Kähler manifolds, Ricci curvature, and hyperkähler metrics

Jeff A. Viaclovsky

June 25-29, 2018

Contents

1 Lecture 1

We will assume a basic familiarity with complex manifolds, and only do a brief review today. Let M be a manifold of real dimension $2n$, and an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$. Then we can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \tag{1.1}$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \tag{1.2}$$

is the i-eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \tag{1.3}$$

is the $-i$-eigenspace of J.

Remark 1.1. For now we will denote $\sqrt{-1}$ by i. However, later we will not do this, because the letter i is sometimes used as an index.

The map J also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{1.4}$$

where

$$\Lambda^{1,0} = \{\alpha - iJ\alpha, \alpha \in T_p^*M\} \tag{1.5}$$

is the i-eigenspace of J, and

$$\Lambda^{0,1} = \{\alpha + iJ\alpha, \alpha \in T_p^*M\} \tag{1.6}$$

is the $-i$-eigenspace of J.

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{1.7}$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{1.8}$$

If (M, J) is a complex manifold, then there exist coordinate systems around any point

$$(z^1, \ldots, z^n) = (x^1 + iy^1, \ldots, x^n + iy^n) \tag{1.9}$$

such that $T^{1,0}$ is spanned by

$$\partial_j \equiv \frac{\partial}{\partial z^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big), \tag{1.10}$$

$T^{0,1}$ is spanned by

$$\partial_{\overline{j}} \equiv \frac{\partial}{\partial \overline{z}^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big), \tag{1.11}$$

$\Lambda^{1,0}$ is spanned by

$$dz^j \equiv dx^j + idy^j, \tag{1.12}$$

and $\Lambda^{0,1}$ is spanned by

$$d\overline{z}^j \equiv dx^j - idy^j, \tag{1.13}$$

for $j = 1 \ldots n$.

1.1 The operators ∂ and $\overline{\partial}$

The real operator $d : \Lambda^k_{\mathbb{R}} \to \Lambda^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \Lambda^k_{\mathbb{C}} \to \Lambda^{k+1}_{\mathbb{C}} \tag{1.14}$$

by complexification.

Proposition 1.2. *On a complex manifold, $d = \partial + \overline{\partial}$ where $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $\overline{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$, and these operators satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{1.15}$$

Proof. On a complex manifold, if α is a (p, q)-form written locally as

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\overline{z}^J, \tag{1.16}$$

then we have the formulas

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k\Big) \wedge dz^I \wedge d\overline{z}^J \tag{1.17}$$

$$\partial\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J \tag{1.18}$$

$$\overline{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J, \tag{1.19}$$

and these formulas give globally defined operators. The other relations follow simply from $d^2 = 0$. □

Definition 1.3. *A form* $\alpha \in \Lambda^{p,0}$ *is holomorphic if* $\overline{\partial}\alpha = 0$*.*

It is easy to see that a $(p,0)$-form is holomorphic if and only if it can locally be written as

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{1.20}$$

where the α_I are holomorphic functions.

Definition 1.4. *The* (p,q) *Dolbeault cohomology group is*

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\{\alpha \in \Lambda^{p,q}(M) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M))}. \tag{1.21}$$

If f is a biholomorphism (one-to-one, onto, with holomorphic inverse), between (M, J_M) and (N, J_N) then the Dolbeault cohomologies of M and N are isomorphic.

1.2 Hermitian and Kähler metrics

We next consider (M, g, J) where J is an almost complex structure and g is a Riemannian metric.

Definition 1.5. An almost Hermitian manifold is a triple (M, g, J) such that

$$g(JX, JY) = g(X, Y). \tag{1.22}$$

The triple is called Hermitian if J is integrable.

We also say that g is J-invariant if condition (1.22) is satisfied. Extend g by complex linearity to a symmetric inner product on $T \otimes \mathbb{C}$.

To a Hermitian metric (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{1.23}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{1.24}$$

Since

$$\omega(JX, JY) = \omega(X, Y), \tag{1.25}$$

this form is a real form of type $(1,1)$, and is called the *Kähler form* or *fundamental 2-form*.

Recall the following definition.

Proposition 1.6. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \tag{1.26}$$

vanishes if and only if J *is integrable.*

Recall that on a Riemannian manifold, there is a unique symmetric connection which is compatible with the metric, and is given explicitly by the following formula

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\Big(& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])\Big). \end{aligned} \tag{1.27}$$

The following proposition gives a fundamental relation between the covariant derivative of J, the exterior derivative of ω and the Nijenhuis tensor.

Proposition 1.7. *Let (M, g, J) be an almost Hermitian manifold (with J not necessarily assumed to be integrable). Then*

$$2g((\nabla_X J)Y, Z) = -d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y,Z), JX), \tag{1.28}$$

where the covariant derivative of J is defined as

$$(\nabla_X J)Y = \nabla_X(JY) - J(\nabla_X Y), \tag{1.29}$$

Proof. Exercise. Hint: use the formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) = & \sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ & + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{1.30}$$

□

Corollary 1.8. *If (M, g, J) is Hermitian, then $d\omega = 0$ if and only if J is parallel.*

Proof. Since $N = 0$, this follows immediately from (1.28). □

Corollary 1.9. *If (M, g, J) is almost Hermitian, $\nabla J = 0$ implies that $d\omega = 0$ and $N = 0$.*

Proof. If J is parallel, then ω is also. The corollary follows from the fact that the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ can be written in terms of covariant differentiation.

$$d\omega(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \dots, \hat{X}_j, \dots, X_p), \tag{1.31}$$

which follows immediately from (1.30) using normal coordinates around a point. This shows that a parallel form is closed, so the corollary then follows from (1.28). □

Definition 1.10. An almost Hermitian manifold (M, g, J) is

- *Kähler* if J is integrable and $d\omega = 0$, or equivalently, if $\nabla J = 0$,
- *Calabi-Yau* if it is Kähler and the canonical bundle $K \equiv \Lambda^{n,0}$ is holomorphically trivial,
- *hyperkähler* is it is Kähler with respect to 3 complex structures I, J, and K satisfiying $IJ = K$.

Note that if (M, g, J) is Kähler, then ω is a parallel $(1,1)$-form.

2 Lecture 2

2.1 Complex tensor notation

Choosing any real basis of the form $\{X_1, JX_1, \dots, X_n, JX_n\}$, let us abbreviate

$$Z_\alpha = \frac{1}{2}\Big(X_\alpha - iJX_\alpha\Big),\ Z_{\overline{\alpha}} = \frac{1}{2}\Big(X_\alpha + iJX_\alpha\Big), \tag{2.1}$$

and define

$$g_{AB} = g(Z_A, Z_B), \tag{2.2}$$

where the indices A and B can be barred or unbarred indices. Notice that

$$\begin{aligned} g_{\alpha\beta} = g(Z_\alpha, Z_\beta) &= \frac{1}{4} g(X_\alpha - iJX_\alpha, X_\beta - iJX_\beta) \\ &= \frac{1}{4}\Big(g(X_\alpha, X_\beta) - g(JX_\alpha, JX_\beta) - i(g(X_\alpha, JX_\beta) + g(JX_\alpha, X_\beta))\Big) \\ &= 0, \end{aligned}$$

since g is J-invariant, and $J^2 = -Id$. Similarly,

$$g_{\overline{\alpha}\overline{\beta}} = 0, \tag{2.3}$$

Also, from symmetry of g, we have

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) = g(Z_{\overline{\beta}}, Z_\alpha) = g_{\overline{\beta}\alpha}. \tag{2.4}$$

However, applying conjugation, since g is real we have

$$\overline{g_{\alpha\overline{\beta}}} = \overline{g(Z_\alpha, Z_{\overline{\beta}})} = g(Z_{\overline{\alpha}}, Z_\beta) = g(Z_\beta, Z_{\overline{\alpha}}) = g_{\beta\overline{\alpha}}, \tag{2.5}$$

which says that $g_{\alpha\overline{\beta}}$ is a Hermitian matrix.

We repeat the above for the fundamental 2-form ω, and define

$$\omega_{AB} = \omega(Z_A, Z_B). \tag{2.6}$$

Note that

$$\omega_{\alpha\beta} = \omega(Z_\alpha, Z_\beta) = g(JZ_\alpha, Z_\beta) = ig_{\alpha\beta} = 0 \tag{2.7}$$

$$\omega_{\overline{\alpha}\overline{\beta}} = \omega(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = -ig_{\overline{\alpha}\overline{\beta}} = 0 \tag{2.8}$$

$$\omega_{\alpha\overline{\beta}} = \omega(Z_\alpha, Z_{\overline{\beta}}) = ig_{\alpha\overline{\beta}} \tag{2.9}$$

$$\omega_{\overline{\alpha}\beta} = \omega(Z_{\overline{\alpha}}, Z_\beta) = -ig_{\overline{\alpha}\beta}. \tag{2.10}$$

The first 2 equations are just a restatement that ω is of type $(1,1)$. Also, note that

$$\omega_{\alpha\overline{\beta}} = ig_{\alpha\overline{\beta}}, \tag{2.11}$$

defines a skew-Hermitian matrix.

On a Hermitian manifold, the fundamental 2-form in holomorphic coordinates takes the form

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta. \tag{2.12}$$

Remark 2.1. Note that for the Euclidean metric, we have $g_{\alpha\overline{\beta}} = \frac{1}{2}\delta_{\alpha\beta}$, so

$$\omega_{Euc} = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{2.13}$$

We note the following local characterizations of Kähler metrics.

Proposition 2.2. *If J is integrable, then (M, g, J) is Kähler if and only if*

- *In local holomorphic coordinates we have*

$$\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} = \frac{\partial g_{k\overline{\beta}}}{\partial z^\alpha}, \tag{2.14}$$

- *For each $p \in M$, there exists an open neighborhood U of p and a function $u : U \to \mathbb{R}$ such that $\omega = i\partial\overline{\partial}u$.*

- *For each $p \in M$, there exists a holomorphic coordinate system around p such that*

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} (\delta_{jk} + O(|z|^2)_{jk}) dz^j \wedge d\overline{z}^k, \tag{2.15}$$

as $|z| \to 0$.

Proof. Exercise. □

2.2 The musical isomorphisms

We recall the following from Riemannian geometry. The metric gives an isomorphism between TM and T^*M,

$$\flat : TM \to T^*M \tag{2.16}$$

defined by

$$\flat(X)(Y) = g(X, Y). \tag{2.17}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp\omega_1, \sharp\omega_2). \tag{2.18}$$

Note that if g has components g_{ij}, then $\langle \cdot, \cdot \rangle$ has components g^{ij}, the inverse matrix of g_{ij}.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{2.19}$$

where

$$X_i = g_{ij} X^j, \tag{2.20}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{2.21}$$

where

$$\omega^i = g^{ij} \omega_j, \tag{2.22}$$

thus the sharp operator "raises" an index.

