Riemannian Curvature Functionals: Lecture II

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Lecture Outline

Today we will discuss the following:

- To understand the local behavior of the functional near an Einstein metric, we only need to look at conformal and transverse-traceless directions, the other directions do not matter.
- Ebin-Palais slice theorem; consider the functional as a mapping from $\mathcal{M}/\mathcal{D}$, the space of Riemannian metrics modulo diffeomorphisms.
- Saddle point structure and the Yamabe invariant.
- Local description of the moduli space of Einstein metrics.

Splitting

We let $\mathcal{K} : T^*M \to S_0^2(T^*M)$ be the conformal Killing operator

$$(\mathcal{K}\alpha)_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i - \frac{1}{2}(\delta\alpha) g_{ij}.$$

Lemma

The space of symmetric 2-tensors admits the following orthogonal decomposition:

$$S^2(T^*M) = \{f \cdot g\} \oplus \{\mathcal{K}(\alpha)\} \oplus \{\delta h = 0, tr_g(h) = 0\}.$$

To prove this, consider the operator $\Box = \delta\mathcal{K}$.

Exercise

(i) Show that this operator is elliptic and self-adjoint.
(ii) Prove that the kernel of $\Box$ is exactly the space of conformal Killing forms.

Proof of Lemma

Given $h \in S^2_0(T^*M)$, consider the 1-form δh. By Fredholm theory, the equation $\Box\alpha = \delta h$ has a solution if and only if δh is orthogonal to the kernel of the adjoint operator, which is exactly the space of conformal Killing 1-forms (by the exercise). If κ is any conformal Killing 1-form, then

$$\int_M \langle \delta h, \kappa \rangle = \frac{1}{2} \int_M \langle h, \mathcal{K}\kappa \rangle = 0.$$

So the equation $\Box\alpha = \delta h$ has a solution, which proves that $h - \mathcal{K}\alpha$ is divergence-free.

Another decomposition

We note that the orthogonal decomposition

$$S^2(T^*M) = \{f \cdot g\} \oplus \{\mathcal{K}(\alpha)\} \oplus \{\delta h = 0, tr_g(h) = 0\}$$

implies the decomposition

$$S^2(T^*M) = \{f \cdot g\} \oplus \{\mathcal{L}(\alpha)\} \oplus \{\delta h = 0, tr_g(h) = 0\}.$$

Proposition

If (M, g) is Einstein, with $Ric = \lambda \cdot g$, then this latter decomposition is a direct sum, unless (M, g) is isometric to (S^4, g_S).

Proof of proposition

We need to show that the spaces $\{f \cdot g\}$ and $\{\mathcal{L}(\alpha)\}$ have intersection $\{0\}$. So if $\mathcal{L}(\alpha) = f \cdot g$, then taking a trace, we have

$$2\delta\alpha = 4f,$$

which implies that $\mathcal{K}(\alpha) = 0$. Taking a divergence, we have

$$\begin{aligned}\nabla_i(\nabla_i\alpha_j + \nabla_j\alpha_i - (1/2)(\delta\alpha)g_{ij}) &= \Delta\alpha_j + \nabla_i\nabla_j\alpha_i - (1/2)\nabla_j(\delta\alpha) \\ &= \Delta\alpha_j + (1/2)\nabla_j(\delta\alpha) + \lambda\alpha_j.\end{aligned}$$

However, we have that

$$\Delta\alpha = -\Delta_H\alpha + Ric * \alpha = (d\delta + \delta d)\alpha + \lambda\alpha.$$

Putting these together, we obtain

$$\Box\alpha = (3/2)d\delta\alpha + \delta d\alpha + 2\lambda\alpha.$$

Proof of proposition continued

Pairing the equation

$$(3/2)d\delta\alpha + \delta d\alpha + 2\lambda\alpha = 0,$$

with α and integrating,

$$-\frac{3}{2}\int_M |\delta\alpha|^2 - \int_M |d\alpha|^2 + 2\lambda\int_M |\alpha|^2 = 0.$$

