COMPLEX GEOMETRY NOTES

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1. Characteristic numbers of hypersurfaces

Let $V \subset \mathbb{P}^n$ be a smooth complex hypersurface. We know that the line bundle $[V] = \mathcal{O}(d)$ for some $d \geq 1$. We have the exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}\mathbb{P}^2\big|_V \to N_V \to 0. \tag{1.1}$$

The adjunction formula says that

$$N_V = \mathcal{O}(d)\big|_V. \tag{1.2}$$

We have the smooth splitting of (1.1),

$$T^{(1,0)}\mathbb{P}^n\big|_V = T^{(1,0)}(V) \oplus \mathcal{O}(d)\big|_V. \tag{1.3}$$

Taking Chern classes,

$$c(T^{(1,0)}\mathbb{P}^n\big|_V) = c(T^{(1,0)}(V)) \cdot c(\mathcal{O}(d)\big|_V). \tag{1.4}$$

From the Euler sequence [GH78, page 409],

$$0 \to \mathbb{C} \to \mathcal{O}(1)^{\oplus(n+1)} \to T^{(1,0)}\mathbb{P}^n \to 0, \tag{1.5}$$

it follows that

$$c(T^{(1,0)}\mathbb{P}^n) = (1 + c_1(\mathcal{O}(1)))^{n+1}. \tag{1.6}$$

Note that for any divisor D,

$$c_1([D]) = \eta_D, \tag{1.7}$$

where η_D is the Poincaré dual to D. That is

$$\int_D \xi = \int_{\mathbb{P}^n} \xi \wedge \eta_D, \tag{1.8}$$

for all $\xi \in H^{2n-2}(\mathbb{P})$, see [GH78, page 141]. So in particular $c_1(\mathcal{O}(1)) = \omega$, where ω is the Poincaré dual of a hyperplane in $\mathbb{P}^n$ (note that ω is integral, and is some multiple of the Fubini-Study metric). Therefore

$$c(T^{(1,0)}\mathbb{P}^n) = (1 + \omega)^{n+1}. \tag{1.9}$$

Also $c_1(\mathcal{O}(d)) = d \cdot \omega$, since $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. The formula (1.4) is then

$$(1 + \omega)^{n+1}\big|_V = (1 + c_1 + c_2 + \dots)(1 + d \cdot \omega\big|_V). \tag{1.10}$$

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2. Dimension $n = 2$

Consider a curve in $\mathbb{P}^2$. The formula (1.10) is

$$(1 + 3\omega)\big|_V = (1 + c_1)(1 + d \cdot \omega\big|_V), \tag{2.1}$$

which yields

$$c_1 = (3 - d)\omega\big|_V. \tag{2.2}$$

The top Chern class is the Euler class, so we have

$$\chi(V) = \int_V (3 - d)\omega \tag{2.3}$$

$$= (3 - d) \int_{\mathbb{P}^2} \omega \wedge (d \cdot \omega) \tag{2.4}$$

$$= d(3 - d) \int_{\mathbb{P}^2} \omega^2 = d(3 - d). \tag{2.5}$$

Here we used the fact that $d \cdot \omega$ is Poincaré dual to V, and ω^2 is a positive generator of $H^4(\mathbb{P}^2, \mathbb{Z})$. Equivalently, we can write

$$\int_V \omega = \int_{\mathbb{P}^2} \omega \wedge \eta_V = \int_{\mathbb{P}^2} \eta_H \wedge \eta_V. \tag{2.6}$$

Since cup product is dual to intersection under Poincaré duality, the integral simply counts the number of intersection points of V with a generic hyperplane.

In term of the genus g,

$$g = \frac{(d - 1)(d - 2)}{2}. \tag{2.7}$$

3. Dimension $n = 3$

We consider a hypersurface in $\mathbb{P}^3$, which is topologically a 4-manifold. The formula (1.10) is

$$(1 + 4\omega + 6\omega^2)\big|_V = (1 + c_1 + c_2) \cdot (1 + d \cdot \omega\big|_V), \tag{3.1}$$

so that

$$c_1 = (4 - d)\omega\big|_V, \tag{3.2}$$

and then

$$c_2 = (6 - d(4 - d))\omega^2\big|_V. \tag{3.3}$$

Since the top Chern class is the Euler class,

$$\chi(V) = \int_V (6 - d(4 - d))\omega^2 \tag{3.4}$$

$$= (6 - d(4 - d)) \int_{\mathbb{P}^3} \omega^2 \wedge (d \cdot \omega) \tag{3.5}$$

$$= d(6 - d(4 - d)), \tag{3.6}$$

again using the fact that $d \cdot \omega$ is Poincaré dual to V, and that ω^3 is a positive generator of $H^6(\mathbb{P}^3, \mathbb{Z})$.

