

Riemannian Curvature Functionals: Lecture III

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July 18, 2013

Today we will discuss the following:

- Complete the local description of the moduli space of Einstein metrics.
- Rigidity of Einstein metrics and discussion of some basic examples.
- Quadratic functionals
- Hitchin-Thorpe and examples of Einstein metrics in dimension 4.
- Self-dual metrics and their deformation theory.

Recall from the previous lecture that our goal is to prove:

Theorem

Assume g is Einstein with $Ric(g) = \lambda \cdot g$ and $\lambda < 0$. Then the space of Einstein metrics near g modulo diffeomorphism is locally isomorphic to the zero set of a map

$$\Psi : H^1 \rightarrow H^1,$$

where

$$H^1 = \{h \in S^2(T^*M) : \delta_g h = 0, tr_g h = 0, \Delta h + 2Rm * h = 0\}$$

The nonlinear map

Given $\theta \in C^{2,\alpha}(S^2T^*M)$, consider the map

$$P : C^{2,\alpha}(S^2(T^*M)) \rightarrow C^{0,\alpha}(S^2(T^*M))$$

by

$$P_g(\theta) = Ric(g + \theta) - \lambda \cdot (g + \theta) - \frac{1}{2} \mathcal{L}_{g+\theta} \beta_g \theta.$$

Last time we showed the linearization is elliptic. Next, we will examine the zero set of P .

Proposition

Assume that $\lambda < 0$. If $\theta \in C^{3,\alpha}$ is sufficiently near g and satisfies $P(g + \theta) = 0$, then $Ric(g + \theta) = \lambda(g + \theta)$, and $\theta \in C^\infty$.

To prove this, apply the operator $\beta_{g+\theta}$ to the equation $P(g + \theta) = 0$ to obtain

$$\beta_{g+\theta} \mathcal{L}_{g+\theta} \beta_g \theta = 0$$

A computation shows that this is equivalently (exercise):

$$(\Delta_{g+\theta} + Ric(g + \theta))(\beta_g \theta) = 0.$$

Since θ is sufficiently small in $C^{2,\alpha}$ norm, and $Ric(g)$ is strictly negative definite, then $Ric_{g+\theta}$ is also strictly negative definite. Pairing with $\beta_g \theta$ and integrating by parts then shows that $\beta_g \theta = 0$.

Exercise

- *Prove the regularity statement. Hint: Letting $\tilde{g} = g + \theta$, in harmonic coordinates, the Ricci tensor can be written in the form*

$$Ric_{kl}(\tilde{g}) = -\frac{1}{2}\tilde{g}^{ij}\partial_{ij}^2\tilde{g}_{kl} + Q_{kl}(\partial\tilde{g}, \tilde{g})$$

where $Q(\partial\tilde{g}, \tilde{g})$ is an expression that is quadratic in $\partial\tilde{g}$, polynomial in \tilde{g} and has $\sqrt{|\tilde{g}|}$ in its denominator. Use a bootstrap argument in these coordinates.

- *Show that we only need to assume that $\theta \in C^{2,\alpha}$. Hint: instead of differentiating in the first step, integrate by parts.*

We have already seen that zeroes of P are Einstein metrics. Now we have a converse:

Proposition

If \tilde{g} is an Einstein metric near g with Einstein constant λ , then there exists a diffeomorphism $\varphi : M \rightarrow M$ such that $\tilde{\theta} = \varphi^\tilde{g} - g$ satisfies $P_g(\tilde{\theta}) = 0$.*

Proof of proposition

The proof uses a modified Ebin-Palais slice theorem:

Lemma

For each metric g_1 in a sufficiently small $C^{\ell+1,\alpha}$ -neighborhood of g ($\ell \geq 1$), there is a $C^{\ell+2,\alpha}$ -diffeomorphism $\varphi : M \rightarrow M$ such that

$$\tilde{\theta} \equiv \varphi^* g_1 - g$$

satisfies

$$\beta_g(\tilde{\theta}) = 0$$

The proof of the lemma is almost identical to the previous one, and is omitted.

To prove the proposition, if \tilde{g} is Einstein then $\varphi^* \tilde{g}$ is also Einstein. Since

$$\beta_g(\tilde{\theta}) = \beta_g(\varphi^* \tilde{g} - g) = 0,$$

we obviously obtain a zero of P_g .

