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Some Fully Nonlinear Equations in Conformal Geometry

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Abstract. We examine some fully nonlinear partial differential equations that originate in the study of conformal geometry. These equations are conformally invariant, elliptic, and variational. We will discuss the proof of a uniqueness theorem which is a fully nonlinear analogue to Obata's Theorem for the scalar curvature. We also provide a derivation of some conservation laws for these equations, giving a fully nonlinear analogue to the Pohozaev identity for the scalar curvature.

1. Introduction

In the following we let (N, g_0) denote a compact, connected smooth Riemannian manifold of dimension $n \geq 3$. The Ricci tensor and scalar curvature will be denoted by Ric and R, respectively. In this paper we will examine the nonlinear curvature equations

$$\sigma_k \left(Ric_g - \frac{R_g}{2(n-1)} \cdot g \right) = \text{constant}, \tag{1.1}$$

for metrics g in the conformal class of g_0, where we are using the metric g to view the tensor as an endomorphism of the tangent bundle, and σ_k denotes the trace of the induced map on the kth exterior power, i.e., σ_k is the kth elementary symmetric function of the eigenvalues.

The case $k = 1$, R=constant, is known as the Yamabe problem, and has been studied in great depth (see [**LP87**], [**Sch89**]). The solution of this problem has a long and interesting history, and we do not attempt to give a complete history here, but we point out that the first complete solution of this problem used the positive mass theorem of general relativity (see [**Sch84**]). This is a beautiful example of the interaction between mathematical physics and partial differential equations.

We let $\mathcal{M}_1$ denote the set of unit volume metrics in the conformal class $[g_0]$. We will show that these equations have the following variational properties.

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Theorem 1.1. *If $k \neq n/2$ and $(N, [g_0])$ is locally conformally flat, a metric $g \in \mathcal{M}_1$ is a critical point of the functional*

$$\mathcal{F}_k : g \mapsto \int_N \sigma_k \left(Ric_g - \frac{R_g}{2(n-1)} \cdot g \right) dvol_g$$

restricted to $\mathcal{M}_1$ if and only if

$$\sigma_k \left(Ric_g - \frac{R_g}{2(n-1)} \cdot g \right) = C_k$$

for some constant C_k. If N is not locally conformally flat, the statement is true for $k = 1$ and $k = 2$.

We will also prove the following uniqueness result.

Theorem 1.2. *Suppose (N, g_0) is of unit volume and has constant sectional curvature $K \neq 0$. Then for any $k \in \{1, \ldots, n\}$, g_0 is the unique unit volume solution in its conformal class of (1.1) unless N is isometric to S^n with the standard metric. In this case we have an $(n+1)$-parameter family of solutions which are the images of the standard metric under the conformal diffeomorphisms of S^n.*

For $k = 1$, the constant scalar curvature case, the theorem holds just assuming N is Einstein. This is a well-known theorem of Obata (see [**Oba72**]).

To prove this in the case $1 \leq k < n$, we use differential geometric methods to derive an integral formula. This integral formula was first derived in [**Viaa**] using methods from conformal geometry. We then use some inequalities involving the elementary symmetric functions to prove the theorem. We note that the negative curvature case can be handled by a maximum principle argument for $1 \leq k \leq n$, see [**Viaa**]. The only case left is $K > 0$ and $k = n$, the determinant. We outline a PDE proof of the theorem in the $K > 0$ case which works for all $1 \leq k \leq n$. In this case, we need only the result on S^n since the general result follows by passing to the universal cover. The complete details of this proof may be found in [**Viab**]

The conservation laws for the constant scalar curvature equation have played a very important role in the study of its solutions, beginning with the results of [**Poh65**]. Further use of these invariants appears in [**Sch88**] and [**Pol93**]. The conservation laws for a fourth order conformally invariant equation may be found in [**Gur97**]. In the final section of this paper we provide a derivation of some corresponding conservation laws in the locally conformally flat case for the fully nonlinear equations (1.1).

2. Variational Properties

In this section we will prove Theorem 1.1.

Definition 2.1. *Let $A : V \to V$ be a symmetric linear transformation where V is an n-dimensional inner product space. For $0 \leq q \leq n$, the q*th Newton transformation *associated with A is*

$$T_q(A) = \sigma_q(A) \cdot I - \sigma_{q-1}(A) \cdot A + \cdots + (-1)^q A^q,$$

where I is the identity.

It is proved in [**Rei73**] that if A^i_j are the components of A with respect to some basis of V then

$$T_q(A)^i_j = \frac{1}{q!} \delta^{i_1 \ldots i_q i}_{j_1 \ldots j_q j} A^{j_1}_{i_1} \cdots A^{j_q}_{i_q}, \tag{2.1}$$

where $\delta^{i_1 \dots i_q i}_{j_1 \dots j_q j}$ is the generalized Kronecker delta symbol. We also have

$$\sigma_k(A) = \frac{1}{k!}\delta^{i_1 \dots i_k}_{j_1 \dots j_k} A^{j_1}_{i_1} \cdots A^{j_k}_{i_k}. \tag{2.2}$$

We note the following formula

$$\frac{d}{dt}\sigma_k(A(t)) = T_{k-1}(A(t))^i_j \frac{d}{dt}A(t)^j_i = \operatorname{trace}\left(T_{k-1}(A(t)) \circ \frac{d}{dt}A(t)\right), \tag{2.3}$$

i.e., the $(k-1)$-Newton transformation is what we get when we differentiate σ_k.

To simplify notation, we let

$$C(g) = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right). \tag{2.4}$$

Here $C(g)$ is defined as a $(0,2)$ tensor, but in the following we will also use the metric to view $C(g)$ as a $(1,1)$ tensor. When necessary, the distinction will be made using upper and lower indices.

PROPOSITION 2.2. *If (N,g) is locally conformally flat, then for $1 \leq k \leq n$,*

$$\delta T_k(C(g)) = 0.$$

For all (N,g), we have

$$\delta T_1(C(g)) = 0.$$

Here δ denotes the divergence.

PROOF. If (N,g) is locally conformally flat, then the Cotten tensor vanishes identically (see [**Eis97**])

$$B^i_{jl} = C^i_{j;l} - C^i_{l;j} = 0. \tag{2.5}$$

We then have

$$\begin{aligned}\delta T_k(C) = T_k(C)^i_{j;i} &= \frac{1}{k!}\delta^{i_1 \dots i_k i}_{j_1 \dots j_k j}\left(C^{j_1}_{i_1} \cdots C^{j_k}_{i_k}\right)_{;i} \\ &= \frac{1}{(k-1)!}\delta^{i_1 \dots i_k i}_{j_1 \dots j_k j} C^{j_1}_{i_1;i} \cdots C^{j_k}_{i_k} = 0,\end{aligned}$$

from (2.5) and the skew-symmetry of the generalized Kronecker delta symbol in the indices i_1 and i. For the case $T_1(C)$, we have

$$T_1(C)^i_j = \sigma_1(C)\delta^i_j - C^i_j = \frac{1}{(n-2)}\left(-Ric^i_j + \frac{R}{2}\delta^i_j\right), \tag{2.6}$$

so we have

$$\delta T_1(C) = T_1(C)^i_{j;i} = \frac{1}{(2-n)}\left(Ric^i_{j;i} - \frac{1}{2}R_{;j}\right) = 0,$$

by the Bianchi identity. □

PROPOSITION 2.3. *If we let $\tilde{g} = e^{-2u}g$, then we have the following transformation formulas*

$$C(\tilde{g}) = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g + C(g), \tag{2.7}$$

and for any function h,

$$\tilde{\nabla}^2 h = \nabla^2 h + du \otimes dh + dh \otimes du - \langle du, dh\rangle g. \tag{2.8}$$

PROOF. Both of these formulas are standard (see [**Bes87**]), we verify only the second here. We have for the Christoffel symbols

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} - u_i\delta^k_j - u_j\delta^k_i + g_{ij}g^{kl}u_l.$$

Therefore

$$\begin{aligned}(\tilde{\nabla}^2 h)_{ij} &= h_{ij} - \tilde{\Gamma}^k_{ij}h_k \\ &= h_{ij} - (\Gamma^k_{ij} - u_i\delta^k_j - u_j\delta^k_i + g_{ij}g^{kl}u_l)h_k \\ &= (\nabla^2 h)_{ij} + u_ih_j + u_jh_i - g^{kl}u_lh_kg_{ij}.\end{aligned}$$

□

The functional in Theorem 1.1 may then be written as

$$\mathcal{F}_k : \tilde{g} \mapsto \int_N e^{(2k-n)u}\sigma_k\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g + C(g)\right)\mathrm{dvol}_g. \tag{2.9}$$

Now take a curve $u(t)$ with $\dot{u}(0) = h$. We have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_k(u(t))|_{t=0} &= (2k-n)\int h\sigma_k(C(\tilde{g}))\ \mathrm{dvol}_{\tilde{g}} \\ &\quad + \int e^{(2k-n)u}T_{k-1}(C(\tilde{g}))_i^j\left((\nabla^2 h)^i_j + h^iu_j + u^ih_j - u^lh_l\delta^i_j\right)\mathrm{dvol}_g.\end{aligned}$$

Now writing the upper indices with respect to the metric $\tilde{g}$ and using (2.8), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_k(u(t))|_{t=0} &= (2k-n)\int h\sigma_k(C(\tilde{g}))\ \mathrm{dvol}_{\tilde{g}} \\ &\quad + \int \tilde{T}_{k-1}(C(\tilde{g}))_i^j\left((\nabla^2 h)^i_j + h^iu_j + u^ih_j - u^lh_l\delta^i_j\right)\mathrm{dvol}_{\tilde{g}} \\ &= (2k-n)\int h\sigma_k(C(\tilde{g}))\ \mathrm{dvol}_{\tilde{g}} + \int \tilde{T}_{k-1}(C(\tilde{g}))_i^j(\tilde{\nabla}^2 h)^i_j\mathrm{dvol}_{\tilde{g}} \\ &= (2k-n)\int h\sigma_k(C(\tilde{g}))\ \mathrm{dvol}_{\tilde{g}} - \int \tilde{T}_{k-1}(C(\tilde{g}))_{i;j}^j h^i\mathrm{dvol}_{\tilde{g}}.\end{aligned}$$

Using Proposition 2.2, we have the formula

$$\frac{d}{dt}\mathcal{F}_k(u(t))|_{t=0} = (2k-n)\int h\sigma_k(C(\tilde{g}))\ \mathrm{dvol}_{\tilde{g}}.$$

Theorem 1.1 follows by observing that the restriction to unit volume metrics gives a Lagrange multiplier.

Note that when $n = 2k$, we get zero variation. This is because, in the locally conformally flat case, the integrand is the Chern-Gauss-Bonnet integrand (see [**Viaa**]).

3. Ellipticity

In this section we will show that the equations 1.1 are elliptic.

DEFINITION 3.1. *Let $(\lambda_1,\dots,\lambda_n) \in \mathbf{R}^n$. We view the elementary symmetric functions as functions on $\mathbf{R}^n$*

$$\sigma_k(\lambda_1,\dots,\lambda_n) = \sum_{i_1<\cdots<i_k}\lambda_{i_1}\cdots\lambda_{i_k},$$

and we define

$$\Gamma_k^+ = \textit{component of } \{\sigma_k > 0\} \textit{ containing the positive cone.}$$

We also define $\Gamma_k^- = -\Gamma_k^+$.

For a symmetric linear transformation $A : V \to V$, where V is an n-dimensional inner product space, the notation $A \in \Gamma_k^\pm$ will mean that the eigenvalues of A lie in the corresponding set.

The following proposition describes some important properties of the sets $\Gamma_k^\pm$.

Proposition 3.2. *Each set* Γ_k^+ *is an open convex cone with vertex at the origin, and we have the following sequence of inclusions*

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+.$$

For symmetric linear transformations $A \in \Gamma_k^+$, $B \in \Gamma_k^+$, *and* $t \in [0,1]$, *we have the following inequality*

$$\{\sigma_k((1-t)A + tB)\}^{1/k} \geq (1-t)\{\sigma_k(A)\}^{1/k} + t\{\sigma_k(B)\}^{1/k}.$$

Furthermore, if $A \in \Gamma_k^+$, *then* $T_{k-1}(A)$ *is positive definite.*

The proof of this proposition is standard, and may be found in [**CNS85**], [**Går59**], and [**Wan94**]. Note that by replacing A with $-A$, analogous statements hold for Γ_k^-.

Definition 3.3. *A metric* g *is* positive k-admissible, *or* negative k-admissible *if* $C(g)$ *is everywhere in* Γ_k^+ *or* Γ_k^-, *respectively.*

Note that we are using the metric g to view this tensor as an endomorphism of the tangent space at any point, and we take the eigenvalues of this map. Since the Ricci tensor is symmetric, these eigenvalues are real.

Proposition 3.4. *If the background metric is* k*-admissible then the equations for a metric* $\tilde{g} = e^{-2u}g$,

$$\sigma_k(C(\tilde{g})) = \textit{constant}$$

are elliptic at any solution.

Proof. From (2.7), we have that the equations may be written as

$$\sigma_k\left(\nabla^2 u + du \otimes du + C(g) - \frac{|\nabla u|^2}{2} g\right) = e^{-2ku} C_k. \tag{3.1}$$

If the fixed metric has $C(g) \in \Gamma_k^+$, then since N is compact, at a minimum of the function u we have

$$\sigma_k\left(\nabla^2 u + C(g)\right) = e^{-2ku} C_k, \tag{3.2}$$

with $\nabla^2 u$ positive semidefinite. From Proposition 3.2, we then have, at the minimum point, $C(\tilde{g}) \in \Gamma_k^+$ with respect to $\tilde{g}$, since g and $\tilde{g}$ are conformal metrics. Therefore since the cones are connected, by continuity we have $C(\tilde{g}) \in \Gamma_k^+$ everywhere. A similar argument holds in the negative k-admissible case.

We let

$$F[u, \nabla u, \nabla^2 u] = \sigma_k\left(\nabla^2 u + du \otimes du + C(g_0) - \frac{|\nabla u|^2}{2} g_0\right) - C_k e^{-2ku}.$$

From the previous section, we see that the linearizion at the solution u in the direction h is given by

$$F'[u, \nabla u, \nabla^2 u](h) = tr(\tilde{T}_{k-1}(C(\tilde{g})) \circ \tilde{\nabla}^2 h) + 2kC_k e^{-2ku} h.$$

Since $C(\tilde{g}) \in \Gamma_k^{\pm}$, from Proposition 3.2, we are done. □

4. Uniqueness

In this section we prove Theorem 1.2. We first give the differential geometry proof, then we discuss the PDE proof.

4.1. Differential Geometry Proof. We state the following result of Obata (see [**Oba72**])

THEOREM 4.1. *Let (N, g) be a compact Einstein manifold. Then any constant scalar curvature metric in the conformal class of g is also Einstein. Moreover, g is the unique constant scalar curvature metric in its conformal class (up to scaling by a constant) unless (N, g) is isometric to (S^n, g_0), the n-sphere with the standard metric. On S^n, there is an $(n+1)$ parameter family of unit volume Einstein metrics conformal to the standard metric, each of which are the image of the standard metric under a conformal diffeomorphism of S^n.*

If there are 2 distinct unit volume Einstein metrics in the conformal class, then Obata constructs an isometry to (S^n, g_0) to prove the second statement. To prove Theorem 1.2 for σ_k, we just need to show that any other solution is necessarily Einstein. We assume N is oriented, noting that the nonorientable case will follow by passing to the orientable double cover.

We let $g_0 = v^{-2} g$, where g_0 is the constant curvature metric, and g is any other solution. From the transformation formula (2.7) for C, we have

$$C(g_0) = \frac{R_0}{2n(n-1)} g_0 = \frac{\nabla^2 v}{v} - \frac{|\nabla v|^2}{2v^2} g + C(g),$$

where everything on the right hand side is with respect to g, and R_0 is the scalar curvature of g_0. We rewrite this as

$$\frac{R_0}{2n(n-1)v^2} g + \frac{|\nabla v|^2}{2v^2} g = \frac{\nabla^2 v}{v} + C(g).$$

The left hand side is diagonal, so the right hand side must be also, i.e.,

$$\frac{\nabla^2 v}{v} + C(g) = \frac{\Delta v}{nv} g + \frac{\sigma_1(C(g))}{n} g.$$

We therefore have the formula

$$\nabla^2 v = \frac{\Delta v}{n} g + \frac{v\sigma_1(C(g))}{n} g - vC(g). \tag{4.1}$$

We now use the metric g to view this formula as a $(1,1)$ tensor, and using Proposition 2.2, we compute

$$\begin{aligned} 0 &= -\int_N \delta T_k(C)(\nabla v) dvol_g = \int_N tr(T_k(C) \circ \nabla^2 v) dvol_g \\ &= \int_N tr\left(T_k(C) \circ \left(\frac{\Delta v}{n} I + \frac{v\sigma_1(C(g))}{n} I - vC(g)\right)\right) dvol_g \end{aligned}$$

Next we observe that (see [**Rei73**])

$$tr(T_k(C)) = (n-k)\sigma_k(C), \quad \text{and} \tag{4.2}$$

$$tr(T_k(C)\circ C) = (k+1)\sigma_{k+1}(C). \tag{4.3}$$

We therefore have

$$0 = \int_N (n-k)\sigma_k(C)\left(\frac{\Delta v}{n} + \frac{v\sigma_1(C(g))}{n}\right) dvol_g - (k+1)\int_N v\sigma_{k+1}(C) dvol_g,$$

and since $\sigma_k(C)$ is constant, we end up with the integral formula

$$0 = (k+1)\int_N v\left(\sigma_{k+1}(C) - \frac{n-k}{(k+1)n}\sigma_k(C)\sigma_1(C)\right) dvol_g. \tag{4.4}$$

We next have a linear algebra lemma (see [**Viaa**] for the proof).

LEMMA 4.2. *Let A be a symmetric $n\times n$ matrix. If $A \in \Gamma_k^+$ then*

$$\sigma_{k+1}(A) \le \frac{n-k}{(k+1)n}\sigma_k(A)\sigma_1(A),$$

with equality if and only if $A = \lambda I$.

Note that by replacing A with $-A$, there are corresponding inequalities for $A \in \Gamma_k^-$. From Proposition 3.4, we have that g is k-admissible, so we can apply Lemma 4.2. Since v is positive, it follows that the integrand in (4.4) is always nonnegative or always nonpositive. Since the integral is equal to zero, we conclude that the integrand is identically zero and therefore, by the lemma, the metric g is Einstein.

4.2. PDE proof. We restate the theorem for S^n in stereographic coordinates We let δ_{ij} be the Kronecker delta symbol.

THEOREM 4.3. *Let $u(x) \in C^\infty(\mathbf{R}^n)$ be a positive solution to*

$$\sigma_k\left(u\cdot\frac{\partial^2 u}{\partial x^i\partial x^j} - \frac{|\nabla u|^2}{2}\delta_{ij}\right) = constant \tag{4.5}$$

for some $k \in \{1,\dots,n\}$. Suppose that $v(y) = |y|^2\cdot u(\frac{y^1}{|y|^2},\dots,\frac{y^n}{|y|^2})$ is smooth and

$$\lim_{y\to 0} v(y) > 0.$$

Then

$$u = a|x|^2 + b_i x^i + c$$

where a, b_i, and c are constants.

PROOF. We give a very brief outline of the proof. The details may be found in [**Viab**]. First, it is shown that the method of moving planes from [**GNN79**] is applicable. We may therefore assume that our solution is rotationally symmetric about the origin, $u = u(r)$. We then let $\phi(t) = e^{-t}u(e^t)$, which corresponds geometrically to cylindrical coordinates. In [**Viab**] it is shown that ϕ must satisfy the following conservation law

$$1 - (\phi^2 - (\phi^{'})^2)^k = D_{k,n}\phi^n, \tag{4.6}$$

where $D_{k,n}$ is a constant parameterizing the solutions. From this it follows that $\phi(t) = \cosh(t-t_0)$ are the only solutions which extend smoothly to S^n. $\square$

5. Conservation Laws

In this section we derive some conservation laws in the fully nonlinear case when N is locally conformally flat. Our derivation was inspired in part by the papers [**Gur97**] and [**Sch88**].

We assume that $k = 1$, or that (N, g) is locally conformally flat. Define

$$L_k(C) = -T_k(C) + \frac{n-k}{n}\sigma_k(C)I. \tag{5.1}$$

PROPOSITION 5.1. *The symmetric tensor $L_k(C)$ satisfies*

$$\delta L_k(C) = \frac{n-k}{n} d(\sigma_k(C)), \text{ and} \tag{5.2}$$

$$trace(L_k(C)) = 0. \tag{5.3}$$

PROOF. Formula 5.2 follows easily from Proposition 2.2. From (4.3), we have

$$trace(L_k(C)) = -(n-k)\sigma_k(C) + \frac{n-k}{n}\sigma_k(C)n = 0.$$

□

We will now use the metric to view $L_k(C)$ as a $(0, 2)$ tensor.

THEOREM 5.2. *Let (N, g) be a locally conformally flat n-dimensional manifold with boundary ∂N. Then for any conformal Killing vector field X, and $1 \leq k \leq n$, we have*

$$\frac{n-k}{n}\int_N \mathcal{L}_X(\sigma_k(C))\ dvol_N = \int_{\partial N} L_k(C)(X, \nu)\ dvol_{\partial N}, \tag{5.4}$$

where $\mathcal{L}$ denotes the Lie derivative, ν denotes the outward unit normal to ∂N, and the measures are those induced by the metric g.

PROOF. Since X is a conformal Killing vector field, it satisfies

$$\nabla_i X_j + \nabla_j X_i = \frac{2}{n}(\delta X)\delta_{ij},$$

with respect to a local orthonormal frame. From (5.3) above, we have

$$(L_k(C))_{ij}\nabla_i X_j = 0. \tag{5.5}$$

Integrating (5.5) over N, we have

$$\begin{aligned} 0 &= \int_N (L_k(C))_{ij}\nabla_i X_j\ dvol_N \\ &= -\int_N \nabla_i (L_k(C))_{ij} X_j\ dvol_N + \int_{\partial N} L_k(C)(X, \nu)\ dvol_{\partial N} \\ &= -\int_N \langle \delta L_k(C), X\rangle\ dvol_N + \int_{\partial N} L_k(C)(X, \nu)\ dvol_{\partial N}. \end{aligned}$$

From Proposition 5.1, we therefore have

$$0 = -\frac{n-k}{n}\int_N \langle d(\sigma_k(C)), X\rangle\ dvol_N + \int_{\partial N} L_k(C)(X, \nu)\ dvol_{\partial N}.$$

□

We note that for $k = 1$, we do not require N to be locally conformally flat, and our proof is identical to that of Proposition 1.4 in [**Sch88**]. Also note that

$$L_n(C) = -T_n(C) \equiv 0,$$

so the theorem is vacuous in this case.

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