

Riemannian Curvature Functionals: Lecture IV

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Today we will discuss the following:

- Strict local minimization of quadratic functionals in dimension 4.
- A local reverse Bishop's inequality.
- B^t -flat metrics.
- Construction of critical points of quadratic functionals using gluing methods.

We saw that critical points of the Einstein-Hilbert functional in general have a saddle-point structure. However, critical points for certain quadratic functionals have a nicer local variational structure. For example, one result we will discuss today is the following:

Theorem (Gursky-V 2011)

On S^4 , g_S is a strict local minimizer (modulo diffeomorphisms) for

$$\mathcal{F}_\tau = \int |Ric|^2 dV + \tau \int R^2 dV$$

provided that

$$-\frac{1}{3} < \tau < \frac{1}{6}.$$

Just as in the case of the Einstein-Hilbert functional, the second variation is orthogonal with respect to the splitting

$$S^2(T^*M) = \{f \cdot g\} \oplus \{\mathcal{L}(\alpha)\} \oplus \{\delta h = 0, \text{tr}_g(h) = 0\}.$$

Therefore, if h is any symmetric 2-tensor, then decompose h as

$$h = f \cdot g + \mathcal{L}\alpha + z,$$

where z is TT. Then

$$\mathcal{F}''_{\tau}(h, h) = \mathcal{F}''_{\tau}(f \cdot g, f \cdot g) + \mathcal{F}''_{\tau}(z, z).$$

- To check the second variation, we only need to consider conformal variations and TT variations separately.

The Jacobi operator

We will not write out the Euler-Lagrange equations, but point out that Einstein metrics are indeed critical for \mathcal{F}_τ . Let us write the second variation as

$$\mathcal{F}_\tau''(h_1, h_2) = \int_M \langle h_1, Jh_2 \rangle dV.$$

We have:

Proposition

if g is Einstein with $Ric = \lambda \cdot g$ and h is TT, then the Jacobi operator of \mathcal{F}_τ is

$$Jh = \frac{1}{2} \left(\Delta_L + 2\lambda \right) \left(\Delta_L + 4 \left(1 + 2\tau \right) \lambda \right) h,$$

where

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ipjq} h^{pq} - 2\lambda \cdot h_{ij}.$$

The case of (S^4, g_S)

In this case, the Lichnerowicz Laplacian on TT-tensors is

$$\Delta_L h = \Delta h - 8h.$$

Proposition

The least eigenvalue of the rough Laplacian on TT-tensors is 8.

The proof is left as an exercise. Hint: use the inequality

$$\int_{S^4} |\nabla_i h_{jk} + \nabla_j h_{ki} + \nabla_k h_{ij}|^2 dV \geq 0.$$

Consequently, least eigenvalue of the Lichnerowicz Laplacian on TT-tensors is 16. The previous proposition then implies that if

$$\tau < \frac{1}{6},$$

then the Jacobi operator is positive definite when restricted to TT-tensors.

Proposition

if g is Einstein with $Ric = \lambda \cdot g$ and $h = fg$, then

$$tr(Jf) = \frac{4 + 12\tau}{2}(3\Delta + 4\lambda)\Delta f.$$

This implies that the Jacobi operator is non-negative in conformal directions for

$$-\frac{1}{3} < \tau,$$

with the zero eigenvalues given by $h = f \cdot g$, where f is a lowest nontrivial eigenfunction of Δ (by Lichnerowicz' Theorem).

- On (S^4, g_S) the second variation is strictly positive on TT -tensors, and strictly positive in conformal directions (except for lowest nontrivial eigenfunction directions) in the range

$$-\frac{1}{3} < \tau < \frac{1}{6}.$$

- Using a modification of the Ebin-Palais slicing, we can ignore Lie derivative directions (as before), but we can also ignore the conformal zero eigentensors using conformal diffeomorphisms, so the functional is in fact strictly locally minimized at the spherical metric modulo diffeomorphisms.

