MIN-MAX VARIATIONAL PRINCIPLE AND FRONT SPEEDS IN RANDOM SHEAR FLOWS∗

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Abstract. Speed ensemble of bistable (combustion) fronts in mean zero stationary Gaussian shear flows inside two and three dimensional channels is studied with a min-max variational principle. In the small root mean square regime of shear flows, a new class of multi-scale test functions are found to yield speed asymptotics. The quadratic speed enhancement law holds with probability arbitrarily close to one under the almost sure continuity (dimension two) and mean square Hölder regularity (dimension three) of the shear flows. Remarks are made on the quadratic and linear laws of front speed expectation in the small and large root mean square regimes.

Key words. Reaction-Diffusion, Random Media, Front Speeds, Min-Max Principle, Speed Asymptotics

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1. Introduction. We consider propagation speeds of reaction-diffusion fronts in random shear flows. The model equation is:

\[ u_t = \Delta_x u + \delta B \cdot \nabla_x u + f(u), \tag{1.1} \]

where \( x = (x_1, \tilde{x}) \in D = R^1 \times \Omega, \Omega = [0, L]^{n-1}, n = 2, 3; \delta > 0 \) a scaling parameter; the vector field \( B = (b(\tilde{x}, \omega), 0) \), \( b(\tilde{x}, \omega) \) is a scalar stationary Gaussian field with zero ensemble mean and almost surely continuous sample paths; \( f \) is either one of the following:

(1) the combustion nonlinearity with ignition temperature cutoff: \( f(u) = 0, u \in [0, \mu], \mu \in (0, 1); f(u) > 0, u \in (\mu, 1), f(1) = 0, f \in C^2([0, 1]); \)

(2) the bistable nonlinearity: \( f(u) = u(1-u)(u-\mu), \mu \in (0, 1/2). \)

The boundary condition is zero Neumann at \( R^1 \times \partial \Omega \). The initial data \( u_0 \) belongs to the set \( I_s \). The set \( I_s \) for bistable \( f \) consists of bounded continuous functions with limits one and zero at \( x_1 \sim \pm \infty \) respectively. For combustion \( f \), one requires also that \( u_0 \) decay to zero exponentially at \( -\infty \). For each realization of \( B \), there exists a traveling front solution of the form \( u = \Phi(x_1 + ct, \tilde{x}) \), with unique speed \( c \) and profile \( \Phi \) [4]. For \( u_0 \in I_s, u(x, t) \) converges to a traveling front at large times [15].

We are interested in the asymptotic behavior of \( c = c(\delta) \) when \( \delta \) is small or large. Such speed asymptotics have been recently studied for both deterministic fronts [3, 5, 8, 10, 9, 12, 14] and KPP fronts in random shears [13, 18]. See also [17] for related applied science literature. The bistable (combustion) front speed \( c \) has a min-max variational characterization [7, 8]. Following the notation of [8], define the functional:

\[ \psi(v) = \psi(v(x)) = \frac{L v + f(v)}{\partial_{x_1} v} = \frac{\Delta v + \delta b(\tilde{x}) \partial_{x_1} v + f(v)}{\partial_{x_1} v}. \tag{1.2} \]

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The strong form of the min-max variational formula of [7, 8] for \( c(\delta) \) is:

\[
\sup_{v \in K} \inf_{x \in D} \psi(v(x)) = c(\delta) = \inf_{v \in K} \sup_{x \in D} \psi(v(x))
\]

where \( K \) is the set of admissible functions:

\[
K = \{ v \in C^2(D) | \partial_{x_1} v > 0, 0 < v(x) < 1, v \in I_s \}.
\]

The variational principle (1.3) is a powerful tool for getting tight bounds on \( c \), if one can construct an admissible test function \( v \) that approximates well the exact traveling front solution \( \Phi \). This is the case when \( \delta \) is small and the test function is a perturbation of the known traveling front solution when \( \delta = 0 \). In [8], a form of such test function is proposed for deterministic shear flow, and the quadratic speed enhancement law ([14] and references therein) is recovered if \( \delta \) is small enough. To handle an ensemble of shear flows in the random case, one must know which norm of \( b \) controls the smallness of \( \delta \) by analyzing an admissible test function \( v \) and the functional \( \psi(v) \). A new form of multi-scale test function is found for this purpose, and it cures a delicate divergence problem when perturbing the traveling front at \( \delta = 0 \).

