Riemannian Curvature Functionals: Lecture I

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Overview of lectures

The goal of these lectures is to gain an understanding of critical points of certain Riemmannian functionals in dimension 4.

The starting point will be the Einstein-Hilbert functional:

$$\tilde{\mathcal{E}}(g) = Vol(g)^{-1/2} \int_M R_g dV_g,$$

where R_g is the scalar curvature.

Euler-Lagrange equations:

$$Ric(g) = \lambda \cdot g,$$

where λ is a constant.

(M, g) is called an *Einstein manifold*.

- We will study this functional in depth, and also give a local description of the moduli space of Einstein metrics.

Overview: Quadratic curvature functionals

We will also concentrate on linear combinations of the following quadratic curvature functionals:

$$\mathcal{W} = \int |W|^2 \, dV, \quad \rho = \int |Ric|^2 \, dV, \quad \mathcal{S} = \int R^2 \, dV.$$

- Einstein metrics are critical for these functionals, but there are many non-Einstein critical metrics.
- Special critical metrics for $\mathcal{W}$: self-dual metrics. We will study the deformation theory of such metrics, and discuss properties of the moduli space, such as existence of the Kuranishi map.

Overview: Stability properties

Critical points of the Einstein-Hilbert functional in general have a saddle-point structure. However, critical points for certain quadratic functionals have a nicer local variational structure. For example, one result we will discuss is the following:

Theorem (Gursky-V 2011)

On S^4, g_S is a strict local minimizer for

$$\mathcal{F}_\tau = \int |Ric|^2 dV + \tau \int R^2 dV$$

provided that

$$-\frac{1}{3} < \tau < \frac{1}{6}.$$

Overview: An existence theorem

Theorem (Gursky-V 2013)

A critical metric for the functional

$$\mathcal{B}_t = \int |W|^2 dV + t \int R^2 dV$$

exists on the manifolds in the table for some t *near the indicated value of* t_0.

Topology of connected sum	Value(s) of t_0
$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$	$-1/3$
$S^2 \times S^2 \# \overline{\mathbb{CP}}^2 = \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$	$-1/3$, $-(9m_1)^{-1}$
$2\# S^2 \times S^2$	$-2(9m_1)^{-1}$

The constant m_1 is a geometric invariant called the *mass* of an certain asymptotically flat metric: the Green's function metric of the product metric $S^2 \times S^2$.

Outline of today's lecture

- Notation and conventions
- Einstein-Hilbert functional.
- First variation.
- Second variation.
- Conformal variations.
- Transverse-traceless variations.
- Constant curvature examples and $S^2 \times S^2$.

Notation and conventions

$\nabla_X Y$ = covariant derivative on Riemannian manifold (M, g).
Christoffel symbols:

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

Formula for Christoffel symbols:

$$\boxed{\Gamma_{ij}^k = \frac{1}{2} g^{kl} \Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \Big)}$$

Notation and conventions: Curvature

Curvature as a $(1,3)$-tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

In coordinates:

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{l} \partial_l,$$

Curvature as a $(0,4)$-tensor

$$Rm(X,Y,Z,W) \equiv -\langle R(X,Y)Z, W \rangle.$$

In coordinates:

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l),$$

Notation and conventions: Warning

Our convention:

$$R_{ijlk} = R_{ijk}{}^{m} g_{ml},$$

We lower the upper index to the *third* position.

Remark

Some authors choose to lower this index to a different position. One has to be very careful with this, or you might end up proving that S^n has negative curvature!

Notation and conventions: Ricci tensor and scalar curvature

Ricci tensor:

$$R_{ij} = R_{lij}{}^{l} = g^{lm} R_{limj} = R_{ji}$$

Scalar curvature:

$$R = g^{pq} R_{pq} = g^{pq} g^{lm} R_{lpmq}.$$

Einstein-Hilbert functional

$\mathcal{M}$ = the space of Riemannian metrics on a manifold M.

The Einstein-Hilbert functional $\mathcal{E} : \mathcal{M} \to \mathbb{R}$:

$$\mathcal{E}(g) = \int_M R_g dV_g.$$

This is a *Riemannian functional* in the sense that it is invariant under diffeomorphisms:

$$\mathcal{E}(\varphi^* g) = \mathcal{E}(g)$$

Einstein-Hilbert functional: First variation

Proposition

If M is closed and $n \geq 3$, then a metric $g \in \mathcal{M}$ is critical for $\mathcal{E}$ if and only if g is Ricci-flat.

