Math 112B Final Exam practice problems

This preparation sheet contains problems from the second part of the course. The final exam will also contain (less than a half) problems from the first part. In order to prepare for those, use the Practice midterm sheet.

1. Consider the nonhomogeneous ODE:

   \[ u'' - 25u = -f(x), \quad 0 < x < 2, \]

   \[ u'(0) = 2, \quad u(2) = 0. \]

(a) Solve the ODE. (b) Plot the Green’s function \( G(x, \xi) \) as a function of \( x \) for a fixed point \( \xi \). What does it represent?

Solution. (a) Solve for Green’s function \( G(x, \xi) \). For fixed \( \xi \), \( G \) satisfies

\[
\begin{cases}
  \frac{d^2 G}{dx^2} - 25G = 0 & \text{for } x \neq \xi, \\
  \frac{dG}{dx} \bigg|_{x=0} = G \bigg|_{x=2} = 0 \\
  G \big|_{x=\xi+0} - G \big|_{x=\xi-0} = 0 \\
  \frac{dG}{dx} \bigg|_{x=\xi+0} - \frac{dG}{dx} \bigg|_{x=\xi-0} = \frac{-1}{p(\xi)} = -1
\end{cases}
\]

From the first two conditions we have that

\[
G(x, \xi) = \begin{cases}
  c(\xi) \cosh (5x), & x < \xi, \\
  d(\xi) \sinh (5(x-2)), & x > \xi,
\end{cases}
\]

where \( c \) and \( d \) depend on \( \xi \) (but are independent of \( x \)). The symmetry condition on \( G \) (that is, \( G(x, \xi) = G(\xi, x) \)) implies that

\[
G(x, \xi) = \begin{cases}
  A \sinh (5(\xi - 2)) \cosh (5x), & x < \xi, \\
  A \cosh (5\xi) \sinh (5(x-2)), & x > \xi,
\end{cases}
\]

where \( A \) is independent of both \( x \) and \( \xi \). Now the final condition in (3) tells us that

\[
5A \cosh (5\xi) \cosh (5(\xi - 2)) - 5A \sinh (5(\xi - 2)) \sinh (5\xi) = -1
\]

\[
\Rightarrow \quad A = \frac{-1}{5 \cosh 10}.
\]
Thus

\[ G(x, \xi) = \begin{cases} 
-\frac{1}{5 \cosh 10} \sinh (5(\xi - 2)) \cosh (5x), & x < \xi, \\
-\frac{1}{5 \cosh 10} \cosh (5\xi) \sinh (5(x - 2)), & x > \xi. 
\end{cases} \]

Now our solution is

\[ u(x) = \int_0^2 f(\xi) G(x, \xi) d\xi + c_1 e^{5x} + c_2 e^{-5x}. \]

The last two terms, two linearly independent solutions to the homogeneous equation, are necessary since we have non-zero boundary conditions. Solve for \(c_1\) and \(c_2\) by plugging in the boundary conditions (the integral part will be zero when you plug in). Doing so results in \(c_1 = (5e^{10} \cosh 10)^{-1}\) and \(c_2 = (5e^{10} \cosh 10)^{-1} - \frac{2}{5}\). Then plugging our expression for \(G\) into the integral, we have our final answer

\[ u(x) = \int_0^x f(\xi) \frac{-1}{5 \cosh 10} \cosh (5\xi) \sinh (5(x - 2)) d\xi \\
+ \int_x^2 f(\xi) \frac{-1}{5 \cosh 10} \sinh (5(\xi - 2)) \cosh (5x) d\xi \\
+ (5e^{10} \cosh 10)^{-1} e^{5x} + ((5e^{10} \cosh 10)^{-1} - \frac{2}{5}) e^{-5x}. \]

2. Consider the Poisson equation in a square,

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin 2x \sin 3y, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad (4) \]

\[ u(x, y) = 0 \quad \text{on the boundary.} \quad (5) \]

(a) Reduce the problem to inhomogeneous ODEs. (b) Solve the ODEs. (c) Present the solution \(u(x, y)\) is a series form.