It follows from the Lefschetz Hyperplane Theorem [GH78, page 156], that M has $b_1 = 0$, therefore $b^{1,0} = b^{0,1} = 0$.

3.1. **Hirzebruch Signature Theorem.** We think of V as a real 4-manifold, with complex structure given by J. Then the kth Pontrjagin Class is defined to be

$$p_k(V) = (-1)^k c_{2k}(TV \otimes \mathbb{C}) \tag{3.7}$$

Since (V, J) is complex, we have that

$$TV \otimes \mathbb{C} = TV \oplus \overline{TV}, \tag{3.8}$$

so

$$c(TV \otimes \mathbb{C}) = c(TV) \cdot c(\overline{TV}) \tag{3.9}$$

$$= (1 + c_1 + c_2) \cdot (1 - c_1 + c_2) \tag{3.10}$$

$$= 1 + 2c_2 - c_1^2, \tag{3.11}$$

which yields

$$p_1(V) = c_1^2 - 2c_2. \tag{3.12}$$

Consider next the intersection pairing $H^2(V) \times H^2(V) \to \mathbb{R}$, given by

$$(\alpha, \beta) \to \int \alpha \wedge \beta \in \mathbb{R}. \tag{3.13}$$

Let b_2^+ denote the number of positive eigenvalues, and b_2^- denote the number of negative eigenvalues. By Poincaré duality the intersection pairing is non-degenerate, so

$$b_2 = b_2^+ + b_2^-. \tag{3.14}$$

The *signature* of V is defined to be

$$\tau = b_2^+ - b_2^-. \tag{3.15}$$

The Hirzebruch Signature Theorem [MS74, page 224] states that

$$\tau = \frac{1}{3}\int_V p_1(V) \tag{3.16}$$

$$= \frac{1}{3}\int_V (c_1^2 - 2c_2). \tag{3.17}$$

Rewriting this,

$$2\chi + 3\tau = \int_V c_1^2. \tag{3.18}$$

Remark. This implies that S^4 does not admit any almost complex structure, since the left hand side is 4, but the right hand side trivially vanishes.

Pointwise, let

$$\Lambda_2^+ = \{\alpha \in \Lambda^2(M, \mathbb{R}) : *\alpha = \alpha\} \tag{3.19}$$

$$\Lambda_2^- = \{\alpha \in \Lambda^2(M, \mathbb{R}) : *\alpha = -\alpha\}. \tag{3.20}$$

On any Kähler manifold,

$$\Lambda^+_2 \otimes \mathbb{C} = (\mathbb{C} \cdot \omega) \oplus \Lambda^{(2,0)} \oplus \Lambda^{(0,2)}, \tag{3.21}$$

$$\Lambda^-_2 \otimes \mathbb{C} = \Lambda^{(1,1)}_o = (\mathbb{C} \cdot \omega)^{\perp} \cap \Lambda^{(1,1)}. \tag{3.22}$$

(Recall that $\Lambda^{(1,1)}_o$ is by definition the primitive $(1,1)$-forms.) Observe that this is a real decomposition, that is

$$\Lambda^+_2 = (\mathbb{R} \cdot \omega) \oplus \{\Lambda^{(2,0)} \oplus \overline{\Lambda^{(2,0)}}\}, \tag{3.23}$$

$$\Lambda^-_2 = \Lambda^{(1,1)}_{o,\mathbb{R}} = (\mathbb{R} \cdot \omega)^{\perp} \cap \Lambda^{1,1}. \tag{3.24}$$

This decomposition follows from the proof of the Hodge-Riemann bilinear relations [GH78, page 123]. Applying this to the intersection form, we have the identities

$$b^+_2 = 1 + 2b^{2,0} \tag{3.25}$$

$$b^-_2 = b^{1,1} - 1. \tag{3.26}$$

So we have that

$$\chi = 2 + b_2 = 2 + b^{1,1} + 2b^{2,0} \tag{3.27}$$

$$\tau = 2 + 2b^{2,0} - b^{1,1}. \tag{3.28}$$

Remark. Notice that

$$\chi + \tau = 4(1 + b^{2,0}), \tag{3.29}$$

so in particular, the integer $\chi + \tau$ is divisible by 4 on a Kähler manifold with $b_1 = 0$. This is in fact true for *any* almost complex manifold of real dimension 4, this follows from a version of Riemann-Roch Theorem which holds for almost complex manifolds, see [Gil95, Lemma 3.5.3].