Outline of Proof: Let $g(t)$ be a variation, with $h = g'(0)$. Then

$$\begin{aligned} \mathcal{E}(g(t))' &= \int_M (R(g(t)) dV_{g(t)})' \\ &= \int_M R(g(t))' dV_{g(t)} + R(g(t))(dV_{g(t)})' \end{aligned}$$

We use the formulae:

$$\begin{aligned} R' &= -\Delta(trh) + \delta^2 h - R_{lp} h^{lp} \\ (dV_g)' &= \frac{1}{2} tr_g(h) dV_g. \end{aligned}$$

Proof continued

Integrating by parts:

$$\mathcal{E}'_g(h) = \int_M \Big(-\Delta(trh) + \delta^2 h - R^{lp} h_{lp} + \frac{R}{2} tr_g(h) \Big) dV_g$$
$$= \int_M \Big((-R^{lp} + \frac{R}{2} g^{lp}) h_{lp} \Big) dV_g.$$

If this vanishes for all variations h, then

$$Ric = \frac{R}{2} g.$$

If $n > 2$, taking a trace, we find that $R = 0$, so (M, g) is Ricci-flat.

Remark

If $n = 2$ then $\mathcal{E}$ has zero variation, thus is constant. This is not surprising in view of the Gauss-Bonnet Theorem.

Diffeomorphism invariance $\Rightarrow$ Bianchi identity

The tensor that arises in the above calculation

$$G = -Ric + \frac{R}{2} g,$$

is known as the Einstein tensor. By the contracted second Bianchi identity, it is divergence-free. We can actually see this as a consequence of diffeomorphism invariance of the functional:

Let ϕ_t be a path of diffeomorphisms, and let $g_t = \phi_t^* g$. Then $g' = \mathcal{L}_X g$, where X is the tangent vector field of this 1-parameter group of diffeomorphisms at $t = 0$, and $\mathcal{L}$ is the Lie derivative operator. Integrating by parts:

$$\int_M \langle G, \mathcal{L}_X g \rangle dV_g = -\int_M \langle 2\delta G, X \rangle dV_g,$$

for any vector field X, which implies that $\delta G = 0$.

Normalized Einstein-Hilbert functional

The functional $\mathcal{E}$ is not scale-invariant for $n \geq 3$. To fix this we define

$$\tilde{\mathcal{E}}(g) = Vol(g)^{\frac{2-n}{n}} \int_M R_g dV_g.$$

To make things simpler, let's restrict to dimension 4:

$$\tilde{\mathcal{E}}(g) = Vol(g)^{-1/2} \int_M R_g dV_g.$$

Proposition

A metric g is critical for $\tilde{\mathcal{E}}$ under all conformal variations (those of the form $h = f \cdot g$ for $f : M \to \mathbb{R}$) if and only if g has constant scalar curvature. Furthermore, a metric $g \in \mathcal{M}$ is critical for $\tilde{\mathcal{E}}$ if and only if g is Einstein.

Proof of Proposition

$$\tilde{\mathcal{E}}'(h) = Vol(g)^{-1/2}\Big(-\frac{1}{2}Vol(g)^{-1}\int_M \frac{1}{2}(tr_g h)dV_g \cdot \int_M R_g dV_g\Big) + Vol(g)^{-1/2}\int_M \Big(-R^{lp} + \frac{R}{2}g^{lp}\Big)h_{lp}dV_g.$$

If $g(t) = f(t)g$, then

$$\tilde{\mathcal{E}}'(h) = \frac{1}{4}Vol(g)^{-1/2}\Big(\int_M (tr_g h)(R_g - \overline{R})dV_g\Big),$$

where $\overline{R}$ denotes the average scalar curvature. If this is zero for an arbitrary function $tr_g h$, then R_g must be constant. The full variation then simplifies to

$$\tilde{\mathcal{E}}'(h) = Vol(g)^{-1/2}\int_M \Big(-R^{lp} + \frac{R}{4}g^{lp}\Big)h_{lp}dV_g.$$

If this vanishes for all variations, then the traceless Ricci tensor must vanish, so (M, g) is Einstein.

Einstein-Hilbert functional: Second variation

Since the functional is scale invariant, from now on we will always restrict to variations satisfying $\int tr_g(h) dV = 0$.

Proposition

Let g be Einstein with $Ric = \lambda g$. Then the second derivative of $\tilde{\mathcal{E}}$ at $t = 0$ is given by

$$\tilde{\mathcal{E}}'' = Vol(g)^{-1/2} \Big\{ \lambda \int |h|^2 dV + \int \langle G'(h), h \rangle dV \Big\}. \tag{1}$$

The proof is left as an exercise. The main point is that the second derivative of a functional is well-defined at a critical point (it only depends on the tangent to the variation).