**Solution.** (a) Recall that in the homogeneous case of this equation, separation of variables gives us the eigenvalue problem

\[ X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0 \]
which has solutions $X_n = \sin nx$. This motivates the idea to look for a solution $u$ in the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin nx,$$

where $b_n(y) = \frac{2}{\pi} \int_0^\pi u(x, y) \sin nx \, dx$. We now take the sine transform of the whole equation.

$$\frac{2}{\pi} \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx \, dx + \frac{2}{\pi} \int_0^\pi \frac{\partial^2 u}{\partial y^2} \sin nx \, dx = \frac{2}{\pi} \int_0^\pi -\sin 2x \sin 3y \sin nx \, dx$$

$$\Rightarrow -n^2 b_n(y) + b''_n(y) = -F_n(y).$$

The first term comes from integrating by parts, and the second term from switching the derivative with the integral (which we can do because the derivative is on the $y$ variable, but the integration is in $x$). We can also calculate the right hand side.

$$F_n(y) = \frac{2}{\pi} \int_0^\pi 2 \sin 2x \sin 3y \sin nx \, dx = \sin 3y \frac{2}{\pi} \int_0^\pi 2 \sin 2x \sin nx \, dx = \begin{cases} \sin 3y, & \text{if } n = 2, \\ 0, & \text{if } n \neq 2. \end{cases}$$

(The easy way to do that integral is to recognize that $\frac{2}{\pi} \int_0^\pi \sin 2x \sin nx \, dx$ are the sine series coefficients of the function $\sin 2x$.)

As for boundary conditions, $b_n(0) = \frac{2}{\pi} \int_0^\pi u(x, 0) \sin nx \, dx = 0$, and similarly $b_n(\pi) = 0$. Thus our problem reduces to the following system of ODEs:

$$\begin{cases} b''_2(y) - 4b_2(y) = -\sin 3y, & 0 < x < \pi, \ 0 < y < \pi, \\ b_2(0) = b_2(\pi) = 0; \\ b''_n(y) - n^2 b_n(y) = 0, \\ b_n(0) = b_n(\pi) = 0, & n = 1, 3, 4, \cdots. \end{cases}$$

(b) When $n \neq 2$, clearly $b_n(y) \equiv 0$. To solve the equation for $n = 2$, we find the Green’s function in the same manner as in problem 1. We get

$$G(y, \eta) = \begin{cases} \frac{1}{2 \sinh 2\pi} \sinh 2(\pi - \eta) \sinh 2y, & \text{for } y < \eta, \\ \frac{1}{2 \sinh 2\pi} \sinh 2\eta \sinh 2(\pi - y), & \text{for } y > \eta. \end{cases}$$
Then
\[ b_2(y) = \int_0^\pi G(y, \eta) \sin 3\eta d\eta = \frac{1}{2 \sinh 2\pi} \left[ \sinh 2(\pi - y) \int_0^y \sinh 2\eta \sin 3\eta d\eta \right. \\
\left. + \sinh 2y \int_y^\pi \sinh 2(\pi - \eta) \sin 3\eta d\eta \right]. \]

(c) Plugging the solutions of our ODEs into (6), we have
\[ u(x, y) = b_2(y) \sin 2x \]
\[ = \frac{1}{2 \sinh 2\pi} \left[ \sinh 2(\pi - y) \int_0^y \sinh 2\eta \sin 3\eta d\eta \right. \\
\left. + \sinh 2y \int_y^\pi \sinh 2(\pi - \eta) \sin 3\eta d\eta \right]. \]

3. Find the Green’s function for the semicircle, \(0 < r < R, 0 < \theta < \pi\).

\textit{Solution.} What we are looking for is a function \(G(r, \theta; \rho, \phi)\) such that
\[ u(r, \theta) = \int_0^\pi \int_0^R G(r, \theta; \rho, \phi) F(\rho, \phi) \rho d\rho d\phi \]
is a solution to Poisson’s equation on the semicircle:
\[ \begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -F(r, \theta), & \text{for } 0 < r < R, 0 < \theta < \pi, \\ u(R, \theta) = u(r, 0) = u(r, \pi) = 0. \end{cases} \]

Separation of variables on the homogeneous problem gives the eigenvalue problem
\[ \Theta'' + \lambda \Theta = 0, \quad \Theta(0) = \Theta(\pi) = 0 \Rightarrow \Theta_n = \sin n\theta \]
so we take a sine transform of the equation. Our solution therefore is
\[ u(r, \theta) = \sum_{n=1}^\infty b_n(r) \sin n\theta, \]
where $b_n(r) = \frac{2}{\pi} \int_0^\pi u(r, \theta) \sin n \theta d\theta$. After the sine transform, our PDE becomes a system of ODEs:

$$b_n''(r) + \frac{1}{r} b_n'(r) - \frac{n^2}{r^2} b_n(r) = -B_n(r) := -\frac{2}{\pi} \int_0^\pi F(r, \phi) \sin n \phi d\phi$$

These equations can be solved in the manner of problem 1, and their solutions are

$$b_n(r) = \int_0^R G_n(r, \rho) B_n(\rho) \rho d\rho,$$

where

$$G_n(r, \rho) = \begin{cases} \frac{1}{n} \left[ \left( \frac{R}{r} \right)^n - \left( \frac{r}{R} \right)^n \right] \left( \frac{\rho}{R} \right)^n, & \text{for } \rho < r, \\ \frac{1}{n} \left( \frac{r}{R} \right)^n \left[ \left( \frac{R}{\rho} \right)^n - \left( \frac{\rho}{R} \right)^n \right], & \text{for } \rho > r. \end{cases}$$