It turns out that the smallness of \( \delta \) for the quadratic speed enhancement depends on the maximum (Hölder) norm of \( b \) if \( n = 2 \) (3).

Our main results are that the quadratic speed enhancement law holds with probability arbitrarily close to one, under the almost sure continuity (\( n = 2 \)) or mean square Hölder regularity (\( n = 3 \)) conditions of the shear. As \( b \) can be arbitrarily large (though with small probability), our admissible test functions are not valid for the entire ensemble. However, the probability of \( b \) being large can be shown to be small enough that the quadratic enhancement law holds for the expectation of front speed [11]. A similar result holds in the Kolmogorov-Petrovsky-Piskunov (KPP) case (e.g. when \( f(u) = u(1-u) \)), where there is less restriction on test functions and the minimal speed variational principle is minimization over functions on the cross section \( \Omega \) [13].

The paper is organized as follows. In section 2, the new multi-scale test function is presented and used to estimate the front speed and identify the smallness of \( \delta \) in terms of proper norms of \( b \). In section 3, probabilistic estimates based on Borell’s inequality and Karhunen-Loève expansion of mean zero Gaussian fields help to control the maximum (Hölder) norms of \( b \), and lead to the quadratic speed enhancement laws. Remarks are made on the quadratic and linear laws of front speed expectation when \( \delta \) is small and large respectively.

2. Deterministic Front Speed Asymptotics. Let us consider \( b \) to be a given Hölder continuous deterministic function, and assume that \( \langle b \rangle = 1 \| b \|_1 \Omega dy = 0 \). Otherwise, the integral mean contributes a linear term \( \delta < b > \) to the front speed. Let \( U = U(x_1 + c_0 t) \) be the traveling fronts when \( \delta = 0 \), satisfying the equation:

\[
U'' - c_0 U' + f(U) = 0,
\]

\( U(-\infty) = 0, U(+\infty) = 1, U' > 0, U \) approaches zero and one at exponential rates.

In the small \( \delta \) regime, define the test function

\[
v(x) = U(\xi) + \delta^2 \hat{u}(\xi, \hat{x}),
\]

where the variable:

\[
\xi = (1 + \alpha \delta^2) x_1 + \delta \chi,
\]

\( 0 < \alpha < 1 \), and the test function is a perturbation of the known traveling front solution when \( \delta = 0 \) (3).
with $\alpha$ a constant to be determined and $\chi = \chi(\tilde{x})$ solution of:
\[-\Delta \tilde{x} \chi = b, \quad \tilde{x} \in \Omega,\]
subject to zero Neumann boundary condition at $\partial \Omega$. We normalize $\chi$ by setting $\int_{\Omega} \chi \, dy = 0$. The test function (2.1) reduces to the one in [8] if $\alpha = 0$. However, we shall see that $\alpha \neq 0$ is essential in suppressing certain divergence arising in the evaluation of $\psi(v)$, similar to curing secular growth in multiple-scale perturbation expansions.

For Hölder continuous $b(\tilde{x})$, $\chi$ is $C^{2+p}$, $p \in (0,1)$. Assume that $\tilde{u}$ is $C^2$, and decays sufficiently fast at $\xi$ infinities. A straightforward computation shows that:
\[
Lv + f(v) = U'' + f(U)
\]
\[
+ \delta^2 (2 \alpha U'' + \Delta U, \tilde{u} + U''|\nabla \chi|^2 + f'(U) \tilde{u})
\]
\[
+ \delta^3 (b \alpha U' + 2 \nabla \tilde{u}_\xi \cdot \nabla \chi)
\]
\[
+ \delta^4 (\alpha^2 U'' + 2 \alpha \tilde{u}_\xi + \Delta U, \tilde{u} |\nabla \chi|^2)
\]
\[
+ \delta^5 (f(v) - f(U) - f'(U) \delta^2 \tilde{u})
\]
\[
+ \delta^6 \alpha b \tilde{u}_\xi + \delta^6 \alpha^2 \tilde{u}_\xi \xi,
\]
(2.3)

where $f(v) - f(U) - f'(U) \delta^2 \tilde{u} = O(\delta^4 \tilde{u}^2)$ and the $O(\delta)$ terms cancel by choice of $-\Delta \chi = b$. On the other hand, up to an undetermined constant $\gamma$, we have:
\[
(c_0 + \delta^2 \gamma) \partial_{x_1} v = c_0 U' + \delta^2 (c_0 \alpha U' + c_0 \tilde{u}_\xi + \gamma U')
\]
\[
+ \delta^4 (\gamma \alpha U' + c_0 \alpha \tilde{u}_\xi + \gamma \tilde{u}_\xi)
\]
\[
+ \delta^6 (\alpha \gamma \tilde{u}_\xi),
\]
(2.4)