Structure of nonlinear terms

Let us expand

$$P_g(\theta) = P_g(0) + P'_g(\theta) + Q_g(\theta)$$

Proposition

For θ_1, θ_2 sufficiently small, we have

$$\|Q(\theta_1) - Q(\theta_2)\|_{C^{0,\alpha}} \leq C(\|\theta_1\|_{C^{2,\alpha}} + \|\theta_2\|_{C^{2,\alpha}})\|\theta_1 - \theta_2\|_{C^{2,\alpha}}.$$

The proof is left as an exercise, with the following hint: Show that

$$Rm(g+h) = Rm(g) + (g+h)^{-1} * \nabla^2 h + (g+h)^{-2} * \nabla h * \nabla h.$$

Contract with $(g+h)^{-1}$ to get $Ric(g+h)$ and then use

$$(g+h)^{-1} = g^{-1} - g^{-1}(g+h)^{-1}h.$$

Lemma

Let $H : E \rightarrow F$ be a smooth map between Banach spaces. Define $Q = H - H(0) - H'(0)$. Assume that there are positive constants C_1, s_0, C_2 so that the following are satisfied:

- (1) The nonlinear term Q satisfies

$$\|Q(x) - Q(y)\|_F \leq C_1(\|x\|_E + \|y\|_E)\|x - y\|_E$$

for every $x, y \in B_E(0, s_0)$.

- (2) The linearized operator at 0, $H'(0) : E \rightarrow F$ is an isomorphism with inverse bounded by C_2 .

If s and $\|H(0)\|_F$ are sufficiently small (depending upon C_1, s_0, C_2), then there is a unique solution $x \in B_E(0, s)$ of the equation $H(x) = 0$.

Proof of implicit function theorem

The equation $H(x) = 0$ expands to

$$H(0) + H'(0)(x) + Q(x) = 0.$$

If we let $x = Gy$, where G is the inverse of $H'(0)$, then we have

$$H(0) + y + Q(Gy) = 0,$$

or

$$y = -H(0) - Q(Gy).$$

In other words, y is a fixed point of the mapping

$$T : y \mapsto -H(0) - Q(Gy).$$

With the assumptions in the lemma, it follows that T is a contraction mapping, so a fixed point exists by the standard fixed point theorem ($T^n y_0$ converges to a unique fixed point for any y_0 sufficiently small).

We next construct the map

$$\Psi : H^1 \rightarrow H^1,$$

whose zero set is locally isomorphic to the zero set of P :

Consider $H = \Pi \circ P$, where Π is projection to the orthogonal complement of H^1 . This map now has surjective differential. Let G be a right inverse, i.e., $H'(0)G = Id$. Given a kernel element $x_0 \in H^1$, the equation $H(x_0 + Gy) = 0$ expands to

$$H(0) + H'(0)(x_0 + Gy) + Q(x_0 + Gy) = 0.$$

We therefore need to find a fixed point of the map

$$T_{x_0} : y \mapsto -H(0) - Q(x_0 + Gy),$$

and the proof is the same as before.

Kernel of the linearized operator

To finish the proof of the theorem, we need to identify the kernel of the linearized operator:

Proposition

If $\lambda < 0$, then $Ker(P')$ consists exactly of transverse-traceless tensors satisfying

$$\Delta h + 2Rm * h = 0.$$

If $P'(h) = 0$, then h is smooth by elliptic regularity. Also,

$$P'h = Ric'(h) - \lambda h - \frac{1}{2}\mathcal{L}_g\beta_g h.$$

Applying β_g to this equation, yields $\beta_g\mathcal{L}_g\beta_g h = 0$, so $\beta_g h = 0$ by the above argument. Taking a trace, we find that

$$\Delta(tr_g(h)) + 2\lambda \cdot tr_g(h) = 0,$$

so $tr_g(h) = 0$ since $\lambda < 0$.