Second variation continued

Using the formula for the linearization of the Ricci tensor,

$$(Ric')_{ij} = \frac{1}{2}\Big(- \Delta h_{ij} + \nabla_i(\delta h)_j + \nabla_j(\delta h)_i - \nabla_i \nabla_j(trh) \\ - 2R_{iljp}h^{lp} + R_i^p h_{jp} + R_j^p h_{ip}\Big),$$

and letting $(Rm * h)_{ij} = R_{iljp}h^{lp}$, this can be rewritten as follows:

Proposition

Let g be Einstein with $Ric = \lambda g$. Then the second derivative of $\tilde{\mathcal{E}}$ at $t = 0$ is given by

$$\tilde{\mathcal{E}}'' = Vol(g)^{-1/2}\Big\{ \int \Big\langle h, \frac{1}{2}\Delta h - \frac{1}{2}\mathcal{L}(\delta h) + (\delta^2 h)g \\ - \frac{1}{2}(\Delta tr_g h)g - \frac{\lambda}{2}(tr_g h)g + Rm * h \Big\rangle dV \Big\}.$$

Conformal variations

The above formula looks pretty complicated, but let's first look at *conformal variations*, that is, those variations of the form $h = f \cdot g$, for a function $f : M \to \mathbb{R}$.

Corollary

If $h = fg$, then

$$\tilde{\mathcal{E}}'' = Vol(g)^{-1/2} \Big\{ \int (-3\Delta f - 4\lambda f) f dV_g \Big\}$$

This implies the following

Corollary

If g is Einstein, then $\tilde{\mathcal{E}}$ is locally strictly minimizing in the conformal direction, unless g is isometric to (S^4, g_S).

The case $\lambda \leq 0$ is obvious, the case $\lambda > 0$ will follow from a result due to Lichnerowicz on the next slide.

Lichnerowicz eigenvalue estimate

We let λ_1 denote the lowest non-trivial eigenvalue of the Laplacian on functions, that is $\Delta u = -\lambda_1 u$.

Theorem (Lichnerowicz)

If a compact manifold (M^n, g) *satisfies* $Ric \geq (n-1) \cdot g$, *then* $\lambda_1 \geq n$, *with equality if and only if* (M^n, g) *is isometric to* (S^n, g_S).

Proof.

Exercise. Hint: Commuting covariant derivatives, write

$$\int_M (\Delta f)^2 = \int |\nabla^2 f|^2 dV + \int Ric(\nabla f, \nabla f) dV,$$

and use the matrix inequality $|A|^2 \geq (1/n)(tr(A))^2$. Characterize equality in this inequality to analyze the case of $\lambda_1 = n$. □

Conformal variations on S^4

On (S^4, g_S), eigenfunctions corresponding to eigenvalue 4 yield directions with $\tilde{\mathcal{E}}'' = 0$. There is a nice geometric explanation for this fact:

Proposition

Let ϕ_t be a 1-parameter group of conformal automorphisms of g_S which are not isometries. Then

$$\frac{d}{dt}(\phi_t^* g_S)|_{t=0} = fg,$$

where f is a nontrivial lowest eigenfunction satisfying $\Delta f = -4f$.

Exercise

Prove this. Hint: use the Hodge decomposition to write any 1-form α dual to a conformal vector field as $\alpha = df + \omega$, with ω divergence free. Apply the conformal Killing operator to α and use the resulting equation to show that the trace-free Hessian of f vanishes, and that ω is Killing.

Global conformal minimization

Actually, it turns out the something much stronger is true:

Theorem (Obata)

If (M^n, g) is Einstein, then g is the unique constant scalar curvature metric in its conformal class (up to scaling), unless (M, g) is isometric to (S^n, g_S), in which case all critical points are the pull-back of g_S under a conformal diffeomorphism. Furthermore, g is a global minimizer of $\tilde{\mathcal{E}}$ in its conformal class (modulo scalings).

To prove this, assume that $\hat{g}$ is a constant scalar curvature metric which is conformal to g. Letting E denote the traceless Ricci tensor, we recall the transformation formula: if $g = \phi^{-2}\hat{g}$, then

$$E_g = E_{\hat{g}} + (n-2)\phi^{-1}\big(\nabla^2\phi - (\Delta\phi/n)\hat{g}\big),$$

where n is the dimension, and the covariant derivatives are taken with respect to $\hat{g}$. Since g is Einstein, we have

$$E_{\hat{g}} = (2-n)\phi^{-1}\big(\nabla^2\phi - \frac{1}{n}(\Delta\phi)g\big).$$

Proof of Obata's Theorem continued

Integrating,

$$\begin{aligned}\int_M \phi |E_{\hat{g}}|^2 d\hat{V} &= (2-n)\int_M \phi E_{\hat{g}}^{ij}\left\{\phi^{-1}\big(\nabla^2\phi - \frac{1}{n}(\Delta\phi)g\big)_{ij}\right\} d\hat{V} \\ &= (2-n)\int_M E_{\hat{g}}^{ij}\nabla^2\phi_{ij} d\hat{V} \\ &= (n-2)\int_M (\nabla_j E_{\hat{g}}^{ij}\cdot\nabla_i\phi) d\hat{V} = 0,\end{aligned}$$

by the Bianchi identity. Consequently, $\hat{g}$ is also Einstein. If $\hat{g} \neq g$ then (M, g) admits a nonconstant solution of the equation

$$\nabla^2\phi = \frac{\Delta\phi}{n} g.$$

Taking a divergence, this implies that ϕ is an eigenfunction of the Laplacian with eigenvalue n, so (M, g) is isometric to (S^n, g_S) by the same argument in Lichnerowicz' Theorem.