Applying these formulas to our example, we find that

$$2\chi + 3\tau = (4 - d)^2 \int_V \omega^2 = d(4 - d)^2. \tag{3.30}$$

Using the formula for the Euler characteristic form above,

$$\chi = d(6 - d(4 - d)), \tag{3.31}$$

we find that

$$\tau = -\frac{1}{3} d(d + 2)(d - 2). \tag{3.32}$$

Some arithmetic shows that

$$b_2 = d^3 - 4d^2 + 6d - 2 \tag{3.33}$$

$$b_2^+ = \frac{1}{3}(d^3 - 6d^2 + 11d - 3) \tag{3.34}$$

$$b_2^- = \frac{1}{3}(d-1)(2d^2 - 4d + 3) \tag{3.35}$$

$$b^{2,0} = b^{0,2} = \frac{1}{6}(d-3)(d-2)(d-1) \tag{3.36}$$

$$b^{1,1} = \frac{1}{3}d(2d^2 - 6d + 7) \tag{3.37}$$

$$b^{1,0} = b^{0,1} = 0. \tag{3.38}$$

For $d = 2$, we find that $b_2^+ = 1$, $b_2^- = 1$. This is not surprising, as any non-degenerate quadric is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ [GH78, page 478]. The Hodge numbers are $b^{1,1} = 2$, $b^{0,2} = b^{2,0} = 0$.

For $d = 3$, we find $b_2^+ = 1$, $b_2^- = 6$. This is expected, since any non-degenerate cubic is biholomorphic to $\mathbb{P}^2$ blown up at 6 points, and is therefore diffeomorphic to $\mathbb{P}^2 \# 6\overline{\mathbb{P}^2}$ [GH78, page 489]. The Hodge numbers in this case are $b^{1,1} = 7$, $b^{0,2} = b^{2,0} = 0$.

For $d = 4$, this is a $K3$ surface [GH78, page 590]. We find $b_2^+ = 3$, $b_2^- = 19$, so $\chi = 24$, and $\tau = -16$. The intersection form is given by

$$2E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.39}$$

Since $c_1 = 0$, the canonical bundle is trivial. The Hodge numbers in this case are $b^{1,1} = 20$, $b^{0,2} = b^{2,0} = 1$.

For $d = 5$, we find $b_2^+ = 9$, $b_2^- = 44$, so $\chi = 55$, and $\tau = -35$. From Freedman's topological classification of simply-connected 4-manifolds, V must be homeomorphic to $9\mathbb{P}^2 \# 44\overline{\mathbb{P}^2}$, see [FQ90]. By the work of Gromov-Lawson [GL80], this latter smooth manifold admits a metric of positive scalar curvature, and therefore all of its Seiberg-Witten invariants vanish [Wit94]. But V is Kähler, so it has some non-zero Seiberg-Witten invariant [Mor96, Theorem 7.4.4]. We conclude that V is homeomorphic to $9\mathbb{P}^2 \# 44\overline{\mathbb{P}^2}$, but not diffeomorphic.

4. Riemann-Roch Theorem

Instead of using the Hirzebruch signature Theorem to compute these characteristic numbers, we can use the Riemann-Roch formula for complex manifolds.

Let $\mathcal{E}$ be a complex vector bundle over V of rank k. Assume that $\mathcal{E}$ splits into a sum of line bundles

$$\mathcal{E} = L_1 \oplus \cdots \oplus L_k. \tag{4.1}$$

Let $a_i = c_1(L_i)$. Then

$$c(\mathcal{E}) = (1 + a_1) \cdots (1 + a_k), \tag{4.2}$$

which shows that $c_j(\mathcal{E})$ is given by the elementary symmetric functions of the a_i, that is

$$c_j(\mathcal{E}) = \sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k}. \tag{4.3}$$