Notice that for $\tau = -(1/4)$, the functional is $\int |E|^2$, so is obviously strictly minimized for this τ , but our improvement of the range of τ for minimization has an interesting application:

A reverse Bishop's inequality

The classical Bishop's inequality implies that if (M^4, g) is a closed manifold with $Ric(g) \geq Ric(S^4, g_S) = 3g$, then the volume satisfies $Vol(g) \leq Vol(S^4, g_S)$, and equality holds only if (M, g) is isometric to (S^4, g_S) . An interesting consequence of strict local minimization for $\tau = 0$ is that, locally, a “reverse Bishop's inequality” holds:

Corollary (Gursky-V 2011)

*On (S^4, g_S) , there exists a $C^{2,\alpha}$ -neighborhood U of g_S such that if $\tilde{g} \in U$ with $Ric(\tilde{g}) \leq 3\tilde{g}$, then $Vol(\tilde{g}) \geq Vol(g_S)$ with equality if and only if $\tilde{g} = \phi^*g$ for some diffeomorphism $\phi : M \rightarrow M$.*

Interesting questions:

- What is the largest neighborhood U for which this holds?
- Is the functional $\int_{S^4} |Ric|^2 dV$ globally minimized at g_S ?

We will be interested in the functional

$$\mathcal{B}_t[g] = \int |W|^2 dV + t \int R^2 dV.$$

The Euler-Lagrange equations of \mathcal{B}_t are given by

$$B^t \equiv B + tC = 0,$$

where B is the *Bach tensor* defined by

$$B_{ij} \equiv -4 \left(\nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} \right),$$

and C is the tensor defined by

$$C_{ij} = 2\nabla_i \nabla_j R - 2(\Delta R)g_{ij} - 2RR_{ij} + \frac{1}{2}R^2 g_{ij}.$$

Note:

- Any Einstein metric is critical for \mathcal{B}_t .
- We will refer to such a critical metric as a B^t -flat metric.

For $t \neq 0$, by taking a trace of the E-L equations:

$$\Delta R = 0.$$

If M is compact, this implies $R = \text{constant}$.

Consequently, the B^t -flat condition is equivalent to

$$\boxed{B = 2tR \cdot E},$$

where E denotes the traceless Ricci tensor. In other words:

- The Bach tensor is a constant multiple of the traceless Ricci tensor.

Orbifold Limits

The B^t -flat equation can be rewritten as

$$\Delta Ric = Rm * Rc. \quad (1)$$

Theorem (Tian-V)

(M_i, g_i) sequence of 4-dimensional manifolds satisfying (1) and

$$\int |Rm|^2 < \Lambda, \quad Vol(B(q, s)) > Vs^4, \quad b_1(M_i) < B.$$

Then for a subsequence $\{j\} \subset \{i\}$,

$$(M_j, g_j) \xrightarrow{\text{Cheeger-Gromov}} (M_\infty, g_\infty),$$

where (M_∞, g_∞) is a multi-fold satisfying (1), with finitely many singular points.

Rescaling such a sequence to have bounded curvature near a singular point yields non-compact limits called *asymptotically locally Euclidean spaces* (ALE spaces).

Can you reverse this process?

I.e., start with an critical orbifold, “glue on” critical ALE metrics at the singular points, and resolve to a smooth critical metric?

Answer is “no” in general, because this is a self-adjoint gluing problem.

However, the following theorem says that the answer is “YES” in certain cases:

Theorem (Gursky-V 2013)

A B^t -flat metric exists on the manifolds in the table for some t near the indicated value of t_0 .

Topology of connected sum	Value(s) of t_0
$\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$	$-1/3$
$S^2 \times S^2 \# \overline{\mathbb{C}\mathbb{P}^2} = \mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}$	$-1/3, -(9m_1)^{-1}$
$2\#S^2 \times S^2$	$-2(9m_1)^{-1}$

The constant m_1 is a geometric invariant called the *mass* of an certain asymptotically flat metric: the Green's function metric of the product metric $S^2 \times S^2$.