The $\flat$ operator extends to a complex linear mapping

$$\flat : TM \otimes \mathbb{C} \to T^*M \otimes \mathbb{C}. \tag{2.23}$$

We have the following

Proposition 2.3. *The operator $\flat$ is a complex anti-linear isomorphism*

$$\flat : T^{1,0} \to \Lambda^{0,1} \tag{2.24}$$

$$\flat : T^{0,1} \to \Lambda^{1,0}. \tag{2.25}$$

Proof. These mapping properties follow from the Hermitian property of g. Next, for any two vectors X and Y

$$\flat(JX)Y = g(JX, Y), \tag{2.26}$$

while

$$J(\flat X)(Y) = (\flat X)(JY) = g(X, JY) = -g(JX, Y). \tag{2.27}$$

□

In components, we have the following. The metric on $T^*Z \otimes \mathbb{C}$ has components

$$g(dz^\alpha, d\overline{z}^\beta) = g^{\alpha\overline{\beta}} \tag{2.28}$$

where these are the components of the inverse matrix of $g_{\alpha\overline{\beta}}$. We have the identities

$$g^{\alpha\overline{\beta}} g_{\overline{\beta}\gamma} = \delta^\alpha_\gamma, \tag{2.29}$$

$$g^{\overline{\alpha}\beta} g_{\beta\overline{\gamma}} = \delta^{\overline{\alpha}}_{\overline{\gamma}}, \tag{2.30}$$

If $X = X^\alpha Z_\alpha$ is in $T^{(1,0)}$, then $\flat X$ has components

$$(\flat X)_{\overline{\alpha}} = g_{\overline{\alpha}\beta} X^\beta, \tag{2.31}$$

and if $X = X^{\overline{\alpha}} Z_{\overline{\alpha}}$ is in $T^{(0,1)}$, then $\flat X$ has components

$$(\flat X)_\alpha = g_{\alpha\overline{\beta}} X^{\overline{\beta}}, \tag{2.32}$$

Similarly, if $\omega = \omega_\alpha Z^\alpha$ is in $\Lambda^{1,0}$, then $\sharp\omega$ has components

$$(\sharp\omega)^{\overline{\alpha}} = g^{\overline{\alpha}\beta} \omega_\beta, \tag{2.33}$$

and if $\omega = \omega_{\overline{\alpha}} Z^{\overline{\alpha}}$ is in $\Lambda^{0,1}$, then $\sharp X$ has components

$$(\sharp\omega)^\alpha = g^{\alpha\overline{\beta}} \omega_{\overline{\beta}}. \tag{2.34}$$

2.3 Trace

Let us rearrange the basis in the order $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$. Write the matrix associated to and endomorphism $H \in End_{\mathbb{R}}(TM)$ as

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.35}$$

Then in the basis $\{Z_1, \ldots, Z_n, Z_{\bar{1}}, \ldots, Z_{\bar{n}}\}$ of $TM \otimes \mathbb{C}$ the matrix associated to the endomorphism $H \otimes \mathbb{C} \in End_{\mathbb{C}}(TM \otimes \mathbb{C})$ is given by

$$\begin{aligned} H \otimes \mathbb{C} = \frac{1}{2} &\begin{pmatrix} A + D + i(C - B) & 0 \\ 0 & A + D - i(C - B) \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} 0 & A - D + i(B + C) \\ A - D - i(B + C) & 0 \end{pmatrix}. \end{aligned} \tag{2.36}$$

Obviously, we have

$$tr(H) = A + D = tr(H \otimes \mathbb{C}) = \sum_{\alpha} h_{\alpha}^{\alpha} + \sum_{\alpha} h_{\bar{\alpha}}^{\bar{\alpha}}. \tag{2.37}$$

Note that the mixed components $h_{\overline{\beta}}^{\alpha}$ do not contribute to the trace.

If (M, J, g) is almost Hermitian, then the $\flat$ operator gives an identification

$$End_{\mathbb{R}}(TM) \cong T^*M \otimes TM \cong T^*M \otimes T^*M, \tag{2.38}$$

This yields a trace map defined on $T^*M \otimes T^*M$ defined as follows. If

$$h = h_{\alpha\beta} dz^{\alpha} dz^{\beta} + h_{\overline{\alpha}\overline{\beta}} d\overline{z}^{\alpha} d\overline{z}^{\beta} + h_{\alpha\overline{\beta}} dz^{\alpha} d\overline{z}^{\beta} + h_{\overline{\alpha}\beta} d\overline{z}^{\alpha} dz^{\beta}, \tag{2.39}$$

then

$$tr(h) = g^{\alpha\overline{\beta}} h_{\alpha\overline{\beta}} + g^{\overline{\alpha}\beta} h_{\overline{\alpha}\beta}. \tag{2.40}$$

Note that the components $h_{\alpha\beta}$ and $h_{\overline{\alpha}\overline{\beta}}$ do not contribute to the trace. If h is real, then this agrees with the usual Riemannian trace mapping defined as

$$tr(h) = tr\Big(X \mapsto \sharp\big(h(X, \cdot)\big)\Big), \tag{2.41}$$

which in components is

$$tr(h) = \sum_{i,j=1}^{2n} g^{ij} h_{ij}, \tag{2.42}$$

where the above formula involves components in a real basis.

Remark 2.4. In Kähler geometry one sometimes sees the trace of some 2-tensor defined as just the first term in (2.40). If h is the complexification of a real tensor, then this is (1/2) of the Riemannian trace.

2.4 Determinant

Let us assume that J is standard, that is,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{2.43}$$

If H is an endomorphism with $H \circ J = J \circ H$, then H must have the form

$$H = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{2.44}$$

Consequently, the matrix associated to the endomorphism $H \otimes \mathbb{C} \in End_{\mathbb{C}}(TM \otimes \mathbb{C})$ is given by

$$H \otimes \mathbb{C} = \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{2.45}$$

It follows that

$$\det(H) = \det(A + iB) \det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{2.46}$$

This implies that any complex manifold is orientable, and admits a unique orientation compatible with the complex struture.

A formula which will be useful later is the following for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{2.47}$$

3 Lecture 3

3.1 Christoffel symbols of a Kähler metric

In general we have

$$\nabla_A \partial_B = \Gamma^k_{AB} \partial_k + \Gamma^{\bar{k}}_{AB} \partial_{\bar{k}}. \tag{3.1}$$

For a Kähler metric, these symbols simplify a lot:

Proposition 3.1. *The only non-zero Christoffel symbols are* $\Gamma^k_{ij} = \Gamma^k_{ji}$ *and* $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}$.

Proof. First, note that

$$0 = \partial_{\bar{k}} g(\partial_i \partial_j) = g(\nabla_{\bar{k}} \partial_i, \partial_j) + g(\partial_i, \nabla_{\bar{k}} \partial_j), \tag{3.2}$$

so

$$g(\nabla_{\bar{k}} \partial_i, \partial_j) = -g(\partial_i, \nabla_{\bar{k}} \partial_j). \tag{3.3}$$

Next,

$$\partial_i g_{j\bar{k}} = \partial_i g(\partial_j, \partial_{\bar{k}}) = g(\nabla_i \partial_j, \partial_{\bar{k}}) + g(\partial_j, \nabla_i \partial_{\bar{k}}), \tag{3.4}$$

and

$$\partial_j g_{i\bar{k}} = \partial_j g(\partial_i, \partial_{\bar{k}}) = g(\nabla_j \partial_i, \partial_{\bar{k}}) + g(\partial_i, \nabla_j \partial_{\bar{k}}), \tag{3.5}$$

Using symmetry of the Riemannian connection

$$\nabla_{\partial_A} \partial_B - \nabla_{\partial_B} \partial_A = [\partial_A, \partial_B] = 0, \tag{3.6}$$

and the Kähler condition (2.14), these equations imply

$$g(\nabla_i \partial_j, \partial_{\bar{k}}) = \frac{1}{2}\left(\partial_i g_{j\bar{k}} + \partial_j g_{i\bar{k}}\right) = \partial_i g_{j\bar{k}} \tag{3.7}$$

$$g(\nabla_i \partial_{\bar{k}}, \partial_j) = \frac{1}{2}\left(\partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}\right) = 0. \tag{3.8}$$

Other symbols are treated similarly, and left as an exercise. □

Note that (3.7) yield a nice formula for the Christoffel symbols of a Kähler metric

$$\boxed{\Gamma^k_{ij} = g^{k\bar{l}} \partial_i g_{j\bar{l}}} \tag{3.9}$$

3.2 Curvature of a Riemannian metric

The curvature tensor is defined by

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3.10}$$

for vector fields X, Y, and Z. We define

$$Rm(X,Y,Z,W) \equiv -g(\mathcal{R}(X,Y)Z, W). \tag{3.11}$$

We will refer to $\mathcal{R}$ as the curvature tensor of type $(1,3)$ and to Rm as the curvature tensor of type $(0,4)$.

In a coordinate system we define quantities ${R_{ijk}}^l$ by

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = {R_{ijk}}^l \partial_l, \tag{3.12}$$

or equivalently,

$$\mathcal{R} = {R_{ijk}}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l. \tag{3.13}$$

Define quantities R_{ijkl} by

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l), \tag{3.14}$$

or equivalently,

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{3.15}$$

Then

$$R_{ijkl} = -g(\mathcal{R}(\partial_i, \partial_j)\partial_k, \partial_l) = -g(R_{ijk}{}^m \partial_m, \partial_l) = -R_{ijk}{}^m g_{ml}. \tag{3.16}$$

Equivalently,

$$R_{ijlk} = R_{ijk}{}^m g_{ml}, \tag{3.17}$$

that is, we lower the upper index to the *third* position.

In coordinates, the algebraic symmetries of the curvature tensor are

$$R_{ijk}{}^l = -R_{jik}{}^l \tag{3.18}$$
$$0 = R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l \tag{3.19}$$
$$R_{ijkl} = -R_{ijlk} \tag{3.20}$$
$$R_{ijkl} = R_{klij}. \tag{3.21}$$

Of course, we can write the first 2 symmetries as a $(0, 4)$ tensor,

$$R_{ijkl} = -R_{jikl} \tag{3.22}$$
$$0 = R_{ijkl} + R_{jkil} + R_{kijl}. \tag{3.23}$$

Note that using (3.21), the algebraic Bianchi identity (3.23) may be written as

$$0 = R_{ijkl} + R_{iklj} + R_{iljk}. \tag{3.24}$$

Definition 3.2. The *Ricci tensor* is the tensor in $\Gamma(T^*M \otimes T^*M)$ defined by

$$Ric(X, Y) = tr\big(U \to \mathcal{R}(U, X)Y\big). \tag{3.25}$$

The *scalar curvature* is the function

$$R = tr\Big(X \to \sharp\big(Ric(X, \cdot)\big)\Big). \tag{3.26}$$

We clearly have

$$Ric(X, Y) = Ric(Y, X), \tag{3.27}$$

so $Ric \in \Gamma(S^2(T^*M))$. We let R_{ij} denote the components of the Ricci tensor,

$$Ric = R_{ij} dx^i \otimes dx^j, \tag{3.28}$$

where $R_{ij} = R_{ji}$. From the definition,

$$R_{ij} = R_{lij}{}^l = g^{lm} R_{limj}, \tag{3.29}$$

and

$$R = g^{ij} R_{ij}. \tag{3.30}$$

3.3 Curvature of a Kähler metric

Proposition 3.3. *If* (M, g, J) *is Kähler, then*

$$Rm(X, Y, Z, W) = Rm(JX, JY, Z, W) = Rm(X, Y, JZ, JW), \tag{3.31}$$
$$Ric(X, Y) = Ric(JX, JY). \tag{3.32}$$

Proof. We first claim that

$$\mathcal{R}(X, Y)JZ = J(\mathcal{R}(X, Y)Z). \tag{3.33}$$

To see this,

$$\begin{aligned} \mathcal{R}(X, Y)JZ &= \nabla_X \nabla_Y (JZ) - \nabla_Y \nabla_X (JZ) - \nabla_{[X,Y]} JZ \\ &= J(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = J(\mathcal{R}(X, Y)Z), \end{aligned}$$

since J is parallel. Next,

$$\begin{aligned} Rm(JX, JY, U, V) &= -g(\mathcal{R}(JX, JY)U, V) = -g(\mathcal{R}(U, V)JX, JY) \\ &= -g(J(\mathcal{R}(U, V)X), JY) = -g(\mathcal{R}(U, V)X, Y) = Rm(X, Y, U, V). \end{aligned}$$

Next, we compute

$$\begin{aligned} Ric(JX, JY) &= tr\big(U \to \mathcal{R}(U, JX)JY\big) \\ &= tr\big(JU \to \mathcal{R}(JU, JX)JY\big) \\ &= tr\big(JU \to \mathcal{R}(U, X)JY\big) \\ &= tr\big(JU \to J(\mathcal{R}(U, X)Y)\big) \\ &= tr\big(U \to \mathcal{R}(U, X)Y\big) = Ric(X, Y). \end{aligned} \tag{3.34}$$

□

Corollary 3.4. *The only nonzero components of the Kähler curvature tensor are*

$$R_{i\bar{j}k}{}^{l}, R_{i\bar{j}\bar{k}}{}^{\bar{l}}, R_{\bar{i}jk}{}^{l}, R_{\bar{i}j\bar{k}}{}^{\bar{l}}, \tag{3.35}$$

and

$$R_{i\bar{j}k\bar{l}}, R_{i\bar{j}\bar{k}l}, R_{\bar{i}jk\bar{l}}, R_{\bar{i}j\bar{k}l}. \tag{3.36}$$

Proposition 3.5. *The Kähler curvature tensor enjoys the following symmetries:*

$$R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}} = R_{k\bar{l}i\bar{j}} \tag{3.37}$$

Proof. Exercise (use the Bianchi identity). □

Note that

$$R_{i\bar{j}k}{}^{l}\partial_l = \nabla_i \nabla_{\bar{j}} \partial_k - \nabla_{\bar{j}} \nabla_i \partial_k = -\nabla_{\bar{j}}(\Gamma^l_{ik}\partial_l) = -\partial_{\bar{j}}(\Gamma^l_{ik})\partial_l, \tag{3.38}$$

so

$$\boxed{R_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}}(\Gamma^l_{ik})}. \tag{3.39}$$