Elements in the space H^1 are called infinitesimal Einstein deformations. In general it is quite difficult to construct the map Ψ explicitly, but one of the easiest consequences of the above discussion is the following:

Corollary

If $\text{Ric}(g) = \lambda \cdot g$ with $\lambda < 0$, and $H^1 = \{0\}$ then g is rigid (isolated as an Einstein metric). That is, if g_t is a path of Einstein metrics passing through g , all with Einstein constant $\lambda < 0$, then there exist a path of diffeomorphisms $\varphi_t : M \rightarrow M$ such that $g_t = \varphi_t^ g$.*

The negative case

In the hyperbolic case, in Lecture I we proved that $H^1 = \{0\}$, so hyperbolic manifolds are locally rigid as Einstein metrics. In fact, something much stronger is true:

Theorem (Besson-Courtois-Gallot)

If (M^4, g) is compact and hyperbolic, then g is the unique Einstein metric on M , up to scaling.

This is proved using completely different methods than we have discussed here, which we will not have time to go into.

Exercise

Show that any Einstein metric with negative sectional curvature is rigid, that is, $H^1 = \{0\}$.

The positive case

It is possible to modify the above construction so that it works also in the positive Einstein case, but we will omit the details due to time considerations.

We will just mention two problems that come up:

- A positive Einstein metric can admit a nontrivial group of isometries (identity component). However, this is not a serious problem; the end result is that the map Ψ is equivariant with respect to the isometry group, and the actual moduli space is described by $\Psi^{-1}(0)/\text{Isom}(g)$, rather than just $\Psi^{-1}(0)$.
- In the case of the sphere, we run into the problem of first nontrivial eigenfunctions yielding pure trace kernel. However, these can also be “gauged away” since they arise as tangents to conformal diffeomorphisms. Details will be omitted.

Some rigid examples in the positive case

Some rigid examples:

- Any spherical space form S^4/Γ with the round metric g_S . In this case, we saw in Lecture I that

$$\frac{1}{2}\Delta h - h,$$

which obviously has trivial kernel.

- $S^2 \times S^2$ with the product metric $g_{S^2} + g_{S^2}$. In this case, we saw in Lecture I that the first two eigenvalues of $\frac{1}{2}\Delta h + Rm * h$ are -1 and 1 , thus 0 does not occur as an eigenvalue
- $(\mathbb{C}P^2, g_{FS})$, where g_{FS} is the Fubini-Study metric. We will not have time to prove this case in these lectures, since the nicest proof involves the theory of deformations of Kähler-Einstein metrics, and needs a considerable amount of background in complex geometry.

The zero case

We saw that, in the case of a flat metric, H^1 consists of parallel sections. The Kuranishi map turns out to be identically zero in this case, since all of these are “integrable”, corresponding to deformations of the flat structure.

Exercise

Determine the moduli space of flat structures on a 2-torus.

Another special class of Ricci-flat metrics are Calabi-Yau metrics. In this case, the following is known (we will not discuss the proof):

Theorem (Tian-Todorov)

For a Calabi-Yau metric (X, g) , the Kuranishi map $\Psi \equiv 0$. That is, every infinitesimal Einstein deformation integrates to an actual deformation.

Quadratic curvature functionals

A basis for the space of quadratic curvature functionals is

$$\mathcal{W} = \int |W|^2 dV, \quad \rho = \int |\text{Ric}|^2 dV, \quad \mathcal{S} = \int R^2 dV.$$

In dimension four, the Chern-Gauss-Bonnet formula

$$32\pi^2\chi(M) = \int |W|^2 dV - 2 \int |\text{Ric}|^2 dV + \frac{2}{3} \int R^2 dV$$

implies that any one of these can be written as a linear combination of the other two (plus a topological term).

Remark

We are using the tensor norm on $|W|^2$, that is $|W|^2 = W^{ijkl}W_{ijkl}$. This differs from the norm of W as a mapping on 2-forms by a factor of 4.

Curvature in dimension 4

If (M^4, g) is oriented, then $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$, and the curvature operator has the corresponding decomposition:

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{R}{12}I & E \\ \hline E & W^- + \frac{R}{12}I \end{array} \right),$$

where $E = Ric - (R/4)g$ is the traceless Ricci tensor.

Theorem (Hirzebruch)

$$48\pi^2\tau(M) = \int_M |W^+|^2 dV - \int_M |W^-|^2 dV,$$

where $\tau = b_2^+ - b_2^-$ is the signature of M .

Hitchin-Thorpe Inequality

One of the only known obstructions to the existence of Einstein metrics is the following:

Theorem (Hitchin-Thorpe)

If (M^4, g) is Einstein and oriented, then

$$2\chi(M) \geq 3|\tau(M)|,$$

with equality if and only if g is flat or finitely covered by $K3$ with a Ricci-flat metric.