Proof of Obata's Theorem continued

For existence of a global minimizer, in the negative or zero scalar curvature case, one can apply a standard argument from the calculus of variations to show that a minimizing sequence converges.

In the positive case, scale so that $Ric = (n-1)g$. Then

$$\tilde{\mathcal{E}}(g) = n(n-1)Vol(g)^{2/n}.$$

By Bishops' theorem, $Vol(M,g) \leq Vol(S^n, g_S)$ with equality if and only if g is isometric to g_S. So if g is not isometric to g_S, we have

$$Y(M,[g]) = \inf_{\tilde{g}\in[g]} \tilde{\mathcal{E}}(g) < \tilde{\mathcal{E}}(S^n, g_S).$$

This estimate implies that a minimizing sequence converges (no bubbles are possible).

The case of (S^n, g_S) takes some extra work, we will omit.

Transverse-traceless variations

Definition

A symmetric 2-tensor h is called transverse-traceless (TT for short) if $\delta_g h = 0$ and $tr_g(h) = 0$.

The second variation formula simplifies considerably for TT variations:

Proposition

If h is transverse-traceless, then

$$\tilde{\mathcal{E}}'' = Vol(g)^{-1/2}\Big\{ \int \Big\langle h, \frac{1}{2}\Delta h + Rm * h \Big\rangle dV \Big\}.$$

The first term is manifestly negative, which shows that critical metrics for $\tilde{\mathcal{E}}$ always have a saddle point structure. In other words, modulo a finite dimensional space, $\tilde{\mathcal{E}}$ is locally strictly *maximizing* in TT directions.

The case of constant curvature

If (M, g) has constant sectional curvature, then

$$R_{ijkl} = k_0(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{2}$$

The above second variation formula for TT tensors simplifies to

$$\tilde{\mathcal{E}}'' = Vol(g)^{-1/2}\Big\{ \int \Big\langle h, \frac{1}{2}\Delta h - k_o h \Big\rangle dV \Big\}.$$

- If $k_0 > 0$, this is clearly strictly negative.
- If $k_0 = 0$, then strictly negative, except for parallel h. These correspond to deformations of the flat structure.
- If $k_0 < 0$, this is still strictly negative, we will see on next slide.

Hyperbolic 4-manifolds

We consider eigenvalues defined by

$$\frac{1}{2}\Delta h + Rm * h = (-\mu)h.$$

Lemma

If (M, g) is compact and hyperbolic, then the least eigenvalue of the rough Laplacian on TT tensors is at least 4.

Proof.

Exercise. Hint: start with the inequality

$$\int_M |\nabla_i h_{jk} - \nabla_j h_{ik}|^2 dV \geq 0, \tag{3}$$

integrate by parts, commute covariant derivatives, etc. □

So at a hyperbolic metric, $\tilde{\mathcal{E}}$ is also locally strictly maximizing in TT directions.

$S^2 \times S^2$

Proposition

On $S^2 \times S^2$ with the product metric $g_1 + g_2$, both metrics of constant Gaussian curvature 1, the lowest eigenvalue of the operator $\frac{1}{2}\Delta h + Rm * h$ on TT tensors is -1. The corresponding eigenspace is 1-dimensional, and is spanned by $h = g_1 - g_2$. The next largest eigenvalue is 1.

We will not prove this, since the argument is lengthy.

Exercise

(i) Show that $h = g_1 - g_2$ is an eigentensor with eigenvalue -1. Find a constant scalar curvature deformation of the product metric corresponding to h, and which increases the functional $\tilde{\mathcal{E}}$.
(ii) Show that $\alpha_1 \odot \alpha_2$ ($\odot$ = symmetric product), where α_i are 1-forms dual to Killing fields are eigentensors with eigenvalue 1.

Next lecture

Next time we will discuss the following:

- To understand the local behavior of the functional near an Einstein metric, we only need to look at conformal and transverse-traceless directions, the other directions do not matter.
- Ebin-Palais slice theorem; consider the functional as a mapping from $\mathcal{M}/\mathcal{D}$, the space of Riemannian metrics modulo diffeomorphisms.
- Saddle point structure and the Yamabe invariant.
- Local description of the moduli space of Einstein metrics.