All terms in (2.3) and (2.4) are evaluated at $(\xi, \tilde{x})$. Let us choose $\tilde{u}$ to solve the equation:
\[
\tilde{L} \tilde{u} = \Delta_{\xi, \tilde{x}} \tilde{u} - c_0 \tilde{u}_\xi + f'(U) \tilde{u} = -|\nabla \chi|^2 U'' - 2 \alpha U'' + \gamma U' + c_0 \alpha U',
\]
(2.5)
subject to zero Neumann boundary condition at $\partial \Omega$, and exponential decay at $\xi$ infinities. The solvability of (2.5) will be discussed later. Then we would have
\[
Lv + f(v) = (\partial_{x_1} v)(c_0 + \delta^2 \gamma) + R_1.
\]
(2.6)

The remainder $R_1$ is:
\[
R_1 = \delta^3 A + \delta^4 B + \delta^5 D + \delta^6 E + O(\delta^4 \tilde{u}^2),
\]
(2.7)

where
\[
A = b \alpha U' + 2 \nabla \tilde{u}_\xi \cdot \nabla \chi
\]
\[
B = \alpha^2 U'' + 2 \alpha \tilde{u}_\xi + \Delta_{\xi, \tilde{x}} \tilde{u} |\nabla \chi|^2 - \gamma \alpha U' - c_0 \alpha \tilde{u}_\xi - \gamma \tilde{u}_\xi
\]
\[
D = ab \tilde{u}_\xi
\]
\[
E = \alpha^2 \tilde{u}_\xi - \alpha \gamma \tilde{u}_\xi.
\]
(2.8)

The constant $\gamma$ is determined by a solvability condition for (2.5). The right hand side must be orthogonal to the function $U'(\xi) e^{-\alpha \xi}$ which spans the kernel of the
adjoint operator \( \tilde{L}^* v = \Delta_{\xi, x} v + c_0 v + f'(U)v \). Thus,

\[
\gamma \int_R (U')^2 e^{-c_0 \xi} d\xi = \frac{1}{|\Omega|} \int_D ((|\nabla \chi|^2 + 2\alpha) U'' U' - c_0 \alpha (U')^2) e^{-c_0 \xi} d\xi d\tilde{x}
\]

\[
= \frac{c_0}{2} |\nabla \chi|^2 + 2\alpha \int_R (U')^2 e^{-c_0 \xi} d\xi - c_0 \alpha \int_R (U')^2 e^{-c_0 \xi} d\xi
\]

where the braket denotes the integral average over \( \Omega \). We have performed integration by parts once, the boundary terms decay to zero and the integral \( \int_R (U')^2 e^{-c_0 \xi} d\xi \) converges for both bistable and combustion type nonlinearities. It follows that equation (2.5) is solvable when

\[
\gamma = \frac{c_0}{2|\Omega|} \int_{\Omega} |\nabla \chi|^2 d\tilde{x} = \frac{c_0}{2} |\nabla \chi|^2,
\]

regardless of the choice of \( \alpha \). For such \( \gamma \), we may choose \( \alpha = -\frac{1}{2} |\nabla \chi|^2 \) so \( \gamma + c_0 \alpha = 0 \) and the \( \tilde{u} \) equation becomes:

\[
\tilde{L}\tilde{u} = \Delta_{\xi, x} \tilde{u} - c_0 \tilde{u}_x + f'(U)\tilde{u} = (|\nabla \chi|^2 - |\nabla \chi|^2) U''.
\]

Note that the right hand side of (2.10) has zero integral average over \( \Omega \).