- $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ admits an $U(2)$ -invariant Einstein metric called the “Page metric”. M does not admit any Kähler-Einstein metric, but the Page metric is conformal to an extremal Kähler metric.
- $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ admits a toric invariant Einstein metric called “Chen-LeBrun-Weber metric”. Again, M does not admit any Kähler-Einstein metric, but the Chen-LeBrun-Weber metric is conformal to an extremal Kähler metric.
- $M = S^2 \times S^2 \# S^2 \times S^2$ does not admit any Kähler metric, it does not even admit an almost complex structure. Our metric is the first known example of a “canonical” metric on this manifold.

Green's function metric

The first step in the gluing procedure is to construct an asymptotically flat metric, called the Green's function metric. The conformal Laplacian:

$$Lu = -6\Delta u + Ru.$$

If (M, g) is compact and $R > 0$, then for any $p \in M$, there is a unique positive solution to the equation

$$\begin{aligned} LG &= 0 \quad \text{on } M \setminus \{p\} \\ G &= \rho^{-2}(1 + o(1)) \end{aligned}$$

as $\rho \rightarrow 0$, where ρ is geodesic distance to the basepoint p .

- Denote $N = M \setminus \{p\}$ with metric $g_N = G^2 g_M$. The metric g_N is scalar-flat and asymptotically flat of order 2.
- If (M, g) is Bach-flat, then (N, g_N) is also Bach-flat (from conformal invariance) and scalar-flat (since we used the Green's function). Consequently, g_N is B^t -flat for all $t \in \mathbb{R}$.

The approximate metric

- Let (Z, g_Z) and (Y, g_Y) be Einstein manifolds, and assume that g_Y has positive scalar curvature.
- Choose basepoints $z_0 \in Z$ and $y_0 \in Y$.
- Convert (Y, g_Y) into an asymptotically flat (AF) metric (N, g_N) using the Green's function for the conformal Laplacian based at y_0 . As pointed out above, g_N is B^t -flat for any t .
- Let $a > 0$ be small, and consider $Z \setminus B(z_0, a)$. Scale the compact metric to $(Z, \tilde{g} = a^{-4}g_Z)$. Attach this metric to the metric $(N \setminus B(a^{-1}), g_N)$ using cutoff functions near the boundary, to obtain a smooth metric on the connect sum $Z \# \bar{Y}$.

The approximate metric

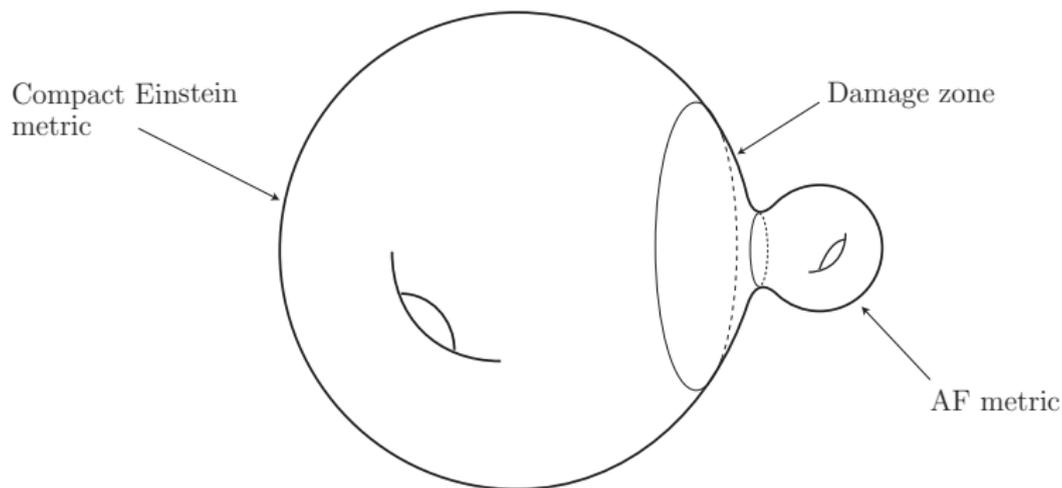


Figure: The approximate metric.