Proposition 3.6. *The components of the Ricci tensor can be written as*

$$R_{i\bar{j}} = R_{p\bar{j}i}{}^{p} = g^{\bar{k}l} R_{i\bar{k}\bar{j}l}, \tag{3.40}$$

and locally

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} (\log \det(g_{k\bar{l}})) \tag{3.41}$$
$$R = -2g^{i\bar{j}} \partial_i \partial_{\bar{j}} (\log \det(g_{k\bar{l}})). \tag{3.42}$$

Proof. Since the Ricci tensor is symmetric, we have

$$R_{i\bar{j}} = R_{\bar{j}i}, \tag{3.43}$$

and since $R_{AB} = R_{CAB}{}^{C} = R_{CBA}{}^{C}$, we have

$$R_{i\bar{j}} = R_{\bar{j}i} = R_{C\bar{j}i}{}^{C} = R_{p\bar{j}i}{}^{p}. \tag{3.44}$$

Also, we have

$$\begin{aligned} R_{i\bar{j}} &= g^{CD} R_{iC\bar{j}D} = g^{\bar{k}l} R_{i\bar{k}\bar{j}l} \\ &= g^{\bar{k}l} R_{l\bar{k}\bar{j}i} = -g^{\bar{k}l} R_{\bar{k}li\bar{j}}. \end{aligned} \tag{3.45}$$

Next, Jacobi's formula says that

$$\partial_i \log \det(g_{k\bar{l}}) = g^{k\bar{l}} \partial_i g_{k\bar{l}}. \tag{3.46}$$

Then

$$\begin{aligned} -\partial_i \partial_{\bar{j}} (\log \det(g_{k\bar{l}})) &= -\partial_{\bar{j}} \partial_i (\log \det(g_{k\bar{l}})) \\ &= -\partial_{\bar{j}} \left(g^{k\bar{l}} \partial_i g_{k\bar{l}} \right) = -\partial_{\bar{j}} \Gamma^{p}_{ip} = -\partial_{\bar{j}} \Gamma^{p}_{pi} = R_{p\bar{j}i}{}^{p} = R_{i\bar{j}}. \end{aligned} \tag{3.47}$$

The last line follows from $R = 2g^{i\bar{j}} R_{i\bar{j}}$, see Remark 2.4. □

3.4 The Ricci form

We can convert the Ricci tensor into a 2-form by

$$\rho(X, Y) \equiv Ric(JX, Y). \tag{3.48}$$

Proposition 3.7. *The Ricci form ρ of a Kähler metric is a real, closed $(1, 1)$ form, and can be written locally as*

$$\rho = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j = -\sqrt{-1} \partial \overline{\partial} \log \det(g_{k\bar{l}}) \tag{3.49}$$

Proof. From Proposition 3.3,

$$\begin{aligned} \rho(Y, X) &= Ric(JY, X) = Ric(X, JY) \\ &= Ric(JX, J^2 Y) = -Ric(JX, Y) = \rho(X, Y), \end{aligned} \tag{3.50}$$

so ρ is indeed a real 2-form. Next,

$$\begin{aligned}\rho(JX, JY) = Ric(J^2X, JY) &= -Ric(X, JY)\\ &= -Ric(JY, X) = -\rho(Y, X) = \rho(X, Y),\end{aligned} \tag{3.51}$$

which implies that ρ is a $(1,1)$-form. Next,

$$\rho_{i\bar{j}} = \rho(\partial_i, \partial_{\bar{j}}) = Ric(J\partial_i, \partial_{\bar{j}}) = \sqrt{-1}R_{i\bar{j}}, \tag{3.52}$$

so then

$$\begin{aligned}\rho &= \rho_{i\bar{j}}dz^i \wedge d\bar{z}^j = \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j\\ &= -\sqrt{-1}\big(\partial_i\partial_{\bar{j}}(\log\det(g_{k\bar{l}}))\big)dz^i \wedge d\bar{z}^j = -\sqrt{-1}\partial\overline{\partial}(\log\det(g_{k\bar{l}})).\end{aligned} \tag{3.53}$$

Finally, closedness of ρ follows easily since $d = \partial + \overline{\partial}$. □

This allows us to make the following definition/proposition.

Proposition 3.8 (First Chern class)**.** *The cohomology class*

$$c_1(M, J) \equiv \frac{1}{2\pi}[\rho] \in H^{1,1}_{\mathbb{R}}(M) \subset H^2(M, \mathbb{R}) \tag{3.54}$$

is independent of the choice of Kähler metric.

Proof. If h is another Kähler metric then

$$\begin{aligned}\rho_g - \rho_h &= -\sqrt{-1}\partial\overline{\partial}\log\det(g_{k\bar{l}}) + \sqrt{-1}\partial\overline{\partial}\log\det(h_{k\bar{l}})\\ &= -\sqrt{-1}\partial\overline{\partial}\log\Big(\frac{\det(g_{k\bar{l}})}{\det(h_{k\bar{l}})}\Big).\end{aligned} \tag{3.55}$$

However, note that for any function f,

$$\begin{aligned}-\sqrt{-1}\partial\overline{\partial}f &= (\partial + \overline{\partial})(-\sqrt{-1}\overline{\partial}f)\\ &= \frac{1}{2}(\partial + \overline{\partial})(-\sqrt{-1}\ \overline{\partial}f + \sqrt{-1}\partial f) = \frac{1}{2}dJdf,\end{aligned} \tag{3.56}$$

hence

$$\rho_g - \rho_h = \frac{1}{2}dJd\log\Big(\frac{\det(g_{k\bar{l}})}{\det(h_{k\bar{l}})}\Big), \tag{3.57}$$

and since the quotient of the determinants is a globally defined function, we are done. □

Remark 3.9. We note that we can write the difference of Ricci forms of any two Kähler metrics as

$$\rho_g - \rho_h = -\sqrt{-1}\partial\overline{\partial}\log\Big(\frac{\omega_g^n}{\omega_h^n}\Big), \tag{3.58}$$

and note also the formula

$$\omega_g^n = n!dV_g, \tag{3.59}$$

where dV_g is the Riemannian volume element.

4 Lecture 4

4.1 Line bundles and divisors

A line bundle over a complex manifold M is a rank 1 complex vector bundle $\pi : E \to M$. The transition functions are defined as follows. A trivialization is a mapping

$$\Phi_\alpha : U_\alpha \times \mathbb{C} \to E \tag{4.1}$$

which maps $x \times \mathbb{C}$ linearly onto a fiber. The transition functions are

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*, \tag{4.2}$$

defined by

$$\varphi_{\alpha\beta}(x) = \frac{1}{v}\pi_2(\Phi_\alpha^{-1} \circ \Phi_\beta(x, v)), \tag{4.3}$$

for $v \neq 0$.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}. \tag{4.4}$$

Conversely, given a covering U_α of M and transition functions $\varphi_{\alpha\beta}$ satifsying (4.4), there is a vector bundle $\pi : E \to M$ with transition functions given by $\varphi_{\alpha\beta}$, and this bundle is uniquely defined up to bundle equivalence, which we will define below. If the transitions function $\varphi_{\alpha\beta}$ are C^∞, then we say that E is a smooth vector bundle, while if they are holomorphic, we say that E is a holomorphic vector bundle. Note that total space of a holomorphic vector bundle over a complex manifold is a complex manifold.

Exercise 4.1. A *section* is a mapping $\sigma : M \to E$ such that $\pi \circ \sigma = id_M$. Show that that is equivalent to a collection of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}$ satisfying

$$\sigma_\alpha = \varphi_{\alpha\beta}\sigma_\beta. \tag{4.5}$$

A vector bundle mapping is a mapping $f : E_1 \to E_2$ which is linear on fibers, and covers the identity map. Assume we have a covering U_α of M such that E_1 has trivializations Φ_α and E_2 has trivializations Ψ_α. Then any vector bundle mapping gives locally defined functions $f_\alpha : U_\alpha \to \mathbb{C}$ defined by

$$f_\alpha(x) = \frac{1}{v}\pi_2(\Psi_\alpha^{-1} \circ F \circ \Phi_\alpha(x, v)) \tag{4.6}$$

for $v \neq 0$. It is easy to see that on overlaps $U_\alpha \cap U_\beta$,

$$f_\alpha = \varphi^{E_2}_{\alpha\beta} f_\beta \varphi^{E_1}_{\beta\alpha}, \tag{4.7}$$

equivalently,

$$\varphi^{E_2}_{\beta\alpha} f_\alpha = f_\beta \varphi^{E_1}_{\beta\alpha}. \tag{4.8}$$

We say that two bundles are E_1 and E_2 are equivalent if there exists an invertible bundle mapping $f : E_1 \to E_2$. This is equivalent to non-vanishing of the local representatives, that is, $f_\alpha : U_\alpha \to \mathbb{C}^*$. A vector bundle is *trivial* if it is equivalent to the trivial product bundle. That is, E is trivial if there exist functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$\varphi_{\beta\alpha} = f_\beta f_\alpha^{-1}. \tag{4.9}$$

Note, by Exercise 4.1, this is equivalent to the existence of a nowhere-vanishing section.

The tensor product $E_1 \otimes E_2$ of two line bundles E_1 and E_2 is again a line bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E_1 \otimes E_2} = \varphi_{\alpha\beta}^{E_1} \varphi_{\alpha\beta}^{E_2}. \tag{4.10}$$

The dual E^* of a line bundle E, is again a line bundle, and has transition functions

$$\varphi_{\alpha\beta}^{E^*} = (\varphi_{\beta\alpha}^{E})^{-1}. \tag{4.11}$$

Note that for any line bundle,

$$E \otimes E^* \cong \mathbb{C}, \tag{4.12}$$

is the trivial line bundle.

An *effective divisor* D is defined to be the zero set of a holomorphic section of a nontrivial line bundle. Conversely, an irreducible holomorphic subvariety of codimension 1 defines a line bundle by taking local defining functions which vanish to order 1 along the divisor to be the transition functions, that is,

$$\varphi_{\alpha\beta} = \frac{f_\alpha}{f_\beta}. \tag{4.13}$$

By Exercise 4.1, we see that such a line bundle always admits a holomorphic section which vanishes exactly on D. However, as we will soon see, not every line bundle admits a nontrivial holomorphic section!

4.2 Hermitian metrics on line bundles

Definition 4.2. A Hermitian metric on a line bundle $L \to M$ is a smooth choice of Hermitian inner product on each fiber. That is

$$h(p) : L_p \otimes L_p \to \mathbb{C}, \tag{4.14}$$

which is linear in the first argument, anti-linear in the second argument, $h(v, w) = \overline{h(w, v)}$, and such that $h(v, v) > 0$ with equality if and only if $v \neq 0$. Equivalently, h is a section of $L^* \otimes \overline{L^*}$, satisfying the latter condition.

An associated metric which is linear in both variables is

$$\langle s_1, s_2 \rangle = h(s_1, \overline{s_2}) \tag{4.15}$$

Recall that a connection is a $\mathbb{C}$-linear mapping

$$\nabla : \Gamma(TM_{\mathbb{C}}) \times \Gamma(L) \to \Gamma(L), \tag{4.16}$$

which satisfies for any function $f : M \to \mathbb{C}$,

$$\nabla_X(fs) = (Xf)s + f\nabla_X s \tag{4.17}$$

$$\nabla_{fX} s = f\nabla_X s. \tag{4.18}$$

We say a connection is compatible with h if

$$Xh(s_1, s_2) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_{\overline{X}} s_2), \tag{4.19}$$

which is equivalent to (exercise).

$$X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle. \tag{4.20}$$

There is a canonical connection associated to a holomorphic line bundle.

Definition 4.3. Given a Hermitian metric h on a line bundle $L \to M$, the Chern connection is the uniquely defined connection satisfying the following properties:

- If s is a locally defined holomorphic section, then $\nabla_{\bar{j}} s = 0$.
- ∇ is compatible with h.