The solution \( \tilde{u} \) of (2.10) can be expanded in eigenfunctions of the Laplacian \( \Delta_{\xi} \):

\[
\tilde{u}(\xi, \tilde{x}) = \sum_{j \in \mathbb{Z}^{n-1}_{+}\cup\{0\}} u_j(\xi) \phi_j(\tilde{x}),
\]

\[
|\nabla_{\xi} \chi|^2 = \sum_{j \in \mathbb{Z}^{n-1}_{+}\cup\{0\}} a_j \phi_j(\tilde{x}),
\]

where \( \mathbb{Z}^{n-1}_{+} \) denotes nonnegative integer vectors in \( \mathbb{R}^{n-1} \) with at least one positive component; \( -\Delta_{\xi} \phi_j = \lambda_j \phi_j \) with zero Neumann boundary condition on \( \partial \Omega \). The eigenfunctions \( \phi_j \) are normalized so that \( \|\phi_j\|_{L^\infty} = 1 \), and the eigenvalues \( \lambda_j = O(|j|^2) \) for large \( |j| \). Clearly, \( \phi_0 = 1 \) and \( \lambda_0 = 0 \), \( a_0 = \langle |\nabla \chi|^2 \rangle \). The functions \( u_j \) solve the equations

\[
L_j u_j = u''_j - \lambda_j u_j - c_0 u'_j + f'(U)u_j = -(a_j - \delta_{j,0} a_0) U''.
\]

The \( j = 0 \) equation has zero right hand side, and we take \( u_0 \equiv 0 \). If \( \alpha \) were zero, equation (2.5) implies that the right hand side of the \( j = 0 \) equation would be \(-a_0 U'' + \gamma U'\). The general solution (zeroth-mode) for \( u_0 \) is \(-\frac{a_0}{\gamma} U'(\xi) + \text{const.} U'(\xi)\). The linear factor \( \xi \) would render \( \nu(x) \) unable to stay in the set \( K \) due to divergence at large \( \xi \) (and \( x_1 \)).

Now consider the equations for \( j \neq 0 \). As shown in [16], the operators \( L_j \) are invertible in part due to \( \lambda_j > 0 \) for \( j \neq 0 \). As in Lemma 2.3 of [16], we have the regularity estimate:

\[
\|u_j\|_{C^2(\mathbb{R})} \leq C_1 |a_j|, \quad j \neq 0,
\]

where the constant \( C_1 \) is independent of \( j \) and depends only on \( U \) and its derivatives.

Next we show that for \( j \neq 0 \), the ratio \( \frac{\lambda_j}{a_j} \) is bounded by \( C_2 |a_j| \), where \( C_2 \) is a positive constant independent of \( j \). It suffices to prove a uniform bound if the \( a_j \)
factor is one. Consider the function $w_j = \beta U' - u_j$ for some positive constant $\beta$. For some $r_0 > 0$ sufficiently large, $f'(U) \leq 0$ whenever $|\xi| \geq r_0$. Choose $\beta$ as

$$\beta \geq \beta_1 \equiv \frac{C_1}{\min_{|\xi| \leq r_0} U'(\xi)},$$

(2.13)

so that by (2.12), $w_j > 0$ in the region $|\xi| \leq r_0$. By equation (2.11) for $u_j$ and the fact that $U'$ solves $L_j U' = -\lambda_j U' < 0$, we have

$$L_j w_j = -\beta \lambda_j U' + U''.$$  

(2.14)

Using the fact that $U''$ is bounded by a constant multiple of $U'$, and that $\inf_{j \neq 0} \lambda_j > 0$, we may further increase $\beta$ if necessary to ensure that $L_j w_j < 0$ in the region $|\xi| > r_0$. Maximum principle implies that $w > 0$ for all $\xi \in R$. Repeating the argument for $-u_j$ and taking into account the $a_j$ factor give:

$$\frac{|u_j|}{U'} \leq C_3 |a_j|,$$ 

(2.15)

where the constant $C_3$ depends only on $U$, and (2.15) holds uniformly in $j \neq 0$. Inequality (2.15) says that there is no resonance when inverting the operator $L_j$ to find $u_j$, for $j \neq 0$, in other words $u_j$ decays same as $U'$ at infinities.

To improve the estimate above, we move the term $f'(U) u_j$ to the right hand side of (2.11) which is then bounded by $C_4 |a_j| U''$, for $C_4$ independent of $j$. The left hand side operator becomes $u''_j - \lambda_j u_j - c_0 u'_j$. An explicit formula can be written for $u_j$, and the asymptotics of $\lambda_j$ imply that

$$\frac{|u_{j}^{(i)}(\xi, \tilde{x})|}{U''(\xi)} \leq C_4 \frac{|a_j|}{(1 + |j|^2)},$$

(2.16)

for a constant $C_4$ depending only on $U$, where $i = 0, 1, 2$ denotes the order of $\xi$ derivatives.