This implies that $\alpha = 0$ if $\lambda < 0$ (so any conformal Killing field vanishes on a negative Einstein metric).
If $\lambda = 0$, we see that $\delta\alpha = 0$ and $d\alpha = 0$. In particular, α is a Killing 1-form, and we are done.
In the case $\lambda > 0$, applying a divergence to the top equation yields

$$(3/2)\Delta(\delta\alpha) + 2\lambda(\delta\alpha) = 0.$$

By Lichnerowicz' Theorem, this implies that $\delta\alpha = 0$ unless (M, g) is isometric to (S^4, g_S), so α is Killing.

Second variation as a bilinear form

Let us recall the second variation is

$$\tilde{\mathcal{E}}''(h,h) = Vol(g)^{-1/2}\int_M \langle h, Jh\rangle dV,$$

where J is the operator

$$J = \lambda h + G'h.$$

Using polarization, the Hessian of $\tilde{\mathcal{E}}$ is the bilinear form given by

$$\tilde{\mathcal{E}}''(h_1,h_2) = Vol(g)^{-1/2}\int_M \langle h_1, Jh_2\rangle dV.$$

Proposition

The decomposition

$$S^2(T^*M) = \{f \cdot g\} \oplus \{\mathcal{L}(\alpha)\} \oplus \{\delta h = 0, tr_g(h) = 0\}$$

is orthogonal with respect to $\tilde{\mathcal{E}}''(\cdot, \cdot)$

First, $\tilde{\mathcal{E}}''(\mathcal{L}(\alpha), \cdot) = 0$ from diffeomorphism invariance. So we just need to check that

$$\tilde{\mathcal{E}}''(f \cdot g, z) = 0$$

if z is TT. To see this,

$$\begin{aligned}
\tilde{\mathcal{E}}''(f \cdot g, z) &= Vol(g)^{-1/2} \int_M \langle f \cdot g, Jz \rangle dV \\
&= Vol(g)^{-1/2} \int_M \langle f \cdot g, \frac{1}{2}\Delta z + Rm * z \rangle dV \\
&= Vol(g)^{-1/2} \int_M f R_{ijip} z_{jp} dV = 0.
\end{aligned}$$

Summary

Summarizing the above discussion:

If h is any symmetric 2-tensor, then decompose h as

$$h = f \cdot g + \mathcal{L}\alpha + z,$$

where z is TT. Then

$$\tilde{\mathcal{E}}''(h, h) = \tilde{\mathcal{E}}''(f \cdot g, f \cdot g) + \tilde{\mathcal{E}}''(z, z).$$

So we have shown that to check the second variation, we really only need to consider conformal variations and TT variations separately.

Ebin-Palais slice theorem

The above discussion was at the level of the "tangent space to the space of Riemannian metrics at g". We will next transfer this statement directly to the space of Riemannian metrics near g *modulo diffeomorphism*.

Theorem

For each metric g_1 in a sufficiently small $C^{\ell+1,\alpha}$-neighborhood of g ($\ell \geq 1$), there is a $C^{\ell+2,\alpha}$-diffeomorphism $\varphi : M \to M$ such that

$$\tilde{\theta} \equiv \varphi^* g_1 - g$$

satisfies

$$\delta_g \Big(\tilde{\theta} - \frac{1}{n} tr_g(\tilde{\theta}) g \Big) = 0.$$

Proof of Ebin-Palais

Let $\{\omega_1, \ldots, \omega_\kappa\}$ denote a basis of the space of conformal Killing forms with respect to g. Consider the map

$$\mathcal{N} : C^{\ell+2,\alpha}(TM) \times \mathbb{R}^\kappa \times C^{\ell+1,\alpha}(S^2(T^*M)) \to C^{\ell,\alpha}(T^*M)$$

given by

$$\begin{aligned}\mathcal{N}(X, v, \theta) &= \mathcal{N}_\theta(X, v) \\ &= \Big(\delta_g\big[\overbrace{\varphi^*_{X,1}(g+\theta)}^{\circ}\big] + \sum_i v_i\omega_i\Big),\end{aligned}$$

where $\varphi_{X,1}$ denotes the diffeomorphism obtained by following the flow generated by the vector field X for unit time, and $\circ$ denotes the traceless part with respect to g.