Exercise

Show that this inequality follows from the Chern-Gauss-Bonnet formula and Hirzebruch signature theorem.

Some examples of Einstein metrics in dimension 4

The only known compact examples with non-negative Einstein constant:

- S^4 or $\mathbb{R}P^4$ with the round metric.
- $S^2 \times S^2$ with the product metric, its orientable $\mathbb{Z}/2\mathbb{Z}$ quotient, and $\mathbb{R}P^2 \times \mathbb{R}P^2$ with the product metric.
- $\mathbb{C}P^2$ with the Fubini-Study metric.
- $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ with the Page metric (an explicit $U(2)$ invariant Einstein metric). Non-orientable quotient $\mathbb{C}P^2 \# \mathbb{R}P^4$.
- $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ with the Chen-LeBrun-Weber metric. Conformal to an extremal Kähler metric.
- $\mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2$, $k = 3, \dots, 8$, admits positive Kähler-Einstein metrics (Tian-Yau, Tian).
- $K3$ with Calabi-Yau Ricci-flat metric (Yau), and its quotients.
- T^4 with a flat metric, and quotients.

Remark

Find a new compact positive Einstein or Ricci-flat metric in dimension 4, and you will be famous.

Anti-self-dual or self-dual metrics

In dimension 4, the curvature condition $W^+ = 0$ is called anti-self-dual (ASD), $W^- = 0$ is called self-dual (SD).

Either condition is conformally invariant.

Proposition

If (M^4, g_0) compact and self-dual, then g_0 is a global minimizer of the functional \mathcal{W} , so is necessarily a critical metric for \mathcal{W} .

Proof.

For any metric g on M , using the signature theorem, we have

$$\begin{aligned}\mathcal{W}(g) &= \int |W|^2 dV_g = \int |W^+|^2 dV_g + \int |W^-|^2 dV_g \\ &= 48\pi^2\tau(M) + 2 \int_M |W^-|^2 dV_g \geq 48\pi^2\tau(M),\end{aligned}$$

with equality if and only if g is self-dual. □

Topological obstruction

The only obvious topological obstruction to existence of a self-dual or anti-self-dual metric comes from the signature theorem:

Proposition

If (M, g) is self-dual (anti-self-dual) then $\tau \geq 0$ ($\tau \leq 0$) with equality if and only if g is locally conformally flat.

One can get a stronger restriction if one assumes the scalar curvature is positive:

Proposition

If (M, g) is self-dual (anti-self-dual) and $R > 0$, then $b_2^- = 0$ ($b_2^+ = 0$).

Proof: On 2-forms in dimension 4, the Weitzenböck formula is

$$(\Delta_H \omega^\pm)_{ij} = -(\Delta \omega^\pm)_{ij} - \sum_{l,m} W_{lmij}^\pm \omega_{lm}^\pm + \frac{R}{3} \omega_{ij}^\pm.$$

Deformation theory of self-dual metrics

We first define

$$H_c^0(M, g) = \{\kappa \in T^*M \mid \mathcal{K}_g \kappa = 0\}.$$

If g is self-dual, we let \mathcal{D}_g denote $(W^-)'_g$. We then define

$$H_c^1(M, g) = \{h \in S_0^2(T^*M) \mid \mathcal{D}_g h = 0, \delta_g h = 0\}.$$

We also define

$$H_c^2(M, g) = \{Z \in S_0^2(\Lambda_-^2) \mid \mathcal{D}_g^* Z = 0\}.$$

Theorem

If g is self-dual, then the moduli space of self-dual metrics near g (modulo diffeomorphism and conformal equivalence), is locally isomorphic to the zero set of a map

$$\Psi : H_c^1(M, g)/H_c^0(M, g) \rightarrow H_c^2(M, g)/H_c^0(M, g).$$

Theorem

If (M, g) is compact and self-dual, then

$$\dim(H_c^0) - \dim(H_c^1) + \dim(H_c^2) = \frac{1}{2}(15\chi(M) - 29\tau(M)).$$

This is proved using the Atiyah-Singer Index Theorem, applied to the elliptic complex

$$\Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S_0^2(T^*M)) \xrightarrow{\mathcal{D}_g} \Gamma(S_0^2(\Lambda_+^2)).$$

Remark

The $H_c^i(M, g)$, for $i = 0, 1, 2$, defined above are simply the cohomology groups of this complex.