Since both g_Z and g_N are B^t -flat, this metric is an “approximate” B^t -flat metric, with vanishing B^t tensor away from the “damage zone”, where cutoff functions were used.

Gluing parameters

In general, there are several degrees of freedom in this approximate metric.

- The scaling parameter a (1-dimensional).
- Rotational freedom when attaching (6-dimensional).
- Freedom to move the base points of either factor (8-dimensional).

Total of 15 gluing parameters.

Lyapunov-Schmidt reduction

These 15 gluing parameters yield a 15-dimensional space of “approximate” kernel of the linearized operator. Using a Lyapunov-Schmidt reduction argument, one can reduce the problem to that of finding a zero of the *Kuranishi map*

$$\Psi : U \subset \mathbb{R}^{15} \rightarrow \mathbb{R}^{15}.$$

- It is crucial to use certain weighted norms to find a bounded right inverse for the linearized operator.
- This 15-dimensional problem is too difficult in general: we will take advantage of various symmetries in order to reduce to only 1 free parameter: the scaling parameter a .

The leading term of the Kuranishi map corresponding to the scaling parameter is given by:

Theorem (Gursky-V 2013)

As $a \rightarrow 0$, then for any $\epsilon > 0$,

$$\Psi_1 = \left(\frac{2}{3}W(z_0) \circledast W(y_0) + 4tR(z_0)\text{mass}(g_N) \right) \omega_3 a^4 + O(a^{6-\epsilon}),$$

where $\omega_3 = \text{Vol}(S^3)$, and the product of the Weyl tensors is given by

$$W(z_0) \circledast W(y_0) = \sum_{ijkl} W_{ijkl}(z_0)(W_{ijkl}(y_0) + W_{ilkj}(y_0)),$$

where $W_{ijkl}(\cdot)$ denotes the components of the Weyl tensor in a normal coordinate system at the corresponding point.

The Fubini-Study metric

$(\mathbb{C}\mathbb{P}^2, g_{FS})$, the Fubini-Study metric, $Ric = 6g$.

Torus action:

$$[z_0, z_1, z_2] \mapsto [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2].$$

Flip symmetry:

$$[z_0, z_1, z_2] \mapsto [z_0, z_2, z_1].$$

The Fubini-Study metric

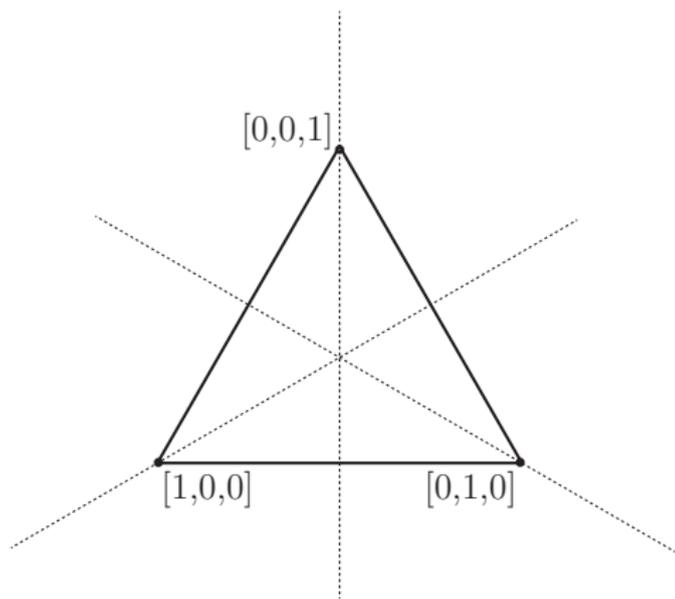


Figure: Orbit space of the torus action on $\mathbb{C}\mathbb{P}^2$.

The product metric

$(S^2 \times S^2, g_{S^2 \times S^2})$, the product of 2-dimensional spheres of Gaussian curvature 1, $Ric = g$.

Torus action:

Product of rotations fixing north and south poles.