Proof of Ebin-Palais continued

linearizing in (X, v) at $(X, v, \theta) = (0, 0, 0)$, we find

$$\mathcal{N}_0'(Y, a) = \frac{d}{d\epsilon}\left(\delta_g\left[\overbrace{\varphi^*_{\epsilon Y,1}(g)}^{\circ}\right] + \sum_i (\epsilon a_i)\omega_i\right)\Big|_{\epsilon=0}$$

$$= \left(\delta_g[\overbrace{\mathcal{L}_g Y^\flat}^{\circ}] + \sum_i a_i \omega_i\right)$$

$$= \left(\Box Y^\flat + \sum_i a_i \omega_i\right),$$

where $Y^\flat$ is the dual one-form to Y.

Proof of Ebin-Palais continued

The adjoint map $(\mathcal{N}_0')^* : C^{m+2,\alpha}(T^*M) \to C^{m,\alpha}(TM) \times \mathbb{R}^\kappa$ is given by

$$(\mathcal{N}_0')^*(\eta) = \Big((\Box\eta)^\sharp, \int \langle \eta, \omega_i \rangle \; dV \Big),$$

where $(\Box\eta)^\sharp$ is the vector field dual to $\Box\eta$.

If η is in the kernel of the adjoint, the first equation implies that η is a conformal Killing form, while the second implies that η is orthogonal (in L^2) to the space of conformal Killing forms. It follows that $\eta = 0$, so the map $\mathcal{N}_0'$ is surjective.

Proof of Ebin-Palais continued

Omitting a few technical details for simplicity, applying an infinite-dimensional version of the implicit function theorem, given $\theta_1 \in C^{\ell+1,\alpha}(S^2(T^*M))$ small enough we can solve the equation $\mathcal{N}_{\theta_1} = 0$; i.e., there is a vector field $X \in C^{\ell+2,\alpha}(TM)$, a $v \in \mathbb{R}^\kappa$, such that

$$\delta_g[\overbrace{\varphi^* g_1}^{\circ}] + \sum_i v_i \omega_i = 0,$$

where $\varphi = \varphi_{X,1}$. Letting $\tilde{\theta} = \varphi^* g_1 - g$, then $\tilde{\theta}$ satisfies

$$\delta_g[\overset{\circ}{\tilde{\theta}}] + \sum_i v_i \omega_i = 0,$$

Pairing with ω_j, for $j = 1 \ldots \kappa$, and integrating by parts, we see that $v_j = 0$, and we are done.

Some exercises

- Verify the above formula for $(\mathcal{N}_0')^*$
- By adding a scaling factor to the map $\mathcal{N}$, modify the above argument to show that we can find a constant c (depending upon g_1), and find

  $$\tilde{\theta} \equiv e^c \varphi^* g_1 - g,$$

  so that in addition to the traceless part of $\tilde{\theta}$ being TT, $\tilde{\theta}$ also satisfies

  $$\int tr_g \tilde{\theta} \, dV_g = 0.$$

  That is, we can also "gauge away" the scale-invariance of the functional. Equivalently, we can look at a slice of unit-volume metrics modulo diffeomorphism.