Suppose that $b$ is H"older continuous, then Schauder estimates give [6]:

$$||\nabla \chi||^{1+p} \leq C \|b\|_{p}^{2},$$

(2.17)

for some $p \in (0, 1)$, $C = C(\Omega)$, $\| \cdot \|_p$ is the standard H"older norm. For rectangular cross section $\Omega$ (dimension $n - 1$), the $\phi_j$’s are trigonometric functions. Then

$$|a_j| \leq C_5 \|b\|_{p}^{2} (1 + |j|)^{-(1+p)}, \quad p \in (0, 1),$$

(2.18)

for some constant $C_5$ [19]. Combining (2.16), (2.17), and (2.18), we see that the eigenfunction expansion of $\bar{u}$

$$\bar{u}(\xi, \tilde{x}) = \sum_{j \in \mathbb{Z}^{n-1}_{+}} u_j(\xi) \phi_j(\tilde{x})$$

(2.19)

converges uniformly in $(\xi, \tilde{x})$. Moreover,

$$\frac{\bar{u}_{i}(\xi, \tilde{x})}{U''(\xi)} \leq \sum_{j \in \mathbb{Z}^{n-1}_{+}} \frac{|u_{j}^{(i)}(\xi)| |\phi_j| |(\tilde{x})|/U''}{U''(\xi)}$$

$$\leq C_4 C_5 \|b\|_{p}^{2} \sum_{j \in \mathbb{Z}^{n-1}_{+}} (1 + |j|)^{-(3+p)} = C_6 \|b\|_{p}^{2},$$

(2.20)
if \( n = 2, 3 \). The mixed derivative term is bounded as:

\[
\left| \frac{\nabla_{\hat{\xi}} \tilde{u}_{\xi}(\xi, \hat{x})}{U'(\xi)} \right| \leq \sum_{j \in \mathbb{Z}_+^{n-1}} \frac{|u_j^{(1)}(\xi)|}{U'} |j| \phi_j(\hat{x})
\]

\[
\leq C_6 \|b\|_p^2 \sum_{j \in \mathbb{Z}_+^{n-1}} (1 + |j|)^{-2+2p} = C_7 \|b\|_p^2, \tag{2.21}
\]

if \( n = 2, 3 \). In case \( n = 2 \), \( \Omega \) is an interval, then \( b \in L^\infty(\Omega) \) suffices, because \( \nabla_{\hat{\xi}} \chi \) is Lipschitz, so \( |a_j| \leq O(\|b\|_\infty^p |j|^{-p}) \) for some \( p \in (0, 1) \) which replaces estimate (2.18). The exponent in (2.21) goes down to \( 1 + p \), yet enough for convergence on \( j \in \mathbb{Z}_+^1 \).

As a result, for \( n = 2, i = 0, 1, 2 \), the estimates:

\[
\frac{|u_j^{(1)}(\xi, \hat{x})|}{U'(\xi)} \leq C_8 \|b\|_\infty^2, \quad \frac{|\nabla_{\hat{\xi}} \tilde{u}_{\xi}(\xi, \hat{x})|}{U'(\xi)} \leq C_9 \|b\|_\infty^2, \tag{2.22}
\]

hold.

As a consequence of these estimates, let us verify the admissibility of test function \( v \). Clearly, \( v \) approaches zero (one) exponentially at \( x_1 = -\infty (+\infty) \). As

\[
v_{x_1} = (1 + \delta^2 \alpha)U' + \delta^2 (1 + \delta^2 \alpha)\tilde{u}_{\xi},
\]

we have for \( n = 3 \):

\[
v_{x_1} \geq \frac{1}{2} U' - \frac{1}{2} \delta^2 |\tilde{u}_{\xi}| \geq \left[ \frac{1}{2} - \delta^2 \frac{C_6}{2} \|b\|_p^2 \right] U' \geq \frac{1}{4} U' > 0, \tag{2.23}
\]

if:

\[
\delta^2 \leq \min \left( \frac{1}{2|\alpha|}, \frac{1}{2} \left( C_6 \|b\|_p^2 \right)^{-1} \right). \tag{2.24}
\]

For \( n = 2 \), in view of (2.22), \( v_{x_1} \geq \frac{1}{4} U' \) if

\[
\delta^2 \leq \min \left( \frac{1}{2|\alpha|}, \frac{1}{2} \left( C_6 \|b\|_\infty^2 \right)^{-1} \right). \tag{2.25}
\]

It follows from the monotonicity of \( v \) in \( x_1 \) and its asymptotics near \( x_1 \) infinities that \( 0 < v < 1 \) for all \( x_1, \hat{x} \).