Plugging our formula for $b_n$ into the formula for $u$, we have

$$u(r, \theta) = \sum_{n=1}^\infty b_n(r) \sin n \theta = \sum_{n=1}^\infty \int_0^R G_n(r, \rho) B_n(\rho) \rho d\rho \sin n \theta$$

Now plug in the definition of $B_n$,

$$= \sum_{n=1}^\infty \int_0^R G_n(r, \rho) \left[ -\frac{2}{\pi} \int_0^\pi F(r, \phi) \sin n \phi d\phi \right] \rho d\rho \sin n \theta$$

and interchange the integrals and the sum,

$$= \int_0^\pi \int_0^R \left[ -\frac{2}{\pi} \sum_{n=1}^\infty G_n(r, \rho) \sin n \phi \sin n \theta \right] F(r, \phi) \rho d\rho d\phi$$

so the Green’s function for the semicircle is

$$G(r, \theta; \rho, \phi) = \frac{2}{\pi} \sum_{n=1}^\infty G_n(r, \rho) \sin n \phi \sin n \theta,$$

where $G_n$ is as given above.
4. Find the double Fourier series for \( f(x, y) = x^2 \sin y \) for \( 0 < x, y < \pi \).

**Solution.** First take the Fourier series of \( f \) in the \( y \) variable.

\[
f(x, y) = x^2 \sin y = \frac{a_0(x)}{2} + \sum_{m=0}^{\infty} \left( a_m(x) \cos my + b_m(x) \sin my \right)
\]

By inspection, we see that \( a_1(x) = x^2 \), and all of the rest of the coefficients are 0. Next take the Fourier series of these coefficients in the \( x \) variable.

\[
a_1(x) = x^2 = \frac{c_{01}}{2} + \sum_{n=1}^{\infty} \left( c_{n1} \cos nx + d_{n1} \sin nx \right)
\]

Solving this in the usual way, we get

\[
a_1(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^2}{n^2} \cos nx.
\]

(Notice that all of the sine terms are 0 since \( x^2 \) is even on the interval \([-\pi, \pi]\).) Now plugging this back into our expression for \( f \), we have our answer.

\[
f(x, y) = x^2 \sin y = \frac{\pi^2}{3} \sin y + 4 \sum_{n=1}^{\infty} \frac{(-1)^2}{n^2} \cos nx \sin y.
\]

5. Solve the problem for the vibrating square membrane,

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad 0 < x, y < \pi, \quad t > 0, \tag{7}
\]

\[
u = 0 \quad \text{on the boundary}, \tag{8}
\]

\[
u(x, y, 0) = \sin 2x \sin 3y, \tag{9}
\]

\[
\frac{\partial u(x, y, 0)}{\partial t} = \sin 7x \sin 8y. \tag{10}
\]

**Solution.** Since we are back to working on a homogeneous equation, we approach this again with separation of variables. Using standard methods, we are left with three ODEs:

\[
T'' - \lambda_1 T = 0 \quad \text{(no boundary conditions)}
\]

\[
X'' - \lambda_2 X = 0, \quad X(0) = X(\pi) = 0
\]

\[
Y'' - \left( \frac{\lambda_1}{c^2} - \lambda_2 \right) Y = 0, \quad Y(0) = Y(\pi) = 0
\]

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The equations in $X$ and $Y$ are our eigenvalue problems, with solutions $X_n = \sin nx, \lambda_2 = -n^2$, and $Y_m = \sin my, \lambda_1/c^2 - \lambda_2 = -m^2$. Then $\lambda_1 = -c^2(n^2 + m^2)$, so our $T$ equation is

$$T''_{nm} + c^2(n^2 + m^2)T = 0$$

and has the solution

$$T_{nm} = \alpha_{nm} \cos (c\sqrt{n^2 + m^2}) + \beta_{nm} \sin (c\sqrt{n^2 + m^2}) t.$$ 

So our solution has the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [\alpha_{nm} \cos (c\sqrt{n^2 + m^2}) + \beta_{nm} \sin (c\sqrt{n^2 + m^2}) t] \sin nx \sin my$$

(Note that at this point we usually only have one constant to solve for. Here we have two because we have two non-zero boundary conditions, but it will work out fine.) Plugging in the first boundary condition on the $t$ variable, we get

$$u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} \sin nx \sin my = \sin 2x \sin 3y.$$ 

Then by inspection, $\alpha_{23} = 1$ and $\alpha_{nm} = 0$ for $(n, m) \neq (2, 3)$. Now we plug in the other boundary condition on $t$.