Any other symmetric polynomial can always be expressed as a polynomial in the elementary symmetric functions. We define the *Chern character* as

$$ch(\mathcal{E}) = e^{a_i} + \cdots + e^{a_k}. \tag{4.4}$$

Re-expressing in terms of the Chern classes, we have the first few terms of the Chern character:

$$ch(\mathcal{E}) = rank(E) + c_1(\mathcal{E}) + \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right) + \ldots. \tag{4.5}$$

The Todd Class is associated to

$$Td(\mathcal{E}) = \frac{a_1}{1 - e^{-a_1}} \cdots \frac{a_k}{1 - e^{-a_k}} \tag{4.6}$$

Re-expressing in terms of the Chern classes, we have the first few terms of the Todd class:

$$Td(\mathcal{E}) = 1 + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}\left(c_1(\mathcal{E})^2 + c_2(\mathcal{E})\right) + \frac{1}{24}c_1(\mathcal{E})c_2(\mathcal{E}) + \ldots. \tag{4.7}$$

For an almost complex manifold V, let $Td(V) = Td(T^{(1,0)}V)$.

Note the following fact: except for ch_0, all of the Chern character and Todd polynomials are independent of the rank of the bundle.

Recall the Dolbeault complex with coefficients in a holomorphic vector bundle,

$$\Omega^p(\mathcal{E}) \xrightarrow{\overline{\partial}} \Omega^{p+1}(\mathcal{E}). \tag{4.8}$$

Let $H^p(V, \mathcal{E})$ denote the pth cohomology group of this complex, and define the *holomorphic Euler characteristic* as

$$\chi(V, \mathcal{E}) = \sum_{p=0}^{k} (-1)^p \dim_{\mathbb{C}}(H^p(V, \mathcal{E})). \tag{4.9}$$

Theorem 4.1. *(Riemann-Roch) Let $\mathcal{E}$ be a holomorphic vector bundle over a complex manifold V. Then*

$$\chi(V, \mathcal{E}) = \int_V ch(\mathcal{E}) \wedge Td(V). \tag{4.10}$$

We look at a few special cases. Let V be a curve, and let $\mathcal{E}$ be a line bundle over V, then we have

$$\dim H^0(V, \mathcal{E}) - \dim H^1(V, \mathcal{E}) = \int_V c_1(\mathcal{E}) + \frac{1}{2}c_1(V). \tag{4.11}$$

Recall that $c_1(V)$ is the Euler class, and $\int_V c_1(\mathcal{E})$ is the degree d of the line bundle. Using Serre duality, this is equivalent to

$$\dim H^0(V, \mathcal{E}) - \dim H^0(V, K \otimes \mathcal{E}^*) = d + 1 - g, \tag{4.12}$$

which is the classical Riemann-Roch Theorem for curves (g is the genus of V).

Next, let V be of dimension 2, and $\mathcal{E}$ be a line bundle, then

$$\begin{aligned} &\dim H^0(V,\mathcal{E}) - \dim H^1(V,\mathcal{E}) + \dim H^2(V,\mathcal{E}) \\ &\qquad = \int_V \frac{1}{2} c_1(\mathcal{E}) c_1(V) + \frac{1}{12}\left(c_1(V)^2 + c_2(V)\right). \end{aligned} \tag{4.13}$$

If $\mathcal{E}$ is the trivial line bundle, then this is

$$1 - b^{0,1} + b^{0,2} = \frac{1}{12}\int_V \left(c_1(V)^2 + c_2(V)\right) \tag{4.14}$$

If V is a hypersurface of degree d in $\mathbb{P}^3$, then this gives

$$b^{0,2} = \frac{1}{6}(d-3)(d-2)(d-1), \tag{4.15}$$

which is of course in agreement with (3.36) above. All of the other characteristic numbers follow from this.

If $\mathcal{E}$ is a rank 2 bundle, then

$$\begin{aligned} &\dim H^0(V,\mathcal{E}) - \dim H^1(V,\mathcal{E}) + \dim H^2(V,\mathcal{E}) \\ &= \int_V \frac{1}{2} c_1(\mathcal{E}) c_1(V) + \frac{1}{6}\left(c_1(V)^2 + c_2(V)\right) + \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right). \end{aligned} \tag{4.16}$$