Proposition 4.4. *The Chern connection exists and is unique.*

Proof. If s is a locally defined holomorphic section over an open set U, since L is a line bundle,

$$\nabla_j s = A_j s, \tag{4.21}$$

for some functions $A_j : U \to \mathbb{C}$. Then by the properties of the the Chern connection,

$$\partial_j |s|_h^2 = \partial_j h(s, s) = h(A_j s, s) = A_j |s|_h^2, \tag{4.22}$$

so we can solve for

$$A_j = |s|_h^{-2} \partial_j |s|_h^2 = \partial_j \log |s|_h^2, \tag{4.23}$$

so the connection is uniquely determined locally, which implies it exists. □

Definition 4.5. The curvature of the Chern connection is the 2-form

$$\Omega_h \in \Gamma(\Lambda^2_{\mathbb{C}}(M) \otimes End(L)) \cong \Gamma(\Lambda^2_{\mathbb{C}}(M)) \tag{4.24}$$

defined by

$$\Omega_h(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s. \tag{4.25}$$

Note that in (4.24), we used that $End(L) \cong L^* \otimes L$ is a trivial bundle to get a globally defined 2-form.

Proposition 4.6. *The curvature of the Chern connection satisifes*

- $\Omega_h \in \Gamma(\Lambda^{1,1}(M))$,
- *For a locally defined nonvanishing holomorphic section s,*

$$\Omega_h = -\partial\overline{\partial}(\log |s|_h^2). \tag{4.26}$$

Proof. For the first part, if s is any local nonvanishing holomorphic section, using (4.23), we have

$$\begin{aligned} \Omega_h(\partial_i, \partial_j)s &= \nabla_i \nabla_j s - \nabla_j \nabla_i s \\ &= \nabla_i (A_j s) - \nabla_j (A_i s) \\ &= \partial_i (A_j)s + A_i A_j s - \partial_j (A_i)s - A_i A_j s \\ &= (\partial_i \partial_j \log |s|_h^2 - \partial_j \partial_i \log |s|_h^2)s = 0. \end{aligned} \tag{4.27}$$

Next,

$$\Omega_h(\partial_{\bar{i}}, \partial_{\bar{j}}) = \nabla_{\bar{i}} \nabla_{\bar{j}} s - \nabla_{\bar{j}} \nabla_{\bar{i}} s = 0. \tag{4.28}$$

For (4.26), again using (4.23), we compute

$$\begin{aligned} \Omega_h(\partial_i, \partial_{\bar{j}})s &= \nabla_i \nabla_{\bar{j}} s - \nabla_{\bar{j}} \nabla_i s \\ &= -\nabla_{\bar{j}}(A_i s) \\ &= -\partial_{\bar{j}}(A_i)s \\ &= -(\partial_{\bar{j}} \partial_i \log |s|_h^2)s = -(\partial_i \partial_{\bar{j}} \log |s|_h^2)s \end{aligned} \tag{4.29}$$

□

Proposition 4.7 (First Chern class of a line bundle)**.** *Let $L \to M$ be a line bundle. The form $\sqrt{-1}\Omega_h$ is a closed real $(1,1)$ form, and the cohomology class*

$$c_1(L) \equiv \frac{\sqrt{-1}}{2\pi}[\Omega_h] \in H^{1,1}_{\mathbb{R}}(M) \subset H^2(M, \mathbb{R}) \tag{4.30}$$

is independent of the choice of Hermitian metric.

Proof. Note that Ω_h is not a real 2-form, but that $\sqrt{-1}\Omega_h$ is real. This is because using the above formula (4.26) and (3.56), we have

$$\begin{aligned} \sqrt{-1}\Omega_h &= -\sqrt{-1}(\partial_i \partial_{\bar{j}} \log |s|_h^2) dz^i \wedge d\bar{z}^j \\ &= -\sqrt{-1}\partial\overline{\partial} \log |s|_h^2 = \frac{1}{2} dJd \log |s|_h^2. \end{aligned} \tag{4.31}$$

Closedness of Ω_h of also follows from the formula. Next, given any other hermitian metric $\tilde{h}$, there exists a function $f : M \to \mathbb{R}$ such that $\tilde{h} = e^{-f}h$. Then it is easy to see that

$$\sqrt{-1}\Omega_{\tilde{h}} = \sqrt{-1}\Omega_h + \sqrt{-1}\partial\overline{\partial} f, \tag{4.32}$$

so the cohomology class is independent of the choice of metric. □

Remark 4.8. The factor of $(1/2\pi)$ is chosen to make this an integral class, we will discuss this again later.

Recall the canonical bundle is the line bundle $K = \Lambda^{n,0}$. This inherits a natural hermitian metric from the Kähler metric defined as follows. Let $e_i \in T^{1,0}$ be a Hermitian orthonormal frame, i.e, $g(e_i, \overline{e_j}) = \delta_{ij}$ and let e^i denote the dual basis of $\Lambda^{1,0}$, for $i = 1 \dots n$. Then declare that

$$e^1 \wedge \cdots \wedge e^n \tag{4.33}$$

has unit norm.

Proposition 4.9. *Denote the curvature form of the canonical bundle by Ω_K. Then*

$$\sqrt{-1}\Omega_K = -\rho, \tag{4.34}$$

where ρ is the Ricci form.

Proof. This follows from above since a local nonvanishing section of K is given by

$$dz^1 \wedge \cdots \wedge dz^n, \tag{4.35}$$

which has norm squared equal to $1/\det(g_{k\bar{l}})$. □

As a corollary, we see that

$$c_1(K) = -c_1(M). \tag{4.36}$$

5 Lecture 5

5.1 Positivity of a line bundle

Definition 5.1. A line bundle $L \to M$ is *positive* if the cohomology class $c_1(L) \in H^{1,1}_{\mathbb{R}}(M)$ has a representative $\gamma \in H^{1,1}_{\mathbb{R}}(M)$ which is positive definite. That is, the symmetric real bilinear form

$$\tilde{\gamma}(X, Y) = \gamma(X, JY), \tag{5.1}$$

is a positive definite symmetric form at each point. A line bundle L is called *negative* if its dual bundle L^* is positive.

Proposition 5.2. *If L is negative, then L does not admit any non-zero holomorphic section.*

Proof. First, we note that $c_1(L) = -c_1(L^*)$, so by assumption there exists a representative γ of $c_1(L)$ which is negative definite. Fix any Hermitian metric h on L. Then

$$\frac{\sqrt{-1}}{2\pi}\Omega_h = \gamma + da, \tag{5.2}$$

for some function $a : M \to \mathbb{R}$. By the $\partial\overline{\partial}$ Lemma which we will prove later, there exists a function $f : M \to \mathbb{R}$ such that

$$\sqrt{-1}\Omega_h = 2\pi\gamma + \sqrt{-1}\partial\overline{\partial} f. \tag{5.3}$$

Given any Hermitian metric h on L, recall from above that if $\tilde{h} = e^f h$, then

$$\sqrt{-1}\Omega_{\tilde{h}} = \sqrt{-1}\Omega_h - \sqrt{-1}\partial\overline{\partial} f, \tag{5.4}$$

so we have found a Hermitian metric $\tilde{h}$ with curvature

$$\sqrt{-1}\Omega_{\tilde{h}} = 2\pi\gamma. \tag{5.5}$$

Next, let $p \in M$ be a point where $|s|$ achieves a maximum. Then $|s| \neq 0$ in some neighborhood of p, so by the formula (4.26), we have

$$2\pi\gamma = -\sqrt{-1}\partial\overline{\partial}(\log |s|^2_h), \tag{5.6}$$

but it is easy to check that the right hand side is positive semidefinite at a maximum of $|s|$, which contradicts that γ is negative definite. □

Exercise 5.3. Show the following stronger result: if there exists a Hermitian metric h on L with

$$tr(\sqrt{-1}\tilde{\Omega}_h) < 0, \tag{5.7}$$

then L admits no non-zero holomorphic sections.

5.2 The Laplacian on a Kähler manifold

Let α and β be sections of $\Lambda^{p,q}$. Then we define the Hermitian inner product of α and β to be

$$(\alpha, \beta) = g(\alpha, \overline{\beta}). \tag{5.8}$$

We next want to compute the formal L^2 adjoints of certain operators. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{p,q+1}), \tag{5.9}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p,q+1}) \xrightarrow{\overline{\partial}^*} \Gamma(\Lambda^{p,q}), \tag{5.10}$$

is defined as follows. For $\alpha \in \Gamma(\Lambda^{p,q+1})$ and $\beta \in \Gamma(\Lambda^{p,q})$, we have

$$\int_M (\alpha, \overline{\partial}\beta) dV = \int_M (\overline{\partial}^* \alpha, \beta) dV, \tag{5.11}$$

where dV denotes the Riemannian volume element. Other adjoints are defined similarly.

Consider the 3 Laplacians

$$\Delta_H = d^*d + dd^*, \tag{5.12}$$

$$\Delta_\partial = \partial^*\partial + \partial\partial^* \tag{5.13}$$

$$\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*, \tag{5.14}$$

where $\cdot^*$ denotes the L^2-adjoint.

First, we consider the case of functions $f : M \to \mathbb{C}$.

Proposition 5.4. *For $f : M \to \mathbb{C}$, we have*

$$\Delta_H f = 2\Delta_\partial f = 2\Delta_{\overline{\partial}} f. \tag{5.15}$$

Proof. In coordinates we have

$$\Delta_H f = (dd^* + d^*d)f = d^*df = -g^{AB}\nabla_A\nabla_B f \tag{5.16}$$

$$\Delta_\partial f = (\partial\partial^* + \partial^*\partial)f = \partial^*\partial f = -g^{\bar{i}j}\nabla_{\bar{i}}\nabla_j f \tag{5.17}$$

$$\Delta_{\overline{\partial}} f = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})f = \overline{\partial}^*\overline{\partial} f = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}} f. \tag{5.18}$$

Then

$$\nabla_i\nabla_{\bar{j}} f = \nabla_i(\partial_{\bar{j}} f) = \partial_i\partial_{\bar{j}} f - \Gamma^A_{i\bar{j}}\partial_A f = \partial_{\bar{j}}\partial_i f = \nabla_{\bar{j}}\nabla_i f, \tag{5.19}$$

since the mixed Christoffel symbols vanish, which implies that $\Delta_\partial = \Delta_{\overline{\partial}}$. Next,

$$\Delta_H f = -g^{AB}\nabla_A\nabla_B f = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}} f - g^{\bar{i}j}\nabla_{\bar{i}}\nabla_j f = \Delta_{\overline{\partial}} f + \Delta_\partial f. \tag{5.20}$$

□

On a Kähler manifold, this occurrence continues to hold for any (p, q)-form.

Proposition 5.5. *For $\alpha \in \Gamma(\Lambda^{p,q})$, if (M, J, g) is Kähler, then*

$$\Delta_H \alpha = 2\Delta_\partial \alpha = 2\Delta_{\overline{\partial}} \alpha. \tag{5.21}$$

Proof. Let L denote the mapping

$$L : \Lambda^p \to \Lambda^{p+2} \tag{5.22}$$

given by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form. Then we have the identities

$$[\overline{\partial}^*, L] = i\partial \tag{5.23}$$

$$[\partial^*, L] = -i\overline{\partial}. \tag{5.24}$$

These are proved proved first in $\mathbb{C}^n$ and then on a Kähler manifold using Kähler normal coordinates. The proposition then follows from these identities (proof omitted). □

Proposition 5.6. *If* (M, J, g) *is a compact Kähler manifold, then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M), \tag{5.25}$$

and

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{q,p}_{\overline{\partial}}(M)^*. \tag{5.26}$$

Consequently,

$$b^k(M) = \sum_{p+q=k} b^{p,q}(M) \tag{5.27}$$

$$b^{p,q}(M) = b^{q,p}(M). \tag{5.28}$$

Proof. This follows because if a harmonic k-form is decomposed as

$$\phi = \phi^{k,0} + \phi^{k-1,1} + \cdots + \phi^{1,k-1} + \phi^{0,k}, \tag{5.29}$$

then

$$0 = \Delta_H \phi = 2\Delta_{\overline{\partial}}\phi^{k,0} + 2\Delta_{\overline{\partial}}\phi^{k-1,1} + \cdots + 2\Delta_{\overline{\partial}}\phi^{1,k-1} + 2\Delta_{\overline{\partial}}\phi^{0,k}, \tag{5.30}$$

therefore

$$\Delta_{\overline{\partial}}\phi^{k-p,p} = 0, \tag{5.31}$$

for $p = 0 \dots k$.

Next,

$$\overline{\Delta_{\overline{\partial}}\phi} = \Delta_{\partial}\overline{\phi}, \tag{5.32}$$

so conjugation sends harmonic forms to harmonic forms.