Weitzenböck formulas

For (M, g) Einstein, with $Ric = \lambda \cdot g$, define the *Lichnerowicz Laplacian* by

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ipjq}h^{pq} - 2\lambda \cdot h_{ij}.$$

A computation shows that

Theorem (Itoh)

If (M, g) is compact and self-dual Einstein with $Ric = \lambda \cdot g$, then

$$\mathcal{D}^* \mathcal{D} h = \frac{1}{2} \left(\Delta_L + 2\lambda \right) \left(\Delta_L + \frac{4}{3}\lambda \right) h$$

$$\mathcal{D} \mathcal{D}^* Z = \frac{1}{12} (3\Delta - 8\lambda) (\Delta - 2\lambda) Z.$$

Hint:

$$(\mathcal{D}^* Z)_{ij} = 2\nabla^k \nabla^l Z_{ikjl}.$$

Compute $H_c^i(M, g)$ for the following examples:

- S^4 with the round metric g_S .
- $\mathbb{C}\mathbb{P}^2$ with the Fubini-Study metric g_{FS} (this is self-dual with respect to the complex orientation).
- $S^1 \times S^3$ with the product metric g (this is not Einstein, so you cannot use the Weitzenböck formulas. But it is locally conformally flat). What is the dimension of the moduli space near g ?

It is known that $n\#\mathbb{C}\mathbb{P}^2$ admits self-dual metrics for any $n \geq 1$ (Donaldson-Friedman, Floer, LeBrun, Poon).

- What is the index for any self-dual metric on $n\#\mathbb{C}\mathbb{P}^2$?

A conjecture

We mention here the following:

Conjecture (Singer 1978)

If (M, g) is self-dual and $R > 0$, then $H_c^2(M, g) = 0$.

A wealth of examples of self-dual metrics have been found since this conjecture was made, and all of the ones with positive scalar curvature have turned out to have $H_c^2(M, g) = 0$. But despite all of the evidence, a proof of this conjecture remains elusive.

K3 metric

The $K3$ surface is defined to be a nondegenerate quartic surface in $\mathbb{C}\mathbb{P}^3$, that is

$$K3 = \{[z_0, z_1, z_2, z_3] \in \mathbb{C}\mathbb{P}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

Topology: $\pi_1(K3) = \{e\}$, $b_2 = 22$, $b_2^+ = 3$, $b_2^- = 19$.

Since $c_1(K3) = 0$, by Yau's solution of the Calabi conjecture, $K3$ admits a Ricci-flat Kähler metric g_Y .

Proposition

$(K3, g_Y)$ is anti-self-dual with respect to the complex orientation.

To see this we use that fact that for any Kähler metric, W^+ is entirely determined by the scalar curvature. In fact,

$$W^+ = \frac{R}{12}(3\omega \odot \omega - I),$$

where ω is the Kähler form.

(Note: this implies that $|W^+|^2 = R^2/6$ for any Kähler metric).

Using the Index Theorem, and the Weitzenböck formulas from above, show that:

- $\dim(H_c^0(K3, g_Y)) = 0.$
- $\dim(H_c^1(K3, g_Y)) = 57.$
- $\dim(H_c^2(K3, g_Y)) = 5.$

In fact, using the isomorphism $S_0^2(T^*M) = \Lambda_+^2 \otimes \Lambda_-^2$, it can be shown that $H_c^1(K3, g_Y)$ has a basis

$$\{\omega_I \otimes \omega_j^-, \omega_J \otimes \omega_j^-, \omega_K \otimes \omega_j^-\},$$

where $\omega_I, \omega_J, \omega_K$ are a basis of the space of self-dual harmonic 2-forms (these are actually Kähler forms for complex structures I, J, K), and $\{\omega_j^-, j = 1, \dots, 19\}$ is a basis of the space of anti-self-dual harmonic 2-forms.

Furthermore, by the Weitzenböck formula, $H_c^1 = H_E^1$ (infinitesimal Einstein deformations) and the moduli space is exactly 57 dimensional; the Kuranishi map $\Psi \equiv 0$ by Tian-Todorov.