Flip symmetry:

$$(p_1, p_2) \mapsto (p_2, p_1).$$

The product metric

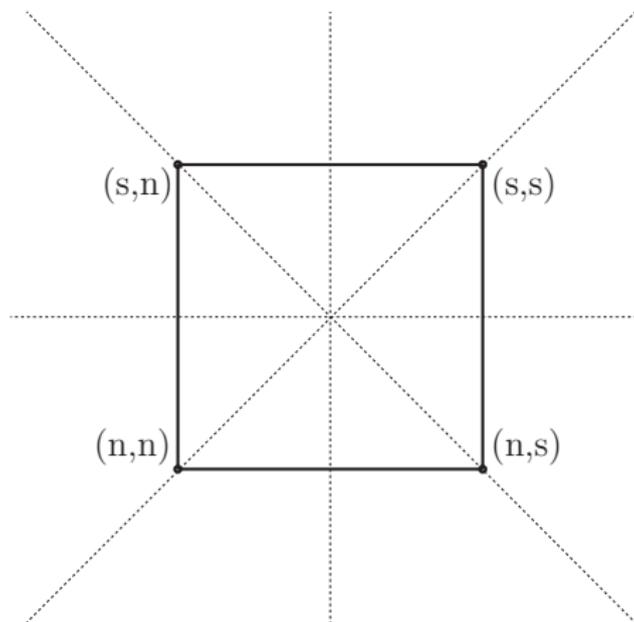


Figure: Orbit space of the torus action on $S^2 \times S^2$.

Mass of Green's function metric

Recall the mass of an AF space is defined by

$$\text{mass}(g_N) = \lim_{R \rightarrow \infty} \omega_3^{-1} \int_{S(R)} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) (\partial_i \lrcorner dV),$$

with $\omega_3 = \text{Vol}(S^3)$.

The Green's function metric of the Fubini-Study metric \hat{g}_{FS} is also known as the Burns metric, and is completely explicit, with mass given by

$$\text{mass}(\hat{g}_{FS}) = 2.$$

Mass of Green's function metric

However, the Green's function metric $\hat{g}_{S^2 \times S^2}$ of the product metric does not seem to have a known explicit description. We will denote

$$m_1 = \text{mass}(\hat{g}_{S^2 \times S^2}).$$

By the positive mass theorem of Schoen-Yau, $m_1 > 0$. Note that since $S^2 \times S^2$ is spin, this also follows from Witten's proof of the positive mass theorem.

Remark

For the curious, the mass $m_1 = .5872\dots$, which implies that $(-9m_1)^{-1} = -.1892\dots$

Remarks on the proof

- We impose the toric symmetry and “flip” symmetry in order to reduce the number of free parameters to 1 (only the scaling parameter). That is, we perform an equivariant gluing.
- The special value of t_0 is computed by

$$\frac{2}{3}W(z_0) \otimes W(y_0) + 4t_0R(z_0)\text{mass}(g_N) = 0.$$

- This choice of t_0 makes the leading term of the Kuranishi map vanish, and is furthermore a *nondegenerate* zero (if $R(z_0) > 0$; $\text{mass}(g_N) > 0$ by the positive mass theorem).

Topology of connected sum	Value(s) of t_0
$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$	$-1/3$

- The compact metric is the Fubini-Study metric, with a Burns AF metric glued on, a computation yields $t_0 = -1/3$.

Topology of connected sum	Value(s) of t_0
$S^2 \times S^2 \# \overline{\mathbb{C}\mathbb{P}^2} = \mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}$	$-1/3, -(9m_1)^{-1}$

- The compact metric is the product metric on $S^2 \times S^2$, with a Burns AF metric glued on, this gives $t_0 = -1/3$.
- Alternatively, take the compact metric to be $(\mathbb{C}\mathbb{P}^2, g_{FS})$, with a Green's function $S^2 \times S^2$ metric glued on. In this case, $t_0 = -(9m_1)^{-1}$.