The Proposition then follows using Hodge Theory to identify the cohomology groups with the corresponding spaces of harmonic forms. □

This yields a topologicial obstruction for a complex manifold to admit a Kähler metric:

Corollary 5.7. *If* (M, J, g) *is a compact Kähler manifold, then the odd Betti numbers of* M *are even.*

Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$

$$(z_1, z_2) \to 2^k(z_1, z_2). \tag{5.33}$$

This is a free and properly discontinuous action, so the quotient $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ is a manifold, which is called a primary Hopf surface. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$, which has $b^1 = 1$, therefore it does not admit any Kähler metric.

5.3 Vanishing theorems

Proposition 5.8. *We have the following vanishing:*

- *(Bochner) If* $\rho_{i\bar{j}} = \sqrt{-1}R_{i\bar{j}}$ *is positive definite, then for* $p > 0$*, any harmonic* $(0,p)$ *or* $(p,0)$ *form vanishes, thus* $b^{0,p} = b^{p,0} = 0$.
- *(Kobayashi-Wu) If* $g^{i\bar{j}}R_{i\bar{j}} > 0$ *then any harmonic* $(0,n)$ *or* $(n,0)$ *form vanishes, thus* $b^{0,n} = b^{n,0} = 0$.

Proof. If $\alpha \in \Gamma(\Lambda^{p,0})$, then

$$\Delta_{\partial}\alpha = \Delta_{\overline{\partial}}\alpha = \overline{\partial}^*\overline{\partial}\alpha. \tag{5.34}$$

Then in components,

$$\begin{aligned}(\Delta_{\overline{\partial}}\alpha)_{i_1\dots i_p} &= -g^{p\bar{i}}\nabla_p\nabla_{\bar{i}}\alpha_{i_1\dots i_p} \\ &= -g^{p\bar{i}}\Big(\nabla_{\bar{i}}\nabla_p\alpha_{i_1\dots i_p} - R_{p\bar{i}i_1}{}^q\alpha_{qi_2\dots i_p} - \dots - R_{p\bar{i}i_1}{}^q\alpha_{i_1\dots i_{p-1}q}\Big).\end{aligned} \tag{5.35}$$

But note that

$$g^{p\bar{i}}R_{p\bar{i}j}{}^q = g^{p\bar{i}}g^{q\bar{l}}R_{p\bar{i}\bar{l}j} = g^{q\bar{l}}R_{\bar{l}j}. \tag{5.36}$$

So we have

$$(\Delta_{\overline{\partial}}\alpha)_{i_1\dots i_p} = -g^{p\bar{i}}\nabla_{\bar{i}}\nabla_p\alpha_{i_1\dots i_p} + R_{\bar{l}i_1}g^{q\bar{l}}\alpha_{qi_2\dots i_p} + \dots + R_{\bar{l}i_p}g^{q\bar{l}}\alpha_{i_1\dots i_{p-1}q}. \tag{5.37}$$

The result then follows by pairing with α and integrating by parts.

In the case $p = n$, it is not difficult to see the only the trace of the Ricci tensor appears in the final integration by parts step, details are left as an exercise.

□

6 Lecture 6

6.1 Kähler class and $\partial\overline{\partial}$-Lemma

Note also that the Kähler form is also a real closed $(1,1)$-form, so we can define the following.

Definition 6.1. The *Kähler class* of ω is

$$[\omega] \in H^{1,1}_{\mathbb{R}} \subset H^2(M,\mathbb{R}) \tag{6.1}$$

In general, this class can be different that the first Chern class defined above.

Proposition 6.2. *If* ω_1 *and* ω_2 *are two real* $(1,1)$*-forms on* (M,J) *and if* $[\omega_1] = [\omega_2] \in H^2(M,\mathbb{R})$*, then there exists a real valued function* $f : M \to \mathbb{R}$ *such that*

$$\omega_2 = \omega_1 + \sqrt{-1}\partial\overline{\partial}f, \tag{6.2}$$

Proof. By assumption, there exists a real 1-form α

$$\omega_2 - \omega_1 = d\alpha \tag{6.3}$$

Next, write $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where $\alpha^{1,0}$ is a 1-form of type $(1,0)$, and $\alpha^{0,1}$ is a 1-form of type $(0,1)$. Since α is real, $\overline{\alpha^{1,0}} = \alpha^{0,1}$. Next,

$$\begin{aligned}\omega_2 - \omega_1 = d\alpha &= \partial\alpha + \overline{\partial}\alpha \\ &= \partial\alpha^{1,0} + \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0} + \overline{\partial}\alpha^{0,1}\end{aligned} \tag{6.4}$$

The first and last terms on the right hand side are forms of type $(2,0)$ and $(0,2)$, respectively. Since the left hand side is of type $(1,1)$, we must have $\overline{\partial}\alpha^{0,1} = 0$, so

$$\omega_2 - \omega_1 = d\alpha = \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0}. \tag{6.5}$$

Next, consider the function $\overline{\partial}^*\alpha^{0,1}$, where $\overline{\partial}^*$ is the L^2-adjoint of $\overline{\partial}$ with respect to the Hermitian inner product

$$\int_M (f_1, f_2) dV_g = \int_M \langle f_1, \overline{f_2} \rangle dV_g. \tag{6.6}$$

Since $\overline{\partial}^*\alpha^{0,1}$ has integral zero, we can solve the equation

$$\Delta \boldsymbol{f} = \overline{\partial}^* \overline{\partial} \boldsymbol{f} = \overline{\partial}^* \alpha^{0,1}, \tag{6.7}$$

equivalently

$$\overline{\partial}^*(\overline{\partial}\boldsymbol{f} - \alpha^{0,1}) = 0. \tag{6.8}$$

But from above, $\overline{\partial}(\overline{\partial}\boldsymbol{f} - \alpha^{0,1}) = 0$, so

$$\Delta_{\overline{\partial}}(\overline{\partial}\boldsymbol{f} - \alpha^{0,1}) \equiv (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})(\overline{\partial}\boldsymbol{f} - \alpha^{0,1}) = 0. \tag{6.9}$$

From Proposition 5.5 this implies that

$$\Delta_{\partial}(\overline{\partial}\boldsymbol{f} - \alpha^{0,1}) \equiv (\partial\partial^* + \partial^*\partial)(\overline{\partial}\boldsymbol{f} - \alpha^{0,1}) = 0, \tag{6.10}$$

which implies that

$$\partial\overline{\partial}\boldsymbol{f} = \partial\alpha^{0,1}. \tag{6.11}$$

Substituting this back into (6.5), we have

$$\omega_2 - \omega_1 = \partial\overline{\partial}\boldsymbol{f} + \overline{\partial}\partial\overline{\boldsymbol{f}} = \sqrt{-1}\partial\overline{\partial}(2Im(\boldsymbol{f})), \tag{6.12}$$

so let $f = 2Im(\mathbf{f})$. □

6.2 Yau's Theorem

The statement of the Calabi conjecture is the following.

Theorem 6.3 (Calabi conjecture, proved by Yau)**.** *Let (M, J, ω) be a compact Kähler manifold, and let ψ be a real $(1, 1)$-form representing $c_1(M)$. Then there exists a unique Kähler form $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega]$ such that $\rho_{\tilde{\omega}} = 2\pi\psi$.*

To prove this, Yau first proved the following statement.

Theorem 6.4 (Yau)**.** *Let (M, J, ω) be a compact Kähler manifold, $F \in C^\infty(M, \mathbb{R})$, with*

$$\int_M e^F \omega^n = \int_M \omega^n. \tag{6.13}$$

Then there exists a unique Kähler metric $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega] \in H^{1,1}(M, \mathbb{R})$ and

$$\tilde{\omega}^n = e^F \omega^n \tag{6.14}$$

To prove this, use the $\partial\overline{\partial}$-Lemma to recast (6.14) as a PDE

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = e^F \omega^n \tag{6.15}$$

This is solved using the method of a priori estimates (details omitted).

Here, we will just show how this implies Theorem 6.3. From above we know that $(1/2\pi)\rho_\omega$ represents $c_1(M)$. So by the $\partial\overline{\partial}$-Lemma, there exists a function $F : M \to \mathbb{R}$ such that

$$\rho_\omega = 2\pi\psi + \sqrt{-1}\partial\overline{\partial}F. \tag{6.16}$$

Adding a constant to F if necessary, let $\tilde{\omega}$ be the solution of (6.14). Applying the operator $-\sqrt{-1}\partial\overline{\partial}\log$ to both sides of (6.14), we obtain

$$\rho_{\tilde{\omega}} = \rho_\omega - \sqrt{-1}\partial\overline{\partial}F = 2\pi\psi. \tag{6.17}$$

Here we note the following immediate corollary.

Theorem 6.5 (Yau)**.** *Let (M, J, ω) be a compact Kähler manifold with $c_1(M) = 0$. Then there exists a unique Kähler form $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega]$ such that $Ric(\tilde{\omega}) = 0$.*

We also mention the following closely related result.

Theorem 6.6 (Aubin-Yau)**.** *Let (M, J, ω_0) be a compact Kähler manifold with $c_1(M, J) < 0$. Then there exists a unique Kähler form ω with $[\omega] = -2\pi c_1(M, J)$ such that $\rho_\omega = -\omega$.*

To reduce this to a PDE, we use the $\partial\overline{\partial}$-Lemma. For any Kähler form $\omega_0 \in -2\pi c_1(M,J)$, we have ρ_{ω_0} is in the same class, so there exists a function $F : M \to \mathbb{R}$ so that

$$\rho_{\omega_0} = -\omega_0 + \sqrt{-1}\partial\overline{\partial}F. \tag{6.18}$$

Now let ω be another Kähler metric in the same class, so that $\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi$. Then

$$\rho_\omega = \rho_{\omega_0} - \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega^n}{\omega_0^n}\right). \tag{6.19}$$

We want to solve $\rho_\omega = -\omega$, which gives

$$-\omega = -\omega_0 - \sqrt{-1}\partial\overline{\partial}\phi = -\omega_0 + \sqrt{-1}\partial\overline{\partial}F - \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega^n}{\omega_0^n}\right), \tag{6.20}$$

which is

$$-\sqrt{-1}\partial\overline{\partial}\phi = \sqrt{-1}\partial\overline{\partial}F - \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega^n}{\omega_0^n}\right). \tag{6.21}$$

This will follow if we can solve the equation

$$-\phi = F - \log\left(\frac{\omega^n}{\omega_0^n}\right), \tag{6.22}$$

or equivalently,

$$(\omega + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{F+\phi}\omega_0^n. \tag{6.23}$$

This equation obeys the maximum principle, so existence of a unique solution follows from the method of a priori estimates.

If $c_1(M,J) > 0$, then a similar discussion shows that the equation $\rho_\omega = \omega$ yields the equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{F-\phi}\omega_0^n. \tag{6.24}$$

This equation does *not* obey the maximum principle, and does not necessarily admit a solution. This case is much more subtle, and we will not go into details here.

Remark 6.7. If there is a Kähler Einstein metric with $\rho_\omega = \pm\omega$, then

$$c_1(M) = \pm 2\pi[\omega], \tag{6.25}$$

So the only possible Kähler class is (a multiple of) the anticanonical class in the positive case, and the canonical class in the negative case.

However, in the case of $c_1(M) = 0$, the Kähler class is not determined.

6.3 The $\overline{\partial}$ operator on holomorphic vector bundles

Recall that the transition functions of a complex vector bundle are locally defined functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$, satisfying

$$\phi_{\alpha\beta} = \phi_{\alpha\gamma}\phi_{\gamma\beta}. \tag{6.26}$$

Definition 6.8. *A vector bundle $\pi : E \to M$ is a* holomorphic vector bundle *if in complex coordinates the transitition functions $\phi_{\alpha\beta}$ are holomorphic.*

Recall that a section of a vector bundle is a mapping $\sigma : M \to E$ satisfying $\pi \circ \sigma = Id_M$. In local coordinates, a section satisfies

$$\sigma_\alpha = \phi_{\alpha\beta}\sigma_\beta, \tag{6.27}$$

and conversely any locally defined collection of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}^m$ satisfying (6.27) defines a global section. A section is *holomorphic* if in complex coordinates, the σ_α are holomorphic.