Summary

Combining the above discussions, given any g_1 sufficiently near g, we can write

$$\varphi^* g_1 = g + \tilde{\theta},$$

with $\tilde{\theta} = f \cdot g + z$, and z is TT. Then

$$\begin{aligned}
\tilde{\mathcal{E}}(g_1) &= \tilde{\mathcal{E}}(\varphi^* g_1) \text{ (from diffeomorphism invariance)} \\
&= \tilde{\mathcal{E}}(g + \tilde{\theta}) \\
&= \tilde{\mathcal{E}}(g) + \tilde{\mathcal{E}}'_g(\tilde{\theta}) + \tilde{\mathcal{E}}''_g(f \cdot g + z, f \cdot g + z) + \text{remainder} \\
&= \tilde{\mathcal{E}}(g) + \tilde{\mathcal{E}}''_g(f \cdot g, f \cdot g) + \tilde{\mathcal{E}}''_g(z, z) + \text{remainder.}
\end{aligned}$$

Theorem

The local behavior of $\tilde{\mathcal{E}}$, when considered as a map on $\mathcal{M}/\mathcal{D}$ (the space of Riemannian metrics modulo diffeomorphism), is determined by the conformal and TT directions (to second order).

Saddle point structure and the Yamabe invariant.

We have seen that the functional $\tilde{\mathcal{E}}$ is minimizing in the conformal directions, but maximizing (modulo a finite-dimensional subspace) in the TT directions. So an Einstein metric is always a saddle point for $\mathcal{E}$. This suggests defining the following min-max type invariant.

First, we minimize in the conformal direction:

$$Y(M,[g]) = \inf_{\tilde{g}\in[g]} \tilde{\mathcal{E}}(g).$$

This is called the *conformal Yamabe invariant*.

From Aubin's estimate,

$$Y(M,[g]) \leq \tilde{\mathcal{E}}(S^n, g_S).$$

It is known that this inequality is strict, unless (M,g) is conformal to (S^n, g_S). Thus:

The smooth Yamabe invariant

Theorem

(The Yamabe problem) If (M^n, g) is compact, then there exists a conformal metric $\tilde{g} \in [g]$ which has constant scalar curvature, and which minimizes $\tilde{\mathcal{E}}$ in its conformal class.

The proof of this is involved, and we will not discuss here, but is due to Aubin, Trudinger, Yamabe, with the most difficult cases solved by Schoen and Schoen-Yau (positive mass theorem).

The min-max invariant is then defined by

$$Y(M) = \sup_{g \in \mathcal{M}} Y(M, [g]),$$

which we will call the *smooth Yamabe invariant* of M, also known as the *σ-invariant* of M. Defined independently by O. Kobayashi and R. Schoen.

Some known cases

We will not focus on smooth Yamabe invariants in this lecture, but only list a few known cases:

- $Y(S^4) = Y(S^4, [g_S]) = 8\pi\sqrt{6}$.
- $Y(S^1 \times S^3) = Y(S^4, [g_S]) = 8\pi\sqrt{6}$.
- $Y(\mathbb{RP}^3) = Y(\mathbb{RP}^3, [g_S])$ (proved by Bray-Neves).
- If (M^3, g_H) is compact hyperbolic, then $Y(M^3) = Y(M^3, [g_H])$ (proved by Perelman).
- $Y(\mathbb{CP}^2) = Y(\mathbb{CP}^2, [g_{FS}]) = 12\pi\sqrt{2}$, where g_{FS} is the Fubini-Study metric (proved by LeBrun).

Some unkown cases

It is a very difficult problem to determine Yamabe invariants in general. Here are a few prominent unknown cases:

- $Y(S^3/\Gamma)$, where S^3/Γ is a spherical space form with $|\Gamma| > 2$. Achieved by the round metric?
- $Y(\mathbb{CP}^2\#\mathbb{CP}^2) = ?$ The only result known is due to O. Kobayashi: $Y(\mathbb{CP}^2\#\mathbb{CP}^2) \geq Y(\mathbb{CP}^2)$.
- $Y(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2) = ?$ Again, the only result known is $Y(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2) \geq Y(\mathbb{CP}^2)$.
- $Y(S^2 \times S^2) = ?$ The only known result is that $Y(S^2 \times S^2) > Y(S^2 \times S^2, g_{S^2} + g_{S^2})$ (strict inequality!). This follows from the above exercise on $S^2 \times S^2$ and an observation of Wang-Ziller that CSC metrics sufficiently near an Einstein metric are also Yamabe minimizers in their conformal class.