Topology of connected sum	Value(s) of t_0
$2\#S^2 \times S^2$	$-2(9m_1)^{-1}$

- The compact metric is the product metric on $S^2 \times S^2$, with a Green's function $S^2 \times S^2$ metric glued on. In this case, $t_0 = -2(9m_1)^{-1}$.

- **Ellipticity and gauging.** The B^t -flat equations are not elliptic due to diffeomorphism invariance. A gauging procedure analogous to the Bianchi gauge is used.
- **Rigidity of g_{FS} and $g_{S^2 \times S^2}$.** Follows from the formulas discussed at the beginning of today's lecture, together with some eigenvalue estimates.
- **Refined approximate metric.** The approximate metric described above is not good enough. It must be improved by matching up leading terms of the metrics by solving certain auxiliary linear equations, so that the cutoff function disappears from the leading term. We will not have time to discuss this today.

Ellipticity and gauging

The linearized operator of the B^t -flat equation is not elliptic, due to diffeomorphism invariance. However, consider the “gauged” nonlinear map P given by

$$P_g(\theta) = (B + tC)(g + \theta) + \mathcal{K}_{g+\theta}[\delta_g \mathcal{K}_g \delta_g \overset{\circ}{\theta}].$$

Let $S^t \equiv P'(0)$ denote the linearized operator at $\theta = 0$.

Proposition

If $t \neq 0$, then S^t is elliptic. Furthermore, if $P(\theta) = 0$, and $\theta \in C^{4,\alpha}$ for some $0 < \alpha < 1$, then $B^t(g + \theta) = 0$ and $\theta \in C^\infty$.

- Proof is an integration-by-parts. It is crucial that the B^t -flat equations are variational (recall \mathcal{B}_t is the functional), so $\delta B^t = 0$. Equivalent to diffeomorphism invariance of \mathcal{B}_t .

For h transverse-traceless (TT), the linearized operator at an Einstein metric is given by

$$S^t h = \left(\Delta_L + \frac{1}{2} R \right) \left(\Delta_L + \left(\frac{1}{3} + t \right) R \right) h,$$

where Δ_L is the Lichnerowicz Laplacian, defined by

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ipjq} h^{pq} - \frac{1}{2} R h_{ij}.$$

- Recall that infinitesimal Einstein deformations given by TT kernel of the operator $\Delta_L + \frac{1}{2} R$. We still have those, but there are now more possibilities of deformations.

For $h = fg$, we have

$$\text{tr}_g(S^t h) = 6t(3\Delta + R)(\Delta f). \quad (1)$$

The rigidity question is then reduced to a separate analysis of the eigenvalues of Δ_L on transverse-traceless tensors, and of Δ on functions.

Theorem (Gursky-V)

On $(\mathbb{C}\mathbb{P}^2, g_{FS})$, $H_t^1 = \{0\}$ provided that $t < 1$.

Theorem (Gursky-V)

On $(S^2 \times S^2, g_{S^2 \times S^2})$, $H_t^1 = \{0\}$ provided that $t < 2/3$ and $t \neq -1/3$. If $t = -1/3$, then H_t^1 is one-dimensional and spanned by the element $g_1 - g_2$.

- Positive mass theorem says that $t_0 < 0$, so luckily we are in the rigidity range of the factors.
- Gauge term is carefully chosen so that solutions of the linearized equation must be in the transverse-traceless gauge. That is, if $S^t h = 0$ then

$$(B^t)'(h) + \mathcal{K}\delta\mathcal{K}\delta\overset{\circ}{h} = 0$$

implies that separately,

$$(B^t)'(h) = 0 \text{ and } \delta\overset{\circ}{h} = 0.$$

The proof shows that there is a dichotomy. Either

- (i) there is a critical metric at exactly the critical t_0 , in which case there would necessarily be a 1-dimensional moduli space of solutions for this fixed t_0 , or
- (ii) for each value of the gluing parameter a sufficiently small, there will be a critical metric for a corresponding value of $t_0 = t_0(a)$. The dependence of t_0 on a will depend on the next term in the expansion of the Kuranishi map.
- Which case actually happens?