Proposition 6.9. *If $\pi : E \to M$ is a holomorphic vector bundle, then there is a first order differential operator*

$$\overline{\partial} : \Gamma(\Lambda^{p,q} \otimes E) \to \Gamma(\Lambda^{p,q+1} \otimes E), \tag{6.28}$$

satisfying the following properties:

- *A section $\sigma \in \Gamma(E)$ is holomorphic if and only if $\overline{\partial}(\sigma) = 0$.*
- *For a function $f : M \to \mathbb{C}$, and a section $\sigma \in \Gamma(\Lambda^{p,q} \otimes E)$,*

$$\overline{\partial}(f \cdot \sigma) = (\overline{\partial} f) \wedge \sigma + f \cdot \overline{\partial}\sigma. \tag{6.29}$$

- $\overline{\partial} \circ \overline{\partial} = 0.$

Proof. Let σ_j be a local basis of holomorphic sections of E in U_α, and write any section $\sigma \in \Gamma(U_\alpha, \Lambda^{p,q} \otimes E)$ as

$$\sigma = \sum s_j \otimes \sigma_j, \tag{6.30}$$

where $s_j \in \Gamma(U_\alpha, \Lambda^{p,q})$. Then define

$$\overline{\partial}\sigma = \sum (\overline{\partial} s_j) \otimes \sigma_j. \tag{6.31}$$

We claim this is a global section of $\Gamma(\Lambda^{p,q+1} \otimes E)$. Choose a local basis σ'_j of holomorphic sections of E in U_β, and write σ as

$$\sigma = \sum s'_j \otimes \sigma'_j. \tag{6.32}$$

Since $\sigma'_j = (\phi^{-1}_{\alpha\beta})_{jl}\sigma_l$, we have that

$$s'_j = (\phi_{\alpha\beta})_{jl} s_l, \tag{6.33}$$

so we can write

$$\sigma = \sum (\phi_{\alpha\beta})_{jl} s_l \otimes \sigma'_j. \tag{6.34}$$

Consequently

$$\begin{aligned} \overline{\partial}\sigma &= \sum (\overline{\partial} s'_j) \otimes \sigma'_j = \sum \overline{\partial}((\phi_{\alpha\beta})_{jk} s_k) \otimes \sigma'_j \\ &= \sum (\phi_{\alpha\beta})_{jk} \overline{\partial}(s_k) \otimes \sigma'_j = \sum (\overline{\partial} s_k) \otimes (\phi_{\alpha\beta})_{jk} \sigma'_j = \sum (\overline{\partial} s_k) \otimes \sigma_k. \end{aligned}$$

The other properties follow immediately from the definition. □

Definition 6.10. The (p, q) Dolbeault cohomology group with coefficients in E is

$$H^{p,q}_{\overline{\partial}}(M, E) = \frac{\{\alpha \in \Lambda^{p,q}(M, E) \mid \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Lambda^{p,q-1}(M, E))}. \tag{6.35}$$

The Dolbeault Theorem says that if M is compact, then

$$H^{p,q}_{\overline{\partial}}(M, E) \cong H^q(M, \Omega^p(E)). \tag{6.36}$$

where the right hand side is a sheaf cohomology group, with $\Omega^p(E)$ the sheaf of holomorphic E-valued p-forms.

7 Lecture 7

7.1 Holomorphic vector fields

Definition 7.1. A *holomorphic vector field* on a complex manifold (M, J) is vector field $Z \in \Gamma(T^{1,0})$ which satisfies Zf is holomorphic for every locally defined holomorphic function f.

In complex coordinates, a holomorphic vector field can locally be written as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}, \tag{7.1}$$

where the Z^j are locally defined holomorphic functions.

From above, there is a first order differential operator

$$\overline{\partial} : \Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}), \tag{7.2}$$

such that a vector field Z is holomorphic if and only if $\overline{\partial}(Z) = 0$.

Letting Θ denote $T^{1,0}$, there is a complex

$$\Gamma(\Theta) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,3} \otimes \Theta) \xrightarrow{\bar{\partial}} \cdots . \tag{7.3}$$

The holomorphic vector fields (equivalently, the automorphisms of the complex structure) are identified with $H^0(M, \Theta)$. The higher cohomology groups $H^1(M, \Theta)$ and $H^2(M, \Theta)$ of this complex play a central role in the theory of deformations of complex structures.

Proposition 7.2. *If Ric is negative definite, then M does not admit any nontrivial holomorphic vector fields.*

Proof. Locally, write $Z = Z^j \partial_j$. Then

$$0 = \partial_{\bar{i}} Z^j = \nabla_{\bar{i}} Z^j. \tag{7.4}$$

Differentiating again, we have

$$0 = \nabla_p \nabla_{\bar{i}} Z^j. \tag{7.5}$$

Trace this on the indices p and $\bar{i}$,

$$0 = g^{p\bar{i}} \nabla_p \nabla_{\bar{i}} Z^j = g^{p\bar{i}} \big(\nabla_{\bar{i}} \nabla_p Z^j + R_{p\bar{i}q}{}^{j} Z^q \big). \tag{7.6}$$

The curvature term is

$$g^{p\bar{i}} R_{p\bar{i}q}{}^{j} = g^{p\bar{i}} g^{j\bar{l}} R_{p\bar{i}\bar{l}q} = g^{j\bar{l}} g^{p\bar{i}} (-R_{p\bar{i}q\bar{l}}) = g^{j\bar{l}} g^{p\bar{i}} (-R_{p\bar{l}q\bar{i}}) = g^{j\bar{l}} R_{\bar{l}q}. \tag{7.7}$$

So we have

$$0 = g^{p\bar{i}} (\nabla_{\bar{i}} \nabla_p Z^j) + g^{j\bar{l}} R_{\bar{l}q} Z^q. \tag{7.8}$$

Note that this is a globally defined tensor equation. Pairing this with $Z_{\bar{k}} = g_{\bar{k}p} Z^p$, and integrating by parts, we obtain

$$0 = -\int_M |\nabla' Z|^2 dV_g + \int_M Ric(Z, \bar{Z}) dV_g, \tag{7.9}$$

where $\nabla' Z$ denotes the $(1,0)$-part of the covariant derivative of Z. So if the *Ric* is negative definite, we conclude that $Z \equiv 0$. □

This result implies that if a Kähler manifold has negative definite Ricci tensor, then the holomorphic automorphsm group is discrete, since the Lie algebra of this group can be identified with the Lie algebra of holomorphic vector fields.

7.2 Serre duality

Note, for this section, we will only assume (M, J) is a compact complex manifold, not necessarily Kähler. But we will choose a metric g which is Hermitian (this can always be done).

For a real oriented Riemannian manifold of dimension n, the Hodge star operator is a mapping

$$* : \Lambda^p \to \Lambda^{n-p} \tag{7.10}$$

defined by

$$\alpha \wedge *\beta = g_{\Lambda^p}(\alpha, \beta) dV_g, \tag{7.11}$$

for $\alpha, \beta \in \Lambda^p$, where dV_g is the oriented Riemannian volume element.

If M is a complex manifold of complex dimension $m = n/2$, and g is a Hermitian metric, then the Hodge star extends to the complexification

$$* : \Lambda^p \otimes \mathbb{C} \to \Lambda^{2m-p} \otimes \mathbb{C}, \tag{7.12}$$

and it is not hard to see that

$$* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}. \tag{7.13}$$

Therefore the operator

$$\overline{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \tag{7.14}$$

is a $\mathbb{C}$-antilinear mapping and satisfies

$$\alpha \wedge \overline{*}\beta = g_{\Lambda^p}(\alpha, \overline{\beta}) dV_g. \tag{7.15}$$

for $\alpha, \beta \in \Lambda^p \otimes \mathbb{C}$.

It turns out that the L^2-adjoint of $\overline{\partial}$ is given by

$$\overline{\partial}^* = -\overline{*}\ \overline{\partial}\ \overline{*}, \tag{7.16}$$

and the $\overline{\partial}$-Laplacian is defined by

$$\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*. \tag{7.17}$$

Letting

$$\mathbb{H}^{p,q}(M, g) = \{\alpha \in \Lambda^{p,q} | \Delta_{\overline{\partial}} \alpha = 0\}, \tag{7.18}$$

Hodge theory tells us that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathbb{H}^{p,q}(M, g), \tag{7.19}$$

is finite-dimensional, and that

$$\Lambda^{p,q} = \mathbb{H}^{p,q}(M, g) \oplus Im(\Delta_{\overline{\partial}}) \tag{7.20}$$

$$= \mathbb{H}^{p,q}(M, g) \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^*), \tag{7.21}$$

with this being an orthogonal direct sum in L^2.

Corollary 7.3. *Let (M, J) be a compact complex manifold of real dimension $n = 2m$. Then*

$$H^{p,q}_{\overline{\partial}}(M) \cong (H^{n-p,n-q}_{\overline{\partial}}(M))^*, \tag{7.22}$$

and therefore

$$b^{p,q}(M) = b^{n-p,n-q}(M) \tag{7.23}$$

Proof. It is easy to see that

$$\bar{*}\ \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}}\ \bar{*}, \tag{7.24}$$

so the mapping $\bar{*}$ preserves the space of harmonic forms, and is invertible. The result then follows from Hodge theory. The dual appears since the operator $\bar{*}$ is $\mathbb{C}$-antilinear. □

A similar argument works with forms taking values in a holomorphic bundle. Choosing a hermitian metric h on E, we have a Hodge star operator

$$\bar{*}_E : \Lambda^{p,q}(E) \to \Lambda^{n-p,n-q}(E^*) \tag{7.25}$$

which is defined by the following. For $\alpha \in \Lambda^{p,q}(E)$. of the form $\alpha_1 \otimes s_1$, where $\alpha_1 \in \Lambda^{p,q}$ and $s_1 \in \Gamma(E)$, and $\gamma \in \Lambda^{n-p,n-q}(E^*)$ of the form $\gamma_1 \otimes s_2$, where $\gamma_1 \in \Lambda^{n-p,n-q}$, and $s_2 \in \Gamma(E^*)$, then

$$\alpha \wedge \gamma \equiv (\alpha_1 \wedge \gamma_1)(s_2(s_1)) \in \Lambda^{n,n}. \tag{7.26}$$

Therefore, we can define an operator

$$\bar{*}_E : \Lambda^{p,q}(E) \to \Lambda^{n-p,n-q}(E^*) \tag{7.27}$$

by insisting that for $\alpha, \beta \in \Lambda^{p,q}(E)$,

$$\alpha \wedge \bar{*}_E \beta = g_{\Lambda^{p,q}(E)}(\alpha, \overline{\beta}) dV_g, \tag{7.28}$$

where dV_g is the Riemannian volume element.

Corollary 7.4. *If $E \to M$ is a holomorphic vector bundle, then*

$$H^{p,q}_{\overline{\partial}}(M, E) \cong \Big(H^{n-p,n-q}(M, E^*)\Big)^* \tag{7.29}$$

Proof. We define $\overline{\partial}_E$ to be the L^2 adjoint of $\overline{\partial}_E$, which is again given by

$$\overline{\partial}^*_E = -\bar{*}_E\ \overline{\partial}_E\ \bar{*}_E, \tag{7.30}$$

and let

$$\Delta_{\overline{\partial}_E} = \overline{\partial}_E \overline{\partial}^*_E + \overline{\partial}^*_E \overline{\partial}_E. \tag{7.31}$$

One can verify easily that

$$\Delta_{\overline{\partial}_E} \bar{*}_E = \bar{*}_E \Delta_{\overline{\partial}_E} \tag{7.32}$$

so $\bar{*}_E$ gives a conjugate linear isomorphism between the corresponding spaces of harmonic form. The results follows from Hodge theory. □

Serre duality is often stated in the following way:

$$H^p(M, \mathcal{O}(E)) \cong \big(H^{n-p}(M, \mathcal{O}(K \otimes E^*))\big)^*, \tag{7.33}$$

where $K = \Lambda^{n,0}$ is the canonical bundle, and these are the cohomologys groups of the sheaf of holomorphic sections of the corresponding bundles. To see this, note by the Dolbeault Theorem,

$$H^p(M, \Omega^q(E)) \cong H^{q,p}_{\bar{\partial}_E}(M, E), \tag{7.34}$$

We then have

$$\begin{aligned} H^p(M, \mathcal{O}(E)) &\cong H^{0,p}_{\bar{\partial}_E}(M, E) \cong \big(H^{n,n-p}_{\bar{\partial}_E}(M, E^*)\big)^* \\ &\cong \big(H^{n-p}(M, \Omega^n(E^*))\big)^* = \big(H^{n-p}(M, \mathcal{O}(K \otimes E^*))\big)^*. \end{aligned} \tag{7.35}$$

8 Lecture 8

8.1 Kodaira vanishing theorem

Proposition 5.8 has a generalization to line bundles.