Moduli space of Einstein metrics

Next, given an Einstein metric g with $Ric = \lambda \cdot g$, we would like understand the space of solutions of the equation

$$Ric(\tilde{g}) = \lambda \cdot \tilde{g} \tag{1}$$

with $\tilde{g}$ near g. This will be infinite-dimensional since $\varphi^* g$ will also be a solution for any diffeomorphism $\varphi : M \to M$.
Therefore, we need to look at the space of Einstein metrics modulo diffeomorphism. Our goal is to prove:

Theorem

Assume g is Einstein with $Ric(g) = \lambda \cdot g$ and $\lambda < 0$. Then the space of Einstein metrics near g modulo diffeomorphism is locally isomorphic to the zero set of a map

$$\Psi : H^1 \to H^1,$$

where

$$H^1 = \{h \in S^2(T^*M) : \delta_g h = 0, tr_g h = 0, \Delta h + 2Rm * h = 0\}.$$

Ellipticity

The diffeomorphism invariance also means that the above equation cannot be elliptic. Indeed, differentiating

$$Ric(\varphi_t^* g) = \varphi_t^*(Ric(g))$$

yields

$$Ric'(\mathcal{L}_X g) = \mathcal{L}_X(Ric(g)) = \lambda \cdot \mathcal{L}_X g.$$

Exercise

Show that this implies that the symbol of Ric' is not elliptic.

We will next describe a procedure called "gauging" which shows in effect, that the diffeomorphism directions are the only obstruction to ellipticity. This is somewhat analogous to the "Coulomb gauge" in electrodynamics.

A gauge choice

Recall that, at an Einstein metric

$$(Ric')_{ij} = \frac{1}{2}\Big(-\Delta h_{ij} + \nabla_i(\delta h)_j + \nabla_j(\delta h)_i - \nabla_i\nabla_j(trh) - 2R_{iljp}h^{lp} + 2\lambda h_{ij}\Big).$$

Define

$$\beta_g h = \delta_g h - \frac{1}{2}d(tr_g h)$$

Exercise

Show that

$$\frac{1}{2}\mathcal{L}\beta_g h = \frac{1}{2}\Big(\nabla_i(\delta h)_j + \nabla_j(\delta h)_i - \nabla_i\nabla_j(trh)\Big).$$

Therefore

$$(Ric' - \frac{1}{2}\mathcal{L}\beta_g)h = \frac{1}{2}\Big(-\Delta h - 2Rm * h + 2\lambda h\Big).$$

The nonlinear map

Given $\theta \in C^{2,\alpha}(S^2T^*M)$, consider the map

$$P : C^{2,\alpha}(S^2(T^*M)) \to C^{0,\alpha}(S^2(T^*M))$$

by

$$P_g(\theta) = Ric(g+\theta) - \lambda \cdot (g+\theta) - \frac{1}{2}\mathcal{L}_{g+\theta}\beta_g\theta.$$

Proposition

The operator P is elliptic.

This is immediate: from the previous slide, the linearized operator at $\theta = 0$ is

$$P'h = \frac{1}{2}\Big(- \Delta h - 2Rm * h\Big),$$

which is clearly elliptic.

Next lecture

Next time we will discuss the following:

- Complete the local description of the moduli space of Einstein metrics.
- Rigidity of Einstein metrics and discussion of some basic examples.
- Quadratic functionals.
- Hitchin-Thorpe Inequality and examples of Einstein metrics in dimension 4.
- Self-dual metrics and their deformation theory.