2 - 6d + 7),\end{aligned}$$

which is of course in agreement with (11.66) from above.

13.2 Hodge numbers of Hopf surface

The Hodge diamond of a Hopf surface is

$$\begin{matrix} & & 1 & & \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{matrix}\quad. \tag{13.17}$$

To see this, obviously $h^{0,0} = 1$ is trivial, and $h^{2,2} = 0$ follows from Serre duality. Next, $h^{1,0} = 0$ and $h^{2,0} = 0$ since there are no holomorphic p-forms on $\mathbb{C}^2$ which are invariant under the group action. By Serre duality, it follows that $h^{1,2} = 0$ and $h^{0,2} = 0$. The Riemann-Roch formula (13.14) yields that

$$h^{0,0} - h^{0,1} + h^{0,2} = 0, \tag{13.18}$$

which implies that $h^{0,1} = 1$. By Serre duality, it follows that $h^{2,1} = 1$. Finally, the Euler characteristic formula (9.19) yields that $h^{1,1} = 0$.

14 Lecture 14

14.1 Moduli of Riemann surfaces

The Riemann-Roch Theorem for a Riemann surface (M, J) and holomorphic line bundle $\mathcal{E}$ says that

$$dim(H^0(M, \mathcal{E})) - dim(H^1(M, \mathcal{E})) = d + 1 - k, \tag{14.1}$$

where d is the the degree of the line bundle, and k is the genus of M. Note the degree is given by counting the zeroes and poles of any meromorphic section.

We apply this to $\mathcal{E} = \Theta$, the holomorphic tangent bundle. The degree of Θ is $2 - 2g$ which is the Euler characteristic. Note by Serre duality, we have

$$H^1(M, \Theta) = H^0(M, \Theta^* \otimes \Theta^*), \tag{14.2}$$

so the Riemann-Roch formula becomes

$$dim(H^0(M, \Theta)) - H^0(M, \Theta^* \otimes \Theta^*) = d + 1 - k. \tag{14.3}$$

14.2 Genus 0

First consider the case of genus 0. In this case, $\Theta^* \otimes \Theta^*$ degree -4, so has no holomorphic section. Riemann-Roch gives

$$dim(H^0(M, \Theta)) = 3. \tag{14.4}$$

This is correct because the complex Lie algebra of holomorphic vector fields is isomorphic to the real Lie algebra of conformal vector fields, and the identity component is

$$SO(3, 1) = PSL(2, \mathbb{C}), \tag{14.5}$$

which is a 6-dimensional real Lie group.

In fact, we have

Corollary 14.1. *If (M, J) is a Riemann surface homeomorphic to S^2 then it is biholomorphic to the Riemann sphere (S^2, J_S).*

Proof. TBC. □

14.3 Genus 1

Next, the case of genus 1. Then the bundles have degree 0, so the space of sections is 1 dimensional, and Riemann-Roch gives $0 = 0$. The moduli space is 1-dimensional. In fact, we have

Corollary 14.2. *Any Riemann surface of genus 1 is biholomorphic to a complex torus.*

Proof. TBC. □

14.4 Genus > 1

We get something new for genus $k > 1$. In this case Θ has negative degree, so has no holomorphic sections. The Riemann-Roch formula yields

$$H^1(M, \Theta) = H^0(M, \Theta^* \otimes \Theta^*) = -(2 - 2k) - 1 + k = 3k - 3, \qquad (14.6)$$

thus the moduli space has complex dimension $3k - 3$. Since $H^2(M, \Theta) = 0$ and $H^0(M, \Theta) = 0$, it is a smooth manifold of real dimension $6k - 6$.

The Riemann-Roch formula implies that all Riemann surfaces are in fact projective, but we will leave this to the interested student to provide a proof.

14.5 Kodaira-Nakano vanishing theorem

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