Proposition 8.1. *Let (M, g, J) be a Kähler manifold, and $L \to M$ a hermitian line bundle with Hermitian metric h. If*

$$\rho_g + \sqrt{-1}\Omega_h > 0, \tag{8.1}$$

then

$$H^q(M, \mathcal{O}(L)) = 0 \textit{ for } q > 0. \tag{8.2}$$

If

$$tr(\rho_g + \sqrt{-1}\Omega_h) > 0, \tag{8.3}$$

then

$$H^n(M, \mathcal{O}(L)) = 0. \tag{8.4}$$

Proof. Similar to the proof of Proposition 5.8 above, each time one commutes a covariant derivative, there is an extra term arising from the curvature of the line bundle h, complete details are left to the reader. □

Theorem 8.2 (Kodaira vanishing)**.** *Let (M, J, g) be Kähler, and let $L \to M$ be a holomorphic line bundle.*

- *If L is positive, then*

$$H^q(M, \mathcal{O}(K \otimes L)) = 0 \textit{ for } 0 < q \le n. \tag{8.5}$$

- *If L is negative, then*

$$H^q(M, \mathcal{O}(L)) = 0 \text{ for } 0 \leq q < n. \tag{8.6}$$

Proof. Since L is positive, there exists a hermitian metric h on L with positive definite curvature form. Let K have the natural metric induced from the Kähler metric g on M. Then we have

$$\sqrt{-1}\Omega_{K\otimes L} = \sqrt{-1}\Omega_K + \sqrt{-1}\Omega_L = -\rho_g + \sqrt{-1}\Omega_L, \tag{8.7}$$

so that

$$\sqrt{-1}\Omega_{K\otimes L} + \rho_g = \sqrt{-1}\Omega_L \tag{8.8}$$

is positive, and therefore $K \otimes L$ satisfies the assumptions of Proposition 8.1.

The second statement follows from Serre duality:

$$H^q(M, \mathcal{O}(L)) \cong \big(H^{n-q}(M, \mathcal{O}(K \otimes L^*))\big)^*. \tag{8.9}$$

Since L is negative, L^* is positive, and the right hand side vanishes for $0 \leq q < n$. □

We also mention the following result.

Theorem 8.3 (Kobayashi-Wu vanishing)**.** *Let (M, J, g) be Kähler, and let $L \to M$ be a holomorphic line bundle.*

- *If L admits a hermitian metric with $tr(\sqrt{-1}\Omega_h) > 0$, then*

$$H^n(M, \mathcal{O}(K \otimes L)) = 0. \tag{8.10}$$

- *If L admits a hermitian metric $tr(\sqrt{-1}\Omega_h) < 0$, then*

$$H^0(M, \mathcal{O}(L)) = 0. \tag{8.11}$$

Proof. This follows from Proposition 8.1. □

There is a more fancy vanishing result called Kodaira-Nakano vanishing which we will not prove:

Theorem 8.4 (Kodaira-Nakano vanishing)**.** *Let (M, J, g) be Kähler, and let $L \to M$ be a holomorphic line bundle.*

- *If L is positive, then*

$$H^p(M, \Omega^q(L)) = 0 \text{ for } n < p + q. \tag{8.12}$$

- *If L is negative, then*

$$H^p(M, \Omega^q(L)) = 0 \text{ for } 0 \leq p + q < n. \tag{8.13}$$

8.2 Complex projective space

Complex projective spaces is defined to be the space of lines through the origin in $\mathbb{C}^{n+1}$. This is equivalent to $\mathbb{C}^{n+1}/\sim$, where $\sim$ is the equivalence relation

$$(z^0, \dots, z^n) \sim (w^0, \dots, w^n) \tag{8.14}$$

if there exists $\lambda \in \mathbb{C}^*$ so that $z^j = \lambda w^j$ for $j = 1 \dots n$. The equivalence class of $(z^0, \dots, z^n)$ will be denoted by $[z^0 : \dots : z^n]$.

The only non-trivial integral cohomology of $\mathbb{CP}^n$ is in even degrees

$$H^{2j}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z} \tag{8.15}$$

for $j = 1 \dots n$.

We next define the tautological bundle

$$\mathcal{O}(-1) = \{([x], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v \in [x]\}. \tag{8.16}$$

The bundle $\mathcal{O}(-1)$ admits a Hermitian metric h by restricting the inner product in $\mathbb{C}^{n+1}$ to a fiber. We define

Definition 8.5. The Fubini-Study metric $\omega_{FS} = -\sqrt{-1}\Omega_h$.

Exercise 8.6. Show the following

- ω_{FS} is positive definite. Consequently, ω_{FS} is a Kähler metric.
- $\int_{\mathbb{P}^1} c_1(\mathcal{O}(-1)) = -1$, for any line $\mathbb{P}^1 \subset \mathbb{P}^n$.

Using Proposition 5.6, it follows that the Hodge numbers are given by

$$b^{p,q}(\mathbb{CP}^n) = \begin{cases} 1 & p = q \\ 0 & p \neq q. \end{cases} \tag{8.17}$$

For a surface, the Hodge diamond is

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} . \tag{8.18}$$

and the Hodge diamond of $\mathbb{CP}^2$ is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{8.19}$$

8.3 Line bundles on complex projective space

If M is any smooth manifold, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^* \to 1. \tag{8.20}$$

where $\mathcal{E}$ is the sheaf of germs of C^∞ functions, and $\mathcal{E}^*$ is the sheaf of germs of non-vanishing C^∞ functions. The associated long exact sequence in cohomology is

$$\begin{aligned} \ldots \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{E}) \to H^1(M,\mathcal{E}^*) \\ \to H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{E}) \to H^2(M,\mathcal{E}^*) \to \ldots. \end{aligned} \tag{8.21}$$

But $\mathcal{E}$ is a flabby sheaf due to existence of partitions of unity in the smooth category, so $H^k(M,\mathcal{E}) = \{0\}$ for $k \geq 1$. This implies that

$$H^1(M,\mathcal{E}^*) \cong H^2(M,\mathbb{Z}). \tag{8.22}$$

Using Čech cohomology, the left hand side is easily seen to be the set of smooth line bundles on M up to equivalence.

Next, if M is a complex manifold, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1. \tag{8.23}$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions, and $\mathcal{O}^*$ is the sheaf of germs of non-vanishing holomorphic functions. The associated long exact sequence in cohomology is

$$\begin{aligned} \ldots \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \\ \xrightarrow{c_1} H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{O}) \to H^2(M,\mathcal{O}^*) \to \ldots. \end{aligned} \tag{8.24}$$

Now $\mathcal{O}$ is not flabby (there are no nontrivial holomorphic partitions of unity!). However

$$\dim(H^k(M,\mathcal{O})) = b^{0,k}. \tag{8.25}$$

Since $b^{0,1} = b^{0,2} = 0$ for $\mathbb{CP}^n$, we have

$$H^1(\mathbb{CP}^n,\mathcal{O}^*) \cong H^2(M,\mathbb{Z}) \cong \mathbb{Z}. \tag{8.26}$$

Again, using Čech cohomology, the left hand side is easily seen to be the set of holomorphic line bundles on M up to equivalence. Consequently, on $\mathbb{CP}^n$ the smooth line bundles are the same as holomorphic line bundles up to equivalence:

Corollary 8.7. *The set of holomorphic line bundles on $\mathbb{CP}^n$ up to equivalence is isomorphic to $\mathbb{Z}$, with the tensor product corresponding to addition.*

The line bundles on $\mathbb{CP}^n$ are denoted by $\mathcal{O}(k)$, where k is the integer obtained under the above isomorphism, which is the first Chern class. Of course, every line bundle must be a tensor power of a generator. If $H \subset \mathbb{CP}^n$ is a hyperplane, then the line bundle corresponding to H, denoted by $[H]$ is $\mathcal{O}(1)$. To see that $[H]$ corresponds to $\mathcal{O}(1)$, use the following:

Proposition 8.8 ([GH78, page 141])**.** *The first Chern class of a complex line bundle L is equal to the Euler class of the underlying oriented real rank 2 bundle, and is the Poincaré dual to the zero locus of a transverse section.*

8.4 Adjunction formula

Let $V \subset M^n$ be a smooth complex hypersurface. The exact sequence

$$0 \to T^{(1,0)}(V) \to T^{(1,0)}M\big|_V \to N_V \to 0, \tag{8.27}$$

defines the holomorphic normal bundle. The adjunction formula says that

$$N_V = [V]|_V. \tag{8.28}$$

To see this, let f_α be local defining functions for V, so that the transition functions of $[V]$ are $g_{\alpha\beta} = f_\alpha f_\beta^{-1}$. Apply d to the equation

$$f_\alpha = g_{\alpha\beta} f_\beta \tag{8.29}$$

to get

$$df_\alpha = d(g_{\alpha\beta}) f_\beta + g_{\alpha\beta} df_\beta. \tag{8.30}$$

Restricting to V, since $f_\beta = 0$ defines V, we have

$$df_\alpha = g_{\alpha\beta} df_\beta. \tag{8.31}$$

Note that df_α is a section of N_V^*. For a smooth hypersurface, the differential of a local defining function is nonzero on normal vectors. Consequently, $N_V^* \otimes [V]$ is the trivial bundle when restricted to V since it has a non-vanishing section.

For any short exact sequence

$$0 \to A \to B \to C \to 0, \tag{8.32}$$

it holds that

$$\Lambda^{\dim(B)}(B) \cong \Lambda^{\dim(A)}(A) \otimes \Lambda^{\dim(C)}(C), \tag{8.33}$$

so the adjunction formula can be rephrased as

$$K_V = (K_M \otimes [V])|_V. \tag{8.34}$$

8.5 del Pezzo surfaces

We begin with the definition.

Definition 8.9. A *del Pezzo surface* or *Fano surface* is a Kähler surface with positive first Chern class, or equivalently, negative canonical bundle.

We will next discuss the following classification.

Proposition 8.10. *A del Pezzo surface is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$ *or* $\mathbb{P}^2$ *blown-up at* $0 \leq k \leq 8$ *distinct points such that no 3 points are on a line, no six are on a conic, and no 8 of them lie on a cubic with one of them a double point.*

Proof. Since the canonical bundle is negative, by Proposition 5.2, it does not admit any non-trivial holomorphic section. Therefore,

$$b^{2,0} = \dim(H^0(M, \mathcal{O}(K)) = 0. \tag{8.35}$$

Since M is Kähler, also $b^{0,2} = 0$. Therefore $H^2(M,\mathbb{R}) = H^{1,1}_{\mathbb{R}}(M)$. The anticanonical class is positive. Some class nearby is rational, and therefore some multiple is an integral, positive class. Therefore M is algebraic by the Kodaira embedding theorem.

Next, by Kodaira's vanishing theorem (Proposition 8.2) and Serre duality (7.33),

$$b^{0,1} = \dim\big(H^1(M, \mathcal{O})\big) = \dim\big(H^1(M, \mathcal{O}(K))\big) = 0, \tag{8.36}$$

and then $b^{1,0} = 0$. Also since K is negative, any power of K is negative, so by Proposition 5.2,

$$\dim(H^0(M, \mathcal{O}(mK))) = 0 \tag{8.37}$$

for any integer $m > 0$.

By Castelnuovo's Theorem M is rational. That is, M is bimeromorphic to $\mathbb{P}^2$. It is not hard to show that if M contains a (-1) curve and has $c_1(M) > 0$, then the blow-down of M, call it M_0 also has $c_1(M_0) > 0$. This is seen by using the fact that

$$c_1(M) = \pi^* c_1(M_0) - [E], \tag{8.38}$$

where E is the exceptional divisor, so that

$$c_1(M)^2 = c_1(M_0)^2 - 1, \tag{8.39}$$

so $c_1(M_0)$ has positive square. Also, it is not hard to see that $c_1(M_0)\cdot[C] > 0$ for any curve C, so $c_1(M_0) > 0$ by Nakai's criterion.

Next, the adjunction formula (8.34), says that for any nonsingular curve D in M,

$$-\chi(D) = -c_1(M)\cdot[D] + [D]^2, \tag{8.40}$$

where $\chi(D)$ is the Euler characteristic of D. Since $c_1(M) > 0$, we obtain the inequality

$$-\chi(D) < [D]^2. \tag{8.41}$$

In particular, if D is a rational curve, we obtain

$$-2 < [D]^2. \tag{8.42}$$

The minimal model of a rational surface is either $\mathbb{P}^2$ or a Hirzebruch surface $F_n = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(n))$. But such a surface contains a rational curve of self-intersection $-n$, so the minimal model is either $F_0 \cong \mathbb{P}^1\times\mathbb{P}^1$, or $\mathbb{P}^2$. Exercise: show that $\mathbb{P}^1\times\mathbb{P}^1$ blown-up at one point is the same as $\mathbb{P}^2$ blown-up at 2 points.