$$u_t(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{nm} (c\sqrt{n^2 + m^2}) \sin nx \sin my = \sin 7x \sin 8y$$

Again by inspection, $\beta_{78} = 1/(c\sqrt{7^2 + 8^2})$ and $\beta_{nm} = 0$ for $(n, m) \neq (7, 8)$. Plugging in the $\alpha$'s and $\beta$'s in, we have

$$u(x, y, t) = \cos (c\sqrt{2^2 + 3^2})t \sin 2x \sin 3y + \frac{1}{(c\sqrt{7^2 + 8^2})} \cos (c\sqrt{7^2 + 8^2})t \sin 7x \sin 8y.$$ 

6. Solve the heat conduction problem,

$$\frac{\partial u}{\partial t} - (1 + x)^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0, \quad (11)$$

$$u(0, t) = u(1, t) = 0, \quad (12)$$

$$u(x, 0) = f(x). \quad (13)$$
**Solution:** Separation of variables gives us

\[ \frac{T'}{T} = (1 + x)^2 \frac{X''}{X} = -\lambda, \]

so our eigenvalue problem is

\[(1 + x)^2 X'' + \lambda X = 0, \quad X(0) = X(1) = 0.\]

The solutions to this equation have the form \(X(x) = (1+x)^\alpha\), and by plugging that solution into the equation, we see that \(\alpha\) must satisfy \(\alpha(\alpha-1) + \lambda = 0\), or \(\alpha = (1 \pm \sqrt{1-4\lambda})/2\). So

\[ X(x) = c_1(1 + x)^{(1 + \sqrt{1-4\lambda})/2} + c_2(1 + x)^{(1 - \sqrt{1-4\lambda})/2}. \]

The boundary conditions give us \(c_1 = -c_2\), and \(2\sqrt{1-4\lambda} = 1\). From the discussion in the book (see pages 169-170) and in lecture, our eigenvalues are

\[ \lambda_n = \frac{n^2\pi^2}{(\log 2)^2} + \frac{1}{4}, \quad n = 1, 2, 3, \ldots \]

and the corresponding eigenfunctions are

\[ X_n = (1 + x)^{1/2} \sin \left( n\pi \frac{\log (1 + x)}{\log 2} \right). \]

Going back to the equation for \(T\), we have solutions

\[ T_n(t) = e^{-\lambda_n t}, \]

so our solution \(u\) looks like

\[ u(x, t) = \sum_{n=1}^{\infty} (1 + x)^{1/2} \sin \left( n\pi \frac{\log (1 + x)}{\log 2} \right) c_n \exp \left\{ -\sqrt{\frac{n^2\pi^2}{(\log 2)^2} + \frac{1}{4}} t \right\} \]

The coefficients \(c_n\) are found as follows:

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} (1 + x)^{1/2} \sin \left( n\pi \frac{\log (1 + x)}{\log 2} \right), \]
therefore $c_n$ are the coefficients in the expansion of the function $f(x)$ in terms of the functions $X_n(x)$:

$$c_n = \frac{\int_0^1 f(x)X_n(x)\,dx}{\int_0^1 X_n^2(x)\,dx}.$$  

7. Consider the eigenvalue problem,

$$u'' - au + \lambda u = 0, \quad u(\alpha) = 0, \quad u'(\beta) = 0.$$  

Find eigenvalues and eigenfunctions.

**Solution:**

Case 1: $a - \lambda > 0$. Then the equation has solution

$$u(t) = c_1 e^{(a-\lambda)(t-\alpha)} + c_2 e^{(\lambda-a)(t-\alpha)}.$$  

$u(\alpha) = 0$ implies $c_1 = -c_2$, so $u = c \sinh t - \alpha$. Then

$$u'(\beta) = c \cosh (\beta - \alpha) = 0.$$  

But $\cosh x$ is never 0, thus $u \equiv 0$ (so this case only gives us a trivial solution).

Case 2: $a - \lambda = 0$. The equation becomes $u'' = 0$, hence $u$ is linear, and so the boundary conditions again imply that $u$ is trivial.

Case 3: $a - \lambda < 0$. Now our equation has solution

$$u(t) = c_1 \cos \left( \sqrt{\lambda - a}(t - \alpha) \right) + c_2 \sin \left( \sqrt{\lambda - a}(t - \alpha) \right).$$  

$u(\alpha) = 0$ implies $c_1 = 0$, and $u'(\beta) = 0$ implies

$$\sqrt{\lambda - a}(\beta - \alpha) = n\pi \Rightarrow \lambda_n = \frac{n^2 \pi^2}{(\beta - \alpha)^2} + a, \quad n = 1, 2, \cdots$$  

and the corresponding eigenfunctions are

$$u_n(t) = \sin \left( \frac{n\pi}{\beta - \alpha}(t - \alpha) \right).$$