For fun, again we let V be a complex hypersurface in $\mathbb{P}^3$, and let $\mathcal{E}$ be $\Omega^1 = \Lambda^{(1,0)} = (T^{(1,0)})^*$, so $c_1(\Omega^1) = -c_1(V)$, and $c_2(\Omega^1) = c_2(V)$. We have $b^{0,1} = b^{1,0} = 0$, and by Serre duality $b^{1,2} = b^{1,0} = 0$. So Riemann-Roch gives

$$\begin{aligned} -b^{1,1} &= \int_V \frac{1}{2} c_1(\Omega^1) c_1(V) + \frac{1}{6}\left(c_1(V)^2 + c_2(V)\right) + \frac{1}{2}\left(c_1(\Omega^1)^2 - 2c_2(\Omega^1)\right) \\ &= \int_V \frac{-1}{2} c_1(V)^2 + \frac{1}{6}\left(c_1(V)^2 + c_2(V)\right) + \frac{1}{2}\left(c_1(V)^2 - 2c_2(V)\right) \\ &= \int_V \left(\frac{1}{6} c_1(V)^2 - \frac{5}{6} c_2(V)\right) \\ &= -\frac{1}{3} d(2d^2 - 6d + 7), \end{aligned}$$

which is of course in agreement with (3.37) from above.

5. Complete Intersections

Let $V^k \subset \mathbb{P}^n$ be a smooth complete intersection of $n-k$ homogeneous polynomials of degree $d_1, \dots, d_{n-k}$. We have the exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}\mathbb{P}^2\big|_V \to N_V \to 0. \tag{5.1}$$

Where N_V is now a bundle of rank $n-k$ bundle. The adjunction formula says that

$$N_V = \mathcal{O}(d_1)\big|_V \oplus \cdots \oplus \mathcal{O}(d_{n-k})\big|_V. \tag{5.2}$$

We have the smooth splitting of (1.1),

$$T^{(1,0)}\mathbb{P}^n\big|_V = T^{(1,0)}(V) \oplus \mathcal{O}(d)\big|_V. \tag{5.3}$$

Taking Chern classes,

$$c(T^{(1,0)}\mathbb{P}^n\big|_V) = c(T^{(1,0)}(V)) \cdot c\left(\mathcal{O}(d_1)\big|_V \oplus \cdots \oplus \mathcal{O}(d_{n-k})\big|_V\right), \tag{5.4}$$

which is

$$(1+\omega)^{n+1}\big|_V = (1+c_1+\cdots+c_k)(1+d_1\cdot\omega\big|_V)\cdots(1+d_{n-k}\cdot\omega\big|_V). \tag{5.5}$$

Let us now just consider the simple case of a complete intersection of $n-1$ hypersurfaces in $\mathbb{P}^n$, of degrees $d_1,\ldots,d_{n-1}$. We have

$$1+(n+1)\omega\big|_V = 1+c_1+(d_1+\cdots+d_{n-1})\omega\big|_V, \tag{5.6}$$

which yields

$$c_1 = (n+1-d_1-\cdots-d_{n-1})\omega\big|_V. \tag{5.7}$$

The Euler characteristic is

$$\chi(V) = (n+1-d_1-\cdots-d_{n-1})\int_V \omega. \tag{5.8}$$

By definition of the Poincaré dual,

$$\int_V \omega = \int_{\mathbb{P}^n} \omega\wedge\eta_V = \int_{\mathbb{P}^n} \eta_H\wedge\eta_V. \tag{5.9}$$

We use some intersection theory to understand the integral. Intersecting cycles is Poincaré dual to the cup product, thus the integral counts the number of intersection points of V with a generic hyperplane. Consequently,

$$\chi(V) = (n+1-d_1-\cdots-d_{n-1})d_1d_2\cdots d_{n-1}. \tag{5.10}$$

The genus g is given by

$$g = 1-\frac{1}{2}(n+1-d_1-\cdots-d_{n-1})d_1d_2\cdots d_{n-1}. \tag{5.11}$$

Corollary 5.1. *A curve of genus* 2 *does not arise as a complete intersection.*

Proof. Assume by contradiction that it does. If any of the $d_i = 1$, then it is a complete intersection in a lower dimensional projective space. So without loss of generality, assume that $d_i \geq 2$. We would then have

$$2 = -(n+1-d_1-\cdots-d_{n-1})d_1d_2\cdots d_{n-1}. \tag{5.12}$$

The right hand side is a product of integers. Since 2 is prime, the only possibility is that $n=2$, and $d_1=2$, in which case the above equation reads

$$2 = -2, \tag{5.13}$$

which is a contradiction. □

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