The blow-up points must be distinct because an interated blow-up contains rational curves of self-intersection strictly less than -1. Indeed, if D is any smooth curve on M_0 passing through the blow-up point, then letting $\tilde{D}$ denote the proper transform of D, we have

$$\begin{aligned}[\tilde{D}] \cdot [\tilde{D}] &= [\pi^* D - E] \cdot [\pi^* D - E] \\ &= [\pi^* D] \cdot [\pi^* D] + [E] \cdot [E] = [D] \cdot [D] - 1.\end{aligned} \tag{8.43}$$

The remaining statement is proved by showing that these "bad" configurations of points are the only configurations which have rational curves of self-intersection strictly less than -1, but we omit the details. □

We see that the Hodge diamond of a del Pezzo surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & k+1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} , \tag{8.44}$$

where k is the number of blow-up points, or $k = 1$ in the case of $\mathbb{P}^1 \times \mathbb{P}^1$. Also, we have that

$$c_1^2(M) = c_1^2(\mathbb{P}^2) - k = 9 - k. \tag{8.45}$$

We call $b = 9 - k$ the degree of the del Pezzo surface. Note that $c_1^2(M) = [T] \cdot [T] = b$, where T is an anticanonical divisor, so the degree of the normal bundle of an anticanonical divisor is $b = 9 - k$.

Note, by Yau's Theorem, all del Pezzo surfaces admit metrics with positive Ricci curvature.

Exercise 8.11. Prove the following.

- The Fubini-Study metric on $\mathbb{P}^n$ is a Kähler-Einstein metric.
- The product metric $(\mathbb{P}^n \times \mathbb{P}^n, g_{FS} + g_{FS})$ is Kähler-Einstein.

For surfaces, Tian proved the following.

Theorem 8.12 (Tian)**.** *For $3 \leq k \leq 8$, any del Pezzo surface $\mathbb{P}^2$ blow-up at k points admits a positive Kähler-Einstein metric in the anticanonical class.*

9 Lecture 9

9.1 Hirzebruch Signature Theorem

We think of a complex surface V as a real 4-manifold, with complex structure given by J. Then the kth Pontrjagin Class is defined to be

$$p_k(V) = (-1)^k c_{2k}(TV \otimes \mathbb{C}) \tag{9.1}$$

Since (V, J) is complex, we have that

$$TV \otimes \mathbb{C} = TV \oplus \overline{TV}, \tag{9.2}$$

so

$$\begin{aligned} c(TV \otimes \mathbb{C}) &= c(TV) \cdot c(\overline{TV}) && (9.3)\\ &= (1 + c_1 + c_2) \cdot (1 - c_1 + c_2) && (9.4)\\ &= 1 + 2c_2 - c_1^2, && (9.5) \end{aligned}$$

which yields

$$p_1(V) = c_1^2 - 2c_2. \tag{9.6}$$

Consider next the intersection pairing $H^2(V) \times H^2(V) \to \mathbb{R}$, given by

$$(\alpha, \beta) \to \int \alpha \wedge \beta \in \mathbb{R}. \tag{9.7}$$

Let b_2^+ denote the number of positive eigenvalues, and b_2^- denote the number of negative eigenvalues. By Poincaré duality the intersection pairing is non-degenerate, so

$$b_2 = b_2^+ + b_2^-. \tag{9.8}$$

The *signature* of V is defined to be

$$\tau = b_2^+ - b_2^-. \tag{9.9}$$

The Hirzebruch Signature Theorem [MS74, page 224] states that

$$\begin{aligned} \tau &= \frac{1}{3} \int_V p_1(V) && (9.10)\\ &= \frac{1}{3} \int_V (c_1^2 - 2c_2). && (9.11) \end{aligned}$$

The Gauss-Bonnet Theorem says that $\int_V c_2 = \chi(V)$, so this can be rewritten as,

$$2\chi + 3\tau = \int_V c_1^2. \tag{9.12}$$

Remark 9.1. This implies that S^4 does not admit any almost complex structure, since the left hand side is 4, but the right hand side trivially vanishes.

9.2 Representations of $U(2)$

As discussed above, some representations which are irreducible for $SO(4)$ become reducible when restricted to $U(2)$. Under $SO(4)$, we have

$$\Lambda^2 T^* = \Lambda^2_+ \oplus \Lambda^2_-, \tag{9.13}$$

where

$$\Lambda_2^+ = \{\alpha \in \Lambda^2(M, \mathbb{R}) : *\alpha = \alpha\} \tag{9.14}$$

$$\Lambda_2^- = \{\alpha \in \Lambda^2(M, \mathbb{R}) : *\alpha = -\alpha\}. \tag{9.15}$$

But under U(2), we have the decomposition

$$\Lambda^2 T^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}. \tag{9.16}$$

Notice that these are the complexifications of real vector spaces. The first is of dimension 2, the second is of dimension 4. Let ω denote the 2-form $\omega(X,Y) = g(JX, Y)$. This yields the orthogonal decomposition

$$\Lambda^2 T^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R} \cdot \omega \oplus \Lambda_0^{1,1}, \tag{9.17}$$

where $\Lambda_0^{1,1} \subset \Lambda^{1,1}$ is the orthogonal complement of the span of ω, and is therefore 2-dimensional (the complexification of which is the space of *primitive* $(1,1)$-forms).

Proposition 9.2. *Under* U(2)*, we have the decomposition*

$$\Lambda^2_+ = \mathbb{R} \cdot \omega \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \tag{9.18}$$

$$\Lambda^2_- = \Lambda_0^{1,1}. \tag{9.19}$$

Proof. We can choose an oriented orthonormal basis of the form

$$\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}. \tag{9.20}$$

Let $\{e^1, e^2, e^3, e^4\}$ denote the dual basis. The space of $(1,0)$ forms, $\Lambda^{1,0}$ has generators

$$\theta^1 = e^1 + ie^2, \quad \theta^2 = e^3 + ie^4. \tag{9.21}$$

We have

$$\begin{aligned} \omega &= \frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 + \theta^2 \wedge \overline{\theta}^2) \\ &= \frac{i}{2}\Big((e^1 + ie^2) \wedge (e^1 - ie^2) + (e^3 + ie^4) \wedge (e^3 - ie^4)\Big) \\ &= e^1 \wedge e^2 + e^3 \wedge e^4 = \omega^1_+. \end{aligned} \tag{9.22}$$

Similarly, we have

$$\frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 - \theta^2 \wedge \overline{\theta}^2) = e^1 \wedge e^2 - e^3 \wedge e^4 = \omega^1_-, \tag{9.23}$$

so ω^1_- is of type $(1,1)$, so lies in $\Lambda^{1,1}_0$. Next,

$$\begin{aligned}\theta^1 \wedge \theta^2 &= (e^1 + ie^2) \wedge (e^3 + ie^4)\\ &= (e^1 \wedge e^3 - e^2 \wedge e^4) + i(e^1 \wedge e^4 + e^2 \wedge e^3)\\ &= \omega^2_+ + i\omega^3_+.\end{aligned} \tag{9.24}$$

Solving, we obtain

$$\omega^2_+ = \frac{1}{2}(\theta^1 \wedge \theta^2 + \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{9.25}$$

$$\omega^3_+ = \frac{1}{2i}(\theta^1 \wedge \theta^2 - \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{9.26}$$

which shows that ω^2_+ and ω^3_+ are in the space $\Lambda^{2,0} \oplus \Lambda^{0,2}$. Finally,

$$\begin{aligned}\theta^1 \wedge \overline{\theta}^2 &= (e^1 + ie^2) \wedge (e^3 - ie^4)\\ &= (e^1 \wedge e^3 + e^2 \wedge e^4) + i(-e^1 \wedge e^4 + e^2 \wedge e^3)\\ &= \omega^2_- - i\omega^3_-,\end{aligned} \tag{9.27}$$

which shows that ω^2_- and ω^3_- are in the space $\Lambda^{1,1}_0$. □

This decomposition also follows from the proof of the Hodge-Riemann bilinear relations [GH78, page 123].

Corollary 9.3. *If (M^4, g) is Kähler, then*

$$b_2^+ = 1 + 2b^{2,0}, \tag{9.28}$$

$$b_2^- = b^{1,1} - 1, \tag{9.29}$$

$$\tau = b_2^+ - b_2^- = 2 + 2b^{2,0} - b^{1,1}. \tag{9.30}$$

Proof. This follows from Proposition 9.2, and Hodge theory on Kähler manifolds, see [GH78]. □

So we have that

$$\chi = 2 + b_2 = 2 + b^{1,1} + 2b^{2,0} \tag{9.31}$$

$$\tau = 2 + 2b^{2,0} - b^{1,1}. \tag{9.32}$$

In the del Pezzo case, from Proposition 8.10, we conclude that

$$b_2 = b^{1,1} = k + 1,\ b_2^+ = 1,\ b_2^- = b^{1,1} - 1 = k, \tag{9.33}$$

and

$$2\chi + 3\tau = 9 - k = b. \tag{9.34}$$

Let us assume that the canonical bundle is trivial and M is simply connected. The signature theorem says that

$$2\chi + 3\tau = 0, \tag{9.35}$$

which is

$$4 + 5b_2^+ - b_2^- = 0. \tag{9.36}$$

Since the canonical bundle is trivial, $b^{0,2} = b^{2,0} = 1$, so we have that

$$b_2^+ = 1 + 2b^{2,0} = 3, \tag{9.37}$$

and therefore we have

$$b_2^- = 19,\ b_2 = 22,\ \chi = 24,\ \tau = -16. \tag{9.38}$$

The Hodge numbers in this case are $b^{1,1} = 20$, $b^{0,2} = b^{2,0} = 1$, so the Hodge diamond of a $K3$ surface is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \quad . \tag{9.39}$$

It follows from the classification of quadratic forms over the integers that the intersection form is given by

$$2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{9.40}$$

For the Ricci-flat metric of Yau on $K3$, it follows that Λ^2_+ is a flat bundle. Since $K3$ is simply connected, there exist 3 orthonormal parallel self-dual 2-forms, call them

$$\omega_I, \omega_J, \omega_K. \tag{9.41}$$

Using the metric, these are Kähler forms associated to orthogonal complex structures I, J, K. That is $I^2 = J^2 = K^2$, and we can assume that $IJ = K$. The metric is Kähler with respect to $aI + bJ + cK$ for $a^2 + b^2 + c^2 = 1$, i.e., $(a, b, c) \in S^2 \subset \mathbb{R}^3$. Such a structure is called a *hyperkähler* structure.

9.3 Examples

Let $V \subset \mathbb{P}^3$ be a nonsingular degree d hypersurface, giving by the vanishing of a homogeneous polynomial of degree d.

Exercise 9.4. Prove that $K_{\mathbb{P}^3} \cong \mathcal{O}(-4)$,

By the adjunction formula (8.34)

$$K_V = (K_M \otimes [V])|_V = (O(-4) \otimes O(d))|_V = (\mathcal{O}(d-4))|_V. \tag{9.42}$$

Note that by the Lefschetz hyperplane theorem, V is simply connected.

- If $d = 1$, then $V \cong \mathbb{P}^2$.
- If $d = 2$, them $c_1(V) = 2c_1([\mathcal{O}(1)])|_V$. This is positive, so V is a del Pezzo surface. Since $\int_M c_1(V)^2 = 4[H] \cdot [V] = 8$, V is a degree 8 del Pezzo. Thus V is either $Bl_p(\mathbb{P}^2)$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Exercise: prove it is the latter case.
- If $d = 3$, $c_1(V) = c_1([\mathcal{O}(1)])_V$. Again, this is positive, so V is a del Pezzo surface. Since $\int_M c_1(V)^2 = [H] \cdot [V] = 3$, V is a degree 3 del Pezzo, and V is biholomorphic to $Bl_{p_1,\ldots,p_6}\mathbb{P}^2$, with the six points not all on a conic.
- For $d = 4$, the canonical bundle is trivial, thus V is a $K3$ surface, and V admits a Ricci-flat metric by Theorem 6.5, which must be hyperkähler.
- For $d > 4$, the first Chern class is negative, so these surfaces admit negative Kähler-Einsten metrics in the canonical class, by Theorem 6.6.

References

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 76.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO, JAPAN
E-mail Address: jviaclov@uci.edu