With \( v \) being an admissible test function, we see from (2.6) that

\[
\psi(v(x)) = \frac{Lv + f(v)}{\partial_{x_1} v} = c_0 + \delta^2 \gamma + \frac{R_1}{\partial_{x_1} v}, \tag{2.26}
\]

with \( R_1 \) defined by (2.7). By (2.24) or (2.25), \( \partial_{x_1} v \geq \frac{1}{4} U' \). Also, each term in \( R_1 \) is bounded by a multiple of \( U' \). It follows that for \( \delta \) satisfying (2.24) or (2.25), \( |\frac{R_1}{\partial_{x_1} v}| \leq |4 \frac{\delta^2 \gamma}{U'}| \leq C_{10} \delta^3 \), where \( C_{10} = C_{10}(U, \|b\|_p), p \in (0, 1) \), if \( n = 3, C_{10} = C_{10}(U, \|b\|_\infty) \) if \( n = 2 \). We have shown:

**Proposition 2.1.** Let \( n = 2 \) or \( 3 \), and let the shear flow profile \( b(\hat{x}) \) be Hölder continuous with exponent \( p \in (0, 1) \) over rectangular domain \( \Omega \subset R^{n-1} \). There is a
positive constant $\delta_0$ depending only on $U$ and the Hölder (maximum) norm of $b$ for $n = 3$ ($n = 2$) such that if $\delta \leq \delta_0$:

$$c(\delta) = c_0 + \frac{C_0}{2|\Omega|} \int_{\Omega} |\nabla \chi|^2 \, dy + O(\delta^3).$$

(2.27)

A specific form of $\delta_0$ follows from (2.24) and (2.25).

At large $\delta$, numerical evidence suggested that $\lim_{\delta \to \infty} c(\delta)/\delta$ exists for bistable and combustion nonlinearities [12]. Such a limit holds for the KPP nonlinearity [3].

3. Random Front Speed Asymptotics. Let us consider $b = b(\tilde{x}, \omega)$ as a stationary mean zero Gaussian field with almost surely continuous sample paths in dimension one and two for $n = 2, 3$ respectively. We are interested in the restriction of $b$ over $\Omega$. In order to apply results of the previous section on each realization of $b$, let us write $b = \tilde{b} + b_1$, where $\tilde{b} = |\Omega|^{-1} \int_{\Omega} b(\tilde{x}, \omega) \, d\tilde{x}$. Correspondingly, $c = c(\delta, \omega) = c_0 - \delta \tilde{b} + c_2(\omega)$, where $c_2$ depends on $b_1$. As the size of $\delta_0$ depends on either the maximum norm or the Hölder norm of $b_1$ which is an unbounded random variable, there is no uniform way to choose a $\delta_0$ for all realizations. On the other hand, the probability of the occurrence of very large maximum (Hölder) norm of $b_1$ is often small, so the random speed asymptotics may hold with probability arbitrarily close to one.

Let us consider $n = 2$ first.

Theorem 3.1 (Two dimensional channel). Let $D = R^1 \times [0, L]$, $\Omega = [0, L]$, and $b(\tilde{x})$ be the restriction on $\Omega$ of a mean zero stationary, Gaussian random process with almost surely continuous sample paths. Let $f$ be a bistable or combustion nonlinearity. Then for each small $\epsilon \in (0, 1/4)$ and $q \in (0, 1)$, there is a deterministic constant $\delta_0 = \delta_0(\epsilon, q)$ such that if $\delta \in (0, \delta_0),$:

$$\text{Prob} \left\{ c(\delta, \omega) - c_0 + \delta b - \delta^2 \frac{1}{2} \left( \int_0^2 b_1(y) \, dy \right)^2 \geq \kappa \delta^{3-q} \right\} < \epsilon.$$

where the constant $\kappa > 0$ is independent of $\delta, \epsilon, q$. As $\epsilon \to 0$, the constant $\delta_0$ can be chosen to satisfy

$$\delta_0(\epsilon, q) \geq C|\log(\epsilon)|^{-\frac{1}{q}}.$$  

(3.1)

Proof. Recall the inequality (Lemma 3.1, p. 62, [2]):

$$E[\sup_{x} b(\tilde{x})] \leq E \sup_{x} |b(\tilde{x})| \leq E|b(0)| + 2E[\sup_{x} b(\tilde{x})],$$

(3.2)

and the Borell inequality on mean zero Gaussian process with almost surely continuous sample paths (Theorem 2.1, p. 43, [2]):

$$\mu \equiv E[\sup_{x} b(\tilde{x})] < \infty$$

$$P(|\sup_{x} b(\tilde{x}) - \mu| > \lambda) \leq 2e^{-\lambda^2/(2\sigma^2)},$$

(3.3)

where $\sigma^2 = E[b^2]$. With the choice of $\lambda = \sqrt{-2\sigma^2 \log(\epsilon/4)}$, inequality (3.3) implies that

$$P(|b|_\infty > \lambda + \mu) \leq P(\sup_{x} b(\tilde{x}) - \mu > \lambda) + P(-\inf_{x} b(\tilde{x}) - \mu > \lambda)$$

$$\leq 4e^{-\lambda^2/(2\sigma^2)} \leq \epsilon.$$

(3.4)
We now see that for $\epsilon > 0$ sufficiently small, $\lambda > 1$; and there is a constant $C_{11}$ independent of $\epsilon$ such that if

$$\delta < \delta_0(\epsilon) = C_{11} \left( \frac{1}{\mu + \lambda} \right)^{\frac{q}{4}} \leq C_{11} \frac{1}{\mu + \lambda}$$  \hspace{1cm} (3.5)$$

then (2.25) holds with probability at least $1 - \epsilon$. Therefore, by Proposition 2.1, the front speed asymptotics hold with probability $1 - \epsilon$. Using (2.7), (2.8), and (2.22) we see that the remainder in the expansion satisfies

$$\left| \frac{R_1}{\partial_{x_1} v} \right| \leq C_{12} \delta^3 \|b\|_{\infty}^4 \leq C_{13} \delta^3 \delta^n (\mu + \lambda)^4$$

$$= C_{13} \delta^{3-q} \delta^n (\mu + \lambda)^4$$

$$\leq C_{14} \delta^{3-q}$$

with probability $1 - \epsilon$. The estimate on $\delta_0(\epsilon)$ follows from (3.5).

**Remark 3.1.** An example of such $b$ process is the mean zero stationary Ornstein-Uhlenbeck (O-U) process which is both Gaussian and Markov. The O-U sample paths are almost surely continuous.

The $n = 3$ case requires a probabilistic estimate of Hölder norm of $b$. For a mean zero Gaussian field, this is related to the structure of the covariance function $R(t, s) = E[b(t)b(s)]$, $t, s \in \Omega$. If the covariance function is continuous, positive and non-negative definite, there exists a Gaussian process with this covariance [2].

The symmetric integral operator: $\phi \rightarrow \int_{\Omega} R(t, s) \phi(t) \, dt$ generates a complete set of orthonormal eigenfunctions $\phi_j$ on $L^2(\Omega)$ with nonnegative eigenvalues $\lambda_j$, $j = 1, 2, \ldots$.

Define:

$$p(u) = \max_{\|s-t\|\leq u \sqrt{2}} \left[ E[|b(s) - b(t)|^2] \right]^{1/2}.$$  \hspace{1cm} (3.6)$$

Consider the partial sum ($m$ a positive integer):

$$X^{(m)}(t, \omega) = \sum_{j=1}^{m} \sqrt{\lambda_j} \phi_j(t) \theta_j(\omega),$$

where $\theta_j$’s are independent unit Gaussian random variables. The convergence of the partial sum to $b$ is given by Garsia’s Theorem (Theorem 3.3.2, p. 52, [1]). It says that if $\int_0^1 (-\log u)^{1/2} \, dp(u) < \infty$, then with probability one, $X^{(m)}(t)$ are almost surely equicontinuous and converge uniformly on $[0, 1]^2$. The resulting infinite series is the celebrated Karhunen-Loève expansion. Moreover, the following estimate holds for all $m$:

$$|X^{(m)}(s) - X^{(m)}(t)| \leq 16 \sqrt{2} |\log B|^{1/2} p(\|s - t\|)$$

$$+ 32 \sqrt{2} \int_0^{\|s-t\|} (-\log u)^{1/2} \, dp(u),$$

where $B = B(\omega)$ is a positive random variable, $E[B^2] \leq 32$. 

Suppose that $p(u)$ is Hölder continuous with exponent $s \in (0, 1)$, then $b$ is almost surely Hölder continuous with exponent $s$, and the Hölder norm of $b$ is bounded by $\alpha_1 \log B^{1/2} + \alpha_2$, where $\alpha_1, \alpha_2$ are two positive deterministic constants. Chebyshev’s inequality gives:

$$\text{Prob} \left( \left| \log B^{1/2} \right| \geq \lambda \right) = \text{Prob} \left( B \geq e^{\lambda^2} \right) \leq E(B^2) / e^{2\lambda^2} \leq 32 e^{-2\lambda^2}. \quad (3.8)$$

This implies that

$$\text{Prob} \left( \left\| b \right\|_s > \lambda \right) \leq 32 e^{-2 \left( \frac{\lambda - \alpha_2}{\alpha_1} \right)^2} \leq C_{15} e^{-2\lambda^2} \leq \epsilon,$$  

if $\lambda = \sqrt{\log \frac{1}{15}}$. For $\epsilon > 0$ sufficiently small, $\lambda > 1$ and we take

$$\delta < \delta_0(\epsilon) = C \left( \frac{1}{\lambda} \right)^{\frac{1}{4}} \leq C \frac{1}{\lambda},$$

so that (2.24) holds with probability at least $1 - \epsilon$. As before we use (2.7), (2.8), and (2.22) to conclude that the remainder in the expansion satisfies

$$\left| \frac{R_1}{\partial_{x_1} v} \right| \leq C_{12} \delta^{\beta} \left\| b \right\|_*^4 \leq C_{16} \delta^{3-q}$$

with probability $1 - \epsilon$. Similar results hold on $\Omega = [0, L]^2$. We now conclude from Proposition 2.1:

**Theorem 3.2** (Three dimensional channel). Let $D = R^1 \times \Omega$, $\Omega = [0, L]^2$, and $b$ be a mean zero Gaussian process such that the function $p(u)$ is Hölder continuous. Then $b$ has almost surely Hölder continuous sample paths. For each small $\epsilon \in (0, 1/5)$ and $q \in (0, 1)$ there is a deterministic constant $\delta_0 = \delta_0(\epsilon, q)$ such that if $\delta \in (0, \delta_0)$,

$$\text{Prob} \left\{ \left| c(\delta, \omega) - c_0 + \delta b - \delta^2 \frac{1}{2} \left| \nabla \tilde{x} \chi \right|^2 \right| \geq \kappa \delta^{3-q} \right\} < \epsilon,$$

where $\chi$ satisfies: $-\Delta \tilde{x} \chi = b_1$, subject to zero Neumann boundary condition at $\partial \Omega$. The constant $\kappa > 0$ is independent of $\delta$, $\epsilon$, and $q$. As $\epsilon \to 0$, the constant $\delta_0$ can be chosen to satisfy

$$\delta_0(\epsilon, q) \geq C |\log(\epsilon)|^{-\frac{2}{q}}. \quad (3.10)$$

**Remark 3.2.** Suppose that $\lim_{\delta \to \infty} c(\delta, \omega)/\delta$ exists almost surely and that $E(\left\| b \right\|_\infty)$ is finite. The dominated convergence theorem implies that $\lim_{\delta \to \infty} E[c(\delta, \omega)]/\delta$ exists and is finite. This argument is same as for the KPP case, see [13] for details.

**Remark 3.3.** In the proofs of Theorem 3.1 and Theorem 3.2, the test functions are admissible provided $\delta$ is small enough, depending on a proper (Hölder or supremum) norm of $b$. By Borel inequality (3.4) and similar bound (3.8) on the Hölder norm, the probability of having inadmissible test functions is exponentially small in terms of deviation. The remainder in the expansion of speed expectation can thus be controlled and shown to be higher order than quadratic. So the quadratic enhancement law of speed expectation holds for combustion and bistable random fronts as for KPP fronts. Complete proof is presented in [11].
4. Concluding Remarks. Bistable and combustion front speeds in mean zero Gaussian random shear flows have been studied with the min-max variational principle \cite{7, 8}. The quadratic speed enhancement law is valid with probability arbitrarily close to one in both two and three dimensional channels under the almost sure continuity and the mean square Hölder regularity conditions of the Gaussian shear flows.

It would be interesting to extend results here to the case of general convex cross section $\Omega$ with smooth boundary, or to the case of $\Omega$ in dimension